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Author(s): Martin J. Osborne and Al Slivinski

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# A MODEL OF POLITICAL COMPETITION WITH CITIZEN-CANDIDATES\*

MARTIN J. OSBORNE AND AL SLIVINSKI

We develop a model of electoral competition in which citizens choose whether or not to run as candidates. A winner implements her favorite policy. The equilibrium number of candidates depends negatively on the cost of running and positively on the benefits of winning. For some parameter values all equilibria under plurality rule have exactly two candidates, whose positions are distinct. Two-candidate elections are more likely under plurality rule than under a runoff system (cf. Duverger's Law). The candidates' positions are less differentiated under a runoff system. There exist equilibria under both systems in which some candidates have no chance of winning.

## I. INTRODUCTION

In this paper we develop a novel spatial model of electoral competition and use it to study the outcomes of elections in which the winner is the candidate who obtains the most votes (plurality rule) and in which the winner is determined by majority rule under a two-ballot "runoff" system.<sup>1</sup>

The distinguishing feature of our model is the notion of a "citizen-candidate." There is a population of *citizens*, each of whom has preferences over a one-dimensional set of *policies* or *positions*. Each citizen chooses whether to become a *candidate* in the election. Running as a candidate is costly. The winner of the election can implement her favorite policy (subject to the constraints she faces as officeholder). In addition, she reaps a direct benefit from being in office—the "spoils of office" ("ego-rents" in Rogoff's [1990] terminology). (Two respects in which our model departs from Hotelling's [1929] seminal model are that the number of candidates is determined endogenously and the candidates care about the policy carried out.)

Our model provides an explanation for the great variation

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1. Besley and Coate [1995] independently develop a similar model (which we discuss in Section V).

observed across political competitions in the number of candidates and the dispersion in these candidates' policy positions. The main explanatory variables are the nature of the electoral system, the cost of running as a candidate in the election, and the benefit of winning. Our main results are the following.

- The number of candidates who enter a political competition depends negatively on the cost of running for office and positively on the benefits of winning the election.
- For a range of parameter values all equilibria under plurality rule have exactly two candidates, whose positions are distinct.
- Two-candidate elections are, in a strong sense, more likely under plurality rule than under a runoff system.
- Multicandidate elections are, in a weaker sense, less likely under plurality rule than under a runoff system.
- For a wide range of parameter values, the maximal dispersion in the candidates' positions in two-candidate equilibria is less under a runoff system than under plurality rule.
- For some parameter values there exist equilibria under both systems in which candidates with no chance of winning enter an election simply to affect the identity of the winner, even though such entry is both optional and costly.

The most prominent hypothesis concerning the number of candidates in an election is Duverger's Law, which states that plurality rule fosters a two-party system, while both proportional representation and a runoff system favor the existence of many parties [Duverger 1954, pp. 217, 239]. In this paper we compare the outcomes of plurality rule and runoff systems. Many plurality rule elections involve more than two candidates, but evidence suggests that such elections involve fewer candidates than do elections held under a runoff system [Wright and Riker 1989]. Our results are consistent with this finding. Previous work (discussed in detail in Section V) offers two primary explanations for the predominance of two-candidate elections under plurality rule, both of which rest on the strategic behavior of voters. The logic underlying our result concerns the strategic behavior of candidates.

We find that under either system only one candidate runs when the benefit of winning is small relative to the cost of running (as in an election for the chair of an academic department, or in a Republican gubernatorial primary in an overwhelmingly Democratic U. S. state). As benefits increase relative to costs, the number of candidates who can coexist in an equilibrium rises.

Equilibria in which many candidates take the same position are possible under a runoff system, but not under plurality rule. In an equilibrium in which some candidate is certain to lose, the winning candidate's position is much more desirable to a sure-loser than the position of the candidate who would win if that sure-loser were to withdraw. This strategic calculation appears to correspond to that of some actual candidates, such as H. Ross Perot in the 1992 U. S. Presidential election, the intensity of whose campaign seemed to be positively related to George Bush's perceived chance of winning.

The next section presents the model more formally. Sections III and IV, respectively, present our results for plurality rule and runoff systems, and Section V discusses previous work. All proofs are in the Appendix.

## II. THE MODEL

Each of a continuum of *citizens* has single-peaked preferences over the set of *policy positions*, which we take to be the real line  $\mathbb{R}$ . The distribution function of the citizens' favorite (ideal) positions on  $\mathbb{R}$  is  $F$ , which we assume to be continuous and to have a unique median  $m$ . Each citizen can choose to enter the competition ( $E$ ) or not ( $N$ ). If she enters, then she proposes her ideal position (she cannot commit to a different position). A citizen who chooses  $E$  is referred to as a *candidate*. After all citizens have simultaneously made their entry decisions, they cast their votes. Voting is "sincere": a candidate whose position  $x_j$  is occupied by  $k$  candidates (including herself) attracts the fraction  $1/k$  of the votes of the citizens whose ideal points are closer to  $x_j$  than to any other occupied position. Under *plurality rule* the winner of the election is the candidate who obtains the most votes. If two or more candidates tie for first place, then each wins with equal probability. Under a *runoff system* the winner is determined as follows. If some candidate obtains a majority (more than half the votes), then she is the winner. If no candidate obtains a majority, then the winner is the candidate who obtains a majority in a second election between the two candidates who obtained the most votes in the first round.<sup>2</sup> In both cases ties are dealt with via an equal-probability rule.

