Hawaiian earring

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Hawaiian earring is the topological space $A \subseteq \mathbb{R}^2$ with infinite circles:

$$A := \bigcup_{n=1}^{\infty} \left\{ (x,y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\} = \bigcup_{n=1}^{\infty} \partial B_{(\frac{1}{n},0)}\left(\frac{1}{n}\right)$$

equipped with the subspace topology from $\mathbb{R}^2$. It has strange structures so that it is not homotopy equivalent to the wedge sum of countably infinite many circles:

$$B := \bigvee_{n=1}^{\infty} S^1$$

equipped with the topology where a subset is open if and only if its intersection with each S^1 is open. We will use fundamental groups to show this.

Proposition 1. *We can construct an explicit continuous bijection $f : B \to A$.*

Proof. Denote circles in B to be $S_n^1 (n \geqslant 1)$, define $f(x_0) = (0,0)$ and f maps each S_n^1 identically to $\partial B_{(\frac{1}{n},0)}\left(\frac{1}{n}\right)$, then f is obviously bijective. We show that f is continuous. If $(0,0) \neq (x_0, y_0) \in \partial B_{(\frac{1}{k},0)}\left(\frac{1}{k}\right)$ for some k, then (x_0, y_0) has a open neighborhood U in A such that $U \subseteq \partial B_{(\frac{1}{k},0)}\left(\frac{1}{k}\right)$. Thus $f^{-1}(U)$ is just the corresponding arc in S_k^1, which is open in B. If $(x_0, y_0) = (0,0)$, then any open neighborhood V of $(0,0)$ contains entirely all but finitely many circles. Explicitly, there exist $m \in \mathbb{N}$, such that for all $n \geqslant m$, $\partial B_{(\frac{1}{m},0)}\left(\frac{1}{m}\right) \subseteq V$. For $n < m$,

$$f^{-1}(V) \cap S_n^1 = f^{-1}\left(V \cap \partial B_{(\frac{1}{n},0)}\left(\frac{1}{n}\right)\right)$$

which is open, and for $n \geqslant m$,

$$f^{-1}(V) \cap S_n^1 = S_n^1$$

Thus we have shown that the preimage of an open set in B is an open set in A, hence f is continuous. $\square$

But f is not a homeomorphism because f^{-1} is not continuous. For example, we take U to be an open set in B such that $U \cap S_n^1$ is a half circle containing x_0 for all n. Then by the discussion above we know that $f(U)$ contains $(0,0)$ but it is not open in A. Hence f^{-1} is not continuous.

Now we study the simpler space B.

Proposition 2. *Each loop in B can only go around finitely many circles*

Proof. Suppose there exists a continuous loop $f : I \to B$ that go around infinitely many circles. Consider the open set U as before, we prove that $f^{-1}(U)$ is not an open set in I.

Note that for each $t_\alpha \in f^{-1}(\{x_0\}) \subseteq f^{-1}(U)$, we can find an open interval $I_\alpha \subseteq f^{-1}(U)$ and these intervals to be pairwise disjoint. But $f^{-1}(\{x_0\})$ is an infinite set in $[0,1]$, hence there exists a limit point t_0. $t_0 \in f^{-1}(U)$ since f is continuous. Thus there exists an open interval I_0, such that $t_0 \in I_0, I_0 \subseteq f^{-1}(U)$. However, by the discussion above, I_0 and any I_α are disjoint, thus $t_\alpha \notin I_0$, which contradicts with the fact that t_0 is a limit point of $f^{-1}(\{x_0\})$. $\square$

Moreover, by the van Kampen Theorem, we directly get:

Theorem 3. *The fundamental group of B is the free product of countably many Z. That is to say,*

$$\pi_1(B) = *_\infty \mathbb{Z}$$

Proposition 4. *$\pi_1(B)$ has only countably many elements.*

Proof. We only need to study $G = *_\infty \mathbb{Z}$. Every element in G is reduced word of finite length. For each fixed length n, every position has countably many choices, hence cardinality of elements of length n is more than ω^n, which is just ω by "diagonal method". Thus we get a countable set by union of all lengths. $\square$

Until now, we understand the structure of B. Let's turn to the Hawaiian earring A.

Proposition 5. *Loops in A can wind around infinitely many circles.*

Proof. We construct a map $f : I \to A$ as follows. Between time $[1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$, we exactly go around the circle $\partial B_{(\frac{1}{n},0)}(\frac{1}{n})$ once, and we let $f(1) = (0, 0)$. It's easy to check this map is continuous. $\square$

Since each circle in A gives a $\pi_1(S^1) \cong \mathbb{Z}$ summand of $\pi_1(A)$, use the same method as the last proof, we immediately get:

Proposition 6. *There is a surjection $\pi_1(A) \to \prod_\infty \mathbb{Z}$.*

Then we can know:

Proposition 7. *$\pi_1(A)$ has uncountably many elements.*

Proof. We only need to study $G = \prod_\infty \mathbb{Z}$. The cardinality of $|G|$ is ω^ω. Note that

$$\mathbb{R} = 2^\omega \leqslant \omega^\omega \leqslant (2^\omega)^\omega = 2^{\omega^2} = 2^\omega = \mathbb{R}$$

Thus G has uncountably many elements. $\square$

Until know, we have shown that the fundamental groups of A and B are distinct. Hence:

Theorem 8. *The Hawaiian earring A and the wedge of infinite circles B are not homotopy equivalent.*

Now we sketch a combinatorial construction of $\pi_1(A)$. Let g_n be the free group generated by $g_1, \cdots, g_n$ corresponding to the biggest n circles and $p_{n+1} : F_{n+1} \to F_n$ identifies the letter g_{n+1} to the identity element. Then we know that $\pi_1(A)$ is a subgroup of the inverse limit of such $\{F_n, p_n\}$, i.e.,

$$\varprojlim(\cdots \to F_{n+1} \to F_n \to \cdots \to F_2 \to F_1)$$

To describe $\pi_1(A)$ more precisely, we make the following definition.

Definition 9. *Suppose $g_{k_1}^{\varepsilon_1} g_{k_2}^{\varepsilon_2} \cdots g_{k_m}^{\varepsilon_m} \in F_n$ is a reduced word in $g_1, \cdots, g_n$. The **k-weight of** w is*

$$\#_k(w) = \sum_{k_m = k} |\varepsilon_{k_m}|$$

If $(w_n) \in \varprojlim F_n$, then the sequence $\#_k(w_n)$ for fixed k is non-decreasing since the projections p_n only delete letters. Let

$$\#_\mathbb{N}\mathbb{Z} = \{(w_n) \in \varprojlim F_n | \lim_{n\to\infty} \#_k(w_n) < \infty, \forall k\}$$

Then by the discussion as in proposition 1, we get

Theorem 10. *$\pi_1(A) \cong \#_\mathbb{N}\mathbb{Z}$.*