Symplectic Geometry

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1 Symplectic Forms

Definition 1.1. *Let V be a vector space over $\mathbb{R}$, a bilinear form $\Omega : V \times V \to \mathbb{R}$ is called **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$ for all $u, v \in V$.*

Theorem 1.2 (Standard Form)**.** *Let Ω be a skew-symmetric bilinear map on V. Then there exists a basis $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots f_n$ of V such that*

$$\begin{aligned}\Omega(u_i, v) &= 0, \ \forall i, v \\ \Omega(e_i, e_j) &= 0 = \Omega(f_i, f_j), \ \forall i, j \\ \Omega(e_i, f_j) &= \delta_{ij}, \ \forall i, j\end{aligned}$$

Definition 1.3. *n is an invariant of (V, Ω) and $2n$ is called the **rank** of Ω.*

Definition 1.4. *The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by $\Omega(v)(u) = \Omega(v, u)$. Let U be the kernel of $\widetilde{\Omega}$.*

Definition 1.5. *A skew-symmetric bilinear map Ω is **symplectic**(or **nondegenerate**) if $\widetilde{\Omega}$ is bijective.Then (V, Ω) is called a **symplectic vector space**.*

We can see that $\dim V = 2n$ is even.

Definition 1.6. *A subspace W is called **symplectic** if $\Omega|_W$ is nondegenerate. W is called **isotropic** if $\Omega|_W = 0$.*

Definition 1.7. *A **symplectomorphism** φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\cong} V'$ such that $\varphi^*\Omega' = \Omega$. If a symplectomorphism exists, (V, Ω) and (V', Ω') are said to be **symplectomorphic**.*

By theorem 1.2, every $2n-$dimensional symplectic vector space (V, Ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n}, \Omega_0)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

Definition 1.8. *A 2-form ω on a manifold M is called **symplectic** if ω is closed and ω_p is symplectic for all $p \in M$.*

If ω is symplectic, then $\dim T_pM = \dim M$ is even.

Definition 1.9. *A **symplectic manifold** is a pair (M, ω) where M is a manifold and ω is a symplectic form.*

Example 1.10. *Let $M = \mathbb{R}^{2n}$ with coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$, the form*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic.

Example 1.11. *Let $M = \mathbb{C}^n$ with coordinates $z_1, \cdots, z_n$, the form*

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic.

Example 1.12. *Let $M = S^2$. Tangent vectors to S^2 at p can be identified with vectors orthogonal to p. The standard symplectic form on S^2 is induced by the inner and exterior products:*

$$\omega_p(u, v) := \langle p, u \times v \rangle, \ u, v \in T_pS^2 = \{p\}^\perp$$

This form is closed because it is of top degree and it is nondegenerate.

Example 1.13. *Smooth semi-projective variety over $\mathbb{C}$.*

Definition 1.14. *Let (M_1, ω_1) and (M_2, ω_2) be $2n-$dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a **symplectomorphism** if $\varphi^*\omega_2 = \omega_1$.*

Theorem 1.15 (Darboux)**.** *Let (M, ω) be a $2n-$dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(\mathcal{U}, x_1, \cdots, x_n, y_1, \cdots, y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

*Such a chart is called a **Darboux chart**.*

2 Symplectic Form on the Cotangent Bundle

Let X be any $n-$dimensional manifold. If the structure on X is described by coordinate charts $(\mathcal{U}, x_1, \cdots, x_n)$ and $\xi \in T_x^*X$, then $\xi = \sum\limits_{i=1}^{n} \xi_i (dx_i)_x$, this induces

$$\begin{aligned} T^*\mathcal{U} &\longrightarrow \mathbb{R}^{2n} \\ (x, \xi) &\longmapsto (x_1, \cdots, x_n, \xi_1, \cdots, \xi_n) \end{aligned}$$

The coordinates $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$ are the **cotangent coordinates** associated to the coordinates $x_1, \cdots, x_n$.

Define a 2-form ω on $T^*\mathcal{U}$ by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

We need to check that this definition is coordinate-independent. First consider the 1-form on $T^*\mathcal{U}$

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i$$

then $\omega = -d\alpha$.

Claim 2.1. *α is intrinsically define(and hence ω is also intrinsically defined).*

Proof. Let $(\mathcal{U}, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ and $(\mathcal{U}', x_1', \cdots, x_n', \xi_1', \cdots, \xi_n')$ be two cotangent coordinate charts. On $\mathcal{U} \cap \mathcal{U}'$, we have $\xi_j' = \sum\limits_{i=1}^{n} \xi_i \frac{\partial x_i}{x_j'}$ and $dx_i = \sum\limits_{j=1}^{n} \frac{\partial x_i}{\partial x_j'} dx_j'$, hence

$$\alpha = \sum_{i=1}^{n} \xi_i dx_i = \sum_{j=1}^{n} \xi_j' dx_j' = \alpha'$$

$\square$

Definition 2.2. *The 1-form α is called the **tautological form** or **Liouville 1-form** and the 2-form ω is called the **canonical symplectic form**.*

Now let's give the tautological form a coordinate-free definition. Let

$$\begin{aligned} \pi : M = T^*X &\longrightarrow X \\ p = (x, \xi) &\longmapsto x \end{aligned}$$

be the natural projection. The tautological 1-form α can be defined pointwise by

$$\alpha_p = (d\pi_p)^* \xi$$

Equivalently,

$$\alpha_p(v) = \xi(d\pi_p(v)),\ v \in T_pM$$

Claim 2.3. *Let $(\mathcal{U}, x_1, \cdots, x_n)$ be a chart on X with associated cotangent coordinates $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$. Then on $T^*\mathcal{U}$, $\alpha = \sum\limits_{i=1}^{n} \xi_i dx_i$.*

The **canonical symplectic 2-form** ω on T^*X is defined as $\omega = -d\alpha$.

Let X_1 and X_2 be $n-$dimensional manifolds with cotangent bundles $M_1 = T^*X_1$ and $M_2 = T^*X_2$, and tautological 1-form α_1 and α_2. Suppose $f : X_1 \to X_2$ is a diffeomorphism. Then there is a natural diffeomorphism

$$f_\# : M_1 \to M_2$$

which lifts f: if $p_1 = (x_1, \xi_1) \in M_1$, we define

$$f_\#(p_1) = p_2 = (x_2, \xi_2) \text{ with } x_2 = f(x_1) \text{ and } \xi_1 = (df_{x_1})^* \xi_2$$

Proposition 2.4. *The lift $f_\#$ of a diffeomorphism $f : X_1 \to X_2$ pulls the tautological form on T^*X_2 back to the tautological form on T^*X_1.*

Proof. Need to prove

$$(df_{\#})^*_{p_1}(\alpha_2)_{p_2} = (\alpha_1)_{p_1}$$

where $p_2 = f_{\#}(p_1)$. We have a commutative diagram

$$\begin{array}{ccc} T^*X_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow{\scriptstyle f_{\#}} & & \downarrow{\scriptstyle f} \\ T^*X_2 & \xrightarrow{\pi_2} & X_2 \end{array}$$

Hence

$$(df_{\#})^*_{p_1}(\alpha_2)_{p_2} = (df_{\#})^*_{p_1}(d\pi_2)^*_{p_2}(\xi_2) = (d(\pi_2 \circ f_{\#}))^*_{p_1}(\xi_2) = (d(f \circ \pi_1)^*_{p_1}(\xi_2) = (d\pi_1)^*_{p_1}(df_{x_1})^*(\xi_2) = (\alpha_1)_{p_1}$$

□

Corollary 2.5. *The lift $f_{\#}$ of a diffeomorphism $f : X_1 \to X_2$ is a symplectomorphism.*

In summary, a diffeomorphism of manifolds induces a canonical symplectomorphism of cotangent bundles.

Example 2.6. *Let $X_1 = X_2 = S^1$, then T^*S^1 is an infinite cylinder $S^1 \times \mathbb{R}$. The canonical 2-form ω is the area form $\omega = d\theta \wedge d\xi$. If $f : S^1 \to S^1$ is any diffeomorphism, then $f_{\#} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is a symplectomorphism, that is an area-preserving diffeomorphism of the cylinder.*

In terms of the group $\mathrm{Diff}(X)$ of diffeomorphisms of X and the group $\mathrm{Sympl}(M,\omega)$ of symplectomorphisms of (M,ω), the map

$$\begin{aligned} \mathrm{Diff}(X) &\longrightarrow \mathrm{Sympl}(M,\omega) \\ f &\longmapsto f_{\#} \end{aligned}$$

is a group homomorphism.

Exercise 2.7. *Show that if $n > 1$ there are no symplectic structures on the sphere S^{2n}.*

Proof. We use the following steps:

(1) Let (M,ω) be a compact $2n-$dimensional symplectic manifold, and ω^n be the volume form. Then the de Rham cohomology class $[\omega^n] \in H^{2n}(M;\mathbb{R})$ is non-zero.(Stokes' theorem)

(2) $[\omega]$ itself is non-zero.

(3) $H^2(S^{2n};\mathbb{R}) = 0$ when $n > 1$.