2. This mechanism is used in U. S. gubernatorial elections that employ a runoff system. Other runoff mechanisms are used in other elections, and it would be of interest to determine whether our results hold for these alternative systems.

Each citizen's payoff depends on the distance between her ideal point and that of the winner of the election, on whether she is a candidate or not, and on her probability of winning. The preferences over policies of a citizen with ideal point  $a$  are represented by the function  $-|x - a|$ . A citizen who chooses  $E$  incurs the (utility) cost  $c > 0$  and, if she wins, derives the benefit  $b > 0$ . Thus, if a citizen with ideal position  $a$  chooses  $N$  and the ideal position of the winner is  $w$ , then her payoff is

$$-|w - a|.$$

A citizen with ideal position  $a$  who chooses  $E$  obtains the payoff,

$$\begin{cases} b - c & \text{if she wins outright} \\ -|w - a| - c & \text{if she loses outright and the winner's ideal position is } w. \end{cases}$$

If no citizen enters, then all obtain the payoff of  $-\infty$ . Each citizen's preferences over lotteries are represented by her expected payoff. Note that  $b$  is the return to a citizen's holding office *over and above* her payoff to implementing her favorite policy. Note also that a noncandidate whose favorite policy is implemented by some other citizen obtains the payoff of zero.

In summary, we study the strategic game in which the set of players is the set of citizens, the set of actions of each player is  $\{E, N\}$ , and the preferences of each player are those given above. The solution notion that we use is Nash equilibrium, which we henceforth refer to simply as "equilibrium." We refer to a distribution of the candidates' ideal positions on  $\mathbb{R}$  as a "configuration."

Before presenting our results, we comment on the interpretation of the model. First, while we follow the literature in referring to the elements of  $\mathbb{R}$  as policy positions, another interpretation is consistent with the fact that the winner of the election is an *officeholder* who is given the right (for some period of time) to make decisions that affect the well-being of all citizens. For example, the model applies to the election of legislative representatives from single-member districts as well as to elections for executive offices, such as the U. S. Presidency, state governorships, city mayors, and many judgeships, state and county prosecutors, and even the chairs of academic departments. In this alternative interpretation the elements of the set  $\mathbb{R}$  index the decision strategies or objective functions that each citizen could use

if she held the office. Each citizen's preferences order these decision strategies. The winner of an election cannot do "whatever she wishes," but can only implement her preferred objective function subject to the constraints that the office carries with it, constraints that vary with the type of office being contested.<sup>3</sup>

Second, we note that while the notion of citizen-candidates is central to our formulation, all our results continue to hold if one posits instead a separate population of potential candidates whose distribution of ideal points has the same support as does the distribution of the citizens' ideal points.

Finally, in most equilibria of our model under either electoral system, elections in which there is more than one candidate involve tie votes. This feature, which our model shares with many other models in the literature, is an artifact of our simplifying assumption of complete information. If candidates are uncertain about the distribution of ideal points or the set of citizens who vote is determined randomly, then equilibria exist in which the candidates receive different numbers of votes.

### III. RESULTS FOR PLURALITY RULE

In this section we derive the conditions under which different numbers and configurations of candidates can arise in plurality-rule elections. Some elections are won by acclamation. Proposition 1 shows that our model predicts such an outcome, independently of the distribution of the voters' preferences, if the spoils of office are sufficiently small relative to the cost of running.

Much of the study of elections focuses on two-candidate contests. Proposition 2 characterizes the set of parameters for which a two-candidate election occurs, and gives the form of such equilibria. It shows, in particular, that two candidates' positions are never the same: if they were, then a third citizen who could win outright would enter.

Proposition 2 reveals another motivation for the entry of a third candidate: altering which of the two other candidates wins, even when the third entrant cannot possibly win herself. This motivation reemerges in Proposition 3, which shows that one possible three-candidate equilibrium entails entry by a candidate with no chance of winning; her entry causes the winner to be her

3. This approach is used widely in the political budget/business cycle literature (see, for example, Rogoff [1990] or Tabellini and Alesina [1990]).

favorite of the other two candidates. The motivations of third candidates who contest elections they are sure to lose are no doubt complex, but our model captures at least one rationale for such behavior: a desire to favorably influence which of the other two candidates wins.

Elections with three or more candidates are not uncommon, and Propositions 4 and 5 contain our results on their occurrence. These results support the simple intuition that the number of candidates is related positively to the spoils of office and negatively to the cost of running.

To present our results precisely, we begin by eliminating as equilibria some, though not all, configurations in which some candidate loses with certainty.

**LEMMA 1.** In equilibrium a candidate does not lose with certainty if either (i) there are other candidates with the same ideal position as hers or (ii) the ideal positions of all other candidates are on the same side of her ideal position.

