

Solutions to Qiuzhen Qualifying Exam-2025 Autumn GT*

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1. Consider the topological space X given by the gluing of the following polygonal pieces:

146, 52153, 2346

Here the word 146 mean the labeling scheme for the 3-gon is $e_1e_4e_6^{-1}$, and so for the other words. Each $e_i, 1 \leq i \leq 6$ is oriented counter-clock-wisely and the underline means the orientation of the edge is clockwise.

Compute the homotopy groups $\pi_i(X)$ and singular homology groups $H_i(X; \mathbb{Z})$ for all $i \geq 1$.

Proof. We use cellular homology to calculate $H_i(X; \mathbb{Z})$. First we can derive that all vertices of edges are attached to a single point v and there are 6 edges and 3 disks.

Then consider the cellular chain of X :

$$\begin{array}{rcl} \mathbb{Z}^3 & \xrightarrow{\partial_2} & \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z} \\ A & \longmapsto & e_1 + e_4 - e_6 \\ B & \longmapsto & -e_1 + e_2 + e_3 + 2e_5 \\ C & \longmapsto & -e_2 - e_3 - e_4 + e_6 \\ & & e_1 \sim e_6 \longmapsto 0 \end{array}$$

Thus we conclude that

$$\begin{aligned} H_2(X; \mathbb{Z}) &= \ker \partial_2 = 0 \\ H_1(X; \mathbb{Z}) &= \frac{\ker \partial_1}{\text{im } \partial_2} \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2 \\ H_n(X; \mathbb{Z}) &= 0, \quad n \geq 3 \end{aligned}$$

For the homotopy groups,

$$\begin{aligned} \pi_1(X) &\cong \langle e_1, e_2, e_3, e_4, e_5, e_6 \mid e_1e_4e_6^{-1}, e_5e_2e_1^{-1}e_5e_3, e_2^{-1}e_3^{-1}e_4^{-1}e_6 \rangle \\ &\cong \langle a, b, c, d \mid daca^{-1}b^{-1}c^{-1}db \rangle \end{aligned}$$

Note that every edge appears twice among all polygons, hence X is a surface. We know that a surface which has infinite π_1 is a $K(\pi_1, 1)$ (see Hatcher 1.B). Thus

$$\pi_n(X) = 0, \quad n \geq 2$$

□

2. Let M be a Riemannian manifold and \mathcal{K} be a set of isometries of M . Suppose that F is a nonempty subset of M which is fixed by every element in \mathcal{K} . Show that each connected component of F is a closed totally geodesic submanifold of M .

Proof. To state the proof conveniently, we may assume \mathcal{K} consists of a single isometry ϕ . For any $p \in F$, then $\phi(p) = p$, and there exists a normal neighborhood U of p where \exp_p is a diffeomorphism

$$\exp_p : B_\varepsilon \subseteq T_p M \longrightarrow U \subseteq M$$

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Let

$$V = \{v \in T_p M \mid d\phi_p(v) = v\}$$

we claim that

$$F \cap U = \exp_p(B_\varepsilon \cap V)$$

This is because for any $q \in U$, we can write $q = \exp_p(v)$ for some $v \in T_p M$. Let γ be the unique geodesic starting at p with $\gamma'(0) = v$, then $\gamma(1) = q$. Since ϕ is an isometry, $\phi \circ \gamma$ is the unique geodesic starting at $\phi(p) = p$ with $(\phi \circ \gamma)' = d\phi_p(v)$. Hence $\phi(q) = q$ if and only if $d\phi_p(v) = v$.

Since \exp_p is a diffeomorphism and V is a linear subspace of $T_p M$, F is a submanifold of dimension $\dim V$. F is closed since ϕ is continuous.