□

3 Lagrangian Submanifolds

Definition 3.1. *A **closed embedding** is a proper injective immersion.*

Example 3.2. *A map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.*

Definition 3.3. *A **submanifold** of M is a manifold X with a closed embedding $i : X \hookrightarrow M$.*

Definition 3.4. *Let (M,ω) be a $2n-$dimensional symplectic manifold. A submanifold Y of M is a **Lagrangian submanifold** if, at each $p \in Y$, T_pY is a Lagrangian subspace of T_pM, that is $\omega_p|_{T_pY} = 0$ and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is **Lagrangian** if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.*

The **zero section** of T^*X is an $n-$dimensional submanifold of T^*X. Clearly $\alpha = \xi_i dx_i$ vanishes on $X_0 \cap T^*U$. In particular, if $i_0 : X_0 \to T^*X$ is the inclusion map, we have $i_0^*\alpha = 0$. Hence $i_0^*\omega = i_0^*d\alpha = 0$, and X_0 is Lagrangian.

Now consider another section

$$X_\mu = \{(x,\mu_x) | x \in X, \mu_x \in T_x^*X\}$$

where the covector μ_x depends smoothly on x, and $\mu : X \to T^*X$ is a 1-form.

Proposition 3.5. *Denote by $s_\mu : X \to T^*X, x \mapsto (x, \mu_x)$, the 1-form μ regarded exclusively as a map. Let α be the tautological 1-form on T^*X. Then*

$$s_\mu^*\alpha = \mu$$

Proof. By definition, we have

$$(s_\mu^*\alpha)_x = (ds_\mu)_x^*(d\pi_p)^*\mu_x = d(\pi \circ s_\mu)_x^*\mu_x = \mu_x$$

□

Suppose that X_μ is an $n-$dimensional submanifold of T^*X with associated 1-form μ. Then $s_\mu : X \to T^*X$ is an embedding with image X_μ, and there is a diffeomorphism $\tau : X \to X_\mu, \tau(x) := (x, \mu_x)$, such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "s_\mu"] \arrow[dr, "\tau"', "\cong"] & & T^*X \\ & X_\mu \arrow[ur, "i"'] & \end{tikzcd}$$

Hence X_μ is Lagrangian if and only if

$$i^*d\alpha = 0 \Longleftrightarrow \tau^*i^*d\alpha = 0 \Longleftrightarrow (i\circ\tau)^*d\alpha = 0 \Longleftrightarrow s_\mu^*d\alpha = 0 \Longleftrightarrow ds_\mu^*\alpha = 0 \Longleftrightarrow d\mu = 0 \Longleftrightarrow \mu \text{ is closed}$$

Therefore,there is a one-to-one correspondence between the set of Lagrangian submanifolds of T^*X of the form X_μ and the set of closed 1-forms on X.

When X is simply connected, $H^1_{dR}(X) = 0$, so every closed 1-form μ is of the form df for some $f \in C^\infty(X)$. Any such primitive f is called a **generating function** for the Lagrangian submanifold X_μ associated to μ.

Definition 3.6. *Let S be any $k-$dimensional submanifold of an $n-$dimensional manifold X. The* ***conormal space*** *at $x \in S$ is*

$$N_x^*S = \{\xi \in T_x^*X | \xi(v) = 0 \; for \; all \; v \in T_xS\}$$

The ***conormal bundle*** *of S is*

$$N^*S = \{(x,\xi) \in T^*X | x \in S, \xi \in N_x^*S\}$$

Proposition 3.7. *Let $i : N^*S \hookrightarrow T^*X$ be the inclusion, and let α be the tautological 1-form on T^*X. Then*

$$i^*\alpha = 0$$

Proof. Let $(\mathcal{U}, x_1, \cdots, x_n)$ be a coordinate system on X centered at $x \in S$ and adapted to S, so that $\mathcal{U} \cap S$ is described by $x_{k+1} = \cdots = x_n = 0$. Let $(T^*\mathcal{U}, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ be the associated cotangent coordinate system. The submanifold $N^*S \cap T^*\mathcal{U}$ is then described by

$$x_{k+1} = \cdots = x_n = 0, \; \xi_1 = \cdots \xi_k = 0$$

Since $\alpha = \sum \xi_i dx_i$ on $T^*\mathcal{U}$, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = 0$$

□

Corollary 3.8. *For any submanifold $S \subset X$, the conormal bundle N^*S is a Lagrangian submanifold of T^*X.*

Now consider two $2n-$dimensional symplectic manifold. Given a diffeomorphism $\varphi : M_1 \xrightarrow{\cong} M_2$, when is it a symplectomorphism?

Consider projection maps $\pi_{1,2} : M_1 \times M_2 \to M_1, M_2$, then $(\pi_1)^*\omega_1 + (\pi_2)^*\omega_2$ is a 2-form on $M_1 \times M_2$ which is closed:

$$d\omega = (\pi_1)^*d\omega_1 + (\pi_2)^*d\omega_2 = 0$$

and symplectic:

$$\omega^{2n} = \binom{2n}{n}\Big((\pi_1)^*\omega_1\Big)^n \wedge \Big((\pi_2)^*\omega_2\Big)^n \neq 0$$

More generally, if $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$, then $\lambda_1(\pi_1)^*\omega_1 + \lambda_2(\pi_2)^*\omega_2$ is also a symplectic form on $M_1 \times M_2$. Take $\lambda_1 = 1, \lambda_2 = -1$ to obtain the **twisted product form** on $M_1 \times M_2$:

$$\widetilde{\omega} = (\pi_1)^*\omega_1 - (\pi_2)^*\omega_2$$

The graph Γ_φ of a diffeomorphism φ is an embedded image of M_1 in $M_1 \times M_2$, the embedding being the map

$$\begin{aligned} \gamma : M_1 &\longrightarrow M_1 \times M_2 \\ p &\longmapsto (p, \varphi(p)) \end{aligned}$$

Proposition 3.9. *A diffeomorphism φ is a symplectomorphism if and only if Γ_φ is a Lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.*

Proof. Γ_φ is Lagrangian if and only if $\gamma^*\widetilde{\omega} = 0$. But

$$\gamma^*\widetilde{\omega} = \gamma^*(\pi_1)^*\omega_1 - \gamma^*(\pi_2)^*\omega_2 = (\pi_1 \circ \gamma)^*\omega_1 - (\pi_2 \circ \gamma)^*\omega_2$$

where $\pi_1 \circ \gamma$ is the identity map on M_1 and $\pi_2 \circ \gamma = \varphi$. Therefore

$$\gamma^*\widetilde{\omega} = 0 \Longleftrightarrow \varphi^*\omega_2 = \omega_1$$

□

4 Generating Functions

Let X_1, X_2 be $n-$dimensional manifolds, with cotangent bundles $M_1 = T^*X_1, M_2 = T^*X_2$, tautological 1-forms α_1, α_2, and canonical 2-forms ω_1, ω_2.

Under the natural identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 \cong T^*(X_1 \times X_2)$$

the tautological 1-from on $T^*(X_1 \times X_2)$ is

$$\alpha = (\pi_1)^*\alpha_1 + (\pi_2)^*\alpha_2$$

The canonical 2-from on $T^*(X_1 \times X_2)$ is

$$\omega = -d\alpha = (\pi_1)^*\omega_1 + (\pi_2)^*\omega_2$$

In order to describe the twisted form $\widetilde{\omega} = (\pi_1)^*\omega_1 - (\pi_2)^*\omega_2$, ew define an involution of $M_2 = T^*X_2$ by

$$\begin{aligned} \sigma_2 : M_2 &\longrightarrow M_2 \\ (x_2, \xi_2) &\longmapsto (x_2, -\xi_2) \end{aligned}$$

which yields $\sigma_2^*\alpha_2 = -\alpha_2$. Let $\sigma = \mathrm{id}_{M_1} \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$. Then

$$\sigma^*\widetilde{\omega} = \pi_1^*\omega_1 + \pi_2^*\omega_2 = \omega$$

If Y is a Lagrangian submanifold of $(M_1 \times M_2, \omega)$, then its **twist** $Y^\sigma := \sigma(Y)$ is a Lagrangian submanifold of $(M_1 \times M_2, \widetilde{\omega})$.

Recipe for producing symplectomorphisms $M_1 = T^*X_1 \to M_2 = T^*X_2$:

(1) Start with a Lagrangian submanifold Y of $(M_1 \times M_2, \omega)$.

(2) Twist it to obtain a Lagrangian submanifold Y^σ of $(M_1 \times M_2, \widetilde{\omega})$.

(3) Check whether Y^σ is the graph of some diffeomorphism $\varphi : M_1 \to M_2$.

(4) If it is, then φ is a symplectomorphism by proposition 3.9.

Let $i : Y^\sigma \hookrightarrow M_1 \times M_2$ be the inclusion map, Step 3 amounts to checking whether $\pi_1 \circ i$ and $\pi_2 \circ i$ are diffeormorphisms. If yes, then $\varphi := (\pi_2 \circ i) \circ (\pi_1 \circ i)^{-1}$ is a diffeomorphism. To find Lagrangian submanifolds of $M_1 \times M_2$, we need the method of generating functions.

