Tianyi Li
1000929445

<u>CSC373 Fall 2017 Assignment 3</u>

1. In the same way that we can express LPs in standard form, we can express an IP and its dual in standard form. Show that weak duality holds for integer programs in standard form.
   - **An integer programming or IP is a linear programming problem with integer values.**
   - **The standard form for an integer program is:**

     Maximize $c^T x$

     Subject to $Ax \leq b$

     $x \geq 0, x \in \mathbb{Z}^n$
   - **The dual for an integer program is:**

     Minimize $\mathrm{y^T b}$

     Subject to $y^T \mathrm{A} \leq c^T$

     $y \geq 0, y \in \mathbb{Z}^n$
   - **Let $\bar{x}$ be a feasible solution to the primal integer program,**
   - **Let $\bar{y}$ be a feasible solution to the dual integer program,**
   - **To show that the weak duality holds for integer programs in standard form, that means we need to prove** $c^T x \leq \mathrm{y}^T \mathrm{b}$.
   - **Since $\bar{x}$ is feasible, and it is primal**,
   - **Then** $A\bar{x} \leq b$,

     $\bar{y}^T A\bar{x} \leq \bar{y}^T b$.
   - **Since $\bar{y}$ is dual,**
   - **Then** $c^T \leq \bar{y}^T A$,

     $c^T \bar{x} \leq \bar{y}^T A\bar{x}$.
   - **From both above**$c^T \bar{x} \leq \bar{y}^T A\bar{x}$**, and** $\bar{y}^T A\bar{x} \leq \bar{y}^T b$,
   - **Then** $c^T \bar{x} \leq \bar{y}^T A\bar{x} \leq \bar{y}^T b$.
   - **Therefore,** $c^T \bar{x} \leq \bar{y}^T b$.

2. Consider the following primal IP and LP in standard form:

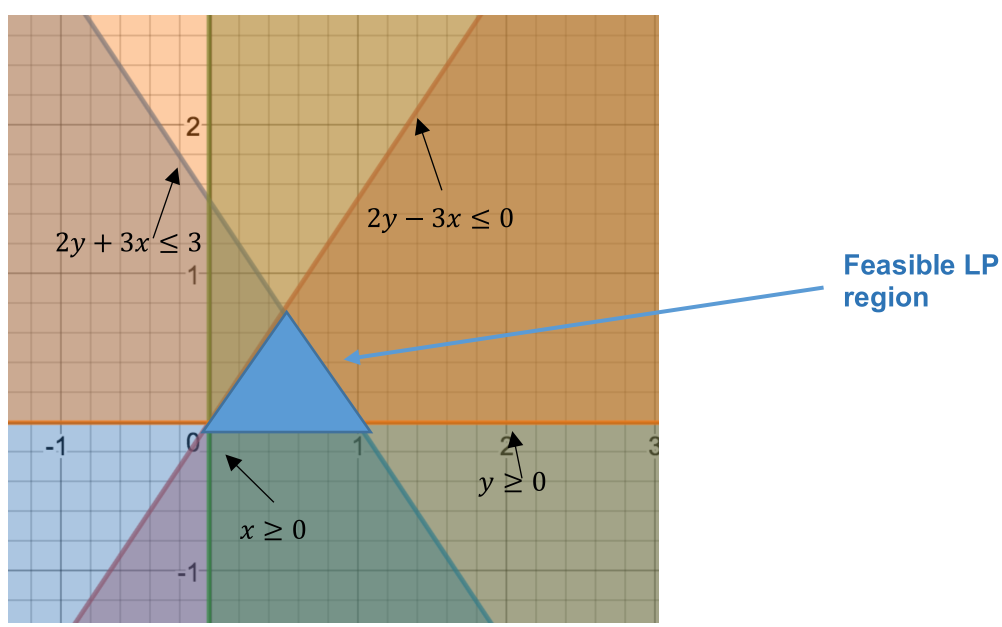
   Maximize y

   Subject to $2y - 3x \leq 0$

   $2y + 3x \leq 3$

   $y, x \geq 0$

   For the IP, add the constraints that x and y are integral. Draw the feasible LP region.



