HOMOLOGY THEORY - CONNECTION BETWEEN ALGEBRA AND GEOMETRY

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1. Introduction

In this paper, we will talk about homology theory, which serves as a bridge connecting algebra and geometry.

To begin with, we will take an overall view of the history of homology theory. For convenience, we then talk about singular homology theory, which is also the most intuitive. We can learn many basic techniques in solving homology problems from it. As a transition, we then sketch the axioms for homology theory. From this section, our discussion will turn from geometry into algebra, delving into the essential properties of homology theory. Next section we extend the singular homology theory to general homological algebra using the language of categories, mainly abelian categories. In the last section, we introduce topology K-theory to broaden our understanding of homology theory.

2. History of Homology Theory

Homology theory can be seen as a tool to solve lots of topological problems when first introduced by topologists. So let's first talk about some history of topology.

Topology is a subject that studies the behavior of geometric figures under the continuous deformation. Although it is the product of 20th century, its origin can be dated back to Euler and Leibniz in 17th century, and even earlier, Descartes. Euler concentrated on graph theory, and he was the first to solve the seven bridges problem. Besides, he proposed Euler's theorem of convex polyhedron:

Theorem 2.0.1 (Euler)**.** *For a convex polyhedron, denote numbers of its vertexes, edges and faces by V, E, F respectively. Then*

$$V - E + F = 2$$

It is worth mentioning that this theorem was later generalized as Gauss formula in differential geometry. Leibniz tried to study the geometric properties of different figures, and he called this subject "analysis situs", which means analysis of positions. After these two great mathematicians, Listing invented the word "topology", Möbius discovered the Möbius bundle, Riemann studied connectivity orders of surfaces, which is a topological invariant, and Betti extended it to more general cases.

Then we came to the late 19th century and early 20th century, which is the era of Poincaré. The word "homology" was first introduced by Poincaré in his paper *Sur l'Analysis situs*[1] in 1892. However, the definition of homology by Poincaré was quite different from what it is now. Now we introduce it. First he defined the manifold and so called "total boundary" of a manifold.

Definition 2.0.2 (Poincaré)**.** *A set in $\mathbb{R}^n$ which is confined by p equations and q inequalities*

$$\begin{cases} F_1(x_1, x_2, \cdots x_n) = 0 \\ F_2(x_1, x_2, \cdots x_n) = 0 \\ \cdots \\ F_p(x_1, x_2, \cdots x_n) = 0 \\ \varphi_1(x_1, x_2, \cdots x_n) > 0 \\ \varphi_2(x_1, x_2, \cdots x_n) > 0 \\ \cdots \\ \varphi_q(x_1, x_2, \cdots x_n) > 0 \end{cases}$$

*is called an $(n-p)$**-dimensional manifold**, F, φ are C^1 function and the rank of Jacobi matrix of F is p. If we revise any inequality $\varphi_\alpha > 0$ to be $\varphi_\alpha = 0$, then the set confined by this new equation system is called the "**total boundary**" of the manifold.*

Then he defined what homology is .

Definition 2.0.3 (Poincaré)**.** *Given a p-dimensional manifold W and a set of $q-1$-dimensional submanifolds $v_1, v_2, \cdots v_\lambda (q \leqslant p)$ without boundary. If their union consists of the "total boundary" of some connected q-dimensional submanifold V of W, we write*

$$v_1 + v_2 + \cdots + v_\lambda \sim 0$$

*and call this relation a "**homology**" between $v_1, \cdots, v_\lambda$.*

He then write down the more general cases, like

$$k_1 v_1 + k_2 v_2 \sim k_3 v_3 + k_4 v_4$$

It means that there exists a q-dimensional manifold V whose "total boundary" is consisted of k_1 slightly different copies of v_1, k_2 slightly different copies of v_2, k_3

slightly different copies of "opposite manifold" of v_3 and k_4 slightly different copies of "opposite manifold" of v_4. He then claimed that homology relations can be added and subtracted, so the homology relation

$$k_1v_1 + k_2v_2 + \cdots + k_\lambda v_\lambda \sim 0$$

has specific meaning. He also claimed homology relations $\Sigma_j k_j v_j$ and $\Sigma_j ck_j v_j$ are equivalent.

It's a pity that this definition was not rigorous enough, such as "slightly different". Moreover, homology relations $\Sigma_j k_j v_j$ and $\Sigma_j ck_j v_j$ may not be equivalent, which leads to the torsion coefficients.

He meanwhile introduced Betti numbers, fundamental groups, the well-known dual theorem and so on. Although there were many mistakes and contents that were not rigorous as above, he laid a solid foundation for homology theory and homotopy theory without doubt.

The next mathematician springing into fame in topology was Brouwer. In 1910, He introduced the degree of a map. The method he used to define the degree was now called simplicial approximation. This was indeed a great contribution since it made combinatorial topology rigorous. Brouwer then used simplicial approximation to solve the problem of dimension invariance of $\mathbb{R}^n$ which troubled mathematicians for a long time in 19th century. That is:

Theorem 2.0.4 (Brouwer). *For $m \neq n$, there doesn't exist homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$.*

Moreover, Brouwer extended Jordan curve theorem to higher dimensions and used the degree of a map to construct several fixed point theorem. We will talk about them in the next section.

In the last 1920s, Alexander and Veblen put a lot effort in triangulation and extended it to more general spaces — complexes. This homology theory was then called simplicial homology.

After that, a significant event which changed the history of homology theory occurred. With the development of algebra, a lot of mathematicians like Weyl, Hopf, Noether and Vietoris noted that we could simply use algebra to define homology. After repeated modifications, from groups to modules, finally came the definition of homology which is close to now.

Definition 2.0.5. *For any module(group) sequence $C_\bullet = (C_j)_{j\geqslant 0}$ with homomorphisms $b_j : C_j \to C_{j-1} (j \geqslant 0)$ satisfying $b_0 = 0$, $b_{j-1} \circ b_j = 0$ for $j \geqslant 1$. Define the j^{th}* ***homology module(group)*** *to be*

$$H_j = \ker b_j / \operatorname{im} b_{j+1}$$

Using this new concept, Mayer studied the problem proposed by Vietoris: Suppose each C_j has a basis decomposed as the union of two subsets B_j^1 and B_j^2. C_j^1, C_j^2 and C_j^3 are generated by B_j^1, B_j^2 and $B_j^1 \cap B_j^2$ respectively, then what's the relationship between H_j^1, H_j^2 and H_j^3. Mayer finished this study in 1930, which are now called the Mayer-Vietoris sequence and widely used in homology theory.

In the following years, mathematics invented different homology theories for different purposes, including singular homology theory, Čech homology and de Rham cohomology.

Singular homology is an improvement of simiplicial hommology theory. In the simiplicial homology theory, we should divide the continuum into discrete blocks, leading to some difficulties of proving the invariance. In 1915, Alexander used linear combinations of singular simplexes to define a new chain — the singular chain, and introduced singular homology theory. After improvement of Veblen, van der Waerden and Lefschetz, Eilenberg overcame all difficulties and completed this theory in 1943. We will talk about singular homology theory in detail in the next section, which gives credit to Eilenberg.

Čech homology was introduced by Čech in 1932, which is a powerful tool in algebraic geometry now.

de Rham cohomology is based on the differential form on the manifold. Here we skip some basic construction and only sketch the most important part.

Suppose D is compact subset of a manifold M, then the Stokes theorem implies the dual correspondence between the boundary operator ∂ and coboundary operator d. That is, if we denote

$$(\partial D, \omega) = \int_{\partial D} \omega, \ (D, \mathrm{d}\omega) = \int_D \mathrm{d}\omega$$

then the Stokes theorem says that

$$(\partial D, \omega) = (D, \mathrm{d}\omega)$$

If we use $A^r(M)$ denote all r-forms, we can consider $A^r(M)$ as a cochain, and $\mathrm{d} : A^r(M) \to A^{r+1}(M)$ is the coboundary operator of this chain. Poincaré Lemma gives that $\mathrm{d} \circ \mathrm{d} = 0$, thus we can define the r^{th} homology group $H^r(M, \mathbb{R})$ of this chain, called the r^{th} **de Rham cohomology group** of M. Then the famous de Rham theorem says that

Theorem 2.0.6 (de Rham Theorem)**.** *Suppose M is a compact smooth manifold, then its r^{th} de Rham cohomology group is isomorphic to its r^{th} homology group. In particular, if we let*

$$\dim H^r(M, \mathbb{R}) = b_r$$

then b_r is just the r^{th} Betti number of M.

However, it should be noted that when de Rham proposed this theorem, there didn't have the concept of cohomology. Product and cohomology were introduced by Alexander and Kolmogoroff in 1935 independently.

The advantage of cohomology compared to homology is that for any finite simplicial complex K, we can define a bilinear map

$$H^p(K) \times H^q(K) \to H^{p+q}(K)$$

making cohomology into a ring and yielding more structures. Now we introduce the notations proposed by Whitney.

Given a Euclidean simplicial complex K, and an order ω of its vertices, define a bilinear map between cochains

$$S^p(K) \times S^q(K) \to S^{p+q}(K)$$

written $(f, g) \mapsto f \cup g$. For any ordered $(p+q)$-dimensional simplex $(x_0, x_1, \cdots, x_{p+q})$, let

$$(f \cup g)(x_0, \cdots x_{p+q}) = f(x_0, \cdots x_p) g(x_p, \cdots x_{p+q})$$

We can check that if f, g are cocycles, then $f \cup g$ is a cocycle, and the homology class of $f \cup g$ is only dependent of the homology class a of f and the homology class b of g, so we can write $a \cup b$. Moreover, $a \cup b$ is independent of the order ω of vertices of K, thus we define a linear map

$$H^p(K) \times H^q(K) \to H^{p+q}(K)$$
$$(a,b) \mapsto a \cup b$$

which is called the **cup product**. The **cap product**

$$H_{p+q}(X) \times H^p(X) \to H_q(X)$$
$$(u,a) \mapsto u \cap a$$

is defined similarly using the duality of homology and cohomology.

Now we turn to some algebraic aspects of homology. Since Noether introduced groups,modules, rings and their homomorphisms into topology in 1920s, algebraic topology had been developed rapidly. Until the mid-1940s, a new branch of algebra emerged — homological algebra.

This began with the discussions about exact sequences by Hurewicz in 1941. He first put cohomology groups into a sequence

$$\cdots \longrightarrow H^q(A) \xrightarrow{j^*} H^q(B) \xrightarrow{\delta} H^{q+1}(A\backslash B) \xrightarrow{v} H^{q+1}(A) \xrightarrow{j^*} H^{q+1}(B) \longrightarrow \cdots$$

where A is a locally compact space and B is a closed subset, $j : B \to A$ is the inclusion map, v maps a cocycle in $A\backslash B$ into its extension in A. δ is defined in the following sense: if $\overline{z}$ is a $q-$cohomology class in B and z is a representative cocycle, consider z as a restriction of $q-$cochain x in A to B, so the restriction of $\delta_q x$ to B is 0, thus can be considered as a $(q+1)-$cocycle in $A\backslash B$. We denote the cohomology class of $\delta_q x$ in $H^{q+1}(A\backslash B)$ as $\delta\overline{z}$. Hurewicz proved that every image of some homomorphism is the kernel of the next homomorphism in the sequence, which is called **exactness** now.

In 1947, Kelley and Pitcher proposed the term "exact sequences" in their book. Moreover, these consequences can be extended to any commutative groups and group homomorphisms, namely "short exact sequences can be extended to a long sequence". We will talk a lot about this in the following sections. Meanwhile, some famous lemmas were proposed at that time, such as the "five lemma" which appeared first in the famous book *Foundations of Algebraic Topology*[3] written by Eilenberg and Steenrod.

Since 1935, Čech had been studying the homology group $H_p(X;G)$ with coefficients in a group G, especially tensor and Tor functors. For a chain complex $C_\bullet = (C_j)$ of free $\mathbb{Z}-$modules, denote the $p-$cycles as Z_p and $p-$boundaries as B_p. Since the kernel and image of $\partial_p : C_p \to Z_{p-1}$ are all free, there exists a free submodule of F_p of C_p such that $C_p = Z_p \oplus F_p$ and hence $C_p \otimes G = (Z_p \otimes G) \oplus (F_p \otimes G)$. The kernel of the boundary map $\partial_p \otimes 1 : C_p \otimes G \to Z_{p-1} \otimes G$ contains $Z_p \otimes G$, but maybe larger. So we write as $Z_p(C_\bullet \otimes G) = Z_p \otimes G \oplus Z'_p$, where $Z'_p = \ker(\partial_p \otimes 1) \cap (F_p \otimes G)$. By exactness of

$$C_{p+1} \xrightarrow{\partial_{p+1}} Z_p \to H_p \to 0$$

we get exactness of

$$C_{p+1} \otimes G \xrightarrow{\partial_{p+1} \otimes 1} Z_p \otimes G \to H_p \otimes G \to 0$$

thus

$$H_p(C_\bullet; G) \cong (H_p \otimes G) \oplus Z'_p$$

Since the image of $F_p \otimes G$ under $\partial_p \otimes 1$ is isomorphic to $B_{p-1} \otimes G$, thus

$$0 \to Z'_p \to B_{p-1} \otimes G \to Z_{p-1} \otimes G \to H_{p-1} \otimes G \to 0$$

is an exact sequence. We denote Z'_p as $H_{p-1} * G$ or $\mathrm{Tor}(H_{p-1}, G)$, called the **torsion product** of H_{p-1} and G. Then we have the short exact sequence

$$H_p(C_\bullet; G) \cong (H_p \otimes G) \oplus \mathrm{Tor}(H_{p-1}, G)$$

This also shows that the $p-$homology group $H_p(C; G)$ is completely determined by H_p, H_{p-1} and G. So we also call this **the universal coefficients theorem**.

The universal coefficient theorem for cohomology was proposed a few years later. Suppose F is a free commutative group and R is a subgroup. Let $H = F/R$ be the quotient group, then we have the short exact sequence

$$0 \to R \xrightarrow{i} F \xrightarrow{j} H \to 0$$

Suppose G is any commutative group, then

$$0 \to \mathrm{Hom}(H, G) \xrightarrow{j^\#} \mathrm{Hom}(F, G) \xrightarrow{i^\#} \mathrm{Hom}(R, G)$$

is a exact sequence. Note that Hom functor is not right exact, hence $i^\#$ is not surjective in general. Thus we can add a term $\mathrm{Ext}(H, G) = \mathrm{Hom}(R, G)/\mathrm{im}\ i^\#$ to construct a complete exact sequence:

$$0 \to \mathrm{Hom}(H, G) \xrightarrow{j^\#} \mathrm{Hom}(F, G) \xrightarrow{i^\#} \mathrm{Hom}(R, G) \longrightarrow \mathrm{Ext}(H, G) \to 0$$

This new functor can be used to state the universal coefficient theorem for cohomology. Suppose $C_\bullet$ is a chain complex of free $\mathbb{Z}-$modules, G is any commutative group, then $\mathrm{Hom}(C_\bullet, G)$ is a cochain complex. We denote its homology by $H^\bullet(C_\bullet, G)$. Then

$$H^p(C_\bullet, G) \cong \mathrm{Hom}(H_p(C_\bullet), G) \oplus \mathrm{Ext}(H_{p-1}(C_\bullet), G)$$

Now let's talk about another algebraist making great contributions to homology algebra — Eilenberg. After Elienberg and Maclane applied the Ext functor to homology theory, they went further to consider the general case without using groups or modules. They published a paper in which they introduced the term "**category**" to represent a type of mathematical objects having the same structure. These objects have "**morphisms**" between, and different categories have **functors** between. In particular, homology is a covariant functor from the category of chain complexes to the category of abelian groups. These concepts became the foundations of modern homological algebra and we will talk about them in the following sections.

Besides, Eilenberg introduced the chain homotopy for chain complexes. Suppose $C_\bullet = (C_j)$ and $C'_\bullet = (C'_j)$ are two chain complexes, $u_\bullet = (u_j)$ and $v_\bullet = (v_j)$ are two chain maps from $C_\bullet$ to $C'_\bullet$. A **chain homotopy** from $u_\bullet$ to $v_\bullet$ is a homomorphism sequence $h_\bullet = (h_j),\ h_j : C_j \to C'_{j+1}$, such that for each $j \in \mathbb{Z}$, we have

$$v_j - u_j = \partial'_{j+1} \circ h_j + h_{j-1} \circ \partial_j$$

A property is that $u_\bullet$ and $v_\bullet$ induce the same homomorphism $H_\bullet(u_\bullet) = H_\bullet(v_\bullet)$ between $H_\bullet(C_\bullet)$ and $H_\bullet(C'_\bullet)$.

More importantly, Eilenberg and Steenrod defined the axiomatic system of homology theory and cohomology theory in [3]. Then this was extended to generalized homology and cohomology theory in 1959. Two famous applications of the generalized theory are the K-theory and corbodism theory in differential topology. The advantage of generalized homology and cohomology theory is that if we know a system satisfying several axioms, then we have all properties of homology theory without complicated inference.

After the 1960s, the development of homological algebra step into modernization using the language of category theory. The main achievements were relied on the abelian categories and sheaf cohomology done by Grothendieck and Serre. Advanced topics involving derived categories and triangulated categories are still ongoing now.

3. Singular Homology Theory

Now let's talk about the singular homology theory. The reason why to choose singular homology theory has three: firstly singular homology is easiest to understand and the most intuitive, secondly it preserves lots of the techniques and methods in homological algebra though it is easy, thirdly what the author encountered first when studying homology theory was just singular homology.

3.1. Chain Complexes and Chain Maps.

We begin by setting the stages for chain complexes and chain maps due to completeness although we have mentioned some concepts and properties in the last section.

Definition 3.1.1. *A **chain complex** $C = \{C_q, \partial_q\}$ is a sequence of Abelian groups C_q(called the $q-$**chain group**) and homomorphisms $\partial_q : C_q \to C_{q-1}$(called the $q-$**boundary operator**)*

$$\cdots \longrightarrow C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} C_{q-2} \longrightarrow \cdots$$

satisfying: for each q, we have $\partial_q \circ \partial_{q+1} = 0$.

Definition 3.1.2. *For a chain complex $C = \{C_q, \partial_q\}$, define the $q-$**cycle group** to be*

$$Z_q(C) := \ker\ \partial_q$$

*and its elements are called the $q-$**cycles**. Define the $q-$**boundary group** to be*

$$B_q(C) := \operatorname{im} \partial_{q+1}$$

*and its elements are called the $q-$**boundaries**. Since $\partial^2 = 0$, we have $B_q \subset Z_q \subset C_q$. We call the quotient group*

$$H_q(C) := Z_q(C)/B_q(C)$$

*$q-$**homology group** and its elements $q-$**homology classes** of C. We use the notation $[z_q]$ to denote the homology class of $z_q \in Z_q(C)$.*

Definition 3.1.3. *Suppose C, D are chain complexes, a **chain map** $f : C \to D$ is a sequence of homomorphisms $f = \{f_q : C_q \to D_q\}$, satisfying: for each q*

$$\partial_q \circ f_q = f_{q-1} \circ \partial_q$$

That is, the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_{q+1} & \xrightarrow{\partial_{q+1}} & C_q & \xrightarrow{\partial_q} & C_{q-1} & \xrightarrow{\partial_{q-1}} & C_{q-2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{q+1}} & & \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_{q-1}} & & \downarrow{\scriptstyle f_{q-2}} & & \\
\cdots & \longrightarrow & D_{q+1} & \xrightarrow{\partial_{q+1}} & D_q & \xrightarrow{\partial_q} & D_{q-1} & \xrightarrow{\partial_{q-1}} & D_{q-2} & \longrightarrow & \cdots
\end{array}$$

Proposition 3.1.4. *The chain map $f : C \to D$ induces a homomorphism between homology groups $f_* : H_\bullet(C) \to H_\bullet(D)$, defined as:*

$$f_*([z_q]) = [f_q(z_q)],\ \forall z_q \in Z_q(C)$$

Proof. Since the chain map commutes with boundary operators, it maps cycles to cycles and boundaries to boundaries. That is, using the previous commutative diagram, f_q maps $Z_q(C)$ to $Z_q(D)$ and $B_q(C)$ to $B_q(D)$. Thus it induces homomorphisms of quotient groups $f_* : H_q(C) \to H_q(D)$. □

Chain complexes and chain maps make up a category, called the category of chain complexes for short and written as {chain complexes, chain maps}.

A sequence of Abelian group $G_\bullet = \{G_q | q \in \mathbb{Z}\}$ is called a **graded group**. A homomorphism $\phi_* : G_\bullet \to G'_\bullet$ means a homomorphism sequence $\{\phi_q : G_q \to G'_q\}$. Then we have a category consisted of graded groups and homomorphisms. Thus we have constructed a covariant **homology functor** $H_\bullet$: {chain complexes, chain maps} →{graded groups, homomorphisms}.

