TWO-CANDIDATE ELECTIONS WITHOUT MAJORITY RULE EQUILIBRIA

An Experimental Study

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The "median voter theorem" in spatial models of election competition states, roughly, that if a two-candidate election concerns a single issue and if all but indifferent voters vote for their preferred candidate, the preference of the median voter on the election issue is an equilibrium strategy for both candidates. That is, the median voter's preference is the unique issue position that, if adopted by a candidate, ensures the candidate of doing no worse than attaining a tie in the election. The importance of this theorem, of course, is that it informs us of a "central tendency" in single-issue majority rule processes—a tendency for candidates to converge to the median.

Unfortunately, this theorem does not extend readily to elections that concern more than one issue, and substantial theorizing yields the incontrovertible conclusion that unless especially restrictive assumptions are imposed about, for example, voter preferences, candidate mobility, and the form of nonvoting, equilibrium strategies for candidates are unlikely to exist (see Ordeshook, 1976). In fact, we now know that in the event of spatial preferences, the absence of such equilibrium is "total"—the social preference is wholly intransitive over the

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entire issue space, thereby opening the possibility that candidates might wander endlessly about the issue space (McKelvey, 1976).

Two-candidate elections, however, can be modeled as two-person zero-sum games, in which case the median voter theorem tells us that under the assigned conditions, the selection by both candidates of the median voter's preference as their campaign platforms is a unique pure strategy equilibrium (minmax) to the corresponding game. Election games without such equilibria are simply games that, mathematically at least, must be solved by the introduction of mixed strategies. There is, however, considerable reluctance to hypothesize that candidates abide by mixed minmax strategies. This reluctance can be attributed to two considerations: First, the assumption in game theory that strategies are revealed simultaneously by the players appears untenable when applied to real election processes unless we radically alter the generally static form of election models. Second, owing to the substantial numerical complexities of computing mixed minmax strategies, it seems incomprehensible that even if all other theoretical preconditions are satisfied, candidates could ever compute and abide by such solutions.

This article reports on a series of experiments that address this second reservation and that provide evidence bearing on the proposition that people will abide by mixed minmax strategies even if they cannot directly compute them. Our experiments, then, model simple two-candidate elections, and with them we can observe whether subjects, acting as candidates, adopt strategies that correspond to the theoretical mixed strategy distribution for the corresponding election game.

THEORETICAL SETTINGS

To model an election as a two-person (candidate) zero-sum game we must first construct the candidate payoff function.

Briefly, we assume there is a set N of voters and an alternative space X which comprises the alternative policies candidates can advocate as election strategies. In our experiments, X consists either of a finite list of alternatives or, in the case of spatial elections, the points on a two-dimensional grid. Each voter $i \epsilon N$ is assumed to possess a binary relation P_i that represents his or her strict preferences over the alternatives. We assume there are two candidates, denoted by $j = 1$ or 2, and that candidate j adopts as his election strategy the alternative or issue position $\theta_j \epsilon X$. Candidate 1's payoff function, $M(\theta_1,\theta_2)$, is +1 if he wins the election (if more voters prefer θ_1 to θ_2 than prefer θ_2 to θ_1), 0 if the election is a tie, and −1 if he loses the election. Formally

$$M(\theta_1,\theta_2) = \begin{array}{l} 1 \text{ if} \mid i\epsilon N \mid \theta_1 P_i \theta_2 \mid > \mid i\epsilon N \mid \theta_2 P_i \theta_1 \mid \\ 0 \text{ if} \mid i\epsilon N \mid \theta_1 P_i \theta_2 \mid = \mid i\epsilon N \mid \theta_2 P_i \theta_1 \mid \\ -1 \text{ if} \mid i\epsilon N \mid \theta_1 P_i \theta_2 \mid < \mid i\epsilon N \mid \theta_2 P_i \theta_1 \mid \end{array}$$

The function $M(\theta_1,\theta_2)$, then, is the payoff function of a two-person zero-sum game, representing a two-candidate, majority rule election. $M(\theta_1,\theta_2)$ is the payoff to candidate 1 while $-M(\theta_1,\theta_2)$ is the payoff to candidate 2.

In this article we investigate two specific experimental designs—the first involves a finite set of alternatives and the second corresponds to the usual spatial models of elections with two issues. In the finite design, then, a candidate's election strategy is simply the choice of an alternative while "voters" vote for the candidate who chooses the alternative that stands highest in their preference order.

In the second design we assume that $X \subseteq R^2$, and that voters' preferences are based on Euclidian distance. That is, each voter $i \epsilon N$ possesses an ideal point $v_i \epsilon X$, and the further one moves from this point, the lower are the corresponding alternatives on the voter's preference order (i.e., for all $x,y \epsilon X$, $xP_iy \Leftrightarrow \|x-v_i\| < \|y-v_i\|$). In our spatial experiments the issue space can be represented by a simple grid, where the horizontal dimension corresponds to "issue 1" and the vertical dimension corresponds to "issue 2." A candidate's election strategy, then, corresponds to a point on this grid while "voters" vote for the candidate closest to their ideal point.

In our experiments we consider a total of five preference configurations for voters: two configurations use finite alternatives and three configurations use spatial alternatives. The configurations are presented in Figures 1-5. Briefly, Figures 1 and 3 give the preferences of five hypothetical voters over a list of alternatives denoted A, B, C, . . . while Figures 2, 4, and 5 give the ideal points of hypothetical voters in a two-issue space.[1] The finite configuration in Figure 1 and the spatial configuration in Figure 2 act as controls for our experimental procedures in that both possess an election equilibrium. In Figure 1, G is a unique alternative that cannot be defeated in a majority vote; in Figure 2, any candidate located at voter 5's ideal cannot be defeated. Both of these configurations are employed elsewhere using different experimental procedures with the finding that the equilibrium is almost always attained (McKelvey and Ordeshook, 1981; Berl et al., 1976). Naturally, we expect that subjects acting as candidates would converge to the equilibrium as well.