To prove that F is totally geodesic, let $p \in F$ and $v \in T_p F$, then $d\phi_p(v) = v$ by previous discussion. Hence the M -geodesic starting at p with initial velocity v will stay in F in some interval $(-\delta, \delta)$. Hence F is totally geodesic. \square

3. Let X be a compact Lie group of dimension ≥ 1 . Show that the Euler characteristic of X vanishes.

Proof. Since X admits a nowhere vanishing vector field induced by multiplication, by Poincaré-Hopf theorem, $\chi(X) = 0$. \square

4. Let $\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ be the upper half-space with the metric defined by $ds^2 = \frac{1}{x_n^2}(dx_1^2 + \dots + dx_n^2)$. Let Σ be a k -dimensional minimal submanifold of \mathbb{H}^n , where $k < n$. Show that $\text{Ric}(X) \leq 0$, i.e., for every vector field X on Σ , we have $\text{Ric}(X, X) \leq 0$.

Proof. Fix a point $p \in \Sigma$, we may assume $X_p \neq 0$, Normalize X such that $|X_p| = 1$. We may choose a local orthonormal basis $\{e_1, \dots, e_k\}$ at p such that $e_1 = X_p$. Thus we only need to prove that $\text{Ric}(e_1, e_1) = 0$. By Gauss equation,

$$\begin{aligned} \text{Ric}(e_1, e_1) &= \sum_{i=1}^k Rm(e_i, e_1, e_1, e_i) \\ &= \sum_{i=1}^k \left(\widetilde{Rm}(e_i, e_1, e_1, e_i) + \langle \Pi(e_i, e_i), \Pi(e_1, e_1) \rangle - \langle \Pi(e_i, e_1), \Pi(e_1, e_i) \rangle \right) \\ &= \sum_{i=1}^k \left(\widetilde{Rm}(e_i, e_1, e_1, e_i) - |\Pi(e_i, e_1)|^2 \right) + \left\langle \sum_{i=1}^k \Pi(e_i, e_i), \Pi(e_1, e_1) \right\rangle \end{aligned}$$

where \widetilde{Rm} denotes the curvature of \mathbb{H}^n . Since \mathbb{H}^n has negative constant sectional curvature,

$$\widetilde{Rm}(e_i, e_1, e_1, e_i) = \sec(e_i, e_1) \leq 0, \quad \forall i$$

Since Σ is a minimal submanifold,

$$0 = \text{tr}_g \Pi = \sum_{i=1}^k \Pi(e_i, e_i)$$

Thus we conclude that $\text{Ric}(e_1, e_1) \leq 0$. \square

5. Suppose X is contractible, compact manifold of dimension $n \geq 1$. Prove that $H_i(\partial X; \mathbb{Z}) = H_i(S^{n-1}; \mathbb{Z})$ for all i .

Proof. Consider the long exact sequence of pair $(X, \partial X)$:

$$\dots \longrightarrow \widetilde{H}_i(\partial X) \longrightarrow \widetilde{H}_i(X) \longrightarrow H_i(\partial X, X) \longrightarrow \widetilde{H}_{i-1}(\partial X) \longrightarrow \dots$$

Since X is contractible, $\widetilde{H}_i(X) = 0$ for all i , and $X/\partial X \simeq S^n$. Hence

$$\widetilde{H}_i(\partial X) \cong H^{i+1}(\partial X, X) \cong \widetilde{H}_{i+1}(S^n) \cong \widetilde{H}_i(S^{n-1})$$

Since ∂X and S^{n-1} are all connected,

$$H_i(\partial X) \cong H_i(S^{n-1}), \quad \forall i$$

\square

6. Let (M, g) be a Riemannian manifold and $\pi : \widetilde{M} \rightarrow M$ be a covering map with the pullback metric π^*g on \widetilde{M} .

- (a) Show that M is complete if and only if \widetilde{M} is complete.
- (b) Is the conclusion in (a) true if π is only assumed to be a local isometry? If so, provide a proof. If not, provide a counterexample.