In each case a candidate who loses with certainty prefers to withdraw, since her doing so either has no effect on the outcome or causes the winning position to be that of a candidate whose position is closest to hers. Note that the result does not rule out the possibility of a candidate's losing an election with certainty when she is the sole proponent of a position between those of two other candidates.

A call for individuals to run for some elected office sometimes results in a single citizen offering herself as a candidate and thus winning the election by acclamation. This was so, for example, in over 25 percent of the plurality-rule gubernatorial Democratic primaries in the United States between 1950 and 1982 [Wright and Riker 1989, p. 161]. The next result shows that such an outcome is consistent with our model: if  $b$  is small enough relative to  $c$ , then regardless of the nature of the distribution  $F$  of the citizens' ideal points there is an equilibrium in which a single candidate runs unopposed. Further, if  $b$  is sufficiently small relative to  $c$ , then this candidate's ideal position need not be the median  $m$  of  $F$ . This result expresses the idea that, if the payoff to being in office is sufficiently small, then even a single candidate who could be beaten by the entry of an appropriate citizen will run unopposed, unless she has relatively extreme preferences.

PROPOSITION 1 (one-candidate equilibria under plurality rule).

There is a one-candidate equilibrium if and only if  $b \leq 2c$ . If  $c \leq b \leq 2c$ , then the candidate's ideal position is  $m$ , while if  $b < c$ , then it may be any position within the distance  $(c - b)/2$  of  $m$ .

The intuition for this result is as follows. If there is a single candidate whose position is different from the median, then a citizen whose ideal point is the median can enter and win outright, obtaining a payoff of  $b - c$ . Hence for such a situation to be an equilibrium, we need  $b < c$ . If there is a single candidate whose position is the median, then another citizen with the same ideal position can enter and win with probability  $\frac{1}{2}$ , obtaining an expected payoff of  $\frac{1}{2}b - c$ . Thus, for this situation to be an equilibrium, we need  $b \leq 2c$ .

Of course, many elections are contested. The next result completely characterizes the set of parameters for which a two-candidate election is an equilibrium outcome. To state the result, we need the following definitions. Suppose that there are two candidates, with ideal positions  $m - \epsilon$  and  $m + \epsilon$  for some  $\epsilon > 0$ , so that each receives half of the votes. Let  $s(\epsilon, F)$  be the position between  $m - \epsilon$  and  $m + \epsilon$  with the property that if a citizen with this ideal position enters the competition then the numbers of votes received by each of the two original candidates remain equal:

$$F\left[\frac{1}{2}(m - \epsilon + s(\epsilon, F))\right] = 1 - F\left[\frac{1}{2}(m + \epsilon + s(\epsilon, F))\right].$$

If  $\epsilon$  is small, then no citizen with ideal position in  $(m - \epsilon, m + \epsilon)$  can enter the competition and obtain sufficiently many votes to win, while if  $\epsilon$  is large, then there is such a citizen who can win. Let  $e_p(F)$  be the critical value of  $\epsilon$  below which all such entrants lose and above which some such entrant wins.<sup>4</sup> (Note that  $e_p(F) > 0$  for any distribution  $F$ .)

PROPOSITION 2 (two-candidate equilibria under plurality rule).

2(i) Two-candidate equilibria exist if and only if  $b \geq 2(c - e_p(F))$ .

4. If the density of  $F$  is single-peaked and symmetric about its median, then

$$s(\epsilon, F) = m, \text{ and } e_p(F) = 2(m - F^{-1}(\frac{1}{3})).$$



- 2(ii) In any two-candidate equilibrium the candidates' ideal positions are  $m - \epsilon$  and  $m + \epsilon$  for some  $\epsilon \in (0, e_p(F)]$ .
- 2(iii) An equilibrium in which the candidates' positions are  $m - \epsilon$  and  $m + \epsilon$  exists if and only if  $\epsilon > 0$ ,  $\epsilon \geq c - b/2$ ,  $c \geq |m - s(\epsilon, F)|$ , and either  $\epsilon < e_p(F)$  or  $\epsilon = e_p(F) \leq 3c - b$ .

This result shows, in particular, that in any two-candidate equilibrium the candidates' positions are neither identical nor too dispersed. Further, if  $c > b/2$ , they are not too similar. If they are identical, then entry by a third candidate is inevitable. If they are too dispersed, then the entry of a citizen whose ideal position is between those of the two candidates causes her to win outright, an outcome that she prefers to that in which she does not enter the competition. If  $c > b/2$  and the positions are too similar (but not identical), then either candidate prefers to exit and let the other candidate win outright rather than pay the entry cost and obtain her ideal position with probability one-half. Note that since by Proposition 1 a one-candidate equilibrium exists whenever  $b \leq 2c$ , it follows from the result that for any distribution  $F$  the model has an equilibrium for all values of  $b$  and  $c$ .