Definition 4.1. *For any $f \in C^\infty(X_1 \times X_2)$, the* ***Lagrangian submanifolds generated by*** *f is*

$$Y_f := \{((x, y), (df)_{(x,y)}) | (x, y) \in X_1 \times X_2\}$$

Denote

$$\begin{aligned} d_x f &:= (df)_{(x,y)} \text{ projected to } T_x^*X_1 \times \{0\} \\ d_y f &:= (df)_{(x,y)} \text{ projected to } \{0\} \times T_y^*X_2 \end{aligned}$$

then

$$Y_f = \{(x, y, d_x f, d_y f) | (x, y) \in X_1 \times X_2\}$$

and

$$Y_f^\sigma = \{(x, y, d_x f, -d_y f) | (x, y) \in X_1 \times X_2\}$$

When Y_f^σ is the graph of a diffeomorphism $\varphi : M_1 \to M_2$, we call φ the **symplectomorphism generated by** f, and call f the **generating function** of φ.

Now we use coordinates to calculate φ. Let $(\mathcal{U}_1, x_1, \cdots, x_n), (\mathcal{U}_2, y_1, \cdots, y_n)$ be coordinate charts for X_1, X_2 with associated charts $(T^*\mathcal{U}_1, x_1, \cdots, x_n, \xi_1, \cdots, \xi_n), (T^*\mathcal{U}_2, y_1, \cdots, y_n, \eta_1, \cdots \eta_n)$ for M_1, M_2. The set Y_f^σ is the graph of $\varphi : M_1 \to M_2$ if and only if $\forall (x, \xi) \in M_1$ and $(y, \eta) \in M_2$, we have

$$\varphi(x, \xi) = (y, \eta) \Longleftrightarrow \xi = d_x f \text{ and } \eta = -d_y f$$

Therefore, we must solve the **Hamilton equations**

$$\begin{cases} \xi_i = \frac{\partial f}{\partial x_i}(x, y) \\ \eta_i = -\frac{\partial f}{\partial y_i}(x, y) \end{cases}$$

In order to solve the first set of equations locally for y in terms of x and ξ, by the implicit function theorem, we need the condition

$$\det \Big[\frac{\partial}{\partial y_j}\Big(\frac{\partial f}{\partial x_i}\Big)\Big]_{i,j=1}^n \neq 0$$

This is a necessary condition for f to generate a symplectomorphism φ and locally this is also sufficient.

Example 4.2. *Let $X_1 = X_2 = \mathbb{R}^n$, and $f(x, y) = -\frac{|x-y|^2}{2}$, then the Hamilton equations are*

$$\begin{cases} \xi_i = \frac{\partial f}{\partial x_i} = y_i - x_i \\ \eta_i = -\frac{\partial f}{\partial y_i} = y_i - x_i \end{cases} \Longrightarrow \begin{cases} y_i = x_i + \xi_i \\ \eta_i = \xi_i \end{cases}$$

The symplectomorphism generated by f is

$$\varphi(x, \xi) = (x + \xi, \xi)$$

Suppose that (X, g) is a geodesically convex Riemannian manifold. Consider the function

$$f : X \times X \to \mathbb{R}, \ f(x, y) = -\frac{d(x, y)^2}{2}$$

What is the symplectomorphism $\varphi : T^*X \to T^*X$ generated by f?

The metric g induces an identification $\widetilde{g}_x : T_x X \xrightarrow{\cong} T_x^* X$. Use $\widetilde{g}$ to translate φ into a map $\widetilde{\varphi} : TX \to TX$, then we need to solve

$$\begin{cases} \widetilde{g}_x(v) = \xi_i = d_x f(x, y) \\ \widetilde{g}_y(w) = \eta_i = -d_y f(x, y) \end{cases}$$

for (y, η) in terms of (x, ξ) in order to find φ, or equivalently, for (y, w) in terms of (x, v) in order to find $\widetilde{\varphi}$.

Let γ be the geodesic with initial conditions $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then the symplectomorphism φ corresponds to the map

$$\begin{aligned} \widetilde{\varphi} : TX &\longrightarrow TX \\ (x, v) &\longmapsto (\gamma(1), \frac{d\gamma}{dt}(1)) \end{aligned}$$

called the **geodesic flow** on X.

5 Recurrence

Let X be an $n-$dimensional manifold and $M = T^*X$ with canonical symplectic form ω.

Suppose we have a smooth function $f : X \times X\mathbb{R}$ which generates a symplectomorphism $\varphi : M \to M$, $\varphi(x, d_x f) = (y, -d_y f)$. What are the fixed point of φ?

Define $\psi : X \to \mathbb{R}$ by $\psi(x) = f(x, x)$.

Proposition 5.1. *There is a one-to-one correspondence between the fixed points of φ and the critical points of ψ.*

Proof. At $x_0 \in X$, $d_{x_0}\psi = (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}$. Let $\xi = d_x f|_{(x,y)=(x_0,x_0)}$.

$$x_0 \text{ is a critical point of } \psi \Longleftrightarrow d_{x_0}\psi = 0 \Longleftrightarrow d_y f|_{(x,y)=(x_0,x_0)} = -\xi$$

Hence the point in Γ_f^σ corresponding to $(x,y) = (x_0,x_0)$ is (x_0,x_0,ξ,ξ). But Γ_f^σ is the graph of φ, so $\varphi(x_0,\xi) = (x_0,\xi)$ is a fixed point. $\square$

Consider the iterates of φ

$$\varphi^{(N)} = \varphi \circ \cdots \varphi : M \to M,\ N = 1,2,\cdots$$

each of which is a symplectomorphism of M. By the previous proposition, if $\varphi^{(N)}$ is generated by $f^{(N)}$, then there is a one-to-one correspondence between $\{$fixed points of $\varphi^{(N)}\}$ and $\{$critical points of $\psi^{(N)} = f^{(N)}(x,x) : X \to \mathbb{R}\}$.

Knowing that φ is generated by f, does $\varphi^{(2)}$ have a generating function?

Yes!

Fix $x, y \in X$. Define a map

$$\begin{aligned} X &\longrightarrow \mathbb{R} \\ z &\longmapsto f(x,z) + f(z,y) \end{aligned}$$

Suppose that this map has a unique critical point z_0, and that z_0 is nondegenerate. Let

$$f^{(2)}(x,y) := f(x,z_0) + f(z_0,y)$$

Proposition 5.2. *The function $f^{(2)} : X \times X \to \mathbb{R}$ is smooth and is a generating function for $\varphi^{(2)}$ if we assume that for each $\xi \in T_x^*X$, there is a unique $y \in X$ for which $d_x f^{(2)} = \xi$.*

Proof. z_0 is given by $d_y f(x,z_0) + d_x f(z_0,y) = 0$. The nondegeneracy condition is

$$\det\Big[\frac{\partial}{\partial z_i}\Big(\frac{\partial f}{\partial y_j}(x,z) + \frac{\partial f}{\partial x_j}(z,y)\Big)\Big] \neq 0$$

By the implicit function theorem, $z_0 = z_0(x,y)$ is smooth.

$f^{(2)}(x,y)$ is a generating function for $\varphi^{(2)}$ if and only if

$$\varphi^{(2)}(x, d_x f^{(2)}) = (y, -d_y f^{(2)})$$

Since φ is generated by f, and z_0 is critical, we obtain

$$\begin{aligned} &\varphi^{(2)}(x, d_x f^{(2)}(x,y)) = \varphi(\varphi(x, d_x f^{(2)}(x,y))) = \varphi(\varphi(x, d_x f(x,z_0))) = \varphi(z_0, -d_y f(x,z_0)) \\ =&\varphi(z_0, d_x f(z_0,y)) = (y, -d_y f(z_0,y)) = (y, -d_y f^{(2)}(x,y)) \end{aligned}$$

$\square$

Theorem 5.3 (Poincaré Recurrence Theorem)**.** *Suppose that $\varphi : A \to A$ is an area-preserving diffeomorphism of a finite-area manifold A. Let $p \in A$, and $\mathcal{U}$ be a neighborhood of p. Then there is $q \in \mathcal{U}$ and a positive integer N such that $\varphi^{(N)}(q) \in \mathcal{U}$.*

Theorem 5.4 (Poincaré-Birkhoff theorem)**.** *Suppose $\varphi : A \to A$ is an area-preserving diffeomorphism of the closed annulus $A = \mathbb{R}/\mathbb{Z} \times [-1,1]$ which preserves the two components of the boundary, and twists them in opposite directions. Then φ has at least two fixed points.*

6 Local forms

Theorem 6.1 (Darboux)**.** *Let (M,ω) be a symplectic manifold, and let p be any point in M. Then we can find coordinate system $(\mathcal{U}, x_1, \cdots, x_n, y_1, \cdots, y_n)$ centered at p such that on $\mathcal{U}$*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

Theorem 6.2 (Weinstein Lagrangian Neighborhood Theorem)**.** *Let M be a $2n-$dimensional manifold, X a compact Lagrangian submanifold of both (M,ω_0) and (M,ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that the following diagram commutes and $\varphi^*\omega_1 = \omega_0$.*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

Theorem 6.3 (Coisotropic Embedding Theorem). *Let M be a manifold of dimension $2n$, X a submanifold of dimension $k \geqslant n$, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M such that $i^*\omega_0 = i^*\omega_1$ and X is coisotropic for both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that the following diagram commutes and $\varphi^*\omega_1 = \omega_0$.*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

Theorem 6.4 (Weinstein Tubular Neighborhood Theorem). *Let (M, ω) be a symplectic manifold, X a compact Lagrangian submanifold, ω_0 the canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the Lagrangian embedding as the zero section, and $i : X \hookrightarrow M$ the Lagrangian embedding given by inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in T^*X,$\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that the following diagram commutes and $\varphi^*\omega = \omega_0$.*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U} \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

6.1 Tangent Space to the Group of Symplectomorphisms

The symplectomorphisms of a symplectic manifold (M, ω) from a group

$$\mathrm{Sympl}(M, \omega) = \{f : M \xrightarrow{\cong} M | f^*\omega = \omega\}$$

What is $T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega))$? What does a neighborhood of id in $\mathrm{Sympl}(M, \omega)$ look like?