The configuration in Figure 3, on the other hand, possesses an equilibrium only in mixed strategies. That is, no lettered alternative guarantees a candidate of doing no worse than attaining a tie—every alternative is beaten in a majority vote by at least one other alternative. In this instance, then, we can observe how subjects acting as candidates respond to disequilibrium in a finite alternative setting. For spatial (two-issue) preferences we consider two configurations without an equilibrium as portrayed in Figures 4 and 5. The properties of mixed minmax solutions to these infinite games are only imprecisely known. We know that such strategies exist (see Kramer, 1978; Rosker, 1981), but their domain cannot be specified generally nor are there any simple procedures for calculating them (McKelvey, 1981). However, since experimental subjects never

Preferences of Voters
(ranked from best to worst)

Player 1	Player 2	Player 3	Player 4	Player 5
X	Y	X	J	V
H	U	E	N	W,T
U	N	M	W	L,P
E	L,S	Q	Z	G
F,Q	B	G	G	Y,J,I,O
B	J,R	F,O	K,L,R	C
I,O	T	H,A	A	E,N
M	H,K	B,I	B,Y	A,D,U,K
K	I	D,R,W	M,O,V	B,M
R	G,Z	N,T	T,E	S,Z,F,Q
G,P	D,V	P,K	S,H	R,X
D	E	Z,C	C,Q	H
N	F,P	J,V,S	U	
S,Z	M,W	U,Y	D,P,X	
T,W	C	L	I	
C	O,X		F	
J,Y	A			
A	Q			
V				
L				

Figure 1 Preference Configuration for Finite Alternative Election with Equilibrium at "G"

consider proposals finer than the grid presented to them, we can use this grid and the corresponding finite game to approximate the mixed minmax strategy.[2] The corresponding mixed minmax strategies are also presented in Figures 4 and 5 where the numbers refer to the relative weight the minmax strategy assigns the corresponding point on the grid.

Note now that the solution portrayed in Figure 5 is much tighter than the solution given in Figure 4. With this difference,

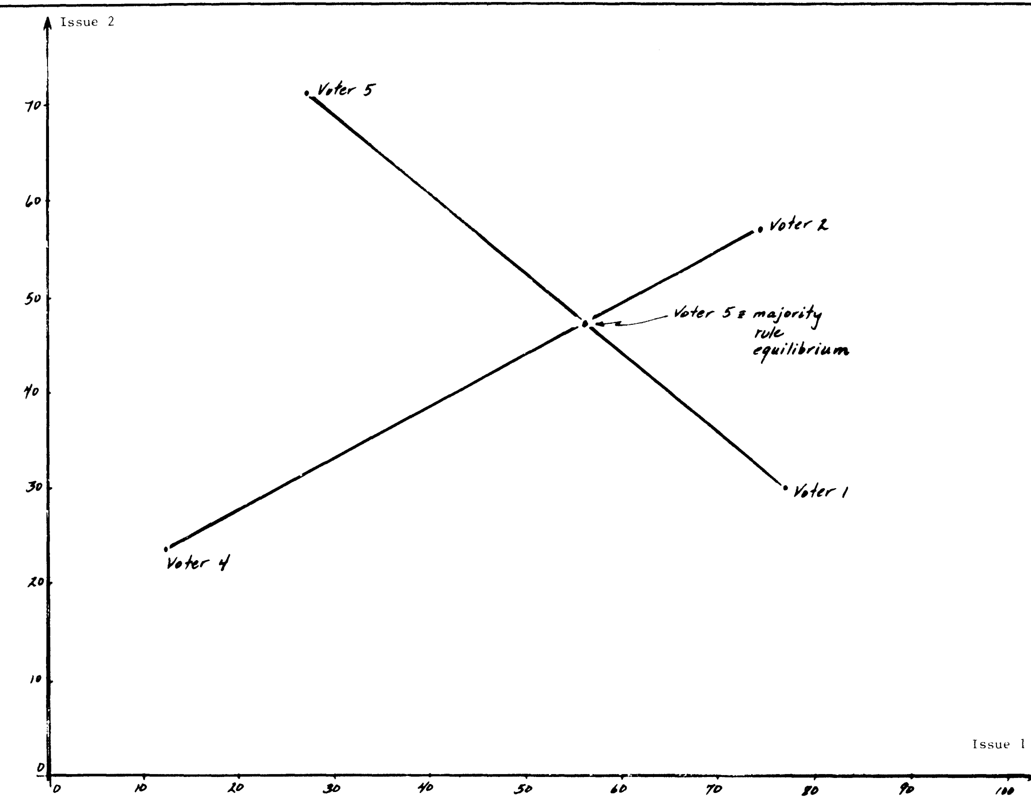

Figure 2 Preference Configuration for Spatial Election with Equilibrium at Voter 5's Ideal

Preferences of Voters
(ranked from best to worst)

Player 1	Player 2	Player 3	Player 4	Player 5
N	J	B	L	B
J	O	H	E	A
F	M	A,F	D	E,G,H
I	E,F	I	A,G,O	K
K	I	L,N	M	Q
G,O,H	K,D	P	I	D
P,Q	G,B	Q	P	M
D,M	H,C,P	K,M,D,J	K,B	O,C
B	L,N,Q	E,C	F,C	P
A,C	A	G	N,J,Q	L,N,J
L		O	H	F,I
E				

Figure 3 Configuration for Finite Alternative Election with No Equilibrium

then, we can assess the extent to which our experimental subjects respond to differences in the distribution of voter ideal points.