It is instructive to consider in more detail the logic underlying the result. By Lemma 1 each candidate must win with probability  $\frac{1}{2}$  in any two-candidate equilibrium, from which it follows that the candidates' positions must be symmetric about the median. Suppose that the candidates' positions are the same. Then a third citizen with a different ideal position can enter and win for sure. For the original two candidates to be willing to enter, we need  $b \geq 2c$ , so that the third candidate's payoff of  $b - c$  is positive, exceeding the negative payoff that she obtains if she stays out of the race. Thus, the inevitable entry of a successful third candidate eliminates the existence of two-candidate equilibria in which both candidates' positions are the median.

If entry by two candidates with positions symmetric about the median is to be an equilibrium, it is necessary that neither prefer to exit and guarantee victory for the other. Letting  $\epsilon$  be each candidate's distance from the median, this requirement implies that  $\frac{1}{2}b + \frac{1}{2}(-2\epsilon) - c \geq -2\epsilon$ , or  $\epsilon \geq c - b/2$ .

For an equilibrium we require also that no other citizen wishes to enter the race. Now, if  $\epsilon > e_p(F)$ , then a citizen whose ideal point is between those of the two candidates wins outright if she enters, obtaining a payoff of  $b - c$ . She obtains  $-\epsilon$  if

she stays out, so for an equilibrium we need  $b - c \leq -\epsilon$ , or  $\epsilon \leq c - b$ . But the requirement  $\epsilon \geq c - b/2$  implies that  $\epsilon > c - b$  if  $b > 0$ . Thus, if  $\epsilon > e_p(F)$ , a citizen whose ideal point is between those of the two candidates is better off if she enters. We conclude that  $\epsilon < e_p(F)$  in an equilibrium.<sup>5</sup>

Finally, we require that no citizen wishes to enter in order to change the identity of the winner, even though she herself has no chance of winning. The entry of a citizen whose ideal position is not between those of the candidates clearly cannot alter the outcome favorably. The entry of a citizen whose ideal position is between those of the candidates may lead to certain victory for one of the candidates, and the entering citizen may prefer this outcome to that in which the two tie. The condition  $c \geq |m - s(\epsilon, F)|$  ensures that the cost of entry is high enough that no third citizen wishes to enter for this reason. (Note that if the density of  $F$  is single-peaked and symmetric about  $m$  then, since  $m = s(\epsilon, F)$ , this condition is not binding.) The possibility of entry by a third candidate who is certain to lose reemerges in the sequel.

Equilibria involving more than two candidates are possible. Since there have been many significant three-party competitions, it is of particular interest to determine when a three-candidate equilibrium can occur. The next result shows that in any such equilibrium there is at least some dispersion in the candidates' positions. The idea behind the result is that, if  $b$  is large enough that more than two citizens with the same ideal position want to enter, then a citizen whose ideal point is close by can win outright by entering and hence will do so, causing the existing entrants to lose.

**LEMMA 2.** In any equilibrium at most two candidates share any given position.

A complete characterization of the conditions under which three-candidate equilibria exist is complex. The next result gives some features of these equilibria. Lemma 2 implies that not all three candidates can have the same ideal position, so there remain two possibilities. If two candidates share one position and a third has a different position, then by Lemma 1 each candidate must obtain one-third of the vote. The other possibility is that all

5. If  $b = 0$  (as Besley and Coate [1995] assume), then there is an equilibrium for  $\epsilon = c$ , even if  $c > e_p(F)$ . (If  $c > e_p(F)$ , then in this equilibrium each candidate and any citizen whose ideal point is the median are indifferent between entering and not.)

three candidates have different positions, in which case the two extreme candidates must each have a positive probability of winning and so must obtain the same fraction of the vote, which must be not less than that of the central candidate.

**PROPOSITION 3** (three-candidate equilibria under plurality rule).

Every three-candidate equilibrium takes one of the following forms, where  $t_1 = F^{-1}(\frac{1}{3})$ ,  $t_2 = F^{-1}(\frac{2}{3})$ , and the candidates' positions are  $a_1 \leq a_2 \leq a_3$ .

- 3(i) The positions of the candidates are not all the same, and  $a_1 = t_1 - \epsilon_1$ ,  $a_2 = t_1 + \epsilon_1 = t_2 - \epsilon_2$ , and  $a_3 = t_2 + \epsilon_2$  for some  $\epsilon_i \geq 0$ . Each candidate obtains one-third of the votes. Necessary condition:  $b \geq 3c + 2|\epsilon_1 - \epsilon_2|$ .
- 3(ii) The positions of the three candidates are all different. Candidates 1 and 3 obtain the same fraction of the votes, while candidate 2 obtains a smaller fraction (and hence surely loses). Necessary conditions:  $b \geq 4c$  and  $c < t_2 - t_1$ .

The most striking aspect of this result is the possibility expressed in 3(ii) of an equilibrium in which one of the three candidates is certain to lose.<sup>6</sup> This candidate enters solely because she prefers the resulting equal-probability lottery over her two rivals' positions to certain victory by the candidate who would win if she withdrew. This strategic reasoning appears to correspond to the rationale often provided for actual third-party candidacies.

The necessary conditions for the existence of these three-candidate equilibria, when compared with that for the existence of a two-candidate equilibrium, provide support for the intuition that larger values of  $b$  relative to  $c$  lead to equilibria with greater numbers of candidates.