Proof. (a) First suppose \widetilde{M} is complete. For any $p \in M$ and $v \in T_p M$, let $\tilde{p} \in \widetilde{M}$ and $\tilde{v} \in T_{\tilde{p}}(\widetilde{M})$ be a lift of p and v (v can be lifted because π is a local isometry). Since \widetilde{M} is complete, there exists a geodesic $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}'(0) = \tilde{v}$ defined on \mathbb{R} . Since π is a local isometry, $\gamma = \pi \circ \tilde{\gamma}$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$, which implies that M is complete.

Then suppose M is complete. For any $\tilde{p} \in \widetilde{M}$ and $\tilde{v} \in T_{\tilde{p}}(\widetilde{M})$, let $p = \pi(\tilde{p})$ and $v = d\pi_{\tilde{p}}(\tilde{v})$. Since M is complete, there exists a geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ defined on \mathbb{R} . By the path lifting property, γ can be lifted to $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}'(0) = \tilde{v}$. Hence \widetilde{M} is complete.

- (b) The inclusion is false. For example, we can let $i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ be the inclusion and both manifolds are equipped with the standard Euclidean metric. Then i is a local isometry, and \mathbb{R}^2 is complete, but $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not complete.

□

7. Let X be a connected finite-dimensional CW complex, \tilde{X} its universal cover. Let $K(\pi_1(X), 1)$ be the Eilenberg-MacLane space (i.e., the space whose fundamental group is $\pi_1(X)$ and whose higher homotopy groups vanish). Let $Y = \tilde{X} \times K(\pi_1(X), 1)$. Show that:

- (a) X and Y have isomorphic homotopy groups;
- (b) If $\pi_1(X)$ contains an element of order 2, then X and Y are not homotopy equivalent.

Proof. (a) We have

$$\pi_n(\tilde{X} \times K(\pi_1(X), 1)) \cong \pi_n(\tilde{X}) \times \pi_n(K(\pi_1(X), 1))$$

Since \tilde{X} is the universal cover of X ,

$$\pi_n(\tilde{X}) = \begin{cases} \pi_n(X), & n \geq 2 \\ 0, & n = 1 \end{cases}$$

By definition of $K(G, 1)$,

$$\pi_n(K(\pi_1(X), 1)) = \begin{cases} \pi_1(X), & n = 1 \\ 0, & n \geq 2 \end{cases}$$

Hence we know that $\pi_n(Y) \cong \pi_n(X)$, $\forall n$.

- (b) Since $\pi_1(X)$ contains a \mathbb{Z}_2 summand, we write $\pi_1(X) = \mathbb{Z}_2 \times H$. Then $K(\pi_1(X), 1) \simeq K(\mathbb{Z}_2, 1) \times K(H, 1) \simeq \mathbb{RP}^\infty \times K(H, 1)$, hence

$$Y \simeq \mathbb{RP}^\infty \times \tilde{X} \times K(H, 1)$$

Thus $H^*(Y; \mathbb{Z}_2)$ consists of a copy of $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ which is infinite dimensional. But X is a finite dimension CW complex, hence $H^*(X; \mathbb{Z}_2)$ is finite dimensional. Thus X and Y are not homotopy equivalent.

□

8. Let M be a complete Riemannian manifold with nonnegative sectional curvature and $\gamma : [0, 1] \rightarrow M$ be a minimizing geodesic with $\gamma(0) = p, \gamma(1) = q$. Suppose that X is a parallel vector field along γ such that $X \perp \gamma'$.

- (a) Show that for sufficiently small $t > 0$, $\text{dist}(\exp_p(tX(0)), \exp_q(tX(1))) \leq \text{dist}(p, q)$.
- (b) State the condition for which the equality in (a) holds (no proof needed).