EXPERIMENTAL OUTCOMES

The exact instructions read to subjects are given in the Appendix. Briefly, each experiment employs two subjects, one denoted candidate 1, the other candidate 2, and consists of from 8 to 15 independent trials of the same election game. In each trial both subjects know the preferences over the alternatives of all voters, but each must choose a strategy (a lettered alternative in the experiments corresponding to Figures 1 and 3, a point in the grid for the experiments of Figures 2, 4, and 5) in ignorance of the choice of the other subject. These choices are then announced and, on the basis of the

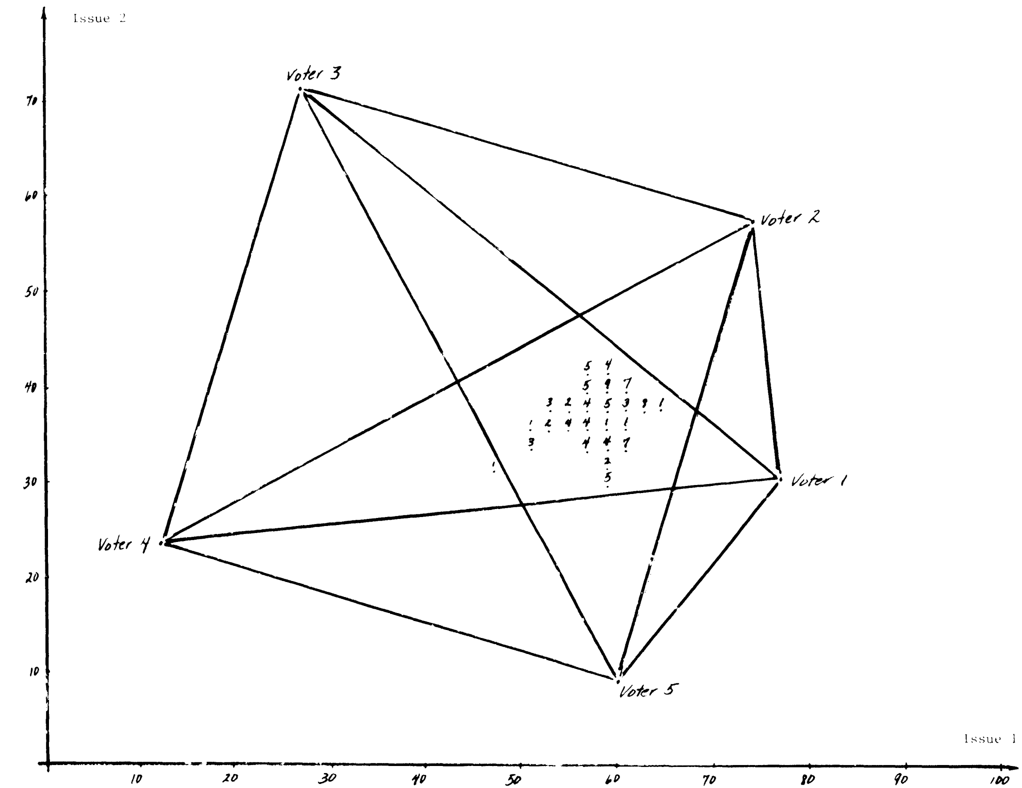

Figure 4 Preference Configuration for Five-Voter Experiment

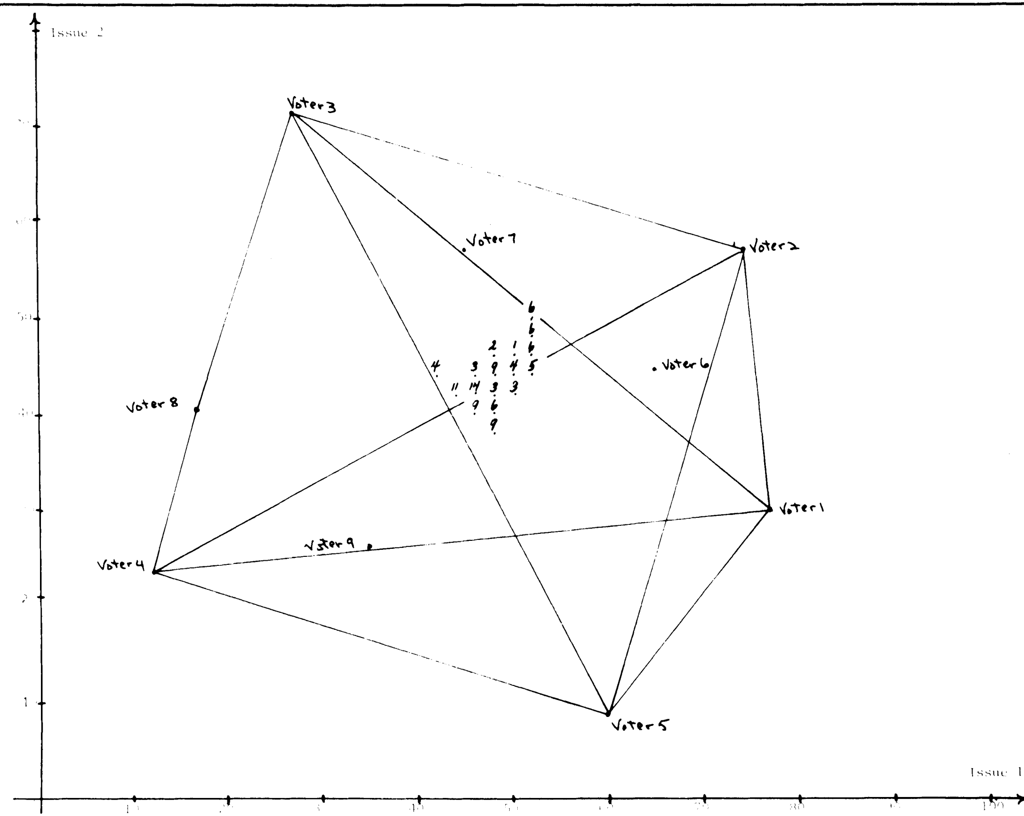

Figure 5 Preference Configuration for Nine-Voter Experiment

voter's preferences for one strategy as against the other, an election winner is determined. For each trial a subject is paid $2 if he wins the election, $1 if he ties, and nothing if the subject loses the election. Subjects are not told how many trials they will be playing and are informed that the experiment is terminated at the completion of the last trial. Overall, 8 experiments are run using the voter preferences of Figure 1, 6 experiments using the preferences of Figure 2, 15 experiments using Figure 3, 12 experiments with Figure 4, and 14 experiments with Figure 5.