The necessary condition for the existence of a three-candidate equilibrium of type 3(i) follows from the requirement that neither of the two candidates with extreme positions prefer to stay out of the election (if either did so, the central candidate would win outright). It turns out that this condition implies also that the central candidate prefers to enter than to stay out. (Note that case 3(i) includes equilibria in which two of the candidates share the same position. For example, if  $\epsilon_2 = 0$ , then  $a_1 = t_1 - \epsilon_1$  and  $a_2 = t_1 = \epsilon_1 = t_2 = a_3$ .)

6. Palfrey [1984] also has three-candidate sure-loser equilibria, but they arise only because the third party has no alternative but to enter.

The requirement that  $b \geq 4c$  in 3(ii) follows from the necessity of keeping the two extreme candidates (each of whom wins with probability  $\frac{1}{2}$ ) from preferring not to enter, given that their nonentry would result in certain victory by the central candidate. As noted before, the sure loser's motivation for entry follows from her preference for an equal-probability lottery over the two extreme candidates to certain victory by the one she least prefers. The condition  $c < t_2 - t_1$  ensures that the cost of entry is low enough to make her entry worthwhile.

The necessary conditions in the result are not sufficient. For example, if the distribution of  $F$  is symmetric, then there is no equilibrium in which one candidate surely loses, because that candidate's withdrawal results in certain victory by the remaining candidate that she most prefers. If in addition the distribution of ideal points is single-peaked, then there is no equilibrium of type 3(i) in which  $\epsilon_i = 0$  for some  $i$  either, since a citizen whose ideal point is close to the position at which there are two candidates can enter and win outright. Thus, any analysis of elections using our model that restricts attention to symmetric single-peaked distributions ignores the two most interesting phenomena captured by Proposition 3.

There are distributions of the citizens' ideal points for which *no* three-candidate equilibrium exists for *any* values of  $b$  and  $c$ . An example is a distribution  $F$  whose density is symmetric about its median and has its mass concentrated at  $t_1$  and  $t_2$ . We argued above that for such a distribution there is no equilibrium of type 3(ii). There is no equilibrium of type 3(i), since an entrant at either  $t_1$  or  $t_2$  can win outright.

We do not have a characterization of the conditions under which an  $n$ -candidate equilibrium exists for an arbitrary value of  $n$ . However, we can show the following.

**PROPOSITION 4.** A necessary condition for the existence of an equilibrium in which  $k \geq 3$  candidates tie for first place is  $b \geq kc$ . A necessary condition for the existence of an equilibrium in which there are three or more candidates is  $b \geq 3c$ .

This result provides further support for the intuition that the number of candidates is positively related to the size of  $b$  relative to  $c$ . (The result is not vacuous. For any single-peaked distribution  $F$  and any value of  $k$ , if  $b$  sufficiently exceeds  $kc$ , then there exists a  $k$ -candidate equilibrium in which the candidates' positions are distinct and each wins with probability  $1/k$ .)

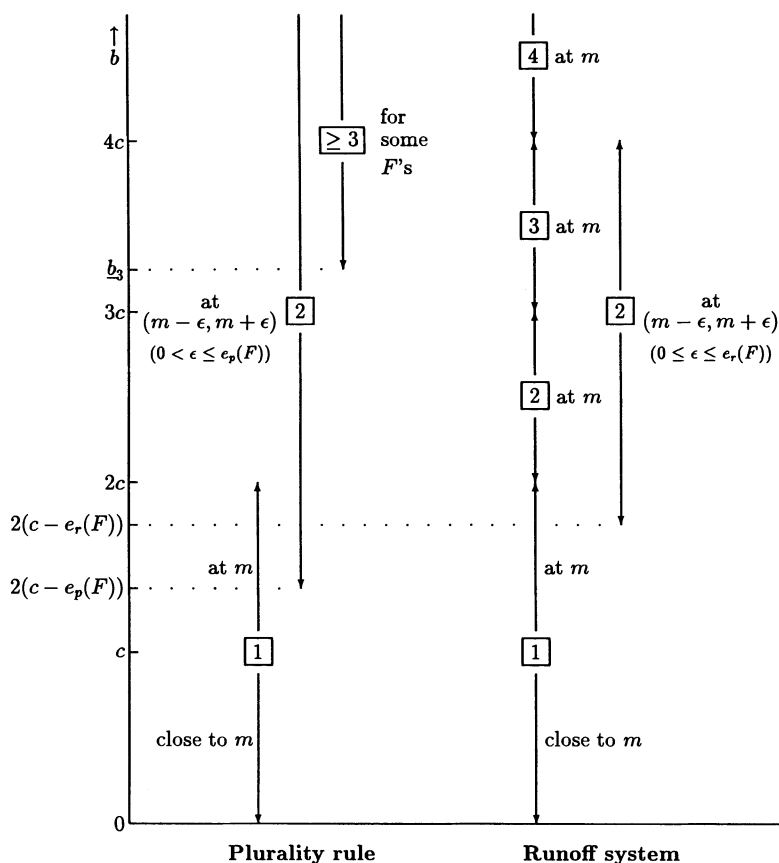


FIGURE I

Numbers of Candidates Possible in Equilibria, as Functions of  $b$ ,  $c$ , and the Electoral System

For  $k \geq 3$ , the  $k$ -candidate equilibria under plurality rule exist only for some distributions  $F$ . The lower limits on  $b$  for the existence of two-candidate equilibria lie between 0 and  $2c$ . The lower limit  $b_3$  on  $b$  for the existence of a three-candidate equilibrium under plurality rule is at least  $3c$ .