Proof. (a) Let F be a variation defined by

$$\begin{aligned} F : (-\varepsilon, \varepsilon) \times [0, 1] &\longrightarrow M \\ (s, t) &\longmapsto \exp_{\gamma(t)}(sX(t)) \end{aligned}$$

where $\exp_{\gamma(t)}$ is a diffeomorphism. Then for fixed $t_0 \in (0, 1)$, $F(-, t_0)$ is a geodesic, and we can view F as a geodesic variation of $F(-, t_0)$ where the variation field is $J(s) = \frac{\partial F}{\partial t}(s, t_0)$ and

$$J(0) = \frac{\partial F}{\partial t}(0, t_0) = \gamma'(t_0)$$

We consider a Jacobi field $\bar{J}(s)$ in Euclidean space with $|\bar{J}(0)| = |\gamma'(t_0)|$. Since $\bar{J}(s)$ is constant, $|\bar{J}(s)| = |\gamma'(t_0)|$ for any s . Now since M has nonnegative sectional curvature, by Rauch comparison theorem, we have

$$|J(s)| \leq |J'(s)| = |\gamma'(t_0)|, \quad \forall s$$

Thus

$$\begin{aligned} \text{dist}(\exp_p(sX(0)), \exp_q(sX(1))) &= \text{dist}(F(s, 0), F(s, 1)) \\ &\leq \int_0^1 \left| \frac{\partial F}{\partial t}(s, t) \right| dt \\ &\leq \int_0^1 |\gamma'(t)| dt \\ &= \text{dist}(p, q) \end{aligned}$$

- (b) If the equality in (a) holds for some t , then by equality condition in Rauch comparison theorem, M has sectional curvature 0 in $F([0, t] \times [0, 1])$, and $J(s)$ is parallel along $F(-, t_0)$ for any $t_0 \leq t$. Hence we conclude that $F([0, t] \times [0, 1])$ is a flat totally geodesic rectangle. \square

9. Let X be the space of “a line contained in a plane in \mathbb{C}^3 ”, namely a point $F \in X$ is described as a sequence

$$\{0\} \subset l \subset P \subset \mathbb{C}^3,$$

where P is a complex plane and l is a complex line.

- (a) The $\text{GL}(3, \mathbb{C})$ -action on \mathbb{C}^3 induces an action on X . Describe the stabilizer of the action and prove that X is a smooth manifold.
(b) Compute the cohomology ring $H^*(X; \mathbb{R})$.

Proof. (a) For a point $F \in X$ and $A \in \text{GL}(3, \mathbb{C})$, if A is in the stabilizer of F , then $A \cdot l = l$, $A \cdot P = P$. Hence if we choose a basis $\{v, w, u\}$ of \mathbb{C}^3 such that

$$l = \mathbb{C}\{v\}, \quad P = \mathbb{C}\{v, w\}$$

then A is represented by an upper-triangle matrix. Conversely, if A is upper-triangle under this basis, then $A \cdot F = F$. Thus the stabilizer G is a Lie group consisted of all upper-triangle matrices acting on $\text{GL}(3, \mathbb{C})$ and X is the orbit space $\text{GL}(3, \mathbb{C})/G$. This action is smooth, free, and proper, hence X is smooth manifold with complex dimension

$$\dim X = \dim \text{GL}(3, \mathbb{C}) - \dim G = 9 - 6 = 3$$

(see Lee smooth manifolds 21.10)

- (b) Consider a map $\pi : X \rightarrow \mathbb{CP}^2$ defined by $\pi(F) = [P]$, then this is a fiber bundle with fiber \mathbb{CP}^1 . By Leray-Hirsch theorem,

$$H^*(X; \mathbb{R}) \cong H^*(\mathbb{CP}^1; \mathbb{R}) \otimes H^*(\mathbb{CP}^2; \mathbb{R})$$

as \mathbb{R} -modules. Suppose

$$\begin{aligned} H^*(\mathbb{CP}^1; \mathbb{R}) &= \mathbb{R}[\alpha]/(\alpha^2), \quad |\alpha| = 2 \\ H^*(\mathbb{CP}^2; \mathbb{R}) &= \mathbb{R}[\beta]/(\beta^3), \quad |\beta| = 2 \end{aligned}$$