FINITE ALTERNATIVES WITH AN EQUILIBRIUM

Table 1 summarizes the 8 finite game experiments with an election equilibrium (Figure 1). This table shows the nearly unanimous choice of the equilibrium alternative "G" over the last 4 trials, whereas the first set of trials portrays a steady increase on the frequency of choice of this equilibrium strategy. In general, then, subjects have little difficulty finding and sticking with the equilibrium strategy.

SPATIAL ALTERNATIVES WITH AN EQUILIBRIUM

Figure 6a describes the first five trials of the six spatial election experiments with an equilibrium at voter 5's ideal point (Figure 2). Figure 6b portrays the last five trials of these experiments. These data tell much the same story as Table 1: Generally, subjects converge rapidly to the equilibrium (most of the deviations from the equilibrium in Figure 6b, in fact, result from a single experiment). What is particularly impressive about these results, moreover, is that despite the visual clues provided by a spatial representation, subjects infrequently choose points at or near the equilibrium in initial trials (see Figure 6a). Thus, eventual convergence to the equilibrium cannot be attributed to such cues or to any prior familiarity with equilibrium concepts. Rather, the choice of the majority

TABLE 1

Percentage of Time "G" Is Chosen in Corresponding Trial (9.25 average trials/game)

1st trial	38%
2nd trial	44
3rd trial	63
4th trial	67
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n-3th trial	81
n-2th trial	88
n-1th trial	94
last play (n)	88

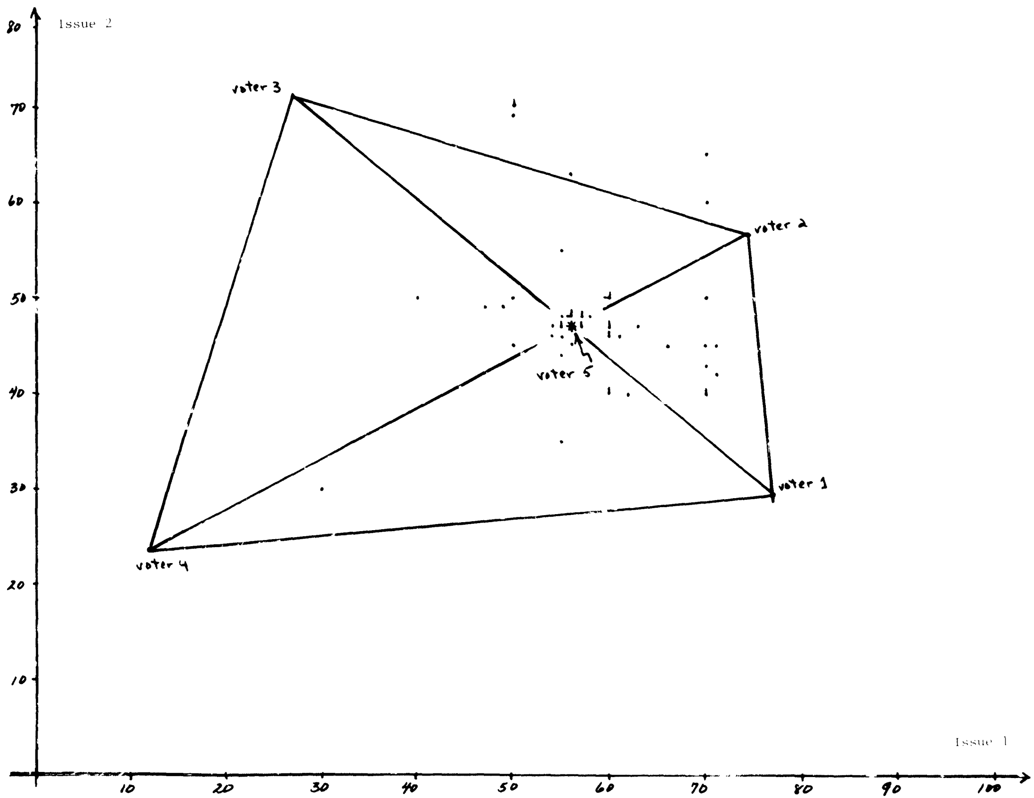

Figure 6a Outcomes of First 5 Trials of Five-Voter Spatial Election with an Equilibrium

NOTE: | denotes two outcomes at that point; ⅃ denotes three outcomes at that point, etc.