Some features of the possible equilibria are summarized in Figure I. Note that, although for most values of the parameters the candidates' equilibrium positions are not uniquely determined, the characteristics of an equilibrium are strongly restricted. In particular, if  $2c < b < b_3$ , then in all equilibria there are exactly two candidates, and if  $b < 2(c - e_p(F))$ , then in all equilibria there is exactly one candidate.

Although a general characterization of equilibria with  $n \geq 4$

candidates is beyond us, the next result, which significantly restricts four-candidate equilibria, is of interest when we compare plurality rule with a runoff system in the next section.

**PROPOSITION 5** (four-candidate equilibria under plurality rule).

For a generic distribution  $F$ , every four-candidate equilibrium takes one of the following four forms.

- 5(i) The candidates' positions are different; the numbers of votes obtained by the two extreme candidates and one of the other candidates are equal and greater than the number received by the remaining candidate, who hence loses. Necessary condition:  $b \geq 3c$ .
- 5(ii) The candidates' positions are different, and each candidate obtains one-quarter of the votes. Necessary condition:  $b \geq 4c$ .
- 5(iii) Exactly two of the candidates' positions are the same, and each candidate obtains one-quarter of the votes. Necessary condition:  $b \geq 4c + F^{-1}(\frac{3}{4}) - F^{-1}(\frac{1}{4})$ .
- 5(iv) Two candidates share a single extreme ideal position, each receiving the same number of votes as does a single candidate at the other extreme, while a lone central candidate receives fewer votes, and hence surely loses. Necessary condition:  $b > \frac{9}{2}$ .

Several aspects of this result are worth noting. First, in the equilibria in parts 5(i) and 5(iv), one of the candidates surely loses. The motivation for this candidate's entry is the same in both cases: she prefers a situation in which each of the other three candidates wins with probability  $\frac{1}{3}$  to the certain victory by one candidate that would result if she exited.

Second, there is no equilibrium in which the candidates are paired, with two sharing one position and the other two sharing another position.

Third, there are distributions  $F$  for which no four-candidate equilibrium exists for *any* values of  $b$  and  $c$ . An example is a distribution whose mass is equally concentrated around three evenly spaced points.

#### IV. RESULTS FOR A RUNOFF SYSTEM

We now analyze majority-rule elections that use a runoff system to decide the winner if no candidate gets a majority on the first ballot. The conditions under which election to an office oc-

curs by acclamation are identical to those for plurality rule, since the entry of a second candidate cannot induce a runoff. However, the conditions under which multicandidate equilibria can arise and the corresponding equilibrium configurations differ markedly between the two systems. Under plurality rule there is no two-candidate equilibrium in which the candidates' ideal positions are the same (Proposition 3) and in no equilibrium are more than two candidates' positions the same (Lemma 2). By contrast, Proposition 6 states that under a runoff system there are equilibria in which all the candidates' positions are the same. Depending on the values of  $b$  and  $c$ , any number of candidates can run in such an election.

Proposition 7 characterizes the parameter values for which differentiated two-candidate elections exist under a runoff system. These conditions, when compared with those for plurality-rule elections (Proposition 2), imply that the model predicts a strong form of Duverger's Law. For any distribution of preferences the set of values of  $b$  and  $c$  that give rise to a two-candidate equilibrium under a runoff system is a subset of those that do so under plurality rule. The fact that an equilibrium in which there is a cluster of three candidates at the median exists under a runoff system for some values of  $b$  and  $c$ , together with the results of Proposition 9 on equilibria in which there is a symmetric clustering of candidates around the median, provide a weaker sense in which elections with three or more candidates are more likely under a runoff system than under majority rule.

Under a runoff system there is no three-candidate equilibrium in which one candidate surely does not get into the second round, since such a candidate's entry has no effect on the winner of the election. It follows from the configuration implied by this requirement that no candidate surely loses in the second round: in contrast to the case of plurality rule, there is no three-candidate sure-loser equilibrium under a runoff system. Proposition 8 describes the three-candidate equilibria in this case.

To present the results in detail, we begin with our result on single-cluster equilibria.

**PROPOSITION 6** (single-cluster multicandidate equilibria under a runoff system). (For any  $k \geq 2$ , there is a  $k$ -candidate equilibrium in which the ideal position of every candidate is  $m$  if and only if  $kc \leq b \leq (k + 1)c$ ).

Under plurality rule, equilibria in which many candidates share the median position are ruled out by the fact that a citizen with an ideal position near  $m$  can enter and win. Under a runoff system entry by such a citizen can result only in her advancing to the second round, where she surely loses. The two inequalities in the result guarantee that no candidate prefers to withdraw and no further citizen with ideal position  $m$  wishes to enter. Note that the result guarantees that for any distribution  $F$  and any values of  $b$  and  $c$  there exists an equilibrium under a runoff system.