then

$$H^*(X; \mathbb{R}) \subseteq \mathbb{R}[\alpha, \beta]/(\alpha^2, \beta^3, \alpha\beta - \beta\alpha) := R$$

as a subring. However, R has dimension 1 at grading 0,6; dimension 2 at grading 2,4; dimension 0 at other gradings, which has already coincided with $H^*(X; \mathbb{R})$. Hence $H^*(X; \mathbb{R})$ doesn't have other relations, that is,

$$H^*(X; \mathbb{R}) = R = \mathbb{R}[\alpha, \beta]/(\alpha^2, \beta^3, \alpha\beta - \beta\alpha), \quad |\alpha| = |\beta| = 2$$

□

10. Let (M, g) be a complete compact oriented Riemannian manifold with dimension $n \geq 2$. We say a vector field X on M is a conformal Killing field if $\mathcal{L}_X g = fg$ for some smooth function f , where \mathcal{L}_X denotes the Lie derivative with respect to X . If $\text{Ric}(M) \leq 0$, show that every conformal Killing field on M is parallel. (Hint: Consider $\nabla_X X - \text{div}(X)X$.)

Proof. Suppose X is a conformal Killing field. For any $Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned} f\langle Y, Z \rangle &= \mathcal{L}_X g(Y, Z) \\ &= X\langle Y, Z \rangle - \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle \\ &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle - \langle \nabla_X Y - \nabla_Y X, Z \rangle - \langle Y, \nabla_X Z - \nabla_Z X \rangle \\ &= \langle \nabla_Y X, Z \rangle + \langle Y, \nabla_Z X \rangle \end{aligned}$$

If we write $A = \nabla X \in \text{End}(\mathfrak{X}(M))$, then the above equality can be written as

$$A + A^T = fI_n$$

By an easy version Bochner formula, we have

$$\nabla_X \text{div} X + \text{tr}(A^2) + \text{Ric}(X) = \text{div} W$$

where $W = \nabla_X X$ (One can derive this from taking trace of the expansion of $\nabla_X A$). Then we have

$$\text{div} X = \text{tr}(\nabla X) = \text{tr}(A) = \frac{1}{2} \text{tr}(A + A^T) = \text{tr}(fI_n) = \frac{nf}{2}$$

and

$$\text{tr}(A^2) = \sum_{i=1}^n \langle A^2 e_i, e_i \rangle = \sum_{i=1}^n \langle (fI_n - A^T) A e_i, e_i \rangle = f \text{tr}(A) - \sum_{i=1}^n \langle A e_i, A e_i \rangle = \frac{nf^2}{2} - |A|^2$$

By expanding div in local coordinates, we have

$$\text{div}(gZ) = g \text{div} Z + Zg, \quad \forall g \in C^\infty(M), Z \in \mathfrak{X}(M)$$

Hence

$$\text{div}(\text{div}(X)X) = \text{div}(X)^2 + \nabla_X \text{div} X = \frac{n^2 f^2}{4} + \nabla_X \text{div} X$$

Finally, we consider

$$\text{div}(\nabla_X X - \text{div}(X)X) = \text{div} W - \text{div}(\text{div}(X)X) = \frac{n(2-n)f^2}{4} + \text{Ric}(X) - |A|^2$$

Integrate both sides on M , and by divergence theorem, we have

$$0 = \int_M \frac{n(2-n)f^2}{4} + \text{Ric}(X) - |A|^2$$

Since $\text{Ric}(M) \leq 0$, $\text{Ric}(X) \leq 0$, thus all three terms in the right side are ≤ 0 . Hence all equal to 0. In particular, $A = 0$, i.e. X is parallel. □

Remark. The proposition doesn't hold for $n = 1$. For example, we can take $M = S^1$, $g = d\theta^2$, $X = \cos \theta \partial_\theta$, we can check that X is a conformal Killing field, but X is not parallel.