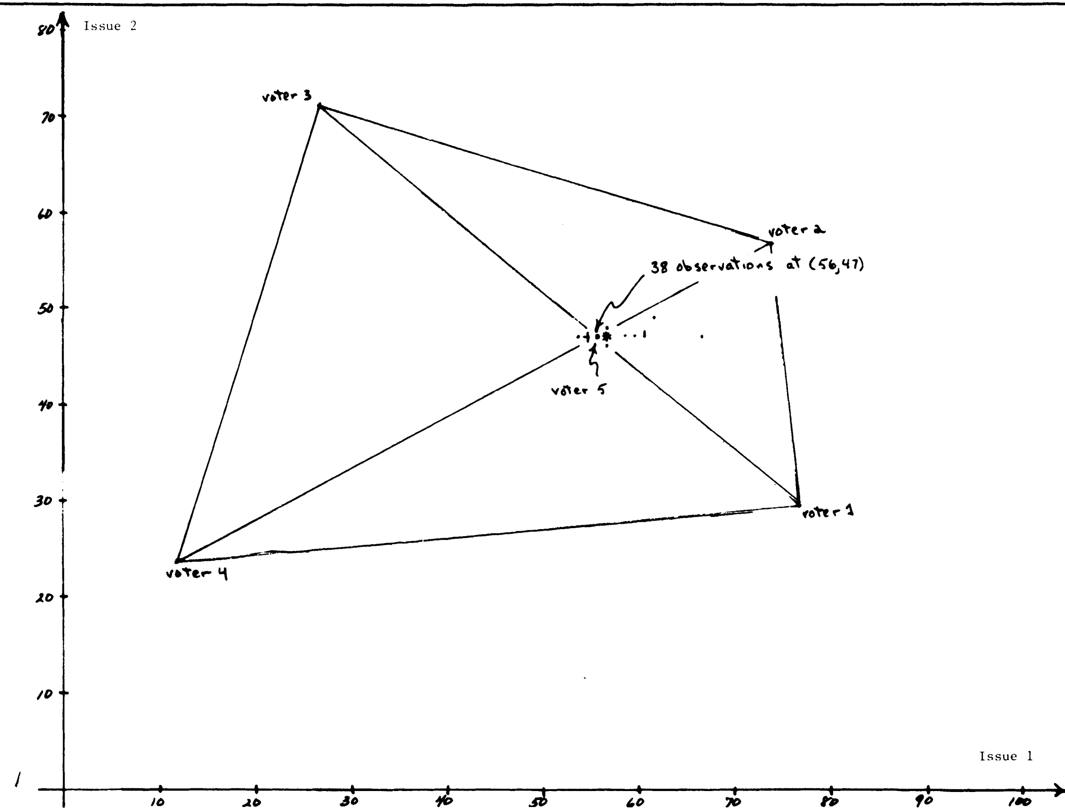

Figure 6b Outcomes of Last 5 Trails of Five-Voter Spatial Election with an Equilibrium

rule equilibrium must be attributed to the dominance properties of this point.

FINITE ALTERNATIVES WITHOUT AN EQUILIBRIUM

Turning now to the 15 experiments corresponding to the finite alternative game without a majority rule equilibrium (Figure 3), Table 2 describes the distribution of subject (candidate) choices over the first five and last five trials, as well as the game's mixed minmax strategy. As we can see, Table 2 tells a different story than does Table 1. While subjects' choices do not appear chaotic, in that over 60% of the choices are in the support set of the minmax strategy, there appears to be no significant tendency to choose these alternatives more frequently as play proceeds. Nor does the distribution of outcomes exhibit any trend toward the distribution describing the minmax solution. At least for this finite game, then, we must reject the hypothesis that subjects exhibit game-theoretic minmax behavior. There appears to be little learning about minmax strategies, and subjects' initial proclivities about acceptable strategies are carried through the entire experiment. And while we can detect some trends in the data (e.g., an increasing frequency of choosing M), we can offer no theoretical insights into these trends.

SPATIAL ALTERNATIVES WITHOUT AN EQUILIBRIUM

Figures 7a and 7b describe the choices of subjects in the first five and the last five trials of the five-voter game portrayed in Figure 4, while Figures 8a and 8b report the corresponding outcomes for the nine-voter game portrayed in Figure 5. These figures also give the convex hull of the support sets (the set of coordinate points assigned a positive probability of being chosen) of the respective computed mixed strategy solution. Table 3 summarizes the data reported in these figures.

TABLE 2
Distribution of Outcomes for Finite Election Game Without Majority Rule Equilibrium

Alternative	Frequency in First Five Trials n	(%)	Frequency in Last Five Trials n	(%)	Expected Minmax Frequency
A	9	(.06)	3	(.02)	0
B	1		1		0
C	14	(.09)	14	(.09)	0
D	32	(.21)	24	(.16)	.188
E	2	(.01)	4	(.03)	.125
F	3	(.02)	11	(.07)	.250
G	0	(0)	1		0
H	13	(.09)	13	(.09)	.063
I	17	(.11)	15	(.10)	.125
J	0	(0)	0	(0)	0
K	4	(.03)	6	(.04)	.125
L	3	(.02)	1		0
M	26	(.17)	34	(.23)	.125
N	14	(.09)	9	(.06)	0
O	5	(.03)	1		0
P	7	(.05)	12	(.08)	0
Q	0	(0)	1		0
% in minmax support set	.65		.71		

These data reveal two principal patterns: (1) The distribution of choices of the last five trials is tighter than the distribution of the first five trials; and (2) the distribution of outcomes for the nine-voter case is considerably tighter than the distribution in the five-voter elections. Further, while 25% and 36% of the first five-trial outcomes fall within the convex hull of the mixed minmax strategy support sets for the five- and nine-voter experiments, respectively, these percentages in-

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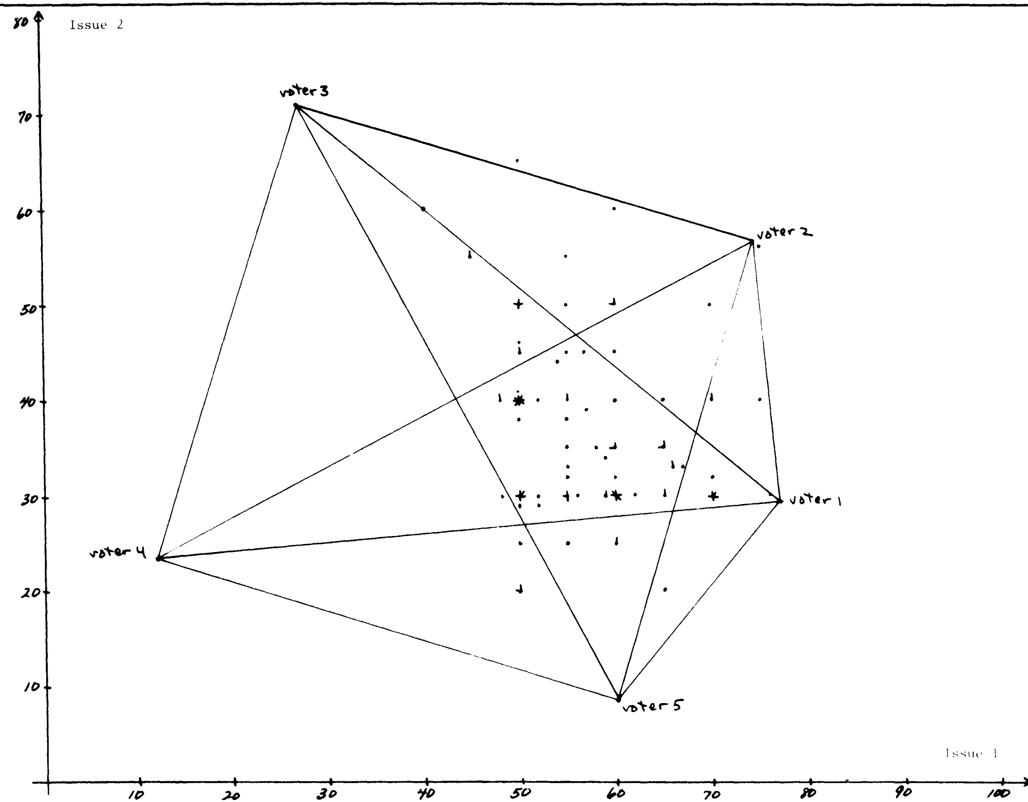