Definition 3.1.5. *Two chain maps $f, g : C \to D$ are called* ***chain homotopic****, if there exists a sequence of homomorphisms $T = \{T_q : C_q \to D_{q+1}\}$, as the following diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, "\partial"] \arrow[d, "g"', "f"] & C_q \arrow[r, "\partial"] \arrow[d, "g"', "f"] \arrow[dl, dashed, "T"'] & C_{q-1} \arrow[r, "\partial"] \arrow[d, "g"', "f"] \arrow[dl, dashed, "T"'] & C_{q-2} \arrow[r] \arrow[d, "g"', "f"] \arrow[dl, dashed, "T"'] & \cdots \\
\cdots \arrow[r] & D_{q+1} \arrow[r, "\partial"] & D_q \arrow[r, "\partial"] & D_{q-1} \arrow[r, "\partial"] & D_{q-2} \arrow[r] & \cdots
\end{tikzcd}$$

satisfying: for each q, we have

$$\partial_{q+1} \circ T_q + T_{q-1} \circ \partial_q = g_q - f_q$$

Then T is called a **chain homotopy** between f and g, written as

$$f \simeq g : C \to D \text{ or } T : f \simeq g : C \to D$$

Remark 3.1.6. *The chain homotopy relation between chain maps in the algebraic category {chain complexes, chain maps} is another version of the homotopy relation between maps in the topological category {spaces, maps}.*

Theorem 3.1.7. *Suppose $f \simeq g : C \to D$, then $f_* = g_* : H_\bullet(C) \to H_\bullet(D)$. That is, homotopic chain maps induce the same homology homomorphisms.*

Proof. Suppose $T : f \simeq g : C \to D$. For $z_q \in Z_q(C)$, we have

$$g_*([z_q]) - f_*([z_q]) = [g_q(z_q) - f_q(z_q)] = [\partial \circ T(z_q) + T \circ \partial(z_q)] = [\partial \circ T(z_q)] = 0$$

Thus $f_* = g_*$. □

Proposition 3.1.8. *The chain homotopy relation is an equivalence relation.*

Definition 3.1.9. *Two chain complexes C, D are called* ***chain homotopy equivalent****, if there exists chain maps $f : C \to D$ and $g : D \to C$, such that*

$$g \circ f \simeq \mathrm{id}_C : C \to C,\quad f \circ g \simeq \mathrm{id}_D : D \to D$$

Then f and g both are called ***chain homotopy equivalences*** *between C, D, written as $C \simeq D$ or $f : C \simeq D$.*

Proposition 3.1.10. *The chain homotopy equivalence induces the isomorphism between homology groups, thus chain complexes that are chain homotopy equivalent have isomorphic homology groups.*

Proof. This is a direct corollary of Theorem 3.1.7. □

Remark 3.1.11. *The chain homotopy equivalence between chain maps in the algebraic category {chain complexes, chain maps} is another version of the homotopy equivalence between maps in the topological category {spaces, maps}.*

3.2. **Singular Homology Groups.**

Denote $\mathbb{R}^{q+1} = \{(x_0, x_1, \cdots, x_q) | x_i \in \mathbb{R}\}$ the $(q+1)-$dimensional Euclidean space. We use e_i to denote the i^{th} standard basis of $\mathbb{R}^{q+1}$.

Definition 3.2.1. *The* ***standard*** $q-$***simplex*** *is the simplex in* $\mathbb{R}^{q+1}$ *whose vertices are* $e_0, e_1 \cdots e_q$

$$\triangle_q := \left\{ (x_0, x_1, \cdots, x_q) \in \mathbb{R}^{q+1} \middle| 0 \leqslant x_i \leqslant 1, \sum_i x_i = 1 \right\}$$

Definition 3.2.2. *A* ***standard*** $p-$***simplex*** *in a topological space* X *is a continuous map from the standard* $q-$*simplex to* X $\sigma : \triangle_q \to X$.

Example 3.2.3. *If* C *is a convex set in a Euclidean space,* $c_0, c_1, \cdots c_q \in C$*, then there exists a unique linear map* $\triangle_q \to C$ *mapping* $e_0, e_1, \cdots, e_q$ *to* $c_0, c_1, \cdots, c_q$ *respectively. We denote this linear map* $(c_0 c_1 \cdots c_q)$,

$$(c_0 c_1 \cdots c_q) : \triangle_q \to C, \ \sum_i x_i e_i \mapsto \sum_i x_i c_i$$

We can view it as a singular $q-$*simplex in* C*, called the* ***linear singular simplex****.*

Now suppose X is a topological space.

Definition 3.2.4. *There is a free Abelian group* $S_q(X)$ *generated by all singular* $q-$*simplexes, called the* ***singular*** $q-$***chain group*** *of* X*. The elements in* $S_q(X)$ *are called* ***singular*** $q-$ ***chains*** *of* X*. Hence, singular chains are linear combinations of singular simplexes with integral coefficients:*

$$c_q = k_1 \sigma_q^{(1)} + \cdots + k_r \sigma_q^{(r)}, \ \ k_i \in \mathbb{Z}, \sigma_q^{(i)} : \triangle_q \to X$$

Besides, we specify singular chain groups of negative dimensions to be 0.

Definition 3.2.5. *The* ***boundary*** *of a singular* $q-$*simplex* $\sigma : \triangle_q \to X$ *in* X *is defined to be the following singular* $(q-1)-$*chain*

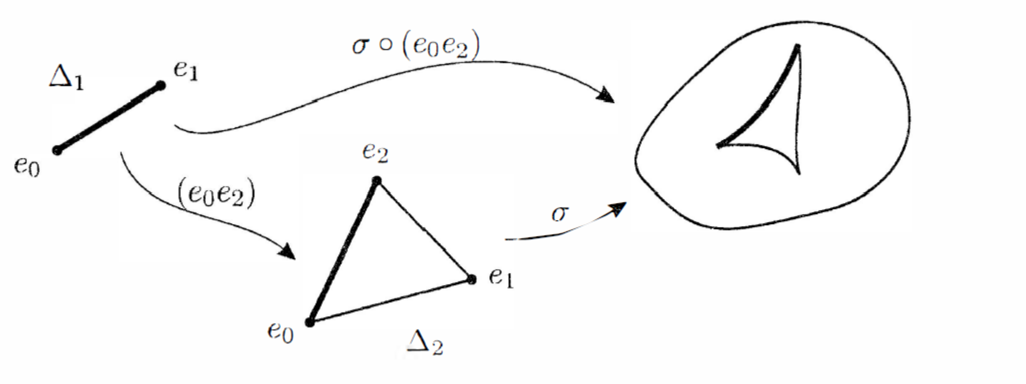
$$\partial \sigma = \partial(\sigma \circ (e_0 \cdots e_q)) := \sum_{i=0}^{q} (-1)^i \sigma \circ (e_0 \cdots \hat{e}_i \cdots e_q)$$

We can do linear extensions and obtain a ***boundary operator*** $\partial_q : S_q(X) \to S_{q-1}(X)$

$$\partial_q (k_1 \sigma_q^{(1)} + \cdots + k_r \sigma_q^{(r)}) := k_1 \partial \sigma_q^{(1)} + \cdots + k_r \partial \sigma_q^{(r)}$$

Besides, we specify boundaries of $0-$*singular chains to be 0.*

Proposition 3.2.6. *Applying the boundary operator* ∂ *twice obtains 0, that is,* $S_\bullet(X) = \{S_q(X), \partial_q\}$ *is a chain complex.*



Proof. First we check the case for a standard simplex. By definition, we have

$$\begin{aligned}
&\partial_{q-1} \circ \partial_q(e_0 \cdots e_q) \\
=&\sum_{i=0}^{q}(-1)^i \partial_{q-1}(e_0 \cdots \hat{e}_i \cdots e_q) \\
=&\sum_{i=0}^{q}(-1)^i \Big\{ \sum_{j<i}(-1)^j(e_0 \cdots \hat{e}_j \cdots \hat{e}_i \cdots e_q) + \sum_{j>i}(-1)^{j-1}(e_0 \cdots \hat{e}_i \cdots \hat{e}_j \cdots e_q) \Big\} \\
=&\sum_{0 \leqslant j<i \leqslant q}(-1)^{i+j}(e_0 \cdots \hat{e}_j \cdots \hat{e}_j \cdots e_q) + \sum_{0 \leqslant i<j \leqslant n}(-1)^{i+j-1}(e_0 \cdots \hat{e}_i \cdots \hat{e}_j \cdots e_q) = \quad 0
\end{aligned}$$

Then we can use $\sigma : \triangle_q \to X$ mapping the formula to X and obtain that $\partial_{q-1} \circ \partial_q(\sigma) = 0$. □

Definition 3.2.7. *The chain complex $S_\bullet = \{S_q(X), \partial_q\}$ is called the* ***singular chain complex*** *of X. The homology groups of $S_\bullet(X)$ are called the* ***singular homology groups*** *of X, denoted by*

$$H_\bullet(X) := H_\bullet(S_\bullet(X))$$

Definition 3.2.8. *Suppose $f : X \to Y$ maps every singular simplex $\sigma : \triangle_q \to X$ in X to a singular simplex in Y*

$$f_\#(\sigma) := f \circ \sigma$$

Then by linear extensions, we get a homomorphism $f_\# : S_q(X) \to S_q(Y)$.

Proposition 3.2.9. *$f_\#$ commutes with the boundary operator, that is, $\{f_\# : S_q(X) \to S_q(Y)\}$ is a chain map $f_\# : S_\bullet(X) \to S_\bullet(Y)$.*

Proof. It is direct due to the composition law of maps. □

Definition 3.2.10. *The homology homomorphism $f_* : H_\bullet(X) \to H_\bullet(Y)$ induced by $f : X \to Y$ is the homology homomorphism $(f_\#)_* : H_\bullet(S_\bullet(X)) \to H_\bullet(S_\bullet(Y))$ induced by $f_\# : S_\bullet(X) \to S_\bullet(Y)$.*

Thus we get a covariant functor $S_\bullet$:{spaces, maps} $\to$ {chain complexes, chain maps} from the category of topological spaces to the category of chain complexes, called the **singular chain functor**. Composing with the homology functor in the last subsection, we get a covariant **singular homology functor** $H_\bullet$:{spaces, maps} $\to$ {graded groups, homomorphisms}.

As the corollary of properties of functors, we immediately obtain:

Proposition 3.2.11 (Topological invariance of singular homology groups)**.** *Homeomorphic topological spaces have isomorphic singular homology groups.*

Proposition 3.2.12. *Use pt to denote the space of a single point, then*

$$H_q(\mathrm{pt}) = \begin{cases} \mathbb{Z}, \ q = 0 \\ 0, \ q \neq 0 \end{cases}$$

Proof. For any $q \geqslant 0$, there is only one singular complex $\sigma_q : \triangle_q \to \mathrm{pt}$. Then $S_q(\mathrm{pt}) = \mathbb{Z}$ for all q. By definition of boundaries, we know that when $q = 0$ or q is odd, $\partial\sigma_q = 0$. For the other q, $\partial\sigma_q = \sigma_{q-1}$. So the chain complex $S_\bullet(\mathrm{pt})$ is like:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ 0\ } \mathbb{Z} \xleftarrow[\cong]{\ 1\ } \mathbb{Z} \xleftarrow{\ 0\ } \mathbb{Z} \xleftarrow[\cong]{\ 1\ } \mathbb{Z}$$

Thus the homology groups are as above. □

For any topological space X, the $0-$singular simplexes are just one point. The sum of coefficients of a $0-$chain $c_0 = k_1 a_1 + \cdots k_r a_r$

$$\varepsilon(c_0) := k_1 + \cdots + k_r$$

is called the Kronecker index of c_0. Then $\varepsilon : S_0(X) \to \mathbb{Z}$ is a homomorphism.

Since the boundary of each point is 0, we have $Z_0(X) = S_0(X)$. Besides, the Kronecker coefficient of the boundary of each $1-$singular complex is 0, thus $B_0(X) \subset \ker(\varepsilon)$. Then $\varepsilon : S_0(X) \to \mathbb{Z}$ induces a surjective homomorphism $\varepsilon : H_0(X) \to \mathbb{Z}$.

Proposition 3.2.13. *Suppose X is path-connected, then $\varepsilon : H_0(X) \to \mathbb{Z}$ is an isomorphism.*

Proof. We only need to prove that $\ker(\varepsilon) \subset B_0(X)$. Choose a basepoint b for X, then for any point a, there exists a path from b to a. That is, there exists a $1-$singular simplex σ such that $\partial\sigma = a - b$. Then for any $0-$chain c_0, we have $c_0 - \varepsilon(c_0) \cdot b \in B_0(X)$. In particular, this implies $\ker(\varepsilon) \subset B_0(X)$. □

For the space that is not path-connected, we can divide it into unions of path-connected components and then discuss its homology groups.

Definition 3.2.14. *For a family of chain complexes $\{C_\lambda | \lambda \in \Lambda\}$, Λ is a set of indexes, $C_\lambda = \{C_{\lambda q}, \partial_{\lambda q}\}$, define the **direct sum** of these chain complexes to be a chain complex*

$$\bigoplus_{\lambda\in\Lambda} C_\lambda := \{ \bigoplus_{\lambda\in\Lambda} C_{\lambda q}, \bigoplus_{\lambda\in\Lambda} \partial_{\lambda q} \}$$

Theorem 3.2.15. *Suppose $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ is the decomposition of path-connected components, then*

$$H_\bullet(X) = \bigoplus_{\lambda\in\Lambda} H_\bullet(X_\lambda)$$

That is, for each dimension q, we have $H_q(X) = \bigoplus\limits_{\lambda\in\Lambda} H_\bullet(X_\lambda)$.

As a corollary of theorem 3.2.13 and 3.2.15, we obtain:

Corollary 3.2.16. *A topological space X is path-connected if and only if $H_0(X) = \mathbb{Z}$.*

We note that almost all homology groups of the space pt of a single point is 0 except the $0-$singular homology group. Now we want to introduce a homology such that all homology groups of pt are 0.

Definition 3.2.17. *The **augmented singular chain complex** $\tilde{S}_\bullet(X) = \{\tilde{S}_q(X), \tilde{\partial}_q\}$ of a topological space X is defined to be*

$$\tilde{S}_q(X) := \begin{cases} S_q(X), & q \neq -1 \\ \mathbb{Z}, & q = -1 \end{cases} \quad \tilde{\partial}_q := \begin{cases} \partial_q, & q \neq 0 \\ \varepsilon, & q = 0 \end{cases}$$

The chain map $f_\# : S_q(X) \to S_q(Y)$ induced by $f : X \to Y$ preserves the Kronecker indexes of $0-$chains, thus it can be augmented to the chain map $f_\# : \tilde{S}_\bullet(X) \to \tilde{S}_\bullet(Y)$, defined to be the same when $q \geqslant 0$ and $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ for $f_\# : \tilde{S}_{-1}(X) \to \tilde{S}_{-1}(Y)$.

Definition 3.2.18. *The **reduced singular homology groups** $\tilde{H}_\bullet(X) = \{\tilde{H}_q(X)\}$ of a topological space X are defined to be the homology groups of the augmented chain complex:*

$$\tilde{H}_\bullet(X) := H_\bullet(\tilde{S}_\bullet(X))$$

The homomorphism $f_* : \tilde{H}_\bullet(X) \to \tilde{H}_\bullet(Y)$ induced by $f : X \to Y$ is defined to be the homology homomorphism induced by the chain map $f_\# : \tilde{S}_\bullet(X) \to \tilde{S}_\bullet(Y)$.

By definition, the reduced singular homology groups are different from the singular homology groups only in the case of 0-dimension. It is straightforward to see that $\tilde{H}_\bullet(pt) = 0$ and $\tilde{H}_0(X) = 0$ if and only if X is path-connected.

Example 3.2.19. *Suppose A is consisted of two points: $A = \{a_0, a_1\}$, then*

$$\tilde{H}_q(A) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0 \end{cases}$$

Now we come to one of the most important properties of homology groups.

Theorem 3.2.20 (Homotopy invariance)**.** *Suppose $f \simeq g : X \to Y$ are homotopic maps, then $f_\# \simeq g_\# : S_\bullet(X) \to S_\bullet(Y)$ are homotopic chain maps, thus inducing the same homology homomorphisms $f_* = g_* : H_*(X) \to H_*(Y)$.*

Before proving the theorem, we consider the cylinder based on standard simplexes since the homotopy relation has close correspondence with the cylinder.

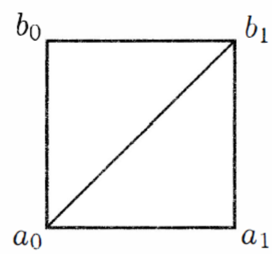
Definition 3.2.21. *Consider the cylinder $\triangle\_q \times I$ based on the standard simplex $\triangle_q$. Denote vertices of the lower bottom $(e_i, 0)$ by a_i and vertices of the upper bottom $(e_i, 1)$ by b_i. Define a $(q+1)-$**cylinder chain** in $\triangle_q \times I$*

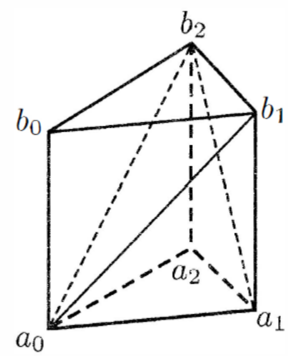
$$P(e_0 \cdots e_q) := \sum_{i=0}^{q} (-1)^i a_0 \cdots a_i b_i \cdots b_q$$

The following shows the cylinder chains for cases $q = 1$ and $q = 2$.

Lemma 3.2.22 (The formula for boundaries of cylinder chains)**.**

$$\partial P(c_0 \cdots c_q) = b_0 \cdots b_q - a_0 \cdots a_q - \sum_{i=0}^{q} (-1)^i P(e_0 \cdots \hat{e}_i \cdots e_q)$$





Proof. Check by direct calculation:

$$
\begin{aligned}
&\partial P(e_0 \cdots e_q) \\
=& \sum_{i=0}^{q} \sum_{j \leqslant i} (-1)^{i+j} a_0 \cdots \hat{a}_j \cdots a_i b_i \cdots b_q + \sum_{i=0}^{q} \sum_{j \geqslant i} (-1)^{i+j+1} a_0 \cdots a_i b_i \cdots \hat{b}_j \cdots b_q \\
=& \sum_{i=0}^{q} a_0 \cdots a_{i-1} b_i \cdots b_q - \sum_{i=0}^{q} a_0 \cdots a_i b_{i+1} \cdots b_q \\
& - \sum_{i=0}^{q} \sum_{i<j} (-1)^{i+j} a_0 \cdots a_i b_i \cdots \hat{b}_j \cdots b_q - \sum_{i=0}^{q} \sum_{i>j} (-1)^{i+j-1} a_0 \cdots \hat{a}_j \cdots a_i b_i \cdots b_q \\
=& \, b_0 \cdots b_q - a_0 \cdots a_q \\
& - \sum_{j=0}^{q} (-1)^j \Big\{ \sum_{i<j} (-1)^i a_0 \cdots a_i b_i \cdots \hat{b}_j \cdots b_q + \sum_{i>j} (-1)^{i-1} a_0 \cdots \hat{a}_j \cdots a_i b_i \cdots b_q \Big\} \\
=& \, b_0 \cdots b_q - a_0 \cdots a_q - \sum_{j=0}^{q} (-1)^j P(e_0 \cdots \hat{e}_j \cdots e_q)
\end{aligned}
$$

□

Now let's proof theorem 3.2.20.

Proof. Let $\iota_0, \iota_1 : X \to X \times I$ denote maps

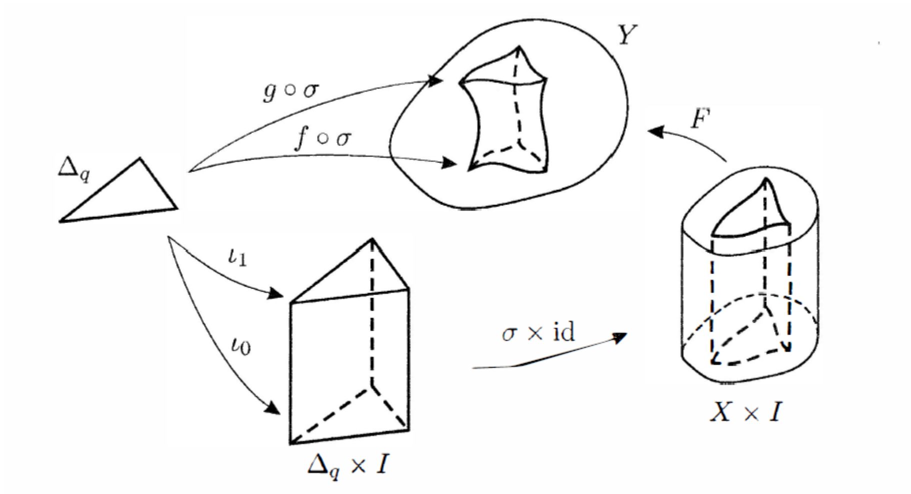
$$\iota_0(x) = (x, 0), \ \iota_1(x) = (x, 1), \ x \in X$$

Then we only need to show $\iota_{0\#} \simeq \iota_{1\#} : S_\bullet(X) \to S_\bullet(X \times I)$.