Let X, Y be manifolds.

Definition 6.5. *A sequence of maps $f_i : X \to Y$ **converges in the C^0-topology** to $f : X \to Y$ if and only if f_i converges uniformly on compact sets.*

Definition 6.6. *A sequence of C^1 maps $f_i : X \to Y$ **converges in the C^1-topology** to $f : X \to Y$ if and only if it and the sequence of derivatives $df_i : TX \to TY$ converge uniformly on compact sets.*

Let (M, ω) be a compact symplectic manifold and $f \in \mathrm{Sympl}(M, \omega)$. Then Graph f and Graph id$= \Delta$ are Lagrangian submanifolds of $(M \times M, \pi_1^*\omega - \pi_2^*\omega)$. By the Weinstein tubular theorem, there exists a neighborhood $\mathcal{U}$ of $\Delta(\sim M)$ in $(M \times, \pi_1^*\omega - \pi_2^*\omega)$ which is symplectomorphic to a neighborhood $\mathcal{U}$, of M in (T^*M, ω_0). Let $\varphi : \mathcal{U} \to \mathcal{U}_0$ be the symplectomorphism satisfying $\varphi(p, p) = (p, 0), \forall p \in M$.

Suppose that f is sufficiently C^1-**close** to id, i.e., f is in some sufficiently small neighborhood of id in the C^1-topology. Then

(1) We can assume that Graphc $f \subseteq \mathcal{U}$. Let $j : M \hookrightarrow \mathcal{U}$ be the embedding as Graph f and $i : M \hookrightarrow \mathcal{U}$ be the embedding as Graph id$= \Delta$.

(2) The map j is sufficiently C^1-close to i.

(3) By the Weinstein theorem, $\mathcal{U} \simeq \mathcal{U}_0 \subseteq T^*M$, so the above j and i induce embeddings $j_0 : M \hookrightarrow \mathcal{U}_0$ and $i_0 : M \hookrightarrow \mathcal{U}_0$. Hence we have the following commutative diagrams.

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\varphi"] & & \mathcal{U}_0 \\ & M \arrow[ul, "i"] \arrow[ur, "i_0"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{U} \arrow[rr, "\varphi"] & & \mathcal{U}_0 \\ & M \arrow[ul, "j"] \arrow[ur, "j_0"'] & \end{tikzcd}$$

where $i(p) = (p, p), i_0(p) = (p, 0), j(p) = (p, f(p)), j_0(p) = \varphi(p, f(p))$ for $p \in M$.

(4) The map j_0 is sufficiently C^1-close to i_0. Thus the image set $j_0(M)$ intersects each Y_p^*M at one point μ_p depending smoothly on p.

(5) The image of j_0 is the image of a smooth section $\mu : M \to T^*M$, that is, a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$.

Therefore, Graph $f \simeq \{(p, \mu_p) | p \in M, \mu_p \in T_p^*M\}$. **Conclusion:** A small C^1-neighborhood of id in $\mathrm{Sympl}(M, \omega)$ is homeomorphic to a C^1-neighborhood of zero in the vector space of closed 1-forms on M. Thus

$$T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega)) \simeq \{\mu \in \Omega^1(M) | d\mu = 0\}$$

In particular, $T_{\mathrm{id}}(\mathrm{Sympl}(M, \omega))$ contains the space of exact 1-forms

$$\{\mu = dh | h \in C^\infty(M)\} \cong C^\infty(M)/\text{locally constant functions}$$

6.2 Fixed Points of Symplectomorphisms

Theorem 6.7. *Let (M,ω) be a compact symplectic manifold with $H^1_{dR}(M) = 0$. Then any symplectomorphism of M which is sufficiently C^1-close to the identity has at least two fixed points.*

Proof. Suppose that $f \in \mathrm{Sympl}(M,\omega)$ is sufficiently C^1-close to id. Then Graph $f \simeq$ closed 1-form μ on M. Since $H^1_{dR}(M) = 0$, $\mu = dh$ for some $h \in C^\infty(M)$.

Since M is compact, h has at least 2 critical points. Then

$$\begin{array}{ccc} \text{Fixed points of } f & \Longrightarrow & \text{Critical points of } h \\ \Downarrow & & \Downarrow \\ \text{Graph } f \cap \Delta & \Longrightarrow & \{p : \mu_p = dh_p = 0\} \end{array}$$

□

Definition 6.8. *A submanifold Y of M is C^1-**close** to X when there is a diffeomorphism $X \to Y$ which is, as a map into M, C^1-close to the inclusion $X \hookrightarrow M$.*

Theorem 6.9. *Let (M,ω) be a symplectic manifold. Suppose that X is a compact Lagrangian submanifold of M with $H^1_{dR}(X) = 0$. Then every Lagrangian submanifold of M which is C^1-close to X intersects X in at least two points.*

7 Contact Forms

Definition 7.1. *A **contact element** on a manifold M is a point $p \in M$, called the **contact point**, together with a tangent hyperspace at p,$H_p \subset T_pM$.*

Suppose that H is a smooth field of contact elements on M. Locally, $H = \ker\alpha$ for some 1-form α, called a **locally defining 1-form** for H.

Definition 7.2. *A **contact structure** on M is a smooth field of tangent hyperspace $H \subset TM$, such that, for any locally defining 1-form α, we have $d\alpha|_H$ non-degenerate(i.e., symplectic). The pair (M,H) is then called a **contact manifold** and α is called a **local contact form**.*

At each $p \in M$,

$$T_pM = \ker\alpha_p \oplus \ker d\alpha_p$$

Therefore, any contact manifold (M,H) has $\dim M = 2n+1$ odd. If α is a global contact form, then $\alpha \wedge (d\alpha)^n$ is a volume form on M.

Proposition 7.3. *Let H be a field of tangent hyperplanes on M. Then H is a contact structure if and only if $\alpha \wedge (d\alpha)^n \neq 0$ for every locally defining 1-form α.*

Proof. $\Leftarrow$ Suppose that $H = \ker\alpha$ locally, we need to show $d\alpha|_H$ nondegenerate.

Take a local trivialization $\{e_1, f_1, \cdots, e_n, f_n, r\}$ of $TM = \ker\alpha \oplus$rest, such that $\ker\alpha = \mathrm{span}\{e_1, f_1, \cdots, e_n, f_n\}$ and rest= $\mathrm{span}\{r\}$. Then

$$(\alpha \wedge (d\alpha)^n)(e_1, f_1, \cdots, e_n, f_n, r) = \alpha(r) \cdot (d\alpha)^n(e_1, f_1, \cdots, e_n, f_n)$$

Thus $(d\alpha)^n(e_1, f_1, \cdots, e_n, f_n) \neq 0$, hence $d\alpha|_H$ is nongenerate. □

Example 7.4. *On $\mathbb{R}^3$ with coordinates (x,y,z), consider $\alpha = xdy + dz$. Since*

$$\alpha \wedge d\alpha = (xdy + dz) \wedge (dx \wedge dy) = dx \wedge dy \wedge dz \neq 0$$

α is contact form on $\mathbb{R}^3$.

This corresponding field of hyperplanes $H = \ker\alpha$ at $(x,y,z) \in \mathbb{R}^3$ is

$$H_{(x,y,z)} = \left\{ v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \middle| bx + c = 0 \right\}$$

Generally, on $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n, z)$, $\alpha = \sum\limits_{i=1}^n x_i dy_i + dz$ is contact.

Example 7.5 (Martinet, 1971)**.** *Any compact orientable 3-manifold admits a contact structure.*

Theorem 7.6. *Let (M, H) be a contact manifold and $p \in M$. Then there exists a coordinate system $(\mathcal{U}, x_1, y_1, \cdots, x_n, y_n, z)$ centered at p such that on $\mathcal{U}$*

$$\alpha = \sum x_i dy_i + dz$$

is a local contact form for H.

Theorem 7.7 (Gray)**.** *Let M be a compact manifold. Suppose that $\alpha_t, t \in [0,1]$ is a smooth family of global contact forms on M. Let $H_t = \ker \alpha_t$. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $H_t = \rho_{t*} H_0$ for all $0 \leqslant t \leqslant 1$.*

8 Contact Dynamics

Let (M, H) be a contact manifold with a contact form α.