Figure 7a Outcomes of First 5 Trails of Five-Voter Election Without an Equilibrium

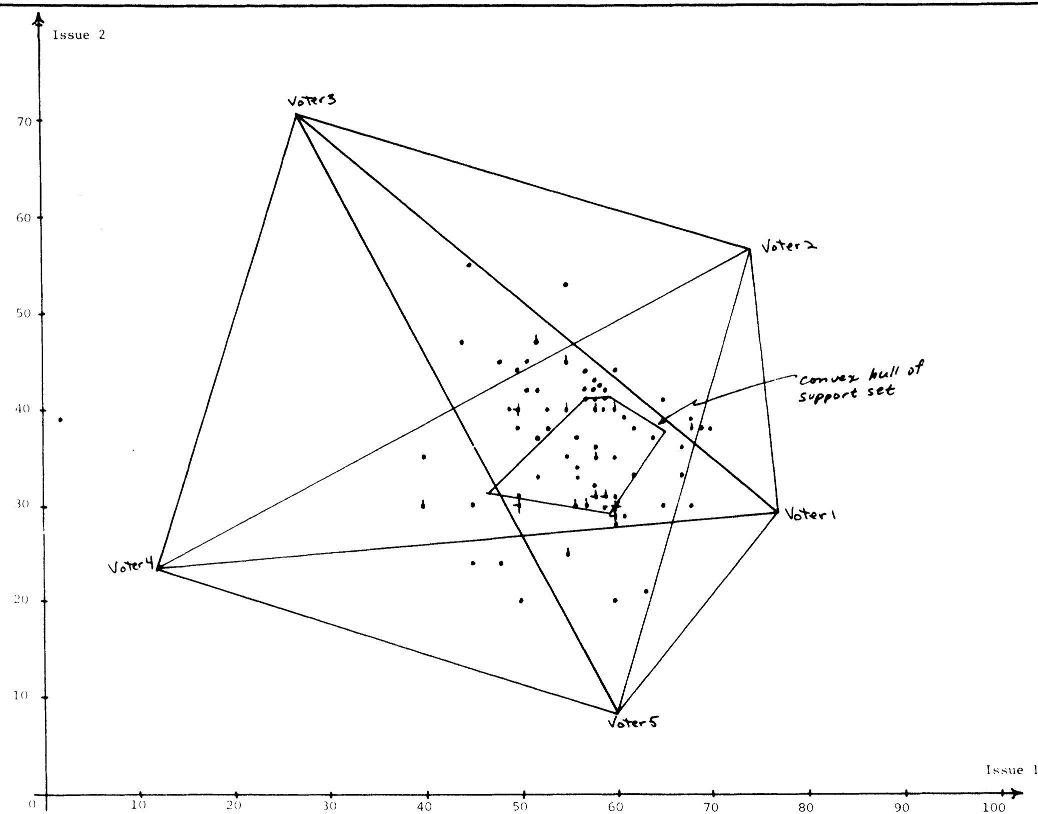

Figure 7b Outcomes of Last 5 Trails of Five-Voter Election Without an Equilibrium