Tianyi Li
1000929445

a) What are the vertices of the feasible region of the primal LP?
- **Let** $2y - 3x = 0$ $2y + 3x = 3$ $y = 0$ $x = 0$
- **Then** $2y = 3x$ $2y = 3 - 3x$
  $y = \frac{3}{2}x$ $y = \frac{3}{2} - \frac{3}{2}x$
- $\begin{cases} y = \frac{3}{2}x \\ y = \frac{3}{2} - \frac{3}{2}x \end{cases} \Longrightarrow (\frac{1}{2}, \frac{3}{4})$
- $\begin{cases} y = \frac{3}{2}x \\ x = 0 \\ y = 0 \end{cases} \Longrightarrow (0, 0)$
- $\begin{cases} y = \frac{3}{2} - \frac{3}{2}x \\ y = 0 \end{cases} \Longrightarrow (1, 0)$
- **Therefore, the vertices of the feasible region are $(\frac{1}{2}, \frac{3}{4})$, $(0, 0)$ and $(1, 0)$.**

b) What is the optimal fractional solution?
- $y = \frac{3}{4}$

c) What is the optimal integral solution?
- $y = 0$

d) Provide the dual IP of the primal IP.
- **Minimize** $3y$
- **Subject to** $3y - 3x \geq 0$
  $2y + 2x \geq 1$
  $y, x \leq 0$

e) Use this example to show that strong duality does not necessarily hold for integer programming.
- **Strong duality: the primal optimal solutions equals to the dual optimal solution.**
- **In this case, the primal optimal solution is $y = 0$.**
- **To calculate the vertices of the feasible region of the dual LP:**
- **Let** $3y - 3x = 0$ $2y + 2x = 1$ $y = 0$ $x = 0$
- **Then** $3y = 3x$ $2y = 1 - 2x$
  $y = x$ $y = \frac{1}{2} - x$
- $\begin{cases} y = x \\ y = \frac{1}{2} - x \end{cases} \Longrightarrow (\frac{1}{4}, \frac{1}{4})$
- $\begin{cases} y = x \\ x = 0 \\ y = 0 \end{cases} \Longrightarrow (0, 0)$
- $\begin{cases} y = \frac{1}{2} - x \\ y = 0 \end{cases} \Longrightarrow (\frac{1}{2}, 0)$
- **Then the vertices of the feasible region are $(\frac{1}{4}, \frac{1}{4})$, $(0, 0)$ and $(\frac{1}{2}, 0)$.**
- **Then the optimal fractional solution is $y = \frac{1}{4}$**
- **Then the optimal integral solution is $3y = \frac{3}{4}$, $y = \frac{1}{4}$**
- **In this case, the primal optimal solution is 0.**
-

3. Consider the following weighted partial vertex cover problem (WPVC):

Tianyi Li
1000929445

Given: A graph $G = (V, E)$ with vertex weights $w: V \rightarrow \mathbb{R}^{\geq 0}$ and edge costs $c: E \rightarrow \mathbb{R}^{\geq 0}$. In the standard vertex cover problem, we need to choose a subset $V' \subseteq V$ so that every edge $e = (u, v)$ has to be covered in the sense that at least one of its endpoints u or v must be in the cover V'. In the WPVC problem, not every edge has to be covered but an uncovered edge e costs $c(e)$. The objective it to minimize the weights of the partial cover plus the costs of uncovered edges.

a) Provide a $\{0, 1\}$ IP for the WPVC problem.
   -
b) Provide an approximation guarantee for WPVC by the IP/LP rounding method.
   -

4. Consider a model of computation where arithmetic operations take $O(1)$ time and choosing a random number in the range $\{1, 2, \ldots, 2n\}$ takes time $O(1)$. Let A, B and C be $n \times n$ matrices over the ring of integers. We want to verify that $AB = C$ using $O(n^2)$ arithmetic operations.
   a) Provide a randomized algorithm ALG satisfying that: If $AB = C$, ALG will always say YES.
      -
   b) Provide a randomized algorithm ALG satisfying that: If $AB \neq C$, ALG will say YES with some constant error probability $p < 1$.
      -
   c) Show how you can verify $AB = C$ with error probability $\frac{1}{n}$. What is the time complexity of your algorithm?
      -

5. Consider the exact Max-3-Sat problem:
   Given a propositional CNF formula $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ where each clause C_i has exactly 3 literals. The objective is to compute a truth assignment so as to maximize the number of satisfied clauses. Provide a polynomial time randomized algorithm that in expectation will satisfy a $\frac{7}{8}$ fraction of the clauses.
   -