Claim 8.1. *There exists a unique vector field R on M such that*

$$\begin{cases} i_R d\alpha = 0 \\ i_R \alpha = 1 \end{cases}$$

Proof. $R \in \ker d\alpha$ which is a line bundle and then normalize R. □

Definition 8.2. *The vector field R is called the* ***Reeb vector field*** *determined by α.*

Claim 8.3. *The flow of R preserves the contact form, i.e., if $\rho_t = \exp tR$ is the isotopy generated by R, then $\rho_t^* \alpha = \alpha$, $\forall t \in \mathbb{R}$.*

Proof. We have

$$\frac{d}{dt}(\rho_t^* \alpha) = \rho_t^*(\mathcal{L}_R \alpha) = \rho_t^*(d i_R \alpha + i_R d\alpha) = 0$$

Hence $\rho_t^* \alpha = \rho_0^* \alpha = \alpha$, $\forall t \in \mathbb{R}$. □

Definition 8.4. *A* ***contactmorphism*** *is a diffeomorphism f of a contact manifold (M, H) which preserves the contact structure(i.e., $f_* H = H$).*

Example 8.5. $\mathbb{R}^{2n+1}$ *with $\alpha = \sum_{i=1}^{n} x_i dy_i + dz$. Then*

$$\begin{cases} i_R \sum dx_i \wedge dy_i = 0 \\ i_R \sum x_i dy_i + dz = 1 \end{cases}$$

implies that $R = \frac{\partial}{\partial z}$ is the Reeb vector field. The contactomorphisms generated by R are translations

$$\rho_t(x_1, y_1, \cdots, x_n, y_n, z) = (x_1, y_1, \cdots, x_n, y_n, z + t).$$

Example 8.6. *Regard the odd sphere $S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ as the set of unit vectors.*

$$\{(x_1, y_1, \cdots, x_n, y_n) \mid \sum (x_i^2 + y_i^2) = 1\}$$

Consider the 1-form on $\mathbb{R}^{2n}$, $\sigma = \frac{1}{2} \sum (x_i dy_i - y_i dx_i)$.

Claim: The form $\alpha = i^* \sigma$ is a contact form on S^{2n-1}.

Proof. We need to show that $\alpha \wedge (d\alpha)^{n-1} \neq 0$. The 1-form on $\mathbb{R}^{2n}$ $\nu = d \sum (x_i^2 + y_i^2) = 2 \sum (x_i dx_i + y_i dy_i)$ satisfies $T_p S^{2n-1} = \ker \nu_p$ at $p \in S^{2n-1}$. We can check that $\nu \wedge \sigma \wedge (d\sigma)^{n-1} \neq 0$. □

The distribution $H = \ker \alpha$ is called the ***standard contact structure*** *on S^{2n-1}. The Reeb vector field is $R = 2 \sum \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)$, and is known as the* ***Hopf vector field*** *on S^{2n-1}.*

Proposition 8.7. *Let (M, H) be a contact manifold with a contact form α. Let $\widetilde{M} = M \times \mathbb{R}$, and let $\pi : \widetilde{M} \to M$, $(p, \tau) \mapsto p$, be the projection. Then $\omega = d(e^\tau \pi^* \alpha)$ is a symplectic form on $\widetilde{M}$.*

Hence, $\widetilde{M}$ has a symplectic form ω canonically determined by a contact form α on M and a coordinate function $\mathbb{R}$. $(\widetilde{M}, \omega)$ is called the **symplectization** of (M, α).

9 Almost Complex Structures

Definition 9.1. *Let V be a vector space. A* ***complex structure*** *on V is a linear map $J : V \to V$ with $J^2 = -\mathrm{id}$. The pair (V, J) is called a* ***complex vector space****.*

A complex structure J is equivalent to a structure of vector space over $\mathbb{C}$ if we identify the map J with multiplication by $\sqrt{-1}$.

Definition 9.2. *Let (V, Ω) be a symplectic vector space. A complex structure V is said to be* ***compatible*** *if*

$$G_J(u, v) := \Omega(u, Jv), \ \forall u, v \in V$$

is a positive inner product on V. That is,

$$\begin{cases} \Omega(Ju, Jv) = \Omega(u, v) \\ \Omega(u, Ju) > 0, \ \forall u \neq 0 \end{cases}$$

Proposition 9.3. *Let (V, Ω) be a symplectic vector space. Then there is a compatible complex structure J on V.*

Proof. Choose a positive inner product G on V. Since Ω and G are nondenegerate,

$$\begin{cases} u \in V \longmapsto \Omega(u, \cdot) \in V^* \\ w \in V \longmapsto G(w, \cdot) \in V^* \end{cases}$$

are isomorphisms between V and V^*. Hence $\Omega(u, v) = G(Au, v)$ for some linear map $A : V \to V$. This map A is skew-symmetric because

$$G(A^*u, v) = G(u, Av) = G(Av, u) = \Omega(v, u) = -\Omega(u, v) = G(-Au, v)$$

Also, AA^* is symmetric and positive. These properties imply that AA^* diagonalizes with positive eigenvalues λ_i

$$AA^* = B \ \mathrm{diag}(\lambda_1, \cdots, \lambda_{2n}) \ B^{-1}$$

Let

$$\sqrt{AA^*} = B \ \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_{2n}}) \ B^{-1}$$

Then $\sqrt{AA^*}$ is symmetric and positive-definite. Let $J = (\sqrt{AA^*})^{-1}A$, the factorization $A = \sqrt{AA^*}J$ is called the **polar decomposition** of A. J is orthogonal and skew-adjoint, hence it is a complex structure on V:

$$J^2 = -JJ^* = -\mathrm{id}$$

Compatibility:

$$\Omega(Ju, Jv) = G(AJu, Jv) = G(JAu, Jv) = G(Au, v) = \Omega(u, v)$$
$$\Omega(u, Ju) = G(Au, Ju) = G(-JAu, u) = G(\sqrt{AA^*}u, u) > 0, \ \forall u \neq 0$$

Therefore, J is a compatible complex structure on V. $\square$

Definition 9.4. *An* ***almost complex structure*** *on a manifold M is a smooth field of complex structures on the tangent spaces:*

$$x \longmapsto J_x : T_xM \to T_xM \text{ linear, and } J_x^2 = -\mathrm{id}$$

The pair (M, J) is then called an ***almost complex manifold****.*

Definition 9.5. *Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called* ***compatible*** *if the assignment*

$$x \longmapsto g_x : T_xM \times T_xM \to \mathbb{R}$$
$$g_x(u, v) : = \omega_x(u, J_xv)$$

is a Riemannian metric on M.

For a manifold M, if M has already had a symplectic form ω, a Riemannian metric g and a almost complex structure J. Then the triple (ω, g, J) is called a **compatible triple** when $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

Proposition 9.6. *Let (M, ω) be a symplectic manifold, and g a Riemannian metric on M. Then there exists a canonical almost complex structure J on M which is compatible.*

Corollary 9.7. *Any symplectic manifold has compatible almost complex structures.*

Proposition 9.8. *Let (M, ω) be a symplectic manifold, and J_0, J_1 two almost complex structures compatible with ω. Then there is a smooth family $J_t, 0 \leqslant t \leqslant 1$, of compatible almost complex structures joining J_0 to J_1.*

Proof. By compatibility, we get $g_0(\cdot,\cdot) = \omega(\cdot, J_0\cdot)$ and $g_1(\cdot,\cdot) = \omega(\cdot, J_1\cdot)$ two Riemannian metrics on M. Their convex combinations

$$g_t(\cdot,\cdot) = (1-t)g_0(\cdot,\cdot) + tg_1(\cdot,\cdot),\ 0 \leqslant t \leqslant 1$$

form a smooth family of Riemannian metrics. Apply the polar decomposition to (ω, g_t) to obtain a smooth family of J_t's joining J_0 to J_1. □

Corollary 9.9. *The set of all compatible almost complex structures on a symplectic manifold is path-connected.*

10 Compatible Triples

In fact, the set of all compatible almost complex structures is even contractible. Let $\mathcal{J}(T_xM, \omega_x)$ be the set of all compatible complex structures on (T_xM, ω_x) for $x \in M$.

Proposition 10.1. *The set $\mathcal{J}(T_xM, \omega_x)$ is contractible, i.e., there exists a homotopy*

$$h_t : \mathcal{J}(T_xM, \omega_x) \longrightarrow \mathcal{J}(T_xM, \omega_x),\ 0 \leqslant t \leqslant 1$$

starting at the identity $h_0 = \mathrm{id}$, finishing at a trivial map $h_1 : \mathcal{J}(T_xM, \omega_x) \to \{J_0\}$, and fixing J_0(i.e., $h_t(J_0) = J_0,\ \forall t$) for some $J_0 \in \mathcal{J}(T_xM, \omega_x)$.

Consider the fiber bundle $\mathcal{J} \to M$ with fiber

$$J_x := \mathcal{J}(T_xM, \omega_x) \ \text{ over } x \in M$$

A compatible almost structure J on (M, ω) is a section of $\mathcal{J}$.