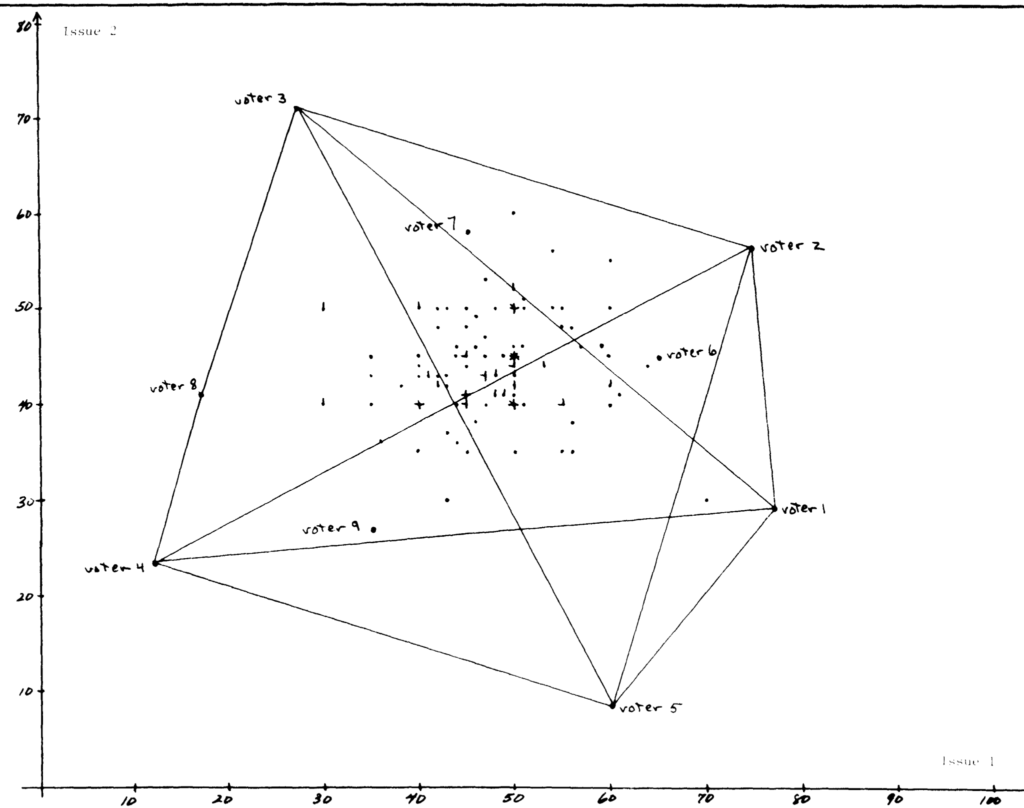

Figure 8a Outcomes of First 5 Trails of Nine-Voter Election Without an Equilibrium

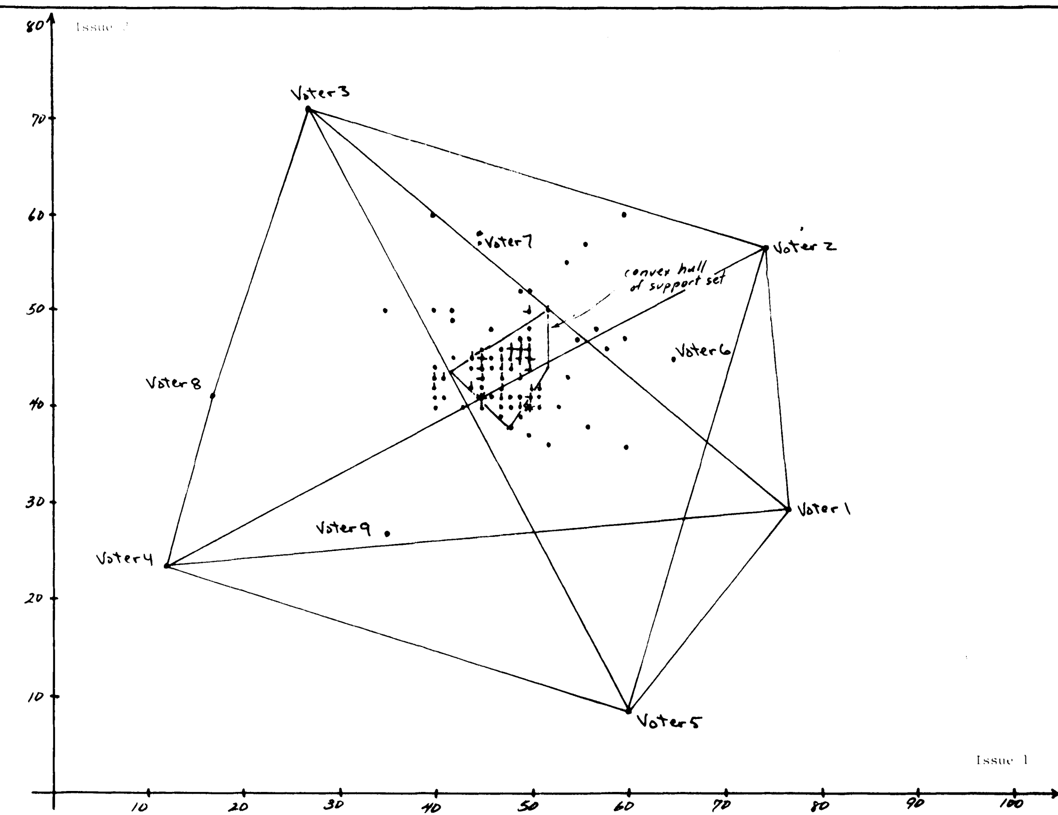

Figure 8b Outcomes of Last 5 Trails of Nine-Voter Election Without an Equilibrium

TABLE 3
Means and Variance-Covariance Matrices for Spatial Experiments

		1st Five Trials	Last Five Trials
Election with a Majority Rule Equilibrium	mean	(57.9, 48.0)	(56.7, 47.0)
	var x	57.23	3.69
	var y	52.17	.10
	cov (x,y)	- 3.66	.18
5-Voter Election	mean	(57.2, 36.7)	(56.4, 35.2)
	var x	84.40	71.60
	var y	97.27	57.60
	cov (x,y)	-30.16	-11.5
9-Voter Election	mean	(47.8, 43.5)	(47.6, 43.9)
	var x	43.44	17.46
	var y	22.91	15.19
	cov (x,y)	1.65	2.25

crease to 38% and 67% for the outcomes of the last five trials. These percentages should be treated with care, since the mixed strategy computations are inexact. A more refined calculation of solutions could dramatically affect the convex hull of the support sets and hence these percentages. Nevertheless, these data provide some evidence that subjects' choices track mixed minmax strategies in that there is a degree of convergence over trials to the region of the mixed strategy solutions.

CONCLUSION

Examination of Figures 7a-8b are especially revealing. Much is made of the theoretical finding that if, for the class of games we consider here, a majority rule equilibrium does not exist, the social preference ordering is wholly intransitive over the entire space of alternatives. Perhaps in the search for strawmen, this result is interpreted by some to mean that spatial election games without equilibria yield chaos or are in some sense indeterminate. The data in Figures 7a-8b contradict such inferences. Neither game possesses a majority rule equilibrium, yet for both games subjects' choices are limited to rather specific domains.

More interestingly, if we base our priors on the properties of minmax solution, we find that these domains are consistent with those priors. Just as the domain of the minmax solution is smaller for the nine-voter than the five-voter design, the distribution of experimental choices is correspondingly smallest in the nine-voter design. And in accordance with our mixed strategy calculations, subjects' choices shift upwards and to the left as we go from the five- to the nine-voter design.