In fact, suppose $F : X \times I \to Y$ is a homotopy connecting f, g, then $f = F \circ \iota_0, \ g = F \circ \iota_1$, thus $\iota_{0\#} \simeq \iota_{1\#}$ implies $f_\# = F_\# \circ \iota_{0\#} \simeq F_\# \circ \iota_{1\#} \simeq g_\#$.

For a singular simplex $\sigma : \triangle_q \to X$ in X, define a singular chain in $X \times I$:

$$P(\sigma) := (\sigma \times \mathrm{id})_\# P(e_0 \cdots e_q)$$



Then we do linear extensions and obtain a homomorphism $P : S_q(X) \to S_{q+1}(X \times I)$. Then lemma 3.2.22 tells us that

$$\begin{aligned}\partial P(\sigma) &= (\sigma \times \mathrm{id})_{\#} \partial P(e_0 \cdots e_q) \\ &= (\sigma \times \mathrm{id})_{\#} \Big(b_0 \cdots b_q - a_0 \cdots a_q - \sum_i (-1)^i P(e_0 \cdots \hat{e}_i \cdots e_q) \Big) \\ &= \iota_{1\#}(\sigma) - \iota_{0\#(\sigma)} - P(\partial\sigma)\end{aligned}$$

Thus $\partial \circ P + P \circ \partial = \iota_{1\#} - \iota_{0\#}$, hence we obtain $\iota_{0\#} \simeq \iota_{1\#} : S_\bullet(X) \to S_\bullet(X \times I)$. $\square$

There are some direct corollaries of theorem 3.2.20.

Corollary 3.2.23. *If X, Y have the same homotopy type $X \simeq Y$, then $H_\bullet(X) = H_\bullet(Y)$.*

Corollary 3.2.24. *If the subspace A is the deformation retract of X, then the inclusion map $i : A \to X$ induces the isomorphism of homology groups $i_* : H_\bullet(A) \cong H_\bullet(X)$. In particular, if X is contractible, then $\tilde{H}_\bullet(X) = 0$.*

3.3. The Mayer-Vietoris Sequence.

Suppose $\mathcal{U}$ is a covering of X, that is, $\bigcup\limits_{U \in \mathcal{U}} U = X$. A singular simplex $\sigma : \triangle_q \to X$ is called $\mathcal{U}-$small, if its image $\sigma(\triangle_q)$ is contained in some $U \in \mathcal{U}$.

All $\mathcal{U}-$small singular simplexes in X generate a subchain complex $S_\bullet^{\mathcal{U}}(X)$ of $S_\bullet(X)$ and the inclusion map $i : S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$ is obviously a chain map.

We have a theorem about the correspondence between $S_\bullet^{\mathcal{U}}(X)$ and $S_\bullet(X)$:

Theorem 3.3.1. *Suppose $\mathcal{U}$ is a family of subsets of X, and its interior*

$$\mathrm{Int}\,\mathcal{U} := \{\mathrm{Int}\,U \big| U \in \mathcal{U}\}$$

is an open covering of X. Then then chain map $S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X)$ is a chain homotopy equivalence. In fact, there exists a chain map $k : S_\bullet(X) \to S_\bullet^{\mathcal{U}}(X)$ such that $k \circ i = \mathrm{id}$, $i \circ k \simeq \mathrm{id}$. Hence $i_* : H_\bullet(S_\bullet^{\mathcal{U}}(X)) \cong H_\bullet(S_\bullet^{\mathcal{U}}(X))$.

The proof of this theorem needs the method barycentric subdivision, thus we won't discuss about this.

Before talking about the Mayer-Vietoris sequence, we need some basic knowledge of homological algebra.

Definition 3.3.2. *A sequence consisted Abelian groups and homomorphisms*

$$C \xrightarrow{f} D \xrightarrow{g} E$$

is called ***exact at*** D*, if the image of* f *is equal to the kernel of* g*, that is,* $\ker g = \operatorname{im} f$*. Moreover, a sequence consisted of Abelian groups and homomorphisms*

$$\cdots \longrightarrow G_{i-1} \xrightarrow{\phi_{i-1}} G_i \xrightarrow{\phi_i} G_{i+1} \longrightarrow \cdots$$

is called an ***exact sequence****, if it is exact at each Abelian group.*

Definition 3.3.3. *A sequence consisted of chain complexes and chain maps*

$$C \xrightarrow{f} D \xrightarrow{g} E$$

is called ***exact at*** D*, if for each dimension* q*, the sequence consisted of Abelian groups and homomorphisms*

$$C_q \xrightarrow{f_q} D_q \xrightarrow{g_q} E_q$$

is exact at D_q*.*

Theorem 3.3.4 (Snake Lemma)**.** *Given a short exact sequence of chain complexes and chain maps*

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$$

we can define a ***boundary homomorphism*** $\partial_* : H_q(E) \to H_{q-1}(C)$ *for each dimension* q *and then obtain a long exact sequence*

$$\cdots \longrightarrow H_{q+1}(E) \xrightarrow{\partial_*} H_q(C) \xrightarrow{f_*} H_q(D) \xrightarrow{g_*} H_q(E) \xrightarrow{\partial_*} H_{q-1}(C) \longrightarrow \cdots$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & \\
0 \arrow[r] & C_{q+1} \arrow[r, "f_{q+1}"] \arrow[d, "\partial_{q+1}"'] & D_{q+1} \arrow[r, "g_{q+1}"] \arrow[d, "\partial_{q+1}"] & E_{q+1} \arrow[r] \arrow[d, "\partial_{q+1}"] & 0 \\
0 \arrow[r] & C_q \arrow[r, "f_q"] \arrow[d, "\partial_q"'] & D_q \arrow[r, "g_q"] \arrow[d, "\partial_q"] & E_q \arrow[r] \arrow[d, "\partial_q"] & 0 \\
0 \arrow[r] & C_{q-1} \arrow[r, "f_{q-1}"] \arrow[d, "\partial"'] & D_{q-1} \arrow[r, "g_{q-1}"] \arrow[d, "\partial"] & E_{q-1} \arrow[r] \arrow[d, "\partial"] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

where every row is exact. For $e_q \in Z_q(E)$, define

$$\partial_* : H_q(E) \to H_{q-1}(C),\ [e_q] \mapsto [f_{q-1}^{-1} \partial_q g_q^{-1}(e_q)]$$

We need to check that the definition is well-defined. The method we use is called diagram chasing.

- g_q is surjective, thus $g_q^{-1}(e_q)$ is well-defined. We denote it by d_q. Then
$$g_{q-1}\partial_q(d_q) = \partial_q g_q(d_q) = \partial_q(e_q) = 0$$
Thus $\partial_q(d_q) \in \ker g_{q-1} = \text{im } f_{q-1}$, which implies that $f_{q-1}^{-1}(\partial_q d_q)$ is well-defined. We denote it by c_{q-1}. Since f_{q-1} is injective, c_{q-1} is uniquely determined. Besides, we need to prove that $c_{q-1} \in Z_{q-1}(C)$. This is because
$$f_{q-2}\partial_{q-1}(c_{q-1}) = \partial_{q-1} f_{q-1}(c_{q-1}) = \partial_{q-1}\partial_q d_q = 0$$
- If there exists $d'_q \in D_q$ such that $e_q = g_q(d'_q)$, and $\partial_q d'_q = f_{q-1}(c'_{q-1})$ for some $c'_{q-1} \in C_{q-1}$, then we need to show that $[c_{q-1}] = [c'_{q-1}]$. We have $g_q(d_q - d'_q) = 0$, thus $d_q - d'_q \in \ker g_q = \text{im } f_q$. Suppose $d_q - d'_q = f_q(c_q),\ c_q \in C_q$, then
$$f_{q-1}(c_{q-1} - c'_{q-1}) = \partial_q f_q(c_q) = f_{q-1}\partial_q(c_q)$$
Since f_{q-1} is injective, we have $c_{q-1} - c'_{q-1} = \partial c_q \in \text{im } \partial_q$, thus $[c_{q-1}] = [c'_{q-1}]$.
- If there exists $e'_q \in Z_q(E)$ such that $[e_q] = [e'_q]$, then we need to prove $[c_{q-1}] = [c'_{q-1}]$. Suppose $e_q - e'_q = \partial_{q+1}(e_{q+1}),\ e_{q+1} \in E_{q+1}$. Since g_{q+1} is surjective, then there exists $d_{q+1} \in D_{q+1}$ such that $g_{q+1}(d_{q+1}) = e_{q+1}$. Then
$$g_q(d_q - d'_q) = e_q - e'_q = \partial_{q+1} g_{q+1}(d_{q+1}) = g_q \partial_{q+1}(d_{q+1})$$
which implies $d_q - d'_q - \partial_{q+1} d_{q+1} \in \ker g_q = \text{im } f_q$. Suppose $d_q - d'_q - \partial_{q+1} d_{q+1} = f_q(c_q),\ c_q \in C_q$, then
$$\begin{aligned} f_{q-1}(c_{q-1} - c'_{q-1}) &= \partial_q(d_q - d'_q) \\ &= \partial_q\partial_{q+1} d_{q+1} + \partial_q f_q(c_q) \\ &= f_{q-1}\partial_q(c_q) \end{aligned}$$
Since f_{q-1} is injective, $c_{q-1} - c'_{q-1} = \partial_q c_q \in \text{im } \partial_q$, which implies $[c_{q-1}] = [c'_{q-1}]$.

Then we need to verify that the long sequence is exact. The short exact sequence immediately gives exactness at $H_q(D)$. Now we prove the exactness at $H_q(E)$, and the exactness at $H_q(C)$ is similar.

- For $d_q \in Z_q(D)$, we have
$$\partial_* g_*[d_q] = [f_{q-1}^{-1}\partial_q(d_q)] = [f_{q-1}(0)] = [0]$$
thus $\text{im } g_* \subset \ker \partial_*$.
- Suppose $e_q \in Z_q(E)$ and $\partial_*[e_q] = [0]$, then there exists $c_{q-1} \in C_q$ such that $f_{q-1}^{-1}\partial_q g_q^{-1}(e_q) = \partial_q(c_q)$, thus
$$\partial_q g_q^{-1}(e_q) = f_{q-1}\partial_q(c_q) = \partial_q f_q(c_q)$$
which implies $g_q^{-1}(e_q) - f_q(c_q) \in \ker \partial_q = \text{im } \partial_{q+1}$. Suppose $g_q^{-1}(e_q) - f_q(c_q) = \partial_{q+1}(d_{q+1}),\ d_{q+1} \in D_{q+1}$, then
$$e_q = g_q f_q(c_q) + g_q \partial_{q+1}(d_{q+1}) = g_q \partial_{q+1}(d_{q+1}) \in \text{im } g_*$$
thus $\ker \partial_* \subseteq \text{im } g_*$.

$\square$

In the previous proof, we state the method of diagram chasing in detail, thus when we meet "diagram chasing" statements after, we will simply skip the proof unless special emphasis.

There is another similar result about exact sequences using diagram chasing.

Theorem 3.3.5 (Five Lemma)**.** *Consider a commutative diagram with exact rows:*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

(i) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.
(ii) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.
(iii) In particular, if f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.

Then we come back to the long exact sequence induced by the short exact sequence. We have the naturality of homology sequences.

Theorem 3.3.6 (Naturality of homology sequences)**.** *Consider a commutative diagram with exact rows of chain complexes and chain maps:*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] \arrow[d, "\alpha"] & D \arrow[r, "g"] \arrow[d, "\beta"] & E \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & C' \arrow[r, "f'"] & D' \arrow[r, "g'"] & E' \arrow[r] & 0 \end{tikzcd}$$

then we have the commutative diagram of their exact homology sequences:

$$\begin{tikzcd} \cdots \arrow[r] & H_q(C) \arrow[r, "f_*"] \arrow[d, "\alpha_*"] & H_q(D) \arrow[r, "g_*"] \arrow[d, "\beta_*"] & H_q(E) \arrow[r, "\partial_*"] \arrow[d, "\gamma_*"] & H_{q-1}(C) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\ \cdots \arrow[r] & H_q(C') \arrow[r, "f'_*"] & H_q(D') \arrow[r, "g'_*"] & H_q(E') \arrow[r, "\partial'_*"] & H_{q-1}(C') \arrow[r] & \cdots \end{tikzcd}$$

Suppose a space X has two subspaces X_1, X_2 such that $X_1 \cup X_2 = X$. Denote the covering $\{X_1, X_2\}$ of X by $\mathcal{U}$, then the subchain complex $S_\bullet^{\mathcal{U}}(X)$ generated by all $\mathcal{U}-$small singular simplexes in $S_\bullet(X)$ is equal to $S_\bullet(X_1) + S_\bullet(X_2)$. We denote the inclusion map by $i : S_\bullet(X_1) + S_\bullet(X_2) \to S_\bullet(X)$.

We have the commutative diagram consisted of inclusion maps:

$$\begin{tikzcd} & X_1 \arrow[dr, "j_1"] & \\ X_1 \cap X_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X_1 \cup X_2 \\ & X_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

Denote all singular simplexes in X_1 by Σ_{X_1}, etc. Then $\Sigma_{X_1 \cap X_2} = \Sigma_{X_1} \cap \Sigma_{X_2}$, and the set of all $\mathcal{U}-$small singular simplexes $\Sigma_X^{\mathcal{U}} = \Sigma_{X_1} \cup \Sigma_{X_2}$. Thus we obtain a short exact sequence of chain complexes and chain maps

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xrightarrow{h_\#} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{k_\#} S_\bullet(X_1) + S_\bullet(X_2) \longrightarrow 0$$

where

$$h_{\#}(x) := (i_{1\#}(x), -i_{2\#}(X)),\ k_{\#}(y,z) := j_{1\#}(y) + j_{2\#}(z)$$

Definition 3.3.7. *Suppose X_1, X_2 are subspaces of X(we don't require $X = X_1 \cup X_2$), if the inclusion map $i : S_\bullet(X_1) + S_\bullet(X_2) \to S_\bullet(X_1 \cup X_2)$ induces an isomorphism of homology groups $H_\bullet(S_\bullet(X_1)+S_\bullet(X_2)) \xrightarrow{\cong} H_\bullet(X_1 \cup X_2)$, then we say $\{X_1, X_2\}$ is a **Mayer-Vietoris couple**.*

Example 3.3.8. *If $\mathrm{Int}X_1 \cup \mathrm{Int}X_2 = X$, then by the theorem 3.3.1, $\{X_1, X_2\}$ is a Mayer-Vietoris couple.*

Theorem 3.3.9 (The Mayer-Vietoris sequence)**.** *Suppose $\{X_1, X_2\}$ is a Mayer-Vietoris couple, then we have the following Mayer-Vietoris exact homology sequence:*

$$\longrightarrow H_q(X_1 \cap X_2) \longrightarrow H_q(X_1) \oplus H_q(X_2) \longrightarrow H_q(X_1 \cup X_2) \xrightarrow{\partial_*} H_{q-1}(X_1 \cap X_2) \longrightarrow$$

Proof. Apply theorem 3.3.4 to the previous short exact sequence. □

Remark 3.3.10. *For augmented chain complexes we have the short exact sequence similarly:*

$$0 \longrightarrow \tilde{S}_\bullet(X_1 \cap X_2) \xrightarrow{h_\#} \tilde{S}_\bullet(X_1) \oplus \tilde{S}_\bullet(X_2) \xrightarrow{k_\#} \tilde{S}_\bullet(X_1) + \tilde{S}_\bullet(X_2) \longrightarrow 0$$

Thus for reduced homology groups, the Mayer-Vietoris sequence

$$\longrightarrow \tilde{H}_q(X_1 \cap X_2) \longrightarrow \tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) \longrightarrow \tilde{H}_q(X_1 \cup X_2) \xrightarrow{\partial_*} \tilde{H}_{q-1}(X_1 \cap X_2) \longrightarrow$$

is also exact.

By the theorem 3.3.6, we immediately get the following theorem.

Theorem 3.3.11 (Naturality of Mayer-Vietoris homology sequences)**.** *Suppose $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are Mayer-Vietoris sequences in X, Y respectively, and $f : X \to Y$ satisfies $f(X_1) \subset Y_1, f(X_2) \subset Y_2$, then the following diagram commutes.*

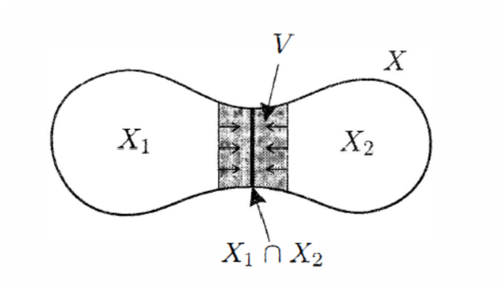
$$\begin{array}{ccccccc}
H_q(X_1 \cap X_2) & \longrightarrow & H_q(X_1) \oplus H_q(X_2) & \longrightarrow & H_q(X_1 \cup X_2) & \xrightarrow{\partial_*} & H_{q-1}(X_1 \cap X_2) \\
\big\downarrow f_* & & \big\downarrow f_* \quad \big\downarrow f_* & & \big\downarrow f_* & & \big\downarrow f_* \\
H_q(Y_1 \cap Y_2) & \longrightarrow & H_q(Y_1) \oplus H_q(Y_2) & \longrightarrow & H_q(Y_1 \cup Y_2) & \xrightarrow{\partial_*} & H_{q-1}(Y_1 \cap Y_2)
\end{array}$$

Corollary 3.3.12. *Suppose X is the union of two closed subspaces X_1, X_2, and $X_1 \cap X_2$ is the deformation retract of some open neighborhood V, then $\{X_1, X_2\}$ is a Mayer-Vietoris couple.*

Proof. Suppose $V_j = V \cup X_j$, then we have known that $\{V_1, V_2\}$ is a Mayer-Vietoris couple by the example 3.3.8 and the inclusion map induces an isomorphism $H_\bullet(S_\bullet(V_1)+S_\bullet(V_2)) \xrightarrow{\cong} H_\bullet(X)$. We need to prove that $H_\bullet(S_\bullet(X_1)+S_\bullet(X_2)) \xrightarrow{\cong} H_\bullet(X)$ is also an isomorphism.

We claim that X_j is the deformation retract of V_j. In fact, suppose $F : V \times I \to V$ is the deformation retraction from V to $X_1 \cap X_2$, then the deformation retraction $F_j : V_j \times I \to V_j$ from V_j to X_j can be defined as

$$F_j(x,t) = \begin{cases} x, & x \in X_j \\ F(x,t), & x \in V - X_j \end{cases}$$



Now we consider the commutative diagram:(use the notations $S_+(X) := S_\bullet(X_1) + S_\bullet(X_2)$ and $S_+(V) := S_\bullet(V_1) + S_\bullet(V_2)$)

$$\begin{tikzcd}
H_q(X_1 \cap X_2) \arrow[r] \arrow[d] & H_q(X_1) \oplus H_q(X_2) \arrow[r] \arrow[d] & H_q(X_1 \cup X_2) \arrow[r, "\partial_*"] \arrow[d] & H_{q-1}(X_1 \cap X_2) \arrow[d] \\
H_q(V_1 \cap V_2) \arrow[r] & H_q(V_1) \oplus H_q(V_2) \arrow[r] & H_q(V_1 \cup V_2) \arrow[r, "\partial_*"] & H_{q-1}(V_1 \cap V_2)
\end{tikzcd}$$

Then the deformation retractions tell us that the vertical arrows are all isomorphisms except for one. Thus by the Five Lemma 3.3.5, we obtain that $H_q(S_+(X)) \to H_q(S_+(V))$ is an isomorphism. Hence $i_* : H_\bullet(S_+(X)) \to H_\bullet(X)$ is an isomorphism. $\square$

3.4. **Examples and Applications.**

It is more important and useful of applications of the Mayer-Vietoris sequence then the proof itself. We first calculate the homology groups of $n-$sphere S^n.