Proposition 10.2. *Let (M, J) be an almost complex manifold. Suppose that J is compatible with two symplectic structures ω_0, ω_1. Then ω_0, ω_1 are deformation-equivalent, that is, there exists a smooth family $\omega_t,\ 0, \leqslant t \leqslant 1$, of symplectic forms joining ω_0 to ω_1.*

Proof. Let $\omega_t = (1-t)\omega_0 + t\omega_1$, then ω_t is closed and ω_t is nondenerate:

$$g_t(\cdot,\cdot) := \omega(\cdot, J\cdot) = (1-t)g_0(\cdot,\cdot) + tg_1(\cdot,\cdot)$$

□

Definition 10.3. *A submanifold X of an almost complex manifold (M, J) is an* ***almost complex submanifold*** *when $J(TX) \subseteq TX$.*

Proposition 10.4. *Let (M, ω) be a symplectic manifold equipped with a compatible almost complex structure J. Then any almost complex submanifold X of (M, J) is a symplectic submanifold of (M, ω).*

11 Dolbeault Theory

Let (M, J) be an almost complex manifold. The complexified tangent bundle of M is the bundle

$$TM \otimes \mathbb{C} \to M$$

with fiber $(TM \otimes \mathbb{C})_p = T_pM \otimes \mathbb{C}$ at $p \in M$. We may extend J linearly to $TM \otimes \mathbb{C}$:

$$J(v \otimes c) = Jv \otimes c,\ v \in TM, c \in \mathbb{C}$$

Since $J^2 = -\mathrm{id}$, on the complex vector space $(TM \otimes \mathbb{C})_p$, the linear map J_p has eigenvalues $\pm i$. Let

$$T_{1,0} = \{v \in TM \otimes \mathbb{C} | Jv = iv\} = \{v \otimes 1 - Jv \otimes i | v \in TM\}$$

called the ($J-$)**holomorphic tangent vectors**.

$$T_{0,1} = \{v \in TM \otimes \mathbb{C} | Jv = -iv\} = \{v \otimes 1 + Jv \otimes i | v \in TM\}$$

called the ($J-$)**anti-holomorphic tangent vectors**. Let

$$\begin{aligned} \pi_{1,0} : TM &\longrightarrow T_{1,0} \\ v &\longmapsto \frac{1}{2}(v \otimes 1 - Jv \otimes i) \end{aligned}$$

be a real bundle isomorphism such that $\pi_{1,0} \circ J = i\pi_{1,0}$, and

$$\begin{aligned} \pi_{0,1} : TM &\longrightarrow T_{0,1} \\ v &\longmapsto \frac{1}{2}(v \otimes 1 + Jv \otimes i) \end{aligned}$$

also be a bundle isomorphism such that $\pi_{0,1} \circ J = -i\pi_{0,1}$. Extending $\pi_{1,0}$ and $\pi_{0,1}$ to projections of $TM \otimes \mathbb{C}$, we obtain an isomorphism

$$(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \xrightarrow{\cong} T_{1,0} \oplus T_{0,1}$$

Similarly, the complexified cotangent bundle splits as

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\cong} T^{1,0} \otimes T^{0,1}$$

where

$$\begin{aligned} T^{1,0} = (T_{1,0})^* &= \{\eta \in T^* \otimes \mathbb{C} | \eta(J\omega) = i\eta(\omega),\ \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 - (\xi \circ J) \otimes i | \xi \in T^*M\} \end{aligned}$$

called the **complex-linear cotangent vectors** and

$$\begin{aligned} T^{0,1} = (T_{0,1})^* &= \{\eta \in T^* \otimes \mathbb{C} | \eta(J\omega) = -i\eta(\omega),\ \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 + (\xi \circ J) \otimes i | \xi \in T^*M\} \end{aligned}$$

called the **complex-antilinear cotangent vectors** and $\pi^{1,0}, \pi^{0,1}$ are the two natural projections

$$\begin{aligned} \pi^{(1,0)} : T^*M \otimes \mathbb{C} &\longrightarrow T^{1,0} \\ \eta &\longmapsto \eta^{1,0} := \frac{1}{2}(\eta - i\eta \circ J) \\ \pi^{(0,1)} : T^*M \otimes \mathbb{C} &\longrightarrow T^{0,1} \\ \eta &\longmapsto \eta^{0,1} := \frac{1}{2}(\eta + i\eta \circ J) \end{aligned}$$

For an almost complex manifold (M, J), let

$$\Omega^k(M; \mathbb{C}) := \text{sections of } \Lambda^k(T^*M \otimes \mathbb{C})$$

called the **complex-valued $k-$forms on M**, where

$$\Lambda^k(T^*M \otimes \mathbb{C}) := \Lambda^k(T^{1,0} \oplus T^{0,1}) = \bigoplus_{l+m=k} (\Lambda^l T^{1,0}) \wedge (\Lambda^m T^{0,1}) = \bigoplus_{l+m=k} \Lambda^{l,m}$$

Definition 11.1. *The **differential forms of type** (l, m) on (M, J) are the sections of $\Lambda^{l,m}$:*

$$\Omega^{l,m} := \textit{sections of } \Lambda^{l,m}$$

Then

$$\Omega^k(M; \mathbb{C}) := \bigoplus_{l+m=k} \Omega^{l,m}$$

Let $\pi^{l,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{l,m}$ be the projection map, where $l + m = k$. The usual exterior derivative d composed with two of these projections induces differential operators ∂ and $\bar{\partial}$ on forms of type (l, m):

$$\begin{aligned} \partial &:= \pi^{l+1,m} \circ d : \Omega^{l,m}(M) \longrightarrow \Omega^{l+1,m}(M) \\ \bar{\partial} &:= \pi^{l,m+1} \circ d : \Omega^{l,m}(M) \longrightarrow \Omega^{l,m+1}(M) \end{aligned}$$

If $\beta \in \Omega^{l,m}(M)$ with $k = l + m$, then $d\beta \in \Omega^{k+1}(M; \mathbb{C})$:

$$d\beta = \sum_{r+s=k+1} \pi^{r,s} d\beta = \pi^{k+1,0} d\beta + \cdots + \partial\beta + \bar{\partial}\beta + \cdots + \pi^{0,k+1} d\beta$$

Definition 11.2. *A function $f : M \to \mathbb{C}$ is $(J-)$**holomorphic at** $x \in M$ if df_p is complex linear, i.e., $df_p \circ J = idf_p$. A function f is $J-$**holomorphic** if it is holomorphic at all $p \in M$.*

Definition 11.3. *A function f is $(J-)$anti-holomorphic at $p \in M$ if df_p is complex antilinear, i.e., $df_p \circ J = -idf_p$.*

Definition 11.4. *On functions, $d = \partial + \bar{\partial}$, where*

$$\partial := \pi^{1,0} \circ d, \ \bar{\partial} := \pi^{0,1} \circ d$$

Then f is holomorphic $\iff \bar{\partial} f = 0$ and f is anti-holomorphic $\iff \partial f = 0$.
For higher differential forms, suppose $d = \partial + \bar{\partial}$, i.e.,

$$d\beta = \partial\beta + \bar{\partial}\beta, \ \forall \beta \in \Omega^{l,m}$$

Then for any form β,

$$0 = d^2\beta = \partial^2\beta + \partial\bar{\partial}\beta + \bar{\partial}\partial\beta + \bar{\partial}^2\beta$$

which implies

$$\begin{cases} \bar{\partial}^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial = 0 \\ \bar{\partial} = 0 \end{cases}$$

Since $\bar{\partial}^2 = 0$, the chain

$$0 \longrightarrow \Omega^{l,0} \xrightarrow{\bar{\partial}} \Omega^{l,1} \xrightarrow{\bar{\partial}} \Omega^{l,2} \xrightarrow{\bar{\partial}} \cdots$$

is a differential complex. Its cohomology groups

$$H^{l,m}_{Dolbeault}(M) := \frac{\ker \bar{\partial} : \Omega^{l,m} \longrightarrow \Omega^{l,m+1}}{\operatorname{im} \bar{\partial} : \Omega^{l,m-1} \longrightarrow \Omega^{l,m}}$$

are called the **Dolbeault cohomology** groups.

12 Complex Manifolds

Definition 12.1. *A **complex manifold** of (complex) dimension n is a set M with a complete complex atlas*

$$\mathcal{A} = \{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha), \alpha \in I\}$$

where $M = \bigcup_\alpha \mathcal{U}_\alpha$, the $\mathcal{V}_\alpha$'s are open subsets of $\mathbb{C}^n$, and the maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ are such that the transition maps $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ are biholomorphic as maps on open subsets of $\mathbb{C}^n$.