We cannot infer from this, of course, that if permitted to play long enough, subjects would in fact abide by mixed strategy solutions. It is entirely possible that in the two spatial games run here, other heuristics model subject choice better and that these heuristics correspond approximately to their mixed strategy solutions. We can infer, however, that chaos does not result from the failure of a majority rule equilibrium to exist.

We must, nevertheless, add two caveats to this conclusion. First, it is the case that experiments with majority rule equilibria are far more determinate—even uninteresting—than experiments without such equilibria. Hence, we cannot rule out the many interesting possibilities for manipulation of outcomes by voting procedure, agenda control, and the like that the absence of equilibrium theoretically engenders and that the existence of an equilibrium theoretically precludes.

Second, despite the relatively clear patterns of outcomes reported for the spatial experiments, we cannot ignore the outcomes for the finite alternative experimental design without a majority rule equilibrium. While we might not choose to interpret our experimental results there as "chaotic," such an interpretation is legitimate barring additional experimentation. Certainly, much experimental and theoretical research remains before we can offer satisfactory hypotheses about how people act in generalized alternative spaces.

APPENDIX: INSTRUCTIONS READ TO SUBJECTS

INSTRUCTIONS: FINITE ALTERNATIVE ELECTIONS

You are about to participate in an experiment that will simulate a two-candidate election. If you follow these instructions and if you make appropriate decisions, you may earn a sizable amount of money. You will be paid immediately after the experiment in cash.

In this experiment, each of you has been denoted either candidate 1 or 2, and each of you will be competing against the other. We will be having a number of elections, and in each election one of you will win or both of you will tie. If you win an election you will be paid $2, if you tie, $1, and nothing if you lose. Thus, the more elections you win, the more money you earn.

Whether or not you win an election depends on the platform you and your opponent choose. In this experiment, a platform consists of a letter A, B, C, D, and so on. Voters are presumed to have well-defined preferences over these platforms, as your sample sheet indicates. Looking at the sample, which lists the preference order of each of five voters, we see that voter 1 prefers A to B, B to C, but is indifferent between C and D, and so on. Thus, if one candidate chooses A and the other chooses B, voter 1 votes for the candidate that chose A. Looking at the other voters, we see that 2 prefers A to B, 3 and 4 prefer B to A, but 5 prefers A to B. Thus, the candidate choosing A wins the election three votes to two. In this experiment, if voters are indifferent between the candidates, they abstain.

The actual procedure we will use is simple. Before each election, you will have as much time as you need to select a platform. These you will secretly record on your ballot card [Show card]. After both candidates hand their cards to me, I will compare the results and declare the winner in that election. You will be informed after each election of the choices of you and your opponent, and you should record those choices as well as the outcome on your record sheet.

The number of elections to be run has been predetermined, but you will be informed only at the conclusion of the last election that the experiment is over. At no time will any discussion be allowed. It is in your own best interest to make your choices in secrecy. Any questions?

INSTRUCTIONS: SPATIAL ELECTIONS

You are about to participate in an experiment that will simulate a two-candidate election. If you follow these instructions and if you make the proper decisions, you may earn a sizable amount of money. You will be paid immediately after the experiment in cash.

In this experiment, each of you has been denoted either candidate 1 or candidate 2, and each of you will be competing against the other. We will be having a number of elections, and in each election one of you will win or both of you will tie. You will be paid \$2 each time you win an election, \$1 if the election is a tie, and nothing if you lose. Thus, the more elections you win, the more money you earn.

As a candidate, you will have to choose your positions on two issues, which we call x and y. You may choose any value for x from 0 to 100 and any value for y from 0 to 80. Your opponent has the same choices. How well you do in the election will depend entirely on the issue positions you and your opponent adopt.

Look now at the sample grid before you. In the actual election, you and your opponent will have identical maps similar to this sample. These grids portray the preferences of voters over the issues. Voter 1, for instance, prefers most the point (x,y) = (60,30). Voter 1 is indifferent to all the points on the nearest circle about (60,30), but he prefers all of these to those on the second circle, and so forth. See that voter 1 prefers (x = 70, y = 30) to (x = 80, y = 30), for example. Thus, voter 1's preferences decline the further we move in any direction from his ideal point. The preferences of voters 2 and 3 work similarly.

Suppose an election is held between candidates A and B on the sample. Now if candidate A chooses the position x = 60, y =

40 and B chooses position $x = 40, y = 60$, who wins? Clearly, A is closer to voter 1's ideal than is B. Thus voter 1 votes for A. Voters 2 and 3, however, are closer to B and thus vote for B. Then A gets one vote to B's two votes, and B wins the election.

A simple trick can be used to determine who wins an election. On the sample, draw a line between A's and B's positions. Next, draw a line which bisects and is perpendicular to the first line. Then all voters on the "A" side of this second line will vote for A, all voters on the "B" side will vote for B, and any voter on this line will abstain. (Voters who have to choose between candidates offering positions they are indifferent to will abstain from voting.)

In the experiment, we will use a more complicated chart of the voters. The indifference circles are not marked on this sheet, but the same rules apply. That is, a voter always prefers a position closer to this ideal point. The charts you and your opponent will have are identical.

The actual procedure we will use is very simple. Before each election, you will have as much time as you like to select a position on issues x and y. These you will secretly record on your ballot card. After both candidates hand their cards to me, I will compare the results and declare the winner in that election. *You will be informed after each election of the choices of you and your opponent, and you should keep careful record of those choices as well as of the outcome on your record sheet.* [Show sample record sheet] The number of elections to be run has been predetermined, but you will be informed only at the conclusion of the last election that the experiment is over. At no time will any discussion be allowed. It is to your own best interest to make your choices in secrecy.

NOTES

1. The lines drawn in these and all subsequent figures are contract curves between particular pairs of voters and are included in the figures simply to give the reader a sense of the relative location of data.

2. Actually, to speed calculations a coarser grid is considered—the grid formed by counting in units of two.

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