A runoff system can also give rise to two-candidate equilibria much like those that result under plurality rule. Let  $e_r(F)$  be the supremum of the values of  $\epsilon$  for which there is a position  $d \in (m - \epsilon, m + \epsilon)$  such that a citizen who enters at  $d$  obtains a smaller fraction of the votes than do *both* of the existing candidates. If  $\epsilon > e_r(F)$ , then the configuration in which one candidate is at  $m - \epsilon$  and one is at  $m + \epsilon$  is not an equilibrium since there is a citizen with ideal point in  $(m - \epsilon, m + \epsilon)$  who, if she enters, gets into a runoff, which she wins (so that she prefers to enter).

PROPOSITION 7 (two-candidate equilibria under a runoff system).

- 7(i) Two-candidate equilibria exist if and only if  $2(c - e_r(F)) \leq b \leq 4c$ .
- 7(ii) In any two-candidate equilibrium the candidates' ideal positions are  $m - \epsilon$  and  $m + \epsilon$  for some  $\epsilon \in [0, e_r(F)]$ .
- 7(iii) An equilibrium in which the candidates' positions are  $m - \epsilon$  and  $m + \epsilon$  exists if and only if either (1)  $\epsilon = 0$  and  $2c \leq b \leq 3c$  or (2)  $\epsilon > 0$ ,  $\epsilon \geq c - b/2$ ,  $b \leq 4c$ , and either  $\epsilon < e_r(F)$  or  $\epsilon = e_r(F) \leq 2c - b$ .

As in the case of plurality rule, the requirement that a citizen whose ideal position is between those of the candidates not want to enter implies that the candidates' positions cannot be too far apart. Also, the requirement that one of the candidates not prefer to withdraw limits how close the candidates' positions can be, although if  $b \leq 3c$  then under a runoff system it does not exclude the case in which the positions are the same. Under plurality rule there is no upper bound on the value of  $b$  for which a two-candidate equilibrium exists. The same is not true under a runoff system since a citizen whose ideal point is the same as that of one of the candidates has a positive probability of getting into a runoff, and of ultimately winning, if she enters the competition.



The condition  $b \leq 4c$  is necessary to make entry unattractive to such a citizen.<sup>7</sup>

Note that for any  $F$  we have  $e_r(F) \leq e_p(F)$ , since under a runoff system any citizen with an ideal position in the interval  $(m - \epsilon, m + \epsilon)$  who receives more votes than at least one of the candidates gets into the runoff, which she surely wins. It follows from Propositions 2, 6 (with  $k = 2$ ), and 7(i) that for any distribution  $F$  of ideal points the set of values of  $(b, c)$  for which a two-candidate equilibrium exists under a runoff system is a subset of the set of values for which a two-candidate equilibrium exists under plurality rule. This is the precise (and strong) sense in which our model predicts Duverger's Law.

We have seen that under a runoff system there can exist two-candidate equilibria in which both candidates choose the same position, while no such equilibrium exists under plurality rule. For values of the parameters for which there exist two-candidate equilibria under both electoral systems, we can compare also the maximal amount of dispersion that can exist in the candidates' positions. If  $c \geq e_r(F)$ , then since  $e_r(F) \leq e_p(F)$  the comparison is unambiguous: the maximal amount of dispersion in the candidates' positions is at least as large under plurality rule as it is under a runoff system. If  $c < e_r(F)$ , then because the requirement  $c \geq |m - s(\epsilon, F)|$  in Proposition 2 may rule out equilibria under plurality rule in which  $\epsilon > c$ , the maximal degree of dispersion in the candidates' positions *may* be larger under a runoff system than under plurality rule. However, for any distribution  $F$  that is single-peaked and symmetric about its median, or is not too different from such a distribution, we have  $|m - s(\epsilon, F)| < c$  for *all* values of  $\epsilon$ , and the maximal degree of dispersion is definitely greater under plurality rule.

Turning to three-candidate equilibria, we found that under plurality rule there are distributions  $F$  of ideal points for which no such equilibria exist for any values of  $b$  and  $c$ . Under a runoff system three-candidate equilibria exist for any distribution  $F$  if  $3c \leq b \leq 4c$  (Proposition 6). In this sense, three-candidate equilibria are more likely under a runoff system. The next result shows, however, that for some parameters there are three-candidate

7. The value of the upper bound on  $b$  depends on our assumption that an election in which one candidate obtains exactly one-half of the votes precipitates a runoff. If in such an election the candidate with one-half of the votes wins in the first round (without any runoff) with some positive probability, then the upper bound on  $b$  is higher.

equilibria under plurality rule but not under a runoff system, so that the comparison between the two systems with respect to the likelihood of a three-candidate election is ambiguous.

To determine when a "differentiated" three-candidate equilibrium can exist under a runoff system, note that there is never an equilibrium in which one candidate is sure to lose in the first round, since such a candidate does not affect who gets into a runoff. The next proposition states that if  $b \neq 4c$  then only *one* differentiated three-candidate equilibrium configuration is possible.

**PROPOSITION 8** (three-candidate equilibria under a runoff system). If  $b \neq 4c$ , then in all three-candidate equilibria in which not all the candidates' positions are the same, these positions are different, equal to  $a_1 = m + t_1 - t_2$ ,  $a_2 = t_1 + t_2 - m$ , and  $a_3 = t_2 + m - t_1$ , where  $t_j = F^{-1}(j/3)$ . In such an equilibrium each candidate obtains one-third of the votes in the first ballot. Necessary condition:  $b \geq 6c$ .

The reason that any equilibrium must take this form is that all three candidates must have a positive probability of being the ultimate winner, or else they prefer not to enter. Thus, each must obtain one-third of the first-round vote, and each must have a positive probability of winning in the second round if they reach it. In any configuration that satisfies these conditions and in which two candidates share an ideal position, it is profitable for a fourth candidate who shares the lone candidate's ideal position to enter unless  $b = 4c$ . Thus, all three must have distinct ideal positions if  $b \neq 4c$ . The two extreme candidates surely lose a runoff with the central candidate so they must have a positive probability of winning against one another in a runoff, implying that they are symmetrically positioned about the median. The only configuration with these properties is the one defined in the proposition. For some distributions  $F$  this configuration is not an equilibrium because a fourth citizen has an incentive to enter. Thus, as in the case of plurality rule, for some distributions  $F$  no differentiated three-candidate equilibrium exists.

Some features of the possible equilibria under a runoff system are summarized in Figure I alongside a similar summary for plurality rule.

To further elaborate the differences that our model predicts between the outcomes of plurality rule and a runoff system, consider the possibility of multicandidate equilibria in which there are two clusters of candidates. Define  $s(\epsilon, F)$  as in Section III.

Suppose that  $k \geq 4$  is even and that there are  $k/2$  candidates at  $m - \epsilon$  and  $k/2$  at  $m + \epsilon$ . Let  $e_r^k(F)$  be the smallest value of  $\epsilon$  for which there is a position in  $(m - \epsilon, m + \epsilon)$  that attracts at least as many first-round votes as does the position of any of the  $k$  candidates.

**PROPOSITION 9** (two-cluster multicandidate equilibria under a runoff system).

If  $k \geq 4$  is even and  $\epsilon > 0$ , then there is a  $k$ -candidate equilibrium in which the ideal position of  $k/2$  candidates is  $m - \epsilon$  and the ideal position of the remaining  $k/2$  candidates is  $m + \epsilon$  if and only if  $\epsilon < e_r^k(F)$ ,  $c \geq |m - s(\epsilon, F)|$ , and  $b \geq 4c$  if  $k = 4$  and  $b \geq k(c + \epsilon)$  if  $k \geq 6$ .

The condition  $\epsilon < e_r^k(F)$  ensures that no citizen with an ideal position in  $(m - \epsilon, m + \epsilon)$  can get into a runoff (if she did, she would win). The condition  $c \geq |m - s(\epsilon, F)|$  ensures that a citizen with an ideal position in  $(m - \epsilon, m + \epsilon)$  who enters does not affect the outcome in a way favorable to her. As before, if  $F$  is single-peaked and symmetric about its median, then  $m = s(\epsilon, F)$ , so that the condition  $c \geq |m - s(\epsilon, F)|$  is redundant.

Under plurality rule no position is shared by more than two candidates (Lemma 2) and there is no four-candidate equilibrium in which two positions are each shared by two candidates (Proposition 5). Under a runoff system, on the other hand, there are always equilibria in which there is a single cluster of candidates at the median (Proposition 6) and two clusters of candidates symmetrically around the median (Proposition 9). In this sense, the equilibria under a runoff system are more agglomerated than those under plurality rule.

Further, for a “randomly chosen” distribution  $F$ , only the configuration described in Proposition 5(i) is a possible four-candidate equilibrium under both systems, and then only if  $b > 6c$ .

**PROPOSITION 10** (four-candidate equilibria under both systems).

For a generic distribution  $F$ , if  $b \leq 6c$ , then no four-candidate configuration is an equilibrium under both plurality rule and a runoff system. If  $b > 6c$  then the only four-candidate configuration that may be an equilibrium under both systems is that in which the candidates' positions are different, the two extreme candidates and one of the middle candidates obtain

the same number of votes in the first round, and the remaining candidate obtains fewer votes.

One can divide Duverger's Law into two statements: (1) a two-candidate election is more likely under plurality rule than under a runoff system; (2) an election with  $n$  candidates, for any  $n > 2$ , is more likely under a runoff system than under plurality rule. Our model predicts (1) in the strongest possible sense and predicts (2) for  $n$  equal to 3 or 4 in a weaker sense. Precisely, if the values of  $b$  and  $c$  are appropriate, three- and four-candidate equilibria exist under a runoff system for *any* distribution  $F$ , while for some distributions neither exists under plurality rule for any parameter values.

## V. RELATION WITH PREVIOUS WORK

Hotelling [1929] first suggested that a model of