Proposition 12.2. *Any complex manifold has a canonical almost complex structure.*

Proof. First we define J locally: Let $(\mathcal{U}, \mathcal{V}, \varphi : \mathcal{U} \to \mathcal{V})$ be a complex chart for a complex manifold M with $\varphi = (z_1, \cdots, z_n)$ written in components relative to complex coordinates $z_j = x_j + iy_j$. At $p \in \mathcal{U}$,

$$T_pM = \mathbb{R}\text{-span of } \left\{ \frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial y_j}\Big|_p : j = 1, \cdots, n \right\}$$

Define J over $\mathcal{U}$ by

$$J_p\Big(\frac{\partial}{\partial x_j}\Big|_p\Big) = \frac{\partial}{\partial y_j}\Big|_p, \quad J_p\Big(\frac{\partial}{\partial y_j}\Big|_p\Big) = -\frac{\partial}{\partial x_j}\Big|_p, \ j = 1, \cdots, n$$

Then we need to check J is well-defined globally:

If $(\mathcal{U}, \mathcal{V}, \varphi)$ and $(\mathcal{U}', \mathcal{V}', \varphi')$ are two charts, on $\mathcal{U} \cap \mathcal{U}'$, $\psi \circ \varphi = \varphi'$. If $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$ are coordinates on $\mathcal{U}$ and $\mathcal{U}'$, so that $\varphi = (z_1, \cdots, z_n)$, $\varphi' = (w_1, \cdots, w_n)$, then $\psi(z_1, \cdots, z_n) = (w_1, \cdots, w_n)$. Taking the derivative of a composition

$$\begin{cases} \frac{\partial}{\partial x_k} = \sum_j \left(\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j} \right) \\ \frac{\partial}{\partial y_k} = \sum_j \left(\frac{\partial u_j}{\partial y_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j} \right) \end{cases}$$

Since ψ is holomorphic, each component of ψ satisfies the Cauchy-Riemann equations:

$$\begin{cases} \frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k} \\ \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k} \end{cases} j, k = 1, \cdots, n$$

These equations imply

$$J' \frac{\partial}{\partial x_k} = J' \sum_j \Big(\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j} \Big) = \sum_j \Big(\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial v_j} - \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial u_j} \Big) = \sum_j \Big(\frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j} + \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial u_j} \Big) = \frac{\partial}{\partial y_k}$$

which matches with the equation

$$J \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$$

□

Suppose that M is a complex manifold and J is its canonical complex structure. Let $\mathcal{U} \subseteq M$ be a coordinate neighborhood with complex coordinates $z_1, \cdots, z_n, z_j = x_j + iy_j$, and real coordinates $x_1, y_1, \cdots, x_n, y_n$. At $p \in \mathcal{U}$

$$\begin{aligned} T_pM &= \mathbb{R}\text{-span} \left\{ \frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial y_j}\Big|_p \right\} \\ T_pM \otimes \mathbb{C} &= \mathbb{C}\text{-span} \left\{ \frac{\partial}{\partial x_j}\Big|_p, \frac{\partial}{\partial y_j}\Big|_p \right\} \\ &= \mathbb{C}\text{-span} \left\{ \frac{1}{2}\Big(\frac{\partial}{\partial x_j}\Big|_p - i \frac{\partial}{\partial y_j}\Big|_p \Big) \right\} \oplus \mathbb{C}\text{-span} \left\{ \frac{1}{2}\Big(\frac{\partial}{\partial x_j}\Big|_p + i \frac{\partial}{\partial y_j}\Big|_p \Big) \right\} \end{aligned}$$

Definition 12.3. *Notation:*

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \Big), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \Big)$$

Hence

$$(T_{1,0})_p = \mathbb{C}\text{-span} \left\{ \frac{\partial}{\partial z_j}\Big|_p : j = 1, \cdots, n \right\}, \ (T_{0,1})_p = \mathbb{C}\text{-span} \left\{ \frac{\partial}{\partial \bar{z}_j}\Big|_p : j = 1, \cdots, n \right\}$$

Similarly,

$$\begin{aligned} T^*M \otimes \mathbb{C} &= \mathbb{C}\text{-span}\{dx_j, dy_j : j = 1, \cdots, n\} \\ &= \mathbb{C}\text{-span}\{dx_j + idy_j : j = 1, \cdots, n\} \oplus \mathbb{C}\text{-span}\{dx_j - idy_j : j = 1, \cdots, n\} \end{aligned}$$

Putting

$$dz_j = dx_j + idy_j, \ d\bar{z}_j = dx_j - idy_j$$

we obtain

$$T^{1,0} = \mathbb{C}\text{-span}\{dz_j : j = 1, \cdots, n\}, \ T^{0,1} = \mathbb{C}\text{-span}\{d\bar{z}_j : j = 1, \cdots, n\}$$

On the coordinate neighborhood $\mathcal{U}$, if we use the multi-index notation:

$$\begin{aligned} &J = (j_1, \cdots, j_m), \ 1 \leqslant j_1 < \cdots < j_m \leqslant n \\ &|J| = m \\ &dz_J = dz_{j_1} \wedge dz_{j_2} \wedge \cdots dz_{j_m} \end{aligned}$$

then $(l, m) - forms$

$$\Omega^{l,m} = \left\{ \sum_{|J|=l, |K|=m} b_{J,K} dz_J \wedge d\bar{z}_K \Big| b_{J,K} \in C^\infty(\mathcal{U}; \mathbb{C}) \right\}$$

On a coordinate neighborhood $\mathcal{U}$, a form $\beta \in \Omega^k(M; \mathbb{C})$ may be written as

$$\beta = \sum_{|J|+|K|=k} a_{J,K} dx_J \wedge dy_K, \ a_{J,K} \in C^\infty(\mathcal{U}; \mathbb{C})$$

then using $d = \partial + \bar{\partial}$ on functions, we have

$$d\beta = \partial\beta + \bar{\partial}\beta$$

Conclusion: If M is a complex manifold, then $d = \partial + \bar{\partial}$.(For an almost complex complex this fails because there are no coordinate functions z_j to give a suitable basis of 1-forms.)

When $\beta \in \Omega^{l,m}$, we have

$$\begin{aligned} d\beta &= \partial\beta + \bar{\partial}\beta \\ 0 = d^2\beta &= \partial^2\beta + (\partial\bar{\partial} + \bar{\partial}\partial)\beta + \bar{\partial}^2\beta \end{aligned}$$

Hence, $\bar{\partial} = 0$.

The Dolbeault theorem states that for complex manifolds

$$H^{l,m}_{Dolbeault}(M) = H^m(M; \mathcal{O}(\Omega^{(l,0)}))$$

where $\mathcal{O}(\Omega^{(l,0)})$ is the sheaf of forms of type $(l, 0)$ over M.

Theorem 12.4 (Newlander-Nirenberg,1957)**.** *Let (M, J) be an almost complex manifold. Let $\mathcal{N}$ be the **Nijenhuis tensor**. Then: M is a complex manifold $\iff$ J is integrable $\iff$ $\mathcal{N} \equiv 0$ $\iff$ $d = \partial + \bar{\partial}$ $\iff$ $\bar{\partial}^2 = 0$ $\iff$ $\pi^{2,0}d|_{\Omega^{0,1}} = 0$*

13 Kähler Forms

Definition 13.1. *A **Kähler manifold** is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form ω is then called a **Kähler form**.*

If (M, ω) is Kähler, then M is a complex manifold, then

$$\Omega^k(M; \mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}, \ d = \partial + \bar{\partial}$$

On a complex chart $(\mathcal{U}, z_1, \cdots, z_n)$, $n = \dim_{\mathbb{C}} M$

$$\Omega^{l,m} = \left\{ \sum_{|J|=l,|K|=m} b_{JK} dz_J \wedge \bar{z}_K \middle| b_{JK} \in C^\infty(\mathcal{U}; \mathbb{C}) \right\}$$

On the other hand, (M, ω) Kähler implies that ω is a symplectic form. A Kähler form ω is

(1) a 2-form,

(2) compatible with the complex structure,

(3) closed,

(4) real-valued, and

(5) nondegenerate.

These properties translate into:

(1) $\Omega^2(M; \mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$.
On a local complex chart $(\mathcal{U}, z_1, \cdots, z_n)$,

$$\omega = \sum a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k + \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k$$

for some $a_{jk}, b_{jk}, c_{jk} \in C^\infty(\mathcal{U}; \mathbb{C})$

(2) J is a symplectomorphism, that is, $J^*\omega = \omega$ where $(J^*\omega)(u, v) := \omega(Ju, Jv)$.
Since

$$J^* dz_j = dz_j \circ J = i dz_j, \ J^* d\bar{z}_j = d\bar{z}_j \circ J = -i d\bar{z}_j$$

we have

$$J^*\omega = \sum -a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k - \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k$$

Since $J^*\omega = \omega$, we conclude that $a_{jk} = c_{jk} = 0$, $\forall j, k$, and $\omega \in \Omega^{1,1}$.

(3) $0 = d\omega = \partial\omega + \bar{\partial}\omega \Longrightarrow \partial\omega = \bar{\partial}\omega = 0$.
Hence ω defines a Dolbeault $(1,1)$ cohomology class

$$[\omega] \in H^{1,1}_{Dolbeault}(M)$$

Putting $b_{jk} = \frac{i}{2}h_{jk}$, then

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k,\ h_{jk} \in C^\infty(\mathcal{U};\mathbb{C})$$

(4) ω real-valued $\Longleftrightarrow \omega = \bar{\omega}$.

$$\bar{\omega} = -\frac{i}{2}\sum \overline{h_{jk}} d\bar{z} j \wedge dz_k = \frac{i}{2}\sum \overline{h_{kj}} dz_j \wedge d\bar{z}_k$$

Thus $h_{jk} = \overline{h_{kj}}$, i.e., at every point $p \in \mathcal{U}$, the $n \times n$ matrix $(h_{jk}(p))$ is Hermitian.