Theorem 3.4.1. *The reduced homology groups of S^n are*

$$\tilde{H}_q(S^n) = \begin{cases} \mathbb{Z}, \ q = n \\ 0, \ q \neq n \end{cases}$$

Proof. We prove it by induction on n. When $n = 0$, S^0 is consisted of two points, then by the example 3.2.19 we have done. Now suppose $n > 0$ and the theorem holds for S^{n-1}.

Consider S^n as the union of upper and lower hemispheres $B+ := \{(x_0, \cdots x_n) \in S^n | x_n \geqslant 0\}$, $B_- := \{(x_0, \cdots x_n) \in S^n | x_n \leqslant 0\}$. Then $B_+ \cap B_- = S^{n-1}$ is the deformation retract of its neighborhood, thus the corollary 3.3.12 implies that $\{B_+, B_-\}$ is a Mayer-Vietoris couple.

B_+ and B_- are contractible, thus $\tilde{H}_\bullet(B_+) = \tilde{H}_\bullet(B_-) = 0$. From the Mayer-Vietoris sequence of reduced homology groups

$$\tilde{H}_q(B_+) \oplus \tilde{H}_q(B_-) \longrightarrow \tilde{H}_q(S^n) \xrightarrow{\partial_*} \tilde{H}_{q-1}(S^{n-1}) \longrightarrow \tilde{H}_{q-1}(B_+) \oplus \tilde{H}_{q-1}(B_-)$$

we obtain $\tilde{H}_q(S^n) = \tilde{H}_{q-1}(S^{n-1})$. Hence we complete the induction. $\square$

Corollary 3.4.2. *When $m \neq n$, $S^m \ncong S^n$ and $\mathbb{R}^m \ncong \mathbb{R}^n$.*

This corollary is quite interesting because it seems trivial but we may not have a simple and rigorous proof without using homology.

The next corollary is one of the most famous theorem in the algebraic topology.

Corollary 3.4.3 (Brouwer's Fixed Point Theorem)**.** *Any map $f : D^n \to D^n$ has a fixed point, that is, there exists $x \in D^n$ such that $f(x) = x$.*

Proof. Suppose f has no fixed points for the sake of contradiction. Then for any $x \in D^n$ we have $f(x) \neq x$. Define a map $g : D^n \to S^{n-1}$ in the following way: for any $x \in D^n$, there is a ray starting at $f(x)$ and pointing to x that intersects S^{n-1} at a unique point, which we denote it by $g(x)$.

In particular, if $x \in S^{n-1}$, then $g(x) = x$. Thus suppose $i : S^{n-1} \to D^n$ is the inclusion map, then $g \circ i = \mathrm{id}_{S^{n-1}}$. Hence they induce the same reduced homology groups, which means

$$\tilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{g_*} \tilde{H}_{n-1}(S^{n-1})$$

is an identity. However, $\tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$ by the theorem 3.4.1 and $\tilde{H}_{n-1}(D^n) = 0$ since it is contractible. Thus the composite can't be an identity by the property of functor, which leads to the contradiction. Hence f has a fixed point. □

Brouwer's fixed point theorem is an extraordinary theorem because it requires nothing but continuity about the map and obtains a fixed point. It has many applications in other subjects like physics and economics. And the theorem itself is extended to more general cases such as Lefschetz's fixed point theorem in topology and Schauder's fixed point theorem in functional analysis.

Another important concept in algebra topology is the degree of a map.

Definition 3.4.4. *Suppose $f : S^n \to S^n$ inducing a homomorphism $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$, then there exists a unique integer d such that for any $h \in \tilde{H}_n(S^n)$ we have $f_*(h) = d \cdot h$. This integer d is called the **degree** of f, denoted by $\deg f$.*

Proposition 3.4.5. *The degree of a map has the following properties.*

(i) $\deg(\mathrm{id}_{S^n}) = 1$.
(ii) $\deg(g \circ f) = (\deg f) \cdot (\deg g)$.
(iii) $\deg(\mathrm{const}) = 0$.
(iv) *If $f \simeq g : S^n \to S^n$, then $\deg f = \deg g$.*

A famous theorem by Hopf solves the problem of classification of maps from the sphere to itself.

Theorem 3.4.6 (Hopf)**.** *If $f, g : S^n \to S^n$ and $\deg f = \deg g$, then $f \simeq: S^n \to S^n$.*

The following propositions indicate that the degree of a map has correspondence with the orientation.

Proposition 3.4.7. *The reflection map*

$$r : S^n \to S^n,\ (x_0, x_1, \cdots, x_n) \mapsto (-x_0, x_1, \cdots, x_n)$$

has degree -1.

Proof. We prove it by induction on n. When $n = 0$,the proposition holds by the example 3.2.19. For $n > 0$, we have the following commutative diagram by the naturality of Mayer-Vietoris sequences.

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_q(S^n) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "r_*"] & \tilde{H}_{q-1}(S^{n-1}) \arrow[r] \arrow[d, "r_*"] & 0 \\
0 \arrow[r] & \tilde{H}_q(S^n) \arrow[r, "\partial_*", "\cong"'] & \tilde{H}_{q-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Hence $\deg(r|_{S^n}) = \deg(r|_{S^{n-1}}) = -1$. □

This proposition immediately gives:

Corollary 3.4.8. *The antipodal map*

$$A : S^n \to S^n,\ (x_0, x_1, \cdots x_n) \mapsto (-x_0, -x_1, \cdots - x_n)$$

has degree $(-1)^{n+1}$.

Another application is used in differential geometry.

Theorem 3.4.9. *There doesn't exist a tangent vector field that is non-zero everywhere on an even dimensional sphere* S^{2n}.

Proof. Suppose there exists such a tangent vector field X for the sake of contradiction, then we can construct a map

$$f : S^{2n} \to S^{2n},\ v \mapsto \frac{X(v)}{\|X(v)\|}$$

Then $f(v) \neq \pm v$ for every $v \in S^{2n}$. Thus $f \simeq \mathrm{id}_{S^{2n}}$. In fact, we have the homotopy

$$F : S^{2n} \times I \to S^{2n},\ (x, t) \mapsto \frac{(1-t)f(x) + tx}{\|(1-t)f(x) + tx\|}$$

Similarly, $f \simeq A$, A is the antipodal map. Thus $\mathrm{id}_{\mathrm{S}^{2n}} \simeq A$. However, $\deg \mathrm{id}_{S^{2n}} = 1$ but $\deg A = -1$ in the dimensional case, which is a contradiction. ☐

Remark 3.4.10. *For the case of* S^2*, the theorem is also called the "fairy ball theorem" since one can imagine the hair on his head.*

Remark 3.4.11. *There is always a tangent vector field that is non-zero everywhere on an even dimensional sphere* S^{2n-1}*, such as*

$$(x_1, x_2 \cdots, x_{2n-1}, x_{2n}) \mapsto (x_2, -x_1, \cdots, x_{2n}, -x_{2n-1})$$

3.5. **Advanced Construction and Applications.**

In this subsection, we will apply the exact homology sequence to a mapping cone. This is especially useful when we attach a cell to a space and see how the homology groups change so that we can calculate the homology groups of cell complexes.

However, in order to make the topic more coherent, we will skip basic concepts such as the attaching map, mapping cylinder, mapping cone, mapping suspension and so on. Readers unfamiliar with these can refer to [7] by Hatcher.

Here we list a few properties of the mapping cone.

Proposition 3.5.1. *The mapping cone* Cf *has the following properties:*

(i) $f : X \to Y$ *is nullhomotopy if and only if* Y *is the retract of* Cf.
(ii) *If* $f \simeq g : X \to Y$*, then* $Cf \simeq Cg$.
(iii) *If* $f : X \xrightarrow{\cong} Y$*, then* $Cf \simeq \mathrm{pt}$.

Remark 3.5.2. *It should be aware that the inverse proposition of (iii) above is not always true. There is an interesting counterexample proposed by Poincaré in* [2] *which is known as the Poicaré homology ball. He constructed a space whose homology groups of all dimensions are 0 but having the nontrivial fundamental group, such as the alternating group* A_5.

Now we consider the mapping cone $Cf = Y \cup_f CX$ as the union of two parts, the bottom $C_-f = Y \cup_f X \times [0, \frac{1}{2}]$ and the top $C_+f = X \times [\frac{1}{2}, 1]/X \times 1$. Then C_f is just the mapping cylinder of f and C_+f is the cone over X.

Theorem 3.5.3. *Suppose $f : X \to Y$ is a map, then we have the long exact sequence*

$$\cdots \xrightarrow{\triangle_*} \tilde{H}_q(X) \xrightarrow{f_*} \tilde{H}_q(Y) \xrightarrow{e_*} \tilde{H}_q(Cf) \xrightarrow{\triangle_*} \tilde{H}_{q-1}(X) \longrightarrow \cdots$$

where $e : Y \to Cf$ is the inclusion map, and $\triangle_*$ is the boundary map in the reduced homology Mayer-Vietoris sequence of the couple $\{C_-f, C_+f\}$.

Proof. By corollary 3.3.12, $\{C_-f, C_+f\}$ is a Mayer-Vietoris couple. Consider the Mayer-Vietoris exact sequence of this couple, and note that $\tilde{H}_q(C_+f) = 0$ since it is contractible, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_q(X) \arrow[r] \arrow[d, equal] & \tilde{H}_q(C_-f) \arrow[r] \arrow[d, shift right, "r_*"'] & \tilde{H}(Cf) \arrow[r, "\triangle_*"] \arrow[d, equal] & \tilde{H}_{q-1}(X) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \tilde{H}_q(X) \arrow[r, "f_*"] & \tilde{H}(Y) \arrow[r, "e_*"] \arrow[u, shift right, "\cong"'] & \tilde{H}_q(Cf) \arrow[r, "\triangle_*"] & \tilde{H}_{q-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where $r : C_-f \to Y$ is the deformation retraction, thus r_* is an isomorphism. Then we obtain exactness of the second row from exactness of the first row above. □

Corollary 3.5.4. *Suppose $A \subset X$, then we have the long exact sequence*

$$\cdots \xrightarrow{\triangle_*} \tilde{H}_q(A) \longrightarrow \tilde{H}_q(X) \longrightarrow \tilde{H}_q(X \cup CA) \xrightarrow{\triangle_*} \tilde{H}_{q-1}(A) \longrightarrow \cdots$$

where $X \cup CA$ is obtained from attaching a cone CA to X.

Corollary 3.5.5. *For the suspension ΣX of a space X, we have the isomorphism $\triangle_* : \tilde{H}_{q+1}(\Sigma X) \cong \tilde{H}_q(X)$. Its inverse isomorphism $\sigma_* : \tilde{H}_q(X) \cong \tilde{H}_{q+1}(\Sigma X)$ is called the **suspension isomorphism**.*

Proof. Note that ΣX is the mapping cone of $f : X \to \mathrm{pt}$, then use theorem 3.5.3. □

Another corollary is applied to the cell complexes.Suppose there exists a map $f : \partial D^n = S^{n-1} \to X$, since CS^{n-1} is homeomorphic to D^n, the mapping cone Cf is just $X \cup_f D^n$.

Corollary 3.5.6. *Suppose $f : S^{n-1} \to X$, then we have*

(i) $\tilde{H}_q(X \cup_f D^n) = \tilde{H}_q(X)$, *when* $q \neq n, n-1$.

(ii) the exact sequence

$$0 \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X \cup_f D^n) \xrightarrow{\triangle_*} \tilde{H}_{n-1}(S^{n-1})$$

$$\xrightarrow{f_*} \tilde{H}_{n-1}(X) \longrightarrow \tilde{H}_{n-1}(X \cup_f D^n) \longrightarrow 0$$

This corollary gives us an explicit formula for homology groups of cell complexes. Let's finish singular homology theory by discussing two examples.

Example 3.5.7 (Homology groups of torus)**.** *The 0-cell of torus is just a single point, and the 1-cell is the wedge sum $S^1 \vee S^1$. By corollary 3.3.12, $\tilde{H}_1(S^1 \vee S^1) = \mathbb{Z} \oplus \mathbb{Z}$ and the homology groups of other dimensions are all 0. The torus T^2 can be obtained from attaching a 2-cell onto $S^1 \vee S^1$ and the attaching map $f : S^1 \to S^1 \vee S^1$*

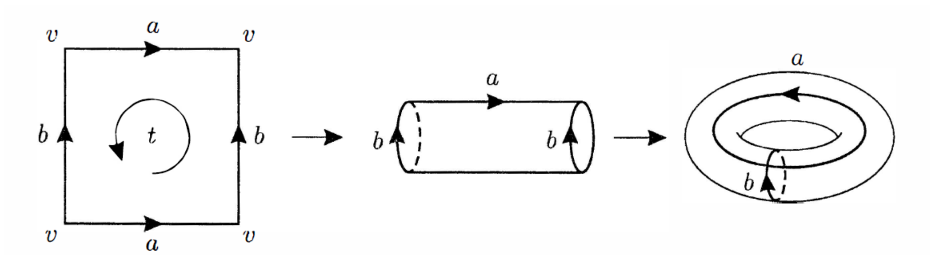
wraps each S^1 in both directions. So $f_* = 0$ in the sense of reduced homology. Thus by corollary 3.5.6, we have the exact sequence

$$0 \longrightarrow \tilde{H}_2(S^1 \vee S^1) \longrightarrow \tilde{H}_2(T^2) \longrightarrow \mathbb{Z} \xrightarrow{0} \tilde{H}_1(S^1 \vee S^1) \longrightarrow H_1(T^2) \longrightarrow 0$$

Thus

$$H_2(T^2) = \mathbb{Z},\ H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z},\ H_0(T^2) = \mathbb{Z}$$

and homology groups of other dimensions are all 0.



Example 3.5.8 (Homology groups of complex projective spaces)**.** *The complex projective space $\mathbb{C}P^n$ has several equivalent definitions:*

(i) all lines passing through the origin in $\mathbb{C}^{n+1}$. That is, $\mathbb{C}P^n = \mathbb{C}^{n+1} - 0/\{z \sim \lambda z, \forall \lambda \neq 0 \in \mathbb{C}, \forall z \in \mathbb{C}^{n+1} - 0\}$.

(ii) the $(2n+1)$-sphere by collapsing each big circle to one point. That is, $\mathbb{C}P^n = S^{2n+1}/\{z \sim e^{i\theta} z, \forall \theta \in \mathbb{R}, \forall z \in S^{2n+1}\}$. Denote the covering map by $\pi_{(n)} : S^{2n+1} \to \mathbb{C}P^n$.

(iii) the $2n$-disk by collapsing each boundary circle to one point. That is, $\mathbb{C}P^n = D^{2n}/\{z' \sim e^{i\theta} z', \forall \theta \in \mathbb{R}, \forall z' \in S^{2n-1}\}$.

From the third definition, we can view $\mathbb{C}P^n$ as attaching a $2n$-cell onto $\mathbb{C}P^{n-1}$ and the attaching map is exactly $\pi_{(n-1)} : S^{2n-1} \to \mathbb{C}P^{n-1}$. Thus we can calculate homology groups of $\mathbb{C}P^n$ by corollary 3.5.6.

We claim

$$\tilde{H}_q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & q \leqslant 2n \text{ and } n \text{ is even} \\ 0, & \text{else} \end{cases}$$

and prove this by induction on n. For $n = 1$, $\mathbb{C}P^1 = S^2$ and the proposition obviously holds. Suppose it holds for $n = k-1$, we want to derive the formula for $n = k$.

From the previous analysis, we have the exact sequence

$$\tilde{H}_q(S^{2k-1}) \xrightarrow{\pi_*} \tilde{H}_q(\mathbb{C}P^{k-1}) \longrightarrow \tilde{H}_q(\mathbb{C}P^k) \longrightarrow \tilde{H}_{q-1}(S^{2k-1}).$$

When $q \neq 2k-1$ or $2k$, we have $\tilde{H}_q(\mathbb{C}P^k) = \tilde{H}_q(\mathbb{C}P^{k-1})$.

When $q = 2k-1$, we have the exact sequence

$$\mathbb{Z} \xrightarrow{\pi_*} 0 \longrightarrow \tilde{H}_{2k-1}(\mathbb{C}P^k) \longrightarrow 0.$$

thus $\tilde{H}_{2k-1}(\mathbb{C}P^k) = 0$.

When $q = 2k$, we have the exact sequence

$$0 \longrightarrow \tilde{H}_{2k}(\mathbb{C}P^k) \longrightarrow \mathbb{Z} \xrightarrow{\pi_*} 0.$$

thus $\tilde{H}_{2k}(\mathbb{C}P^k) = \mathbb{Z}$, *completing the proof.*

Remark 3.5.9. *If we apply the same method to the real projective space* $\mathbb{R}P^n$ *to calculate its homology groups, then it's difficult to determine the homomorphism induced by the attaching map. However, cellular homology theory will work and gives*

$$\tilde{H}_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & q = n = \text{odd} \\ \mathbb{Z}_2, & q = \text{odd and } 0 < q < n \\ 0, & \text{else} \end{cases}$$

but we won't talk about it.

4. Axioms for Homology Theory

In this section, we will talk about remarkable work done by Eilenberg and Steenrod in [3]. They extended the traditional homology theory which needs concrete construction to abstract axiomatic definition for homology theory. Thus it can be applied to many other branches like cobordism theory in differential topology and K-theory in vector bundles. We adopt the primitive notations and introduce this axiomatic system.

Definition 4.0.1. *Define* ***a pair of sets*** (X, A) *to be a set* X *and a subset* A *of* X. *If* $A = 0$, *we write abbreviated by* X. *A* ***map*** $f : (X, A) \to (Y, B)$ *is a function from* X *to* Y *such that* $f(A) \subset B$.

Definition 4.0.2. *Define the* ***lattice*** *of a pair* (X, A) *consisting of the pairs*

$$\begin{tikzcd}
& & (X,0) \arrow[dr] & & \\
(0,0) \arrow[r] & (A,0) \arrow[ur] \arrow[dr] & & (X,A) \arrow[r] & (X,X) \\
& & (A,A) \arrow[ur] & &
\end{tikzcd}$$

all their identity maps, the inclusion maps indicated by arrows, and their compositions.

Note that if $f : (X, A) \to (Y, B)$, then f defines a map of every pair of the lattice of (X, A) into the corresponding pair of the lattice of (Y, B) by restriction.

Definition 4.0.3. *A family* $\mathscr{A}$ *of pairs of topological spaces and maps of such pairs which satisfy the conditions (i) to (v) below is called an* ***admissible category for homology theory****. The pairs and maps of* $\mathscr{A}$ *are called* ***admissible****.*

- *(i) If* $(X, A) \in \mathscr{A}$, *then all pairs and inclusion maps of the lattice of* (X, A) *are in* $\mathscr{A}$.
- *(ii) If* $f : (X, A) \to (Y, B)$ *is in* $\mathscr{A}$, *then* (X, A) *and* (Y, B) *are in* $\mathscr{A}$ *together with all maps that* f *defines of members of the lattice of* (X, A) *into corresponding members of the lattices of* (Y, B) *by restriction.*
- *(iii) If* f_1 *and* f_2 *are in* $\mathscr{A}$ *and their composition* $f_1 f_2$ *is defined, then* $f_1 f_2 \in \mathscr{A}$.
- *(iv) If* $I = [0, 1]$ *is the closed unit interval, and* $(X, A) \in \mathscr{A}$,*then the cartesian product*

$$(X, A) \times I = (X \times I, A \times I)$$

is in $\mathscr{A}$ and the maps

$$g_0, g_1 : (X, A) \to (X, A) \times I$$

given by

$$g_0(x) = (x, 0),\ g_1(x) = (x, 1)$$

are in $\mathscr{A}$.

(v) *There is a space P_0 in $\mathscr{A}$ consisting of a single point. If X, P are in $\mathscr{A}$, if $f : P \to X$, and if P is a single point, then $f \in \mathscr{A}$.*

Example 4.0.4. *The following are examples of admissible categories for homology theory:*

(i) *$\mathscr{A}_1$ = the set of all pairs (X, A) and all maps of such pairs. This is the largest admissible category.*
(ii) *$\mathscr{A}_C$ = the set of all compact pairs and all maps of such pairs.*
(iii) *$\mathscr{A}_{LC}$ = the set of pairs (X, A) where X is locally compact Hausdorff space, A is closed in X, and all maps of such pairs having the property that the inverse images of compact sets are compact sets.*

Definition 4.0.5. *Two maps $f_0, f_1 : (X, A) \to (Y, B)$ in the admissible category $\mathscr{A}$ are said to be* ***homotopic*** *in $\mathscr{A}$ if there is a map*

$$h : (X, A) \times I \to (Y, B)$$

in $\mathscr{A}$ such that

$$f_0 = hg_0,\ f_1 = hg_1$$

or, explicitly,

$$f_0(x) = h(x, 0),\ f_1(x) = h(x, 1)$$

The map h is called a ***homotopy****.*

Now let's start the main topic of axioms for homology theory.

A homology theory H on an admissible category $\mathscr{A}$ is a collection of three functions as follows: The first is a function $H_q(X, A)$ defined for each pair (X, A) in $\mathscr{A}$ and each integer q (positive, negative, or zero). The value of the function is an abelian group. It is called the **q-dimensional relative homology group of X modulo A**.

The second function is defined for each map

$$f : (X, A) \to (Y, B)$$

in $\mathscr{A}$ and each integer q, and attaches to such a pair a homomorphism

$$f_{*q} : H_q(X, A) \to H_q(Y, B)$$

It is called the homomorphism **induced** by f.

The third function $\partial(q, X, A)$ is defined for each (X, A) in $\mathscr{A}$ and each integer q. Its value is a homomorphism

$$\partial(q, X, A) : H_q(X, A) \to H_{q-1}(A)$$

called the **boundary operator**.

In addition, the three functions are required to have the following properties:

Axiom 1. If f = identity, then f_* = identity.

Axiom 2. $(gf)_* = g_* f_*$.

Axiom 3. $\partial f_* = (f|A)_* \partial$.

Explicitly, if $f : (X, A) \to (Y, B)$ is admissible and $f|A : A \to B$ is the map restricted by f, then there is a commutative diagram:

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{f_*} & H_q(Y,B) \\ \big\downarrow \partial & & \big\downarrow \partial \\ H_{q-1}(A) & \xrightarrow{(f|A)_*} & H_{q-1}(B) \end{array}$$

Axiom 4 (Exactness Axiom). If (X, A) is admissible and $i : A \to X,\ j : X \to (X, A)$ are inclusion maps, then the lower sequence of groups and homomorphisms

$$\cdots \xrightarrow{\partial} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \longrightarrow \cdots$$

is exact. This sequence is called the homology sequence of (X, A).

Axiom 5 (Homotopy Axiom). If the admissible maps $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic in $\mathscr{A}$, then, for each q, the homomorphisms f_{0*}, f_{1*} of $H_q(X, A)$ into $H_q(Y, B)$ coincide.

Axiom 6 (Excision Axiom). If U is an open subset of X whose closure $\overline{U}$ is contained in $\mathrm{Int}(A)$, and if the inclusion map $(X-U, A-U) \to (X, A)$ is admissible, then it induces isomorphisms $H_q(X-U, A-U) \cong H_q(X, A)$ for each q.

An inclusion map $i : (X-U, A-U) \subseteq (X, A)$ where U is open in X and $\overline{U}$ is in the interior of A, will be called an **excision map** or just an **excision**.

Axiom 7 (Dimensional Axiom). If P is an admissible space consisting of a single point, then $H_q(P) = 0$ for all $q \neq 0$.

Remark 4.0.6. *The reason why we call Axiom 7 the Dimension axiom is not apparent. Suppose $H_q(X, A), \partial, f_*$ is a homology theory satisfying Axioms 1 through 7. Define $H'_q(X, A) = H_{q-1}(X, A)$. Define ∂' and f_* in the natural way. Then the new homology theory satisfies Axioms 1 through 6. This is also true for the homology theory $H''_q(X, A) = H_q(X, A) + H_{q-1}(X, A)$. Thus, Axiom 7 insures that the dimensional index q shall have a geometric meaning.*

If Axiom 7 isn't satisfied, we call this a generalized homology theory.

The homotopy axiom can be put in the following form, which is sometimes more convenient:

Axiom 5'. If (X, A) is admissible and $g_0, g_1 : (X, A) \to (X, A) \times I$ are defined by $g_0(x) = (x, 0),\ g_1(x) = (x, 1)$, then $g_{0*} = g_{1*}$.

The excision axiom may be reformulated as follows:

Axiom 6'. Let X_1 and X_2 be subsets of a space such that X_1 is closed and $X = \mathrm{Int}X_1 \cup \mathrm{Int}X_2$. If $i : (X_1, X_1 \cap X_2) \subseteq (X_1 \cup X_2, X_2)$ is admissible, then it induces isomorphisms $i_* : H_q(X_1, X_1 \cap X_2) \to H_q(X_1 \cup X_2, X_2)$ for each q.

Given an abelian group G, then singular homology theory with coefficients in G satisfy all the axioms above. Although we didn't define relative singular homology groups in the last section, readers can simply consider these as the singular homology groups on the quotient space X/A.

Moreover, Eilenberg and Steenrod proved the uniqueness of homology theory in the category of finite simplicial complex couples, which needs much effort and we don't talk about it.

5. General Homological Algebra in Category

In this section, we talk about homological algebra in category for the most general cases. This work was done after the 19060s and it needs foundation of basic category theory. For the readers unfamiliar with the basic category theory, the author strongly recommend the book [8] for a glance.

5.1. **Setting the Stage.**

We will begin with the preadditive category which has less structures.

Definition 5.1.1. *A* ***preadditive category*** $\mathscr{C}$ *is a category whose every Hom-set is an abelian group and the composition is biadditive. That is, we have* $(f_1+f_2)\circ g = f_1\circ g + f_2\circ g$ *and* $f\circ(g_1+g_2) = f\circ g_1 + f\circ g_2$ *whenever morphisms can be composed.*

Definition 5.1.2. *Let* $\mathscr{A},\mathscr{C}$ *be two preadditive categories, a functor* $F : \mathscr{A} \to \mathscr{C}$ *is called* ***additive*** *if for all* $X, Y \in \mathrm{ob}(\mathscr{A})$*, the map*

$$F : \mathrm{Hom}_{\mathscr{A}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(F(X), F(Y))$$

is a homomorphism of abelian groups. Thai is, we have $F(f+g) = F(f)+F(g)$ *for any* $f, g \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$.

Proposition 5.1.3. *Let* $\{X_i\}_{i=1}^n$ *be a finite set of objects in a preadditive category* $\mathscr{C}$. *Then the product product* $\prod\limits_{i=1}^{n} X_i$ *exists* $\equiv$ *the coproduct* $\coprod\limits_{i=1}^{n} X_i$ *exists. In both cases, the product equals to the coproduct, and is called the* ***biproduct****.*

Proposition 5.1.4. *Let* $\mathscr{C}$ *be a preadditive category and* $X \in \mathrm{ob}(\mathscr{C})$*. Then the following are equivalent:*

(i) X *is an initial object.*
(ii) X *is a terminal object.*
(iii) $1_X = 0$ *in* $\mathrm{Hom}_{\mathscr{C}}(X,X)$*, where* 0 *stands for the zero element in the abelian group* $\mathrm{Hom}_{\mathscr{C}}(X,X)$.

In either case, we call X *a* ***zero object****.*

Proposition 5.1.5. *Let* $\mathscr{A},\mathscr{C}$ *be two preadditive categories. If* $F : \mathscr{A} \to \mathscr{C}$ *is an additive functor and* X *is a zero object in* $\mathscr{A}$*, then* $F(X)$ *is also a zero object in* $\mathscr{C}$.

The previous three propositions are all easily to check from the definition. Here we give some examples.

Example 5.1.6. *The empty set is the unique initial object in the category* $\mathbf{Set}_\bullet$ *of all sets, and every one-element set(singleton) is a terminal object in* $\mathbf{Set}_\bullet$*. Thus* $\mathbf{Set}_\bullet$ *is not preadditive. Similarly,the category* $\mathbf{Top}_\bullet$ *of all topological spaces is not preadditive.*

Definition 5.1.7. *An* ***additive category*** $\mathscr{C}$ *is a preadditive category admitting all finite biproducts.*

Remark 5.1.8. *From the definition, an additive category* $\mathscr{C}$ *admits the empty biproduct, which is a zero object.*

Example 5.1.9. *The category* $\mathbf{Ab}_\bullet$ *of all abelian groups is an additive category. The zero object is the trivial group, and the biproducts are direct sums.*

First we generalize the construction of kernels and cokernels between abelian groups to additive categories.

Definition 5.1.10. *In an additive category $\mathscr{C}$, a **kernel** of a morphism $f : X \to Y$ is a morphism $\iota : K \to X$ such that $f\iota = 0$ and universal with this property, which means if $j : S \to X$ also satisfies $fj = 0$, then j factors through ι uniquely.*

$$\begin{tikzcd} K \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \\ S \arrow[u, dashed, "\exists!"] \arrow[ur, "j"'] & & \end{tikzcd}$$

Sometimes, we use a pair (K, ι) or simply the object K to denote a kernel of f.

*Dually, a **cokernel** of a morphism $f : X \to Y$ is a morphism $\pi : Y \to Z$ universal with the property $\pi f = 0$.*

*A morphism $f : X \to Y$ is **monic**(or a **monomorphism**) if $fg = 0$ implies $g = 0$ for every morphism $g : W \to X$. Dually, a morphism $f : X \to Y$ is **epic**(or an **epimorphism**) if $gf = 0$ implies $g = 0$ for every $g : Y \to Z$.*

Proposition 5.1.11. *In a category $\mathscr{C}$, every kernel is a monomorphism and every cokernel is an epimorphism.*

Proof. Easy diagram chasing. □

Now we have the definition of abelian categories.

Definition 5.1.12. *A category $\mathscr{C}$ is **abelian** if $\mathscr{C}$ is additive and*

- *every morphism admits a kernel and a cokernel.*
- *every monomorphism is a kernel and every epimorphism is a cokernel.*

Example 5.1.13. *Given a commutative ring R, the category $\mathbf{Mod_R}$ of all left $R-$modules is an abelian category. This is based on the first isomorphism theorem. In particular, the category $\mathbf{Ab_\bullet}$ is an abelian category.*

Proposition 5.1.14. *Let $\mathscr{C}$ be an abelian category. If $\iota : K \to X$ is a monomorphism in $\mathscr{C}$ and $\pi : X \to C$ is an epimorphism in $\mathscr{C}$, then ι is a kernel of π if and only if π is a cokernel of ι.*

Proof. Easy diagram chasing. □

Corollary 5.1.15. *In an abelian category $\mathscr{C}$, a monomorphism is a kernel of its cokernel and an epimorphism is a cokernel of its kernel.*

Proof. Let $f : X \to Y$ be a monomorphism in an abelian category $\mathscr{C}$ and $h : Y \to Z$ is a cokernel of f. Then h is an epimorphism. By the previous proposition, f is a kernel of h. □

Proposition 5.1.16. *Let $\mathscr{C}$ be an additive category and $f : X \to Y$ be a morphism. Then the following are equivalent:*

(i) *f is a monomorphism.*
(ii) $\ker f = 0$.

Dually, g is an epimorphism if and only if $\mathrm{coker} f = 0$.

Proof. Easy diagram chasing. □

These propositions show that monomorphisms(epimorphisms) in abelian categories act just like injectives(surjectives) between abelian groups, and the kernel(cokernel) we defined is also as usual sense.

Definition 5.1.17. *Let $\mathscr{C}$ be an additive category. The **image** of a morphism $f : X \to Y$ is a kernel $\varphi : L \to Y$ of a cokernel $\pi : Y \to C$ of f. The **coimage** of f is a cokernel $\sigma : Z \to X$ of a kernel $\iota : K \to X$.*

$$\begin{tikzcd}
K \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "\sigma"] & Y \arrow[r, "\pi"] & C \\
& Z & L \arrow[u, "\varphi"'] &
\end{tikzcd}$$

Proposition 5.1.18. *Let $\mathscr{C}$ be an abelian category and $f \in \mathrm{Hom}_{\mathscr{C}(X,Y)}$. If $\varphi : L \to Y$ is the image of f and $\sigma : Z \to X$ is the coimage of f, then there exists a unique morphism $\theta : Z \to L$ such that $\varphi\theta\sigma = f$. Moreover, θ is an isomorphism.*

$$\begin{tikzcd}
K \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "\sigma"] & Y \arrow[r, "\pi"] & C \\
& Z \arrow[r, dashed, "\theta"] & L \arrow[u, "\varphi"'] &
\end{tikzcd}$$

Proof. This proof is still a diagram chasing but with some effort.

Since π is a cokernel of f, we have $\pi f = 0$. Since φ is a kernel of π, by the universal property, f factors through φ uniquely, say $f = \varphi g$ for $g : X \to L$. Thus $\varphi g \iota = f \iota = 0$. Since φ is a kernel, it is a monomorphism by proposition 5.1.16, hence $g\iota = 0$. Note that σ is a kernel of ι, by the universal property, g factors through σ uniquely, say, $g = \theta\sigma$ for $\theta : Z \to L$. Thus $f = \varphi g = \varphi\theta\sigma$.

$$\begin{tikzcd}
K \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "\sigma"] \arrow[dr, dashed, "g"] & Y \arrow[r, "\pi"] & C \\
& Z \arrow[r, dashed, "\theta"] & L \arrow[u, "\varphi"'] &
\end{tikzcd}$$

If there exists another $\theta' : Z \to L$ such that $\varphi\theta\sigma = \varphi\theta'\sigma$. Since φ is a monomophism and σ is an epimorphism, we have $\theta = \theta'$. Now we show that θ is an epimorphism and similarly a monomorphism, thus an isomorphism.

Let $h : L \to M$ be any morphism such that $h\theta = 0$, we need to prove that $h = 0$. Let (N, p) be a kernel of h. Then there exists unique $\tau : Z \to N$ such that $\theta = p\tau$. Let (W, q) be a cokernel of φp, then we have

$$qf = q\varphi\theta\sigma = (q\varphi p)\tau\sigma = 0$$

Thus q uniquely factors through π, say $q = \rho\pi$ for $\rho : C \to W$.

$$\begin{tikzcd}
& & W & \\
K \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "\sigma"] \arrow[dr, "g"] & Y \arrow[r, "\pi"] \arrow[u, "q"] & C \arrow[ul, "\rho"'] \\
& Z \arrow[r, "\theta"] \arrow[dr, "\tau"'] & L \arrow[u, "\varphi"'] \arrow[r, "h"] \arrow[d, shift left, "p'"] & M \\
& & N \arrow[u, shift left, "p"] &
\end{tikzcd}$$

Since p, φ are monomorphisms, φp is a monomorphism. Note that q is a cokernel of φp, by 5.1.16 we have φp is a kernel of q. Since $q\varphi = \rho(\pi\varphi) = 0$, φ factors through φp uniquely, say $\varphi = \varphi p p'$ for $p' : L \to N$. Since φ is a monomorphism, we get $pp' = 1_L$. Note that p is a kernel of h, thus $hp = 0$. Hence $h = (hp)p' = 0$, which implies that θ is an epimorphism. □

Remark 5.1.19. *This proposition implies that any morphism f in an abelian category factors as a composition of an epimorphism and a monomorphism. For example, in the abelian category $\mathbf{Mod_R}$, for any $f : M \to N$, we can factor f through* $\operatorname{im} f$.

$$\begin{tikzcd}
\ker f \arrow[r, hook] & M \arrow[rr, "f"] \arrow[dr] & & N \arrow[r, two heads] & N/\operatorname{im} f \\
& & \operatorname{im} f \arrow[ur, hook] & &
\end{tikzcd}$$

The proposition also gives us an alternative definition for the abelian category.

Definition 5.1.20 (Alternative definition)**.** *An **abelian category** $\mathscr{C}$ is an additive category in which for every morphism $f : X \to Y$, there exists a sequence*

$$K \xrightarrow{\iota} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{\pi} C$$

such that

(i) $f = j \circ i$.
(ii) $\iota = \ker f$ *and* $\pi = \operatorname{coker} f$.
(iii) $i = \operatorname{coker}\iota$ *and* $j = \ker \pi$.

Proposition 5.1.21. *Let $\mathscr{A}$ be a small category, and $\mathscr{C}$ an abelian category, then $\mathscr{C}^{\mathscr{A}} = \operatorname{Fun}(\mathscr{A}, \mathscr{C})$ is abelian.*

Proof. For any $\eta, \tau \in \operatorname{Hom}_{\mathscr{C}^{\mathscr{A}}}(F, G)$, define $\eta + \tau : F \to G$ to be the natural transformation given by

$$(\eta + \tau)_X := \eta_X + \tau_X, \ \forall X \in \operatorname{ob}\mathscr{A}$$

Since $\mathscr{A}$ is an abelian category, $\eta_X + \tau_X$ makes sense. Then $\operatorname{Hom}_{\mathrm{C}^{\mathscr{A}}}(F, G)$ has the structure of an abelian group.

Besides, the zero object in $\mathscr{C}^{\mathscr{A}}$ is the zero funtor $\mathscr{A} \xrightarrow{0} \mathscr{C}$ that sends all objects $X \in \operatorname{ob}\mathscr{A}$ to the zero object in $\mathscr{C}$ and all morphisms to the zero morphisms. By diagram chasing, we can check that $\operatorname{Fun}(\mathscr{A}, \mathscr{C})$ satisfies the conditions in the definition 5.1.20, thus an abelian category. □

Definition 5.1.22. *An object P in an abelian category $\mathscr{C}$ is **projective** if for every epimorphism $Y \to Z$ and every morphism $f : P \to Z$. there exists a morphism $h : P \to Y$ such that $f = g \circ h$.*

$$\begin{tikzcd}
& P \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\
Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

*An object I in an abelian category $\mathscr{C}$ is **injective** if for every monomorphism $g : Z \to Y$ and every morphism $f : Z \to I$, there exists a morphism $h : Y \to I$ such* *that $f = h \circ g$.*

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "h"] \\ & I & \end{tikzcd}$$

An abelian category $\mathscr{C}$ has ***enough injectives*** *if for every $X \in \mathrm{ob}\mathscr{C}$, there exists an injective I and a monomorphism $X \to I$.*

An abelian category $\mathscr{C}$ has ***enough projectives*** *if for every $X \in \mathrm{ob}\mathscr{C}$, there exists a projective P and an epimorphism $P \to X$.*

Example 5.1.23. $\mathbf{Mod_R}$ *has enough injectives and enough projectives.*

Definition 5.1.24. *Let $\mathscr{C}$ be a category and Y an object. For any objects W, V in $\mathscr{C}$ and monomorphisms $\iota : W \to Y, \tau : V \to Y$, define an equivalence relation by $(W, \iota) \equiv (V, \tau)$ if there exists an isomorphism $\varphi : W \to V$ such that $\iota = \tau \circ \varphi$.*

$$\begin{tikzcd} W \arrow[r, "\iota"] \arrow[d, "\varphi"'] & Y \\ V \arrow[ur, "\tau"'] & \end{tikzcd}$$

A ***subobject*** *of Y is an equivalence class of (W, ι), where W is an object in $\mathscr{C}$ and $\iota : W \to Y$ is a monomorphism.*

Dually, we can define a ***quotient object*** *of Y.*

Example 5.1.25. *For any morphism $f : X \to Y$ in an abelian category, if both (K, ι) and (K', ι') are kernels of f, then $K = K'$ as subobjects of X. Similarly, if (C, π) and (C', π') are cokernels of f, then $C = C'$ as quotient objects of Y.*

Definition 5.1.26. *A non-zero object X is* ***simple*** *if X doesn't have any subobjects except for 0 and X.*

Definition 5.1.27. *In an abelian category C, a sequence of morphism*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is ***exact*** *at Y if $\ker g = \mathrm{im}\, f$ as subobjects of Y. A* ***short exact sequence*** *in $\mathscr{C}$ is an exact sequence of morphisms*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

That is, f is a monomorphism, g is an epimorphism and $\ker g = \mathrm{im}\, f$.

Definition 5.1.28. *Let $\mathscr{A}, \mathscr{C}$ be two abelian categories and $F : \mathscr{A} \to \mathscr{C}$ an additive functor. Then f is*

- ***left exact*** *if for every short exact sequence in $\mathscr{A}$*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

  the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

  is exact in $\mathscr{C}$.

- ***right exact*** *if for every short exact sequence in* $\mathscr{A}$

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact in $\mathscr{C}$*.*

- ***exact*** *if for every short exact sequence in* $\mathscr{A}$

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact in $\mathscr{C}$*.*

Example 5.1.29. *Let* $\mathscr{C}$ *be an abelian category and* W *an object in* $\mathscr{C}$*. For any short exact sequence in* $\mathscr{C}$

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

we have the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}}(W,X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathscr{C}}(W,Y) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathscr{C}}(W,Z)$$

in $\mathbf{Ab}_\bullet$*. Thus* $\mathrm{Hom}_{\mathscr{C}}(W,-)$ *is a left exact funtor from* $\mathscr{C}$ *to* $\mathbf{Ab}_\bullet$*. Moreover,* $\mathrm{Hom}_{\mathscr{C}}(W,-) : \mathscr{C} \to \mathbf{Ab}_\bullet$ *is an exact functor if and only if* W *is projective in* $\mathscr{C}$*.*

Example 5.1.30. *Let* R *be a ring and* M *an* $R-$*module, then* $-\otimes_R M$ *is a right exact funtor from* $\mathbf{Mod}_{\mathbf{R}}^{\mathbf{op}}$ *to* $\mathbf{Ab}_\bullet$*.* $-\otimes_R M$ *is exact if and only if* M *is flat.*

We have known that the Hom functor and tensor functor are form an adjoint pair, thus the following proposition is a generalization of last two examples.

Proposition 5.1.31. *Let* $F : \mathscr{A} \to \mathscr{C}$ *be an additive functor between two abelian categories. Then*

(i) If F *has a right adjoint, then* F *is right exact.*
(ii) If F *has a left adjoint, then* F *is left exact.*

Proof. Only need to note that for an adjoint pair (G,H), the functor G preserves finite colimits, especially cokernels, and H preserves finite limits, especially kernels. □

Definition 5.1.32. *Let* $\mathscr{C}$ *be an abelian category and* $X \in \mathrm{ob}\mathscr{C}$*. The object* X *is a* ***generator*** *in* $\mathscr{C}$ *if* $\mathrm{Hom}_{\mathscr{C}}(X,-) : \mathscr{C} \to \mathbf{Ab}_\bullet$ *is faithful, which means*

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X,-) : \mathrm{Hom}_{\mathscr{C}}(Y,Z) &\longrightarrow \mathrm{Hom}_{\mathbf{Ab}_\bullet}(\mathrm{Hom}_{\mathscr{C}}(X,Y), \mathrm{Hom}_{\mathscr{C}}(X,Z)) \\ \varphi &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X,-)(\varphi) = \varphi \circ - \end{aligned}$$

is injective for all Y, Z *in* $\mathscr{C}$*.*
Dually, the object X *is a* ***cogenerator*** *in* $\mathscr{C}$ *if* $\mathrm{Hom}_{\mathscr{C}}(-,X) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}_\bullet$ *is faithful.*

Example 5.1.33. *Let* R *be a ring. The* $R-$*module* R *is a generator in* $\mathbf{Mod}_{\mathbf{R}}$*.*

Proof. We need to check that for any $R-$module M, N, any $\varphi \in \mathrm{Hom}_R(M, N), \varphi \neq 0$, then $\mathrm{Hom}_R(R, \varphi) \neq 0$.

$\varphi \neq 0$ implies that there exists $m \in M$ such that $\varphi(m) \neq 0$. We consider

$$\mathrm{Hom}_R(R, M) \xrightarrow{\mathrm{Hom}_R(R,\varphi)} \mathrm{Hom}_R(R, N)$$
$$f \longmapsto \varphi \circ f$$

Denote by f_m the $R-$module homomorphism $R \to M$ sending 1 to m, then

$$\varphi \circ f_m(1) = \varphi(m) \neq 0$$

Thus $\varphi \circ f_m \neq 0$, and then $\mathrm{Hom}_R(R, \varphi) \neq 0$. □

A famous theorem in category theory says that any small abelian category can be embedded into a category of $R-$module. However, the proof is beyond this article.

Theorem 5.1.34 (Freyd-Mitchell)**.** *If $\mathscr{C}$ is a small abelian category, then there is a ring R and a fully faithful exact functor $F : \mathscr{C} \to \mathbf{Mod_R}$.*

From this theorem we can conclude that many categorical statements about a diagram in an abelian category are true if and only if they are true in $\mathbf{Mod_R}$ such as the snake lemma, five lemma and so on.

5.2. **Chain Complexes in Abelian Categories.**

Now we generalize chain complexes in modules to abelian categories. And from now on, we always assume $\mathscr{A}$ to be an abelian category.

Definition 5.2.1. *A **chain complex** $(X_\bullet, d_\bullet)$ is a sequence of objects and morphisms in $\mathscr{A}$*

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 5.2.2. *A **chain map** $f = f_\bullet : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ between two chain complexes is a sequence of morphisms $f_n : X_n \to X'_n$ where $n \in \mathbb{Z}$ such that the following diagram commutes.*

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & X'_{n+1} \arrow[r, "d'_{n+1}"] & X'_n \arrow[r, "d'_n"] & X'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Given any two chain maps $f : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ and $g : (X'_\bullet, d'_\bullet) \to (X''_\bullet, d''_\bullet)$, we can define their composition $g \circ f : (X_\bullet, d_\bullet) \to (X''_\bullet, d''_\bullet)$ by

$$(g \circ f)_n := g_n \circ f_n$$

Besides, for any chain complex $(X_\bullet, d_\bullet)$, we have the identity chain map

$$\left(1_{(X_\bullet, d_\bullet)}\right)_n := 1_{X_n},\ \forall n \in \mathbb{Z}$$

Thus we can define $\mathbf{Ch}(\mathscr{A})$ to be the category of chain complexes. That is, the objects are chain complexes over $\mathscr{A}$ and the morphisms are all chain maps between chain complexes.

Note that $\mathscr{A}$ can be identified with the full subcategory of $\mathbf{Ch}(\mathscr{A})$ in consisted of chain complexes concentrated in degree zero. Hence some concepts are coincide in these two categories. For example, if $f : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ is a chain map, the kernels $\{\ker(f_n)\}$ assemble to form a subcomplex of X denoted by $\ker(f)$. Then $\ker(f)$ is a kernel of f. With this observation, we obtain the following theorem.

Theorem 5.2.3. *If $\mathscr{A}$ is an abelian category, then $\mathbf{Ch}(\mathscr{A})$ is also abelian.*

Now we define the homology of chain complexes.

Definition 5.2.4. *For $(X_\bullet, d_\bullet) \in \mathbf{Ch}(\mathscr{A})$, define the n-**cycles** $Z_n(X_\bullet) = \ker d_n$ and the n-**boundaries** $B_n(X_\bullet) = \operatorname{im} d_{n+1}$. Then define its nth **homology** $H_n(X_\bullet) = Z_n(X_\bullet)/B_n(X_\bullet)$.*

We have discussed a few propositions for chain complexes of modules in section 3. Thanks to the theorem 5.1.34, these propositions are also true for chain complexes in abelian categories. Now we list here.

Proposition 5.2.5. *For each $n \in \mathbb{Z}$, $H_n : \mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$ is an additive functor.*

Theorem 5.2.6. *Let $0 \longrightarrow X'_\bullet \xrightarrow{f} X_\bullet \xrightarrow{g} X''_\bullet \longrightarrow 0$ be an short exact sequence of complexes in $\mathbf{Ch}(\mathscr{A})$. Then for each $n \in \mathbb{Z}$ there is a morphism $\partial_n : H_n(X''_\bullet) \to H_{n-1}(X'_\bullet)$ and we have the following long exact sequence*

$$\cdots \longrightarrow H_{n+1}(X''_\bullet) \xrightarrow{\partial_{n+1}} H_n(X'_\bullet) \xrightarrow{H_n(f)} H_n(X_\bullet) \xrightarrow{H_n(g)} H_n(X''_\bullet) \xrightarrow{\partial_n} H_n(X'_\bullet) \longrightarrow \cdots$$

This theorem is just the snake lemma 3.3.4 in the abelian category.

Proposition 5.2.7. *Given a commutative diagram in $\mathbf{Ch}(\mathscr{A})$*

$$\begin{tikzcd}
0 \arrow[r] & X'_\bullet \arrow[r, "f"] \arrow[d, "\sigma"] & X_\bullet \arrow[r, "g"] \arrow[d, "\rho"] & X''_\bullet \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & Y'_\bullet \arrow[r, "h"] & Y_\bullet \arrow[r, "p"] & Y''_\bullet \arrow[r] & 0
\end{tikzcd}$$

there is a commutative diagram in $\mathscr{A}$ with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X'_\bullet) \arrow[r, "H_n(f)"] \arrow[d, "H_n(\sigma)"] & H_n(X_\bullet) \arrow[r, "H_n(g)"] \arrow[d, "H_n(\rho)"] & H_n(X''_\bullet) \arrow[r, "\partial_n"] \arrow[d, "H_n(\tau)"] & H_{n-1}(X'_\bullet) \arrow[r] \arrow[d, "\sigma_*"] & \cdots \\
\cdots \arrow[r] & H_n(Y'_\bullet) \arrow[r, "H_n(p)"] & H_n(Y_\bullet) \arrow[r, "H_n(p)"] & H_n(Y''_\bullet) \arrow[r, "\partial'_n"] & H_{n-1}(Y'_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Easy diagram chasing. $\square$

We continue our generalization.

Definition 5.2.8. *A chain map $f : X \to X'$ is a **quasi-isomorphism** if $H_n(f) : H_n(X_\bullet) \to H_n(X'_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$. In particular, if the chain map $0 \to X_\bullet$ is a quasi-isomorphism, then X is called **acyclic**.*

Definition 5.2.9. *Let $f, g : (X_\bullet, d_\bullet) \to (X'_\bullet, d'_\bullet)$ be two chain maps. They are **homotopic**, denoted by $f \sim g$, if there is a collection of morphisms $s_n : X_n \to X'_{n+1}$ such that*

$$f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n, \ \forall n \in \mathbb{Z}$$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "g_{n+1}"', "f_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[d, "g_n"', "f_n"] \arrow[dl, dashed, "s_n"'] & X_{n-1} \arrow[r] \arrow[d, "g_{n-1}"', "f_{n-1}"] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & X'_{n+1} \arrow[r, "d'_{n+1}"] & X'_n \arrow[r, "d'_n"] & X'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, if $f \sim 0$, then f is called **null-homotopic**.

Definition 5.2.10. *A chain map $f : X_\bullet \to X'_\bullet$ is called a **homotopy equivalence** if there exists a chain map $g : X'_\bullet \to X_\bullet$ such that $g \circ f \sim 1_{X_\bullet}$ and $f \circ g \sim 1_{X'_\bullet}$. If such f exists, then $X_\bullet$ and $X'_\bullet$ are called **homotopy equivalent**.*

Proposition 5.2.11. *Let $f, g \in \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(X_\bullet, X'_\bullet)$*

(i) If $f \sim g$, then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.
(ii) If f is a homotopy equivalence, then f is a quasi-isomorphism.

Now we define "translation" of chain complexes.

Definition 5.2.12. *For each $k \in \mathbb{Z}$, define a functor*

$$\begin{aligned}
[k] : \mathbf{Ch}(\mathscr{A}) &\longrightarrow \mathbf{Ch}(\mathscr{A}) \\
(X_\bullet, d_\bullet) &\longmapsto (X[k]_\bullet, d_{X[k],\bullet}) \\
f &\longmapsto f[k]_\bullet
\end{aligned}$$

where $X[k]_n := X_{n+k}, d_{X[k],n} := (-1)^k d_{n+k}$ and $f[k]_n = f_{n+k}$, $\forall n \in \mathbb{Z}$.

A simple observation implies that $[0] = 1_{\mathbf{Ch}(\mathscr{A})}$ and $[k] \circ [l] = [k+l]$, $\forall k, l \in \mathbb{Z}$. Hence each $[k] : \mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{A})$ is an auto-equivalence and $[k]^{-1} = [-k]$. Also, $[k]$ can be restricted as a functor from $\mathscr{A}$ to $\mathbf{Ch}(\mathscr{A})$.

Proposition 5.2.13. *$H_{n+k}(X_\bullet)$ can be identified with $H_n(X[k])$.*

Proposition 5.2.14. *Given any $f \in \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(X_\bullet, X_\bullet)$. There is a bijection between the set of homotopies from f to f and $\mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(X_\bullet, X[1]_\bullet)$*

Proof. Let $s = \{s_n : X_n \to X'_{n+1}\}$ be a homotopy from f to f, then

$$0 = f_n - f_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n$$

Thus

$$(-d_{n+1}) \circ s_n = s_{n-1} \circ d_n,\ \forall n \in \mathbb{Z}$$

which implies $s \in \mathrm{Hom}_{\mathbf{Ch}(\mathscr{A})}(X_\bullet, X[1]_\bullet)$.

The contrary is similar. $\square$

Next we generalize the mapping cone and mapping cylinder in the abelian category.

Definition 5.2.15. *Let $f : X_\bullet \to X'_\bullet$ be a chain map, define the **mapping cone** of f:*

$$\mathrm{Cone}(f) := \left(X[-1]_\bullet \oplus X'_\bullet,\ \begin{pmatrix} d_{X[-1],\bullet} & 0 \\ -f[-1] & d'_\bullet \end{pmatrix} \right)$$

which is the chain complex

$$\cdots \longrightarrow X_n \oplus X'_{n+1} \xrightarrow{\begin{pmatrix} -d_n & 0 \\ -f_n & d'_{n+1} \end{pmatrix}} X_{n-1} \oplus X'_n \xrightarrow{\begin{pmatrix} -d_{n-1} & 0 \\ -f_{n-1} & d'_n \end{pmatrix}} X_{n-2} \oplus X'_{n-1} \longrightarrow \cdots$$

Remark 5.2.16. *We should check that the mapping cone defined is indeed a chain complex. In fact, we have*

$$\begin{pmatrix} -d_{n-1} & 0 \\ -f_{n-1} & d'_n \end{pmatrix} \begin{pmatrix} -d_n & 0 \\ -f_n & d'_{n+1} \end{pmatrix} = \begin{pmatrix} d_{n-1}d_n & 0 \\ f_{n-1}d_n - d'_n f_n & d'_n d'_{n+1} \end{pmatrix}$$

And $f_{n-1}d_n - d'_n f_n = 0$ *since* f *is a chain map. Thus the mapping cone satisfies the chain condition.*

There is a short exact sequence of chain complexes

$$0 \longrightarrow X'_\bullet \xrightarrow{\begin{pmatrix} 0 \\ 1_{X'_\bullet} \end{pmatrix}} \mathrm{Cone}(f) \xrightarrow{\begin{pmatrix} -1_{X[-1]_\bullet} & 0 \end{pmatrix}} X[-1]_\bullet \longrightarrow 0$$

By theorem 5.2.6, we have the long exact sequence

$$\cdots \longrightarrow H_{n+1}(\mathrm{Cone}(f)) \longrightarrow H_{n+1}(X[-1]_\bullet) \xrightarrow{\partial_{n+1}} H_n(X'_\bullet) \longrightarrow H_n(\mathrm{Cone}(f)) \longrightarrow \cdots$$

where $H_{n+1}(X[-1]_\bullet) = H_n(X_\bullet)$. Now let's determine the connecting morphism ∂_{n+1}.

By construction, for any $[z_n] \in H_n(X_\bullet)$

$$\partial_{n+1}([z_n]) = \begin{pmatrix} 0 \\ 1_{X'_n} \end{pmatrix}^{-1} \begin{pmatrix} -d_n & 0 \\ -f_n & d'_{n+1} \end{pmatrix} \begin{pmatrix} -1_{X_n} & 0 \end{pmatrix}^{-1} ([z_n]) = f_n([z_n])$$

Thus $\partial_{n+1} = f_n$ and the long exact sequence turns into

$$\cdots \longrightarrow H_{n+1}(\mathrm{Cone}(f)) \longrightarrow H_n(X_\bullet) \xrightarrow{f_n} H_n(X'_\bullet) \longrightarrow H_n(\mathrm{Cone}(f)) \longrightarrow \cdots$$

Using this long exact sequence, we immediately get the following proposition.

Proposition 5.2.17. *A chain map* $f : X_\bullet \to X'_\bullet$ *is a quasi-isomorphism if and only if* $\mathrm{Cone}(f)$ *is acyclic.*

Definition 5.2.18. *Let* $f : X_\bullet \to X'_\bullet$ *be a chain map, define the* ***mapping cylinder*** *of* f*:*

$$\mathrm{Cyl}(f) := \left(X_\bullet \oplus X[-1]_\bullet \oplus X'_\bullet, \begin{pmatrix} d_\bullet & 1_{X[-1]_\bullet} & 0 \\ 0 & d_{X[-1]_\bullet} & 0 \\ 0 & -f & d'_\bullet \end{pmatrix} \right)$$

Similarly, we should check that the mapping cylinder defined is indeed a chain complex by matrix multiplication.

There are many connections between the mapping cone and mapping cylinder just as in algebraic topology. We can construct two simple short exact sequences of chain complexes:

$$0 \longrightarrow X_\bullet \xrightarrow{\begin{pmatrix} 1_{X_\bullet} \\ 0 \\ 0 \end{pmatrix}} \mathrm{Cyl}(f) \xrightarrow{\begin{pmatrix} 0 & 1_{X[-1]_\bullet} & 0 \\ 0 & 0 & 1_{X'_\bullet} \end{pmatrix}} \mathrm{Cone}(f) \longrightarrow 0$$

and

$$0 \longrightarrow X'_\bullet \xrightarrow{\alpha} \mathrm{Cyl}(f) \xrightarrow{\begin{pmatrix} 0 & 1_{X[-1]_\bullet} & 0 \\ 0 & 0 & 1_{X'_\bullet} \end{pmatrix}} \mathrm{Cone}(-1_{X_\bullet}) \longrightarrow 0$$

where α is the inclusion $\begin{pmatrix} 0 \\ 0 \\ 1_{X'_\bullet} \end{pmatrix}$.

By the proposition 5.2.17, $H_n(\mathrm{Cone}(-1_{X_\bullet})) = 0$ for any $n \in \mathbb{Z}$. Thus from the long exact sequence induced by the latter short exact sequence above, we obtain that $H_n(\alpha) : H_n(X'_\bullet) \to H_n(\mathrm{Cyl}(f))$ is an isomorphism for any n, which implies that α is a quasi-isomorphism. Moreover, by the following proposition, we know that α is a homotopy equivalence.

Proposition 5.2.19. *Given a short exact sequence of complexes*

$$0 \longrightarrow X_\bullet \longrightarrow X'_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & X'_\bullet \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Cone}(f) \arrow[r] \arrow[d, equal] & X[-1]_\bullet \arrow[r] & 0 \\
0 \arrow[r] & X_\bullet \arrow[r] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Cone}(f) \arrow[r] \arrow[d, "\varphi"'] & 0 & \\
0 \arrow[r] & X_\bullet \arrow[r, "f"] & X'_\bullet \arrow[r, "g"] & Z_\bullet \arrow[r] & 0 &
\end{tikzcd}$$

where $\beta = (f, 0, 1_{X'_\bullet})$ and $\varphi = (0, g)$. Then we have

(i) $\beta \circ \alpha = 1_{X'_\bullet}$.
(ii) $\alpha \circ \beta \sim 1_{\mathrm{Cyl}(f)}$.
(iii) *φ is a quasi-isomorphism.*

5.3. **Projective Resolutions and Injective Resolutions.**

In this subsection, we will discuss another important concept called projective(injective) resolutions.

Definition 5.3.1. *A **projective resolution** of $X \in \mathrm{ob}\mathscr{A}$ is an exact sequence*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} X \longrightarrow 0$$

where each P_n is projective. Denote by $P_\bullet$ the corresponding chain complex

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

which is also called the projective resolution of X in some literature.

Remark 5.3.2. *If $\mathscr{A} = \mathbf{Mod_R}$ where R is a ring, then a **free resolution** of an $\mathbf{Mod_R}$ X is a projective resolution where each P_n is free; a **flat resolution** of is an exact sequence where each P_n is flat.*

Proposition 5.3.3. *Every $R-$module has a free resolution in $\mathbf{Mod_R}$.*

Proof. Let X be any $R-$module, then there exists a free $R-$module F_0 such that X is a quotient of F_0, saying $\varepsilon : F_0 \to X$. Hence we have the short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\iota_1} F_0 \xrightarrow{\varepsilon} X \longrightarrow 0$$

where $(K_1, \iota_1) = \ker \varepsilon$. Similarly, there exists a free $R-$module F_1 and a surjective $\varepsilon_1 : F_1 \to K_1$. Thus we have the short exact sequence

$$0 \longrightarrow K_2 \xrightarrow{\iota_2} F_1 \xrightarrow{\varepsilon_1} K_1 \longrightarrow 0$$

where $(K_2, \iota_2) = \ker \varepsilon_1$. Going on this procedure, we have the sequence

$$\begin{tikzcd}[column sep=small] \cdots \arrow[rr] & & F_2 \arrow[rr, "d_2"] \arrow[dr, "\varepsilon_2"'] & & F_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & F_0 \arrow[r, "\varepsilon"] & X \arrow[r] & 0 \\ & \cdots \arrow[ur] & & K_2 \arrow[ur, "\iota_2"'] & & K_1 \arrow[ur, "\iota_1"'] & & & \end{tikzcd}$$

where each F_n is a free $R-$module and $d_n := \iota_n \circ \varepsilon_n$. Then

$$\ker d_n = \ker(\iota_n \circ \varepsilon_n) = \ker \varepsilon_n = \text{im } \iota_{n+1}$$
$$\text{im } d_{n+1} = \text{im } (\iota_{n+1} \circ \varphi_{n+1}) = \text{im } \iota_{n+1}$$

Thus $\ker d_n = \text{im } d_{n+1}$. Similarly, $\ker \varepsilon = \text{im } \iota_1 = \text{im } d_1$, hence the sequence is exact. Therefore X has a free resolution. $\square$

Just in the same way, we can prove the following proposition.

Proposition 5.3.4. *If $\mathscr{A}$ is an abelian category with enough projectives, then every $X \in \text{ob}\mathscr{A}$ has a projective resolution.*

Dually, we can define injective resolutions.

Definition 5.3.5. *An **injective resolution** of $X \in \text{ob}\mathscr{A}$ is an exact sequence*

$$0 \longrightarrow X \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$$

where each E^n is injective. Denote by $E^\bullet$ the corresponding cochain complex

$$0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$$

which is also called the injective resolution of X in some literature.

Proposition 5.3.6. *Every $R-$module has an injective resolution.*

Proof. We only need to use the fact that every $R-$module can be embedded as a submodule of an injective $R-$module.(See theorem 3.38 in [4] for reference.) Then the other procedures are the same as proposition 5.3.3. $\square$

Proposition 5.3.7. *If $\mathscr{A}$ is an abelian category with enough injectives, then every $X \in \text{ob}\mathscr{A}$ has an injective resolution.*

Now let's focus on projective resolutions. In general, the projective resolution of an object is not unique. For example, we have the following two projective resolutions of $\mathbb{Z}-$module $\mathbb{Z}_n$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{n\cdot -} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_n \longrightarrow 0$$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix} n\cdot - & 0 \\ 0 & 1_{\mathbb{Z}} \end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{pmatrix} \pi & 0 \end{pmatrix}} \mathbb{Z}_n \longrightarrow 0$$

However, it is unique up to homotopy equivalence by the following theorem.

Theorem 5.3.8 (Comparison Theorem)**.** *Let $\mathscr{A}$ be an abelian category. Given a morphism $f : X \to X'$ in $\mathscr{A}$, consider the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\tilde{f}_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\tilde{f}_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\tilde{f}_0"'] & X \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & P_2' \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] & P_0' \arrow[r, "\varepsilon'"] & X' \arrow[r] & 0 \end{tikzcd}$$

where the rows are chain complexes. If each P_n in the top row is projective and the bottom row is exact, then there exists a chain map $\tilde{f}_\bullet : P_\bullet \to P'_\bullet$ lifting f in the sense that $f \circ \varepsilon = \varepsilon' \circ \tilde{f}_0$. The chain map $\tilde{f}$ is unique up to chain homotopy.

Proof. By induction on $\tilde{f}_n$ and diagram chasing with some effort. □

Corollary 5.3.9. *Let $\mathscr{A}$ be an abelian category with enough projectives, then any two projective resolutions of $X \in \mathrm{ob}\mathscr{A}$ are homotopy equivalent.*

Proof. Assume that $P_\bullet, Q_\bullet$ are two projective resolutions of $X \in \mathrm{ob}\mathscr{A}$. Consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"'] \arrow[dd, bend left, "1_{P_2}"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] \arrow[dd, bend left, "1_{P_1}"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] \arrow[dd, bend left, "1_{P_0}"] & X \arrow[r] \arrow[d, "1_X"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\delta_2"] \arrow[d, "g_2"'] & Q_1 \arrow[r, "\delta_1"] \arrow[d, "g_1"'] & Q_0 \arrow[r, "\eta"] \arrow[d, "g_0"'] & X \arrow[r] \arrow[d, "1_X"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\varepsilon"] & X \arrow[r] & 0
\end{tikzcd}$$

By the comparison theorem 5.3.8, 1_X has lifting chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$. Both $1_{P_\bullet}$ and $g \circ f$ lift the identity map 1_X. Hence $g \circ f \sim 1_P$. Similarly, $f \circ g \sim 1_Q$. Therefore, f is a homotopy equivalence. □

Another important consequence about projective resolutions is the following lemma.

Lemma 5.3.10 (Horseshoe Lemma)**.** *Given a diagram in an abelian category $\mathscr{A}$ with enough projectives*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_2 \arrow[r, "d'_2"] & P'_1 \arrow[r, "d'_1"] & P'_0 \arrow[r, "\varepsilon'"] & X' \arrow[r] \arrow[d, "f"] & 0 \\
 & & & & X \arrow[d, "g"] & \\
\cdots \arrow[r] & P''_2 \arrow[r, "d''_2"] & P''_1 \arrow[r, "d''_1"] & P''_0 \arrow[r, "\varepsilon''"] & X'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the rows are projective resolutions and the column is exact. Set $P_n = P'_n \oplus P''_n$ for $\forall n \in \mathbb{Z}$. Then $P_\bullet$ forms a projective resolution of X. Moreover, the right-hand column lifts to an exact sequence of complexes

$$0 \longrightarrow P'_\bullet \xrightarrow{\iota'} P_\bullet \xrightarrow{\pi''} P''_\bullet \longrightarrow 0$$

where $\iota'_n : P'_n \to P_n$ and $\pi''_n : P_n \to P''_n$ are the natural monomorphism and epimorphism respectively.

Proof. Similar to theorem 5.3.8, by induction and diagram chasing with some effort. □

Remark 5.3.11. *The comparison theorem and Horseshoe lemma have the dual versions for injective resolutions.*

5.4. **Functors.**

Now we introduce some interesting functors in abelian categories such as $\delta-$functors, derived functors and Tor and Ext functors.

The homological $\delta-$functor is generalization of the common homology functor and its connecting morphism in the snake lemma 5.2.6.

Definition 5.4.1. *Given two abelian category $\mathscr{A}$ and $\mathscr{C}$. A **homological** $\delta-$**functor** between $\mathscr{A}$ and $\mathscr{C}$ is a collection of additive functors $T_n : \mathscr{A} \to \mathscr{C}$, $n \in \mathbb{N}$ and a collection of morphisms $\delta_n : T_n(X'') \to T_{n-1}(X')$, $n \in \mathbb{N}$ in $\mathscr{C}$ for each short exact sequence*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in $\mathscr{A}$, such that

(i) for each short exact sequence as above, there is a long exact sequence

$$\cdots \longrightarrow T_n(X'') \xrightarrow{\delta_n} T_{n-1}(X') \longrightarrow T_{n-1}(X) \longrightarrow T_{n-1}(X'') \xrightarrow{\delta_{n-1}} \cdots$$

(ii) for each following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "g"'] & X'' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0
\end{tikzcd}$$

δ gives a commutative diagram

$$\begin{tikzcd}
T_n(X'') \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"] & T_{n-1}(X') \arrow[d, "T_{n-1}(f)"] \\
T_n(Y'') \arrow[r, "\delta_n"] & T_{n-1}(Y')
\end{tikzcd}$$

*Dually, we can define a **cohomological** $\delta-$**functor**.*

Remark 5.4.2. *The convention is that $T_n = 0$ for $n < 0$.*

Example 5.4.3. *Define $\mathbf{Ch}_{\geqslant 0}(\mathscr{A})$ to be the full subcategory of $\mathbf{Ch}(\mathscr{A})$ consisting of chain complexes concentrated in non-negative degrees. For each $n \in \mathbb{Z}$, we have the homology functor $H_n : \mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$. Each H_n can be restricted to an additive functor from $\mathbf{Ch}_{\geqslant 0}(\mathscr{A})$ to $\mathscr{A}$. Thus the homological functor H is a homological $\delta-$functor from $\mathbf{Ch}_{\geqslant 0}(\mathscr{A})$ to $\mathscr{A}$.*

Definition 5.4.4. *A **morphism** $S \to T$ of $\delta-$functors is a system of natural transformation $S_n \xrightarrow{\alpha_n} T_n$ that commute with δ. Namely, for every short exact sequence*

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & S_n(X'') \arrow[r, "\delta_n^S"] \arrow[d, "(\alpha_n)_{X''}"] & S_{n-1}(X') \arrow[r] \arrow[d, "(\alpha_{n-1})_{X'}"] & S_{n-1}(X) \arrow[r] \arrow[d, "(\alpha_{n-1})_{X}"] & S_{n-1}(X'') \arrow[r, "\delta_{n-1}^S"] \arrow[d, "(\alpha_{n-1})_{X''}"] & \cdots \\
\cdots \arrow[r] & T_n(X'') \arrow[r, "\delta_n^T"] & T_{n-1}(X') \arrow[r] & T_{n-1}(X) \arrow[r] & T_{n-1}(X'') \arrow[r, "\delta_{n-1}^T"] & \cdots
\end{tikzcd}$$

Definition 5.4.5. *A homological $\delta-$functor T is **universal** if, given any other $\delta-$functor S and a natural transformation $\alpha_0 : S_0 \to T_0$, there exists unique morphisms $\{\alpha_n : S_n \to T_n\}_{n\in\mathbb{N}}$ of $\delta-$functors that extend α_0.*

Now let's construct a family of universal homological $\delta-$functors.

Definition 5.4.6. *Let $F : \mathscr{A} \to \mathscr{C}$ be a right exact functor between two abelian categories and assume that $\mathscr{A}$ has enough projectives. For any $X \in \mathrm{ob}\mathscr{A}$, choose a projective resolution $P_\bullet$ of X. Define*

$$L_nF(X) := H_n(F(P_\bullet)),\ \forall n \in \mathbb{N}$$

*which is called the **left derived functors**.*

Remark 5.4.7. *The convention is that $L_nF(X) = 0$ for $n < 0$.*

Remark 5.4.8. *We have the projective resolution*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} X \longrightarrow 0$$

Then the sequence

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\varepsilon)} F(X) \longrightarrow 0$$

is exact since F is right exact. Thus

$$L_0F(X) = H_0(F(P_\bullet)) = \mathrm{coker}F(d_1) = F(P_0)/\ker F(\varepsilon) = F(X)$$

We have to check that the definition is well-defined since we can choose different projective resolutions of X.

Proposition 5.4.9. *The object $L_nF(X)$ in $\mathscr{A}$ are well defined up to natural isomorphism. That is, if $Q_\bullet$ is another projective resolution, then there is a canonical isomorphism*

$$L_nF(X) = H_n(F(P_\bullet)) \xrightarrow{\cong} H_n(F(Q_\bullet)) = \tilde{L}_nF(X)$$

In particular, different choices of the projective resolutions would yield new functors $\tilde{L}_nF$, which are naturally isomorphic to the functor L_nF.

Proof. By corollary 5.3.9, the identity map 1_X has lifting chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$. Moreover, $f \circ g \sim 1_{Q_\bullet}$ and $g \circ f \sim 1_{P_\bullet}$.

Since F is additive, we have $F(f) \circ F(g) \sim 1_{F(Q_\bullet)}$ and $F(g) \circ F(f) \sim 1_{F(P_\bullet)}$. Thus

$$H_n(F(f)) \circ H_n(F(g)) = H_n(F(f) \circ F(g)) = H_n(1_{F(Q_\bullet)}) = 1_{H_n(F(Q_\bullet))}$$
$$H_n(F(g)) \circ H_n(F(f)) = H_n(F(g) \circ F(f)) = H_n(1_{F(P_\bullet)}) = 1_{H_n(F(P_\bullet))}$$

which implies that $H_n(F(f)) : H_n(F(P_\bullet)) \to H_n(F(Q_\bullet))$ is an isomorphism.

Let $h : X \to X'$ be any morphism in (A). Choose a projective resolution $P_\bullet$ of

X and a projective resolution $P'_\bullet$ of X'. By the comparison theorem 5.3.8, h can be lifted to a chain map $\tilde{h}_\bullet : P_\bullet \to P'_\bullet$. Then define

$$L_nF(h) := H_n(F(\tilde{h})) : H_n(F(P_\bullet)) \to H_n(F(P'_\bullet))$$

Each $L_nF(h)$ is well-defined since any two lifts are homotopy equivalent.

Choose other projective resolutions $Q_\bullet$ of X and $Q'_\bullet$ of X', which yield new functors $\tilde{L}_nF$ in the sense that $\tilde{L}_nF(X) = H_n(F(Q_\bullet))$ and $\tilde{L}_nF(X') = H_nF(Q'_\bullet)$. Now we check the naturality of the canonical isomorphism $L_nF(X) \cong \tilde{L}_nF(X)$. That is, every such diagram

$$\begin{tikzcd}
L_nF(X) \arrow[r, "\cong"] \arrow[d, "L_nF(h)"] & \tilde{L}_nF(X) \arrow[d, "\tilde{L}_nF(h)"] \\
L_nF(X') \arrow[r, "\cong"] & \tilde{L}_nF(X')
\end{tikzcd}$$

commutes. We choose a lift $\tilde{h} : P_\bullet \to P'_\bullet$ of h to define $L_nF(h)$ and a lift $\rho : Q_\bullet \to Q'_\bullet$ of h to define $\tilde{L}_nF(h)$. Choose a lift $f : P_\bullet \to Q_\bullet$ of 1_X which gives the induced isomorphism $L_nF(X) \cong \tilde{L}_nF(X)$ and a lift $f' : P'_\bullet \to Q'_\bullet$ of 1_X which gives the induced isomorphism $L_nF(X') \cong \tilde{L}_nF(X')$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & X \arrow[r] \arrow[d, "1_X"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\sigma_2"] \arrow[d, "\rho_2"] & Q_1 \arrow[r, "\sigma_1"] \arrow[d, "\rho_1"] & Q_0 \arrow[r, "\eta"] \arrow[d, "\rho_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
\cdots \arrow[r] & Q'_2 \arrow[r, "\sigma'_2"] & Q'_1 \arrow[r, "\sigma'_1"] & Q'_0 \arrow[r, "\eta'"] & X' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\tilde{h}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\tilde{h}_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\tilde{h}_0"] & X \arrow[r] \arrow[d, "h"] & 0 \\
\cdots \arrow[r] & P'_2 \arrow[r, "d'_2"] \arrow[d, "f'_2"] & P'_1 \arrow[r, "d'_1"] \arrow[d, "f'_1"] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "f'_0"] & X' \arrow[r] \arrow[d, "1_{X'}"] & 0 \\
\cdots \arrow[r] & Q'_2 \arrow[r, "\sigma'_2"] & Q'_1 \arrow[r, "\sigma'_1"] & Q'_0 \arrow[r, "\eta'"] & X' \arrow[r] & 0
\end{tikzcd}$$

Note that $h \circ 1_X = h = 1_{X'} \circ h$. Both $\rho \circ f$ and $f' \circ \tilde{h}$ are lifts of h. By the comparison theorem 5.3.8, $\rho \circ f \sim f' \circ \tilde{h}$. Thus $F(\rho) \circ F(f) \sim F(f') \circ F(\tilde{h})$, which induces the same morphism on homology. Thus the required diagram commutes. □

Using the similar technique in the previous proof, we can get the following proposition.

Proposition 5.4.10. *Each L_nF is an additive functor from $\mathscr{A}$ to $\mathscr{C}$.*

The most important property of the derived functor and the motivation to define it are given by the following theorem. Since the proof is too tedious and is just diagram chasing, we skip it.

Theorem 5.4.11. *The derived functors $L_\bullet F$ form a universal homological $\delta-$functor.*

Remark 5.4.12. *In particular, the homology functor $H_\bullet : \mathbf{Ch}_{\geqslant 0}(\mathscr{A}) \to \mathscr{A}$ and cohomology functor $H^\bullet : \mathbf{Ch}^{\geqslant 0}(\mathscr{A}) \to \mathscr{A}$ are universal $\delta-$functors.*

Another special additive functor studied by Grothendieck is the effaceable functor in the following definition, which gives a sufficient condition for a homological $\delta-$functor to be universal.

Definition 5.4.13. *An additive functor $F : \mathscr{A} \to \mathscr{C}$ between two abelian categories is **effaceable** if for each $X \in \mathrm{ob}\mathscr{A}$, there exists a monomorphism $u : X \to Y$ such that $F(u) = 0$. Dually, we can define the **coeffaceable** functor.*

Proposition 5.4.14 (Grothendieck)**.** *Let $T_\bullet$ be a homological $\delta-$functor from $\mathscr{A}$ to $\mathscr{C}$. If each T_n is coeffaceable for $n > 0$, then $T_\bullet$ is universal. Dually, let $T^\bullet$ be a cohomological $\delta-$functor from $\mathscr{A}$ to $\mathscr{C}$. If each T^n is effaceable for $n > 0$, then $T^\bullet$ is universal.*

One can see the Grothendieck's Tôhoku paper [10] for proof.

Dual to the construction of left derived functors, we have the right derived functors.

Definition 5.4.15. *Let $F : \mathscr{A} \to \mathscr{C}$ be a left exact functor between teo abelian categories and assume that $\mathscr{A}$ has enough injectives. For any $X \in \mathrm{ob}\mathscr{A}$, choose an injective resolution $E^\bullet$ of X. Define*

$$R^n F(X) := H^n(F(E^\bullet)),\ \forall n \in \mathbb{N}$$

*which is called the **right derived functors**.*

Remark 5.4.16. *The convention is that $R^n F(X) = 0$ for $n < 0$.*

All the propositions and theorems for left derived functors have the similar version for right derived functors., such as the comparison theorem and particularly the following theorem.

Theorem 5.4.17. *The derived functor $R^\bullet F$ form a universal cohomological $\delta-$functor.*

Now we focus our attention on the category of $R-$modules.

Definition 5.4.18. *Let R be a ring and N a left $R-$module. Then $F := - \otimes_R N$ is a right exact functor from $\mathbf{Mod_R^{op}}$ to $\mathbf{Ab_\bullet}$. Denote by*

$$\mathrm{Tor}_n^R(-, N) := L_n F : \mathbf{Mod_R^{op}} \to \mathbf{Ab_\bullet}$$

called the Tor functor.

Remark 5.4.19. *For any right $R-$module M, choose a projective resolution $P_\bullet$ of M:*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Doing tensor products with N on the right, we get

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \xrightarrow{\varepsilon \otimes 1_N} M \otimes_R N \longrightarrow 0$$

where the last four terms form an exact sequence. Thus

$$\mathrm{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N) = \ker(d_n \otimes 1_N)/\mathrm{im}\,(d_{n+1} \otimes 1_N),\ \forall n \geqslant 1$$
$$\mathrm{Tor}_0^R(M, N) = M \otimes_R N$$

Definition 5.4.20. *Let M be a right $R-$module. Then $G := M \otimes_R -$ is a right exact functor from $\mathbf{Mod_R}$ to $\mathbf{Ab_\bullet}$. Denote by*

$$\mathrm{tor}_n^R(M, -) = L_n G : \mathbf{Mod_R} \to \mathbf{Ab_\bullet}$$

called the tor functor.

Remark 5.4.21. *For any left $R-$module N, choose a projective resolution $P_\bullet$ of N:*

$$\cdots \longrightarrow Q_2 \xrightarrow{\sigma_2} Q_1 \xrightarrow{\sigma_1} Q_0 \xrightarrow{\eta} N \longrightarrow 0$$

Doing tensor products with M on the left, we get

$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes \sigma_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes \sigma_1} M \otimes_R Q_0 \xrightarrow{1_M \otimes \eta} M \otimes_R N \longrightarrow 0$$

where the last four terms form an exact sequence. Thus

$$\mathrm{tor}_n^R(M,N) = H_n(M \otimes_R Q_\bullet) = \ker(1_M \otimes \sigma_n)/\mathrm{im}\,(1_M \otimes \sigma_{n+1}),\ \forall n \geqslant 1$$
$$\mathrm{tor}_0^R(M,N) = M \otimes_R N$$

We calculate an example of the tor functor before advanced discussion.

Example 5.4.22. *Let k, l be two positive integers. Take a projective resolution of $\mathbb{Z}-$module $\mathbb{Z}_l$:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\sigma_1} \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}_l \longrightarrow 0$$
$$r \longmapsto rl$$
$$s \longmapsto \overline{s} = s + l\mathbb{Z}$$

Applying the functor $\mathbb{Z}_k \otimes_{\mathbb{Z}} -$ to the projective resolution, we obtain

$$0 \longrightarrow \mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{1_{\mathbb{Z}_k} \otimes \sigma_1} \mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Z}_l \longrightarrow 0$$

Thus

$$\mathrm{tor}_1^R(\mathbb{Z}_k, \mathbb{Z}_l) = \ker(1_{\mathbb{Z}_k} \otimes \sigma_1) \cong \mathbb{Z}_{(k,l)}$$
$$\mathrm{tor}_0^R(\mathbb{Z}_k, \mathbb{Z}_l) = \mathbb{Z}_k \otimes_{\mathbb{Z}} \mathbb{Z}_l \cong \mathbb{Z}_{(k,l)}$$
$$\mathrm{tor}_n^R(\mathbb{Z}_k, \mathbb{Z}_l) = 0,\ \forall n \neq 0, 1$$

Similarly, one can take a projective resolution $P_\bullet$ of $\mathbb{Z}_k$, apply the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}_l$, calculate the homology and obtain that

$$\mathrm{Tor}_n^R(\mathbb{Z}_k, \mathbb{Z}_l) = \begin{cases} \mathbb{Z}_{(k,l)}, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

In fact, the Tor functor and the tor functor are the same by the following theorem.

Theorem 5.4.23. *Let M be a right $R-$module and N a left $R-$module. Then we have $\mathrm{Tor}_n^R(M,N) \cong \mathrm{tor}_n^R(M,N)$ as abelian groups.*

Proof. See [4] or [5]. □

From now on, we will use $\mathrm{Tor}_n^R(M,-)$ to stand for $\mathrm{tor}_n^R(M,-)$.

Proposition 5.4.24. *The following are equivalent:*

(i) *A right $R-$module M is flat. That is, the functor $M \otimes_R -$ is an exact functor.*
(ii) $\mathrm{Tor}_n^R(M,N) = 0$ *for all $n > 0$ and all left $R-$modules N.*
(iii) $\mathrm{Tor}_1^R(M,N) = 0$ *for all left $R-$modules N.*

Proof. (i)⇒(ii) Take a projective resolution $Q_\bullet$ of N. After doing tensor products with M, the sequence is still exact since M is flat. Thus $\mathrm{Tor}_n^R(M,N) = 0$ for all $n > 0$ and all left $R-$module N.
(ii)⇒(iii) is obvious.
(iii)⇒(i) For any short exact sequence of left $R-$modules:

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

we have the exact sequence of abelian groups

$$\cdots \longrightarrow \mathrm{Tor}_1^R(M,N'') \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$$

By assumption, $\mathrm{Tor}(M,N'') = 0$. Thus we have the exact sequence of abelian groups

$$0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N'' \longrightarrow 0$$

which implies that M is flat. □

Remark 5.4.25. *Similarly, one can write down the equivalent conditions for a left flat $R-$module.*

The dual construction of Tor and tor functors are Ext and ext functors respectively.

Definition 5.4.26. *Let M be a left $R-$module. Then $F := \mathrm{Hom}_R(M,-)$ is a left exact functor from $\mathbf{Mod_R}$ to $\mathbf{Ab_\bullet}$. Denote by*

$$\mathrm{Ext}_R^n(M,-) := R^nF : \mathbf{Mod_R} \to \mathbf{Ab_\bullet}$$

called the Ext functor.

Similarly, we have $\mathrm{Ext}_R^0(M,N) \cong \mathrm{Hom}_R(M,N)$.

Definition 5.4.27. *Let N be a left $R-$module. Then $G := \mathrm{Hom}_R(-,N)$ is a left exact functor from $\mathbf{Mod_R^{op}}$ to $\mathbf{Ab_\bullet}$. Denote by*

$$\mathrm{ext}_n^R(-,N) = R^nG : \mathbf{Mod_R^{op}} \to \mathbf{Ab_\bullet}$$

called the ext functor.

Similarly, we have $\mathrm{ext}_R^0(M,N) \cong \mathrm{Hom}_R(M,N)$.

Example 5.4.28. *We have*

$$\mathrm{Ext}_R^0(\mathbb{Z}_k,\mathbb{Z}_l) = \mathrm{Ext}_R^1(\mathbb{Z}_k,\mathbb{Z}_l) = \mathrm{ext}_R^0(\mathbb{Z}_k,\mathbb{Z}_l) = \mathrm{ext}_R^1(\mathbb{Z}_k,\mathbb{Z}_l) \cong \mathbb{Z}_{(k,l)}$$
$$\mathrm{Ext}_R^n(\mathbb{Z}_k,\mathbb{Z}_l) = \mathrm{ext}_R^n(\mathbb{Z}_k,\mathbb{Z}_l) = 0, \ \forall n \geqslant 2$$

The following consequence is similar to the Tor and tor functors.

Theorem 5.4.29. *Let M,N be two left $R-$modules, then we have*

$$\mathrm{Ext}_R^n(M,N) \cong \mathrm{ext}_R^n(M,N)$$

as abelian groups.

Proposition 5.4.30. *The following are equivalent:*

(i) a left $R-$module M is projective.
(ii) The functor $\mathrm{Hom}_R(M,-)$ is an exact functor.
(iii) $\mathrm{Ext}_R^n(M,N) = 0$ for all $n > 0$ and all left $R-$modules N.

(iv) $\mathrm{Ext}^1_R(M,N)=0$ *for all left* $R-$*modules* N.

Proposition 5.4.31. *The following are equivalent:*

(i) *A left* $R-$*module* N *is injective.*
(ii) *The functor* $\mathrm{Hom}_R(-,N)$ *is an exact functor.*
(iii) $\mathrm{Ext}^n_R(M,N)=0$ *for all* $n>0$ *and all left* $R-$*modules* M.
(iv) $\mathrm{Ext}^1_R(M,N)=0$ *for all left* $R-$*modules* M.

How to describe Tor and Ext functors was always a tough problem for homologists in the last century since it's not easy to calculate them from the definition. They introduced an extension of one module of another, the Baer sum of two extensions and so on. Subject to the author's knowledge and the length of the article, we won't delve into advanced theories. However, the most famous consequence is the following universal coefficient theorem as follows which is worth mentioning.

Definition 5.4.32. *A ring* R *is* ***right(left) hereditary*** *if every submodule of a projective right(left)* $R-$*module is projective.*

Theorem 5.4.33 (Universal Coefficient Theorem for Homology)**.** *Let* R *be a right hereditary ring and* N *a left* $R-$*module. Assume that* $(X_\bullet,d_\bullet)$ *is a chain complex of projective right* $R-$*modules. Then for each* $k\in\mathbb{N}$*, there is a split short exact sequence*

$$0\longrightarrow H_k(X_\bullet)\otimes_R N\longrightarrow H_k(X_\bullet\otimes_R N)\longrightarrow \mathrm{Tor}^R_1(H_{k-1}(X_\bullet),N)\longrightarrow 0$$

That is,

$$H_k(X_\bullet\otimes_R N)\cong H_k(X_\bullet)\otimes_R N\oplus \mathrm{Tor}^R_1(H_{k-1}(X_\bullet),N)$$

Theorem 5.4.34 (Universal Coefficient Theorem for Cohomology)**.** *Let* R *be a ring and* N *a left* $R-$*module. Assume that* $(X_\bullet,d_\bullet)$ *is a chain complex of projective left* $R-$*modules such that* $B_k=\mathrm{im}\, d_{k+1}$ *is also projective. Then*

(i) *for each* $k\in\mathbb{N}$*, there is a short exact sequence*

$$0\longrightarrow \mathrm{Ext}^1_R(H_{k-1}(X_\bullet),N)\longrightarrow H^k(\mathrm{Hom}_R(X_\bullet,N))\longrightarrow \mathrm{Hom}_R(H_k(X_\bullet),N)\longrightarrow 0$$

(ii) *if further* R *is left hereditary, then for each* $k\in\mathbb{N}$*, the above short exact sequence splits. That is,*

$$H^k(\mathrm{Hom}_R(X_\bullet,N))\cong \mathrm{Ext}^1_R(H_{k-1}(X_\bullet),N)\oplus \mathrm{Hom}_R(H_k(X_\bullet),N)$$

6. Introduction to Topology K-Theory

In the last section, we introduce the topology K-theory as an application of homology theory. However, we won't prove most of propositions and theorems. One that is interested in this theory can see Hatcher's book [10] for reference.

First we need the knowledge of vector bundles.

Definition 6.0.1. *An* ***n-dimensional vector bundle*** *is a map* $p:E\to B$ *together with a real vector space structure on* $p^{-1}(b)$ *for each* $b\in B$*, such that the following local triviality condition is satisfied: There is a cover of* B *by open sets* U_α *for each of which there exists a homeomorphism* $h_\alpha:p^{-1}(U_\alpha)\to U_\alpha\times\mathbb{R}^n$ *taking* $p^{-1}(b)$ *to* $b\times\mathbb{R}^n$ *by a vector space isomorphism for each* $b\in U_\alpha$*. Such an* h_α *is called a* ***local trivialization*** *of the vector bundle. The space* B *is called the* ***base space****,* E *is the* ***total space****, and the vector spaces* $p^{-1}(b)$ *are the* ***fibers****.*

If we require $\mathbb{R}^n$ to be $\mathbb{C}^n$, then we obtain the complex vector bundle.

Example 6.0.2. *Here are some examples of vector bundles:*

(i) *The* ***product*** *or* ***trivial*** *bundle* $E = B \times \mathbb{R}^n$ *with* p *the projection onto the first factor.*

(ii) *Let* E *be the quotient space of* $I \times \mathbb{R}$ *under the identifications* $(0,t) \sim (1,-t)$, *then the projection* $I \times \mathbb{R} \to I$ *induces a map* $p : E \to S^1$ *which is a 1-dimensional vector bundle, or* ***line bundle****. Since* E *is homeomorphism to a Möbius band, we call this bundle the* ***Möbius bundle****.*

(iii) *The tangent bundle of the unit sphere* S^n *in* $\mathbb{R}^{n+1}$ *is a vector bundle* $p : E \to S^n$ *where* $E = \{(x,v) \in S^n \times \mathbb{R}^{n+1} | x \perp v\}$, p *sends* (x,v) *to* x.

(iv) *The normal bundle to* S^n *in* $\mathbb{R}^{n+1}$ *is a line bundle* $p : E \to S^n$ *with* E *consisting of pairs* $(x,v) \in S^n \times \mathbb{R}^{n+1}$ *such that* v *is perpendicular to the tangent plane to* S^n *at* x, *that is,* $v = tx$ *for some* $t \in \mathbb{R}$.

(v) *Real projective* n*-space* $\mathbb{R}\mathrm{P}^n = S^n/\{x \sim -x\}$, *so there is a* ***canonical line bundle*** $p : E \to \mathbb{R}\mathrm{P}^n$ *has as its total space* E *the subspace of* $\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$ *consisting of pairs* (l,v) *with* $v \in l$, *and* $p(l,v) = l$.

(vi) *The canonical line bundle over* $\mathbb{R}\mathrm{P}^n$ *has an orthogonal complement, the space* $E^\perp = \{(l,v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} | v \perp l\}$. *The projection* $p : E^\perp \to \mathbb{R}\mathrm{P}^n$, $p(l,v) = l$, *is a vector bundle with fibers the orthogonal subspaces* $l^\perp$,*of dimension* n.

Definition 6.0.3. *An* ***isomorphism*** *between vector bundles* $p_1 : E_1 \to B$ *and* $p_2 : E_2 \to B$ *over the same base space* B *is a homeomorphism* $h : E_1 \to E_2$ *taking each fiber* $p_1^{-1}(b)$ *to the corresponding fiber* $p_2^{-1}(b)$ *by a linear isomorphism. We use the notation* $E_1 \approx E_2$.

Definition 6.0.4. *A* ***section*** *of a vector bundle* $p : E \to B$ *is a map* $s : B \to E$ *assigning to each* $b \in B$ *a vector* $s(b)$ *in the fiber* $p^{-1}(b)$. *The condition can also be written as* $ps = \mathbb{1}$, *the identity map of* B. *Every vector bundle has a canonical section, the* ***zero section*** *whose value is the zero vector in each fiber.*

We have the condition to identify a vector bundle to be isomorphic to a trivial bundle.

Proposition 6.0.5. *An* n*-dimensional bundle* $p : E \to B$ *is isomorphic to the trivial bundle if and only if it has* n *sections* $s_1, \cdots, s_n$ *such that the vectors* $s_1(b), \cdots, s_n(b)$ *are linearly independent in each fiber* $p^{-1}(b)$.

Definition 6.0.6. *Given two vetor bundles* $p_1 : E_1 \to B$ *and* $p_2 : E_2 \to B$ *over the same base space* B, *define the* ***direct sum*** *of* E_1 *and* E_2 *as the space*

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \big| \, p_1(v_1) = p_2(v_2)\}$$

For vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, we can construct the tensor product of these two vector bundles. Let $E_1 \otimes E_2$, as a set, be the disjoint union of the vector spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for $x \in B$. The topology is defined in the following way.

Choose isomorphisms $h_i : p_i^{-1}(U) \to U \times \mathbb{R}^{n_i}$ for each open set $U \subseteq B$ over which E_1 and E_2 are trivial. Then a topology $\mathscr{T}_U$ on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is defined by letting the fiberwise tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ be a homeomorphism. The topology $\mathscr{T}_U$ is independent of the choice of the h_i's since any other choices are obtained by composing with isomorphisms of $U \times \mathbb{R}^{n_1}$

of the form $(x, v) \mapsto (x, g_i(x)(v))$ for continuous maps $g_i : U \to GL_{n_i}(\mathbb{R})$, hence $h_1 \otimes h_2$ changes by composing with analogous isomorphisms of $U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ whose second coordinates $g_1 \otimes g_2$ are continuous maps $U \to GL_{n_1 n_2}(\mathbb{R})$. When we replace U by an open subset V, the topology on $p_1^{-1}(V) \times p_2^{-1}(V)$ induced by $\mathscr{T}_U$ is the same as the topology $\mathscr{T}_V$ since local trivializations over U restrict to local trivializations over V. Hence we get a well-defined topology on $E_1 \otimes E_2$ making it a vector bundle over B.

Proposition 6.0.7. *Given a map $f : A \to B$ and a vector bundle $p : E \to B$, then there exists a vector bundle $p' : E \to A$ with a map $f' : E' \to E$ taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$, and such a vector bundle E' is unique up to isomorphism.*

Remark 6.0.8. *We have a function $f^* : \mathrm{Vect}(B) \to \mathrm{Vect}(A)$ taking the isomorphism class of E to the isomorphism class of E'. Often the vector bundle E' is written as $f^*(E)$ and called the bundle **induced** by f, or the **pullback** of E by f.*

Now we can define the K-group for a topological space.

Let us assume 'vector bundle' as 'complex vector bundle' and the base spaces always to be compact Hausdorff. And we can have a broader definition of 'vector bundle' which allows the fibers of a vector bundle $p : E \to X$ to be vector spaces of different dimensions. We still assume local trivializations of the form $h : p^{-1}(U) \to U \times \mathbb{C}^n$, so the dimensions of fibers must be locally constant over X, but if X is disconnected the dimensions of fibers need not be globally constant.

Consider vector bundles over a fixed base space X. The trivial $n-$dimensional vector bundle we write as $\varepsilon^n \to X$. Define two vector bundles E_1 and E_2 over X to be **stably isomorphic**, written $E_1 \approx_s E_2$, if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n. In a similar vein we set $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^m \approx E_2 \oplus \varepsilon^n$ for some m and n. Then both $\approx_s$ and $\sim$ are equivalence relations. On equivalence classes of either sort the operation of direct sum is well-defined, commutative, and associative. A zero element is the class of ε^0.

Proposition 6.0.9. *If X is compact Hausdorff, then the set of $\sim -$equivalence classes of vector bundles over X forms an abelian group with respect to $\oplus$. This group is called $\tilde{K}(X)$.*

For the direct sum operation on $\approx_s -$equivalence classes, only the zero element, the class of ε^0, can have an inverse since $E \oplus E; \approx_s \varepsilon^0$ implies $E \oplus E' \oplus \varepsilon^n \approx \varepsilon^n$, which can only happen if E and E' are 0-dimensional. However, even though inverses do not exist, we do have the cancellation property that $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ implies $E_2 \approx E_3$ over a compact base space X.

We can form for compact X an abelian group $K(X)$ consisting of formal differences $E - E'$ of vector bundles E and E' over X, with the equivalence relation $E_1 - E_1' = E_2 - E_2'$ if and only if $E_1 \oplus E_2' \approx E_2 \oplus E_1'$. Verifying transitivity of this relation involves the cancellation property, which is why compactness of X is needed. With the obvious addition rule $(E_1 - E_1') + (E_2 - E_2') = (E_1 \oplus E_2) - (E_1' \oplus E_2')$, $K(X)$ is then a group. The zero element is the equivalence class of $E - E$ for any E, and the inverse of $E - E'$ is $E' - E$. Note that every element of $K(X)$ can be represented as a difference $E - \varepsilon^n$.

There is a natural homomorphism $K(X) \to \tilde{K}(X)$ sending $E - \varepsilon^n$ to the $\sim -$class of E. This is well-defined since if $E - \varepsilon^n = E' - \varepsilon^m$ in $K(X)$, then $E \oplus \varepsilon^m \approx_s E' \oplus \varepsilon^n$, hence $E \sim E'$. The map $K(X) \to \tilde{K}(X)$ is obviously surjective, and its kernel consists of elements $E - \varepsilon^n$ with $E \sim \varepsilon^0$, hence $E \approx_s \varepsilon^m$ for some m, so the kernel consists of elements of the form $\varepsilon^m - \varepsilon^n$. This subgroup of $K(X)$ is isomorphic to $\mathbb{Z}$. Thus we have a splitting $K(X) \approx \tilde{K}(X) \oplus \mathbb{Z}$. The group $\tilde{K}(X)$ is sometimes called **reduced**, to distinguish it from $K(X)$.

Besides the additive structure in $K(X)$ there is also a natural multiplication coming from tensor product of vector bundles. For arbitrary elements of $K(X)$ represented by differences of vector bundles, their product in $K(X)$ is defined by the formula

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2'$$

It is routine to verify that this is well-defined and makes $K(X)$ into a commutative ring with identity ε^1, the trivial line bundle. We can simplify notation by writing the element $\varepsilon^n \in K(X)$ just as n. For example, the product nE is the sum of n copies of E.

If we choose a basepoint $x_0 \in X$, then the map $K(X) \to K(x_0)$ obtained by restriction vector bundles to their fibers over x_0 is a ring homomorphism. Its kernel, which can be identified with $\tilde{K}(X)$, is an ideal, hence also a ring in its own right.

The rings $K(X)$ and $\tilde{K}(X)$ can be regarded as functors of X. A map $f : X \to Y$ induces a map $f^* : K(Y) \to K(X)$, sending $E - E'$ to $f^*(E) - f^*(E')$. This is a ring homomorphism since $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$ and $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$. The functor properties $(fg)^* = g^* f^*$ and $\mathbb{1}^* = \mathbb{1}$ as well as the fact that $f \simeq g$ implies $f^* = g^*$ all follow from the corresponding properties for pullbacks of vector bundles. Similarly, we have induced maps $f^* : \tilde{K}(Y) \to \tilde{K}(X)$ with the same properties except that for f^* to be a ring homomorphism we must be in the category of basepointed spaces and basepoint-preserving maps since our definition of multiplication for $\tilde{K}$ required basepoints.

An **external product** $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ can be defined by $\mu(a \otimes b) = p_1^*(a) p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. The tensor product of rings is a ring, with multiplication defined by $(a \otimes b)(c \otimes d) = ac \otimes bd$, and μ is a ring homomorphism.

Let H be the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$, then there is a theorem to define the $K-$group of the sphere S^2, or more generally, $X \times S^2$ for all compact Hausdorff spaces X.

Theorem 6.0.10. *The homomorphism $\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.*

Corollary 6.0.11. *The map $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ is an isomorphism of rings.*

More advanced consequences come from Bott periodicity using the technique of cohomology theory.

Theorem 6.0.12. *The homomorphism $\beta : \tilde{K}(X) \to \tilde{K}(S^2 X)$, $\beta(a) = (H-1) * a$, is an isomorphism for all compact Hausdorff spaces X.*

Corollary 6.0.13. *$\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) \cong \mathbb{Z}$, generated by the $n-$fold reduced external product $(H-1) * \cdots * (H-1)$.*

Theorem 6.0.14. *There exists a map $f : S^{4n-1} \to S^{2n}$ of Hopf invariant ± 1 only when $n = 1, 2$ or 4.*

To distinguish the structure of vector bundles, various characteristic classes are invented. To be precise, characteristic classes are cohomology classes associated to vector bundles. We have the Stiefel-Whitney classes $w_i(E) \in H^i(B;\mathbb{Z}_2)$ for a real vector bundle, Chern classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ for a complex vector bundle, Pontryagin classes $p_i(E) \in H^{4i}(B;\mathbb{Z})$ for a real vector bundle and the Euler class $e(E) \in H^n(B;\mathbb{Z})$ for an oriented $n-$dimensional real vector bundle. These characteristic classes play a significant role in modern geometry research.

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