(5) Nondenegeracy: $\omega^n \neq 0$, i.e., at every $p \in M$, $(h_{jk}(p))$ is a nonsingular matrix.

(6) By the positivity condition: $\omega(v, Jv) > 0$, $\forall v \neq 0$, i.e., at each $p \in \mathcal{U}$, $(h_{jk}(p))$ is positive-definite.

Conclusion: Kähler forms are ∂- and $\bar{\partial}$-closed $(1,1)$-forms, which are given on a local chart $(\mathcal{U}, z_1, \cdots, z_n)$ by

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{jk} dz_j \wedge \bar{z}^k$$

where, at every point $p \in \mathcal{U}$, $(h_{jk}(p))$ is a positive-definite Hermitian matrix.

Theorem 13.2 (Banyaga)**.** *Let M be a compact complex manifold. Let ω_0 and ω_1 be Kähler forms on M. If $[\omega_0] = [\omega_1] \in H^2_{dR}(M)$, then (M,ω) and (M,ω) are symplectomorphic.*

Definition 13.3. *Let M be a complex manifold. A function $\rho \in C^\infty(M;\mathbb{R})$ is **strictly plurisubharmonic**(s.p.s.h.) if, on each local complex charts $(\mathcal{U}, z_1, \cdots, z_n)$, where $n = \dim_{\mathbb{C}} M$, the matrix $\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(\rho)\right)$ is positive-definite at all $p \in \mathcal{U}$.*

Proposition 13.4. *Let M be a complex manifold and let $\rho \in C^\infty(M;\mathbb{R})$ be s.p.s.h.. Then*

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

*is Kähler. Such a function ρ is called a (global) **Kähler potential**.*

Example 13.5. *Let $M = \mathbb{C}^n$. Let*

$$\rho(x_1, y_1, \cdots, x_n, y_n) = \sum_{j=1}^{n}(x_j^2 + y_j^2) = \sum |z_j|^2 = \sum z_j \bar{z}_j$$

Then

$$\frac{\partial}{\partial z_j}\frac{\partial \rho}{\partial \bar{z}_k} = \delta_{jk}$$

so

$$(h_{jk}) = (\delta_{jk}) = id$$

The corresponding Kähler form

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j$$

which is the standard form.

Theorem 13.6. *Let ω be a closed real-valued $(1,1)$-form on a complex manifold M and let $p \in M$. Then there exist a neighborhood $\mathcal{U}$ of p and $\rho \in C^\infty(\mathcal{U};\mathbb{R})$ such that on $\mathcal{U}$*

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho$$

*The functino ρ is then called a (local) **Kähler potential**.*

Proof. Let M be a complex manifold, $\rho \in C^\infty(M;\mathbb{R})$ s.p.s.h., X a complex submanifold, and $i : X \rightarrowtail M$ the inclusion map. Then $i^*\rho$ is s.p.s.h.. $\square$

Corollary 13.7. *Any complex submanifold of a Kähler manifold is also Kähler.*

Definition 13.8. *Let (M,ω) is a Kähler manifold, X a complex submanifold, and $i : X \rightarrowtail M$ the inclusion. Then $(X, i^*\omega)$ is called a **Kähler manifold**.*

Example 13.9. *The complex projective space is $\mathbb{CP}^n$. The Fubini-Study form is Kähler. Therefore, every **non-singular projective variety**(smooth zero locus of a collection of homogeneous polynomials) is a Kähler manifold.*

14 Compact Kähler Manifolds

Theorem 14.1 (Hodge)**.** *On a compact Kähler manifold (M,ω) the Dolbeault cohomology groups satisfy*

$$H^k_{dR}(M;\mathbb{C}) \cong \bigoplus_{l+m=k} H^{l,m}_{Dolbeault}(M)$$

with $H^{l,m} \cong \overline{H^{m,l}}$. In particular, the spaces $H^{l,m}_{Dolbeault}$ are finite-dimensional.

Each tangent space has a positive inner product, which induces the Hodge $*$-operator. The **codifferential** and the **Laplacian** are the operators defined by

$$\delta = (-1)^{n(k+1)+1} * d* : \Omega^k(M) \to \Omega^{k-1}(M)$$
$$\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)$$

The operator Δ is called the **Laplace-Beltrami operator**.

Suppose that M is compact. Define an inner product on forms by

$$\langle \cdot,\cdot \rangle : \Omega^k \times \Omega^k \to \mathbb{R}, \quad \langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$$

This is symmetric, positive-definite and satisfies $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. Therefore, δ is often denoted by d^* and called the adjoint of d.

The **harmonic k-forms** are the elements of $\mathcal{H}^k := \{\alpha \in \Omega^k | \Delta\alpha = 0\}$. Note that $\Delta\alpha = 0 \Longleftrightarrow d\alpha = \delta\alpha = 0$. Since a harmonic form is d-closed, it defines a de Rham cohomology class.

Theorem 14.2 (Hodge)**.** *Every de Rham cohomology class on a compact oriented Riemannian manifold (M,g) posses a unique harmonic representative, i.e.,*

$$\mathcal{H}^k \simeq H^k_{dR}(M;\mathbb{R})$$

In particular, the spaces $\mathcal{H}^k$ are finite-dimensional. We also have the following orthogonal decomposition with respect to $\langle \cdot,\cdot \rangle$:

$$\Omega^k \simeq \mathcal{H}^k \oplus \Delta(\Omega^k(M)) \simeq \mathcal{H}^k \oplus d\Omega^{k-1} \oplus \delta\Omega^{k+1}$$

When M is Kähler, the Laplacian satifies $\Delta = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ and preserves the decomposition according to type, $\Delta : \Omega^{l,m} \to \Omega^{l,m}$. Hence, harmonic forms are also bigraded

$$\mathcal{H}^k = \bigoplus_{l+m=k} \mathcal{H}^{l,m}$$

Theorem 14.3 (Hodge)**.** *Every Dolbeault cohomology class on a compact Kähler manifold (M,ω) possesses a unique harmonic representative, i.e.,*

$$\mathcal{H}^{l,m} \simeq H^{l,m}_{Dolbeault}(M)$$

and the spaces $\mathcal{H}^{l,m}$ are finite-dimensional. Hence we have the following isomorphisms

$$H^k_{dR}(M) \simeq \mathcal{H}^k \simeq \bigoplus_{l+m=k} \mathcal{H}^{l,m} \simeq \bigoplus_{l+m=k} H^{l,m}_{Dolbeault}(M)$$

Let $b^k(M) := \dim H^k_{dR}(M)$ be the usual **Betti numbers** of M, and let $h^{l,m}(M) := \dim H^{l,m}_{Dolbeault}(M)$ be the so-called **Hodge numbers** of M. Then Hodge theorem implies

$$b^k = \sum_{l+m=k} h^{l,m} \; (h^{l,m} = h^{m,l})$$

There are some immediate topological consequences:

(1) On compact Kähler manifolds, the odd Betti numbers are even.

(2) On compact Kähler manifolds, $h^{1,0} = \frac{1}{2}b^1$ is a topological invariant.

(3) On compact symplectic manifolds, even Betti numbers are positive, because ω^k is closed but not exact($k = 0, \cdots, n$).

Proof. If $\omega^k = d\alpha$, by Stokes' theorem, $\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$. This cannot happen since ω^n is a volume form. □

(4) On compact Kähler manifolds, the $h^{l,l}$ are positive.
Claim: $0 \neq [\omega^n] \in H^{l,l}_{Dolbeault}(M)$.

Proof. If $\omega^l = \bar{\partial}\beta$ for some $\beta \in \Omega^{l-1,l}$, then

$$\omega^n = \omega^l \wedge \omega^{n-l} = \bar{\partial}(\beta \wedge \omega^{n-l}) \Longrightarrow 0 = [\omega^n] \in H^{n,n}_{Dolbeault}(M)$$

But $[\omega^n] \neq 0$ in $H^{2n}_{dR} \simeq H^{n,n}_{Dolbeault}(M)$ since it is a volume form. □

Example 14.4. *Some examples and counterexamples to aware:*

(1) Not all smooth even-dimensional manifolds are almost complex. For example, S^4, S^8, S^10.

(2) If M is both symplectic and complex, it may not be Kähler.

(3) There are symplectic manifolds which do not admit an complex structure.

(4) Given a complex structure on M, there may not exist a symplectic structure. For example, $S^1 \times S^3$.

(5) Almost complex manifolds may be neither complex nor symplectic. For example, $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$.

Example 14.5. *Main Kähler manifolds:*

(1) Compact Riemannian surfaces.

(2) Stein manifolds.

Definition 14.6. *A **Stein manifold** is a Kähler manifold (M, ω) which admits a global proper Kähler potential, i.e., $\omega = \frac{i}{2}\partial\bar{\partial}\rho$ for some proper function $\rho : M \to \mathbb{R}$.*

(3) Complex tori $M = \mathbb{C}^n/\mathbb{Z}^n$.

(4) Complex projective spaces.

(5) Products of Kähler manifolds.

(6) Complex submanifolds of Kähler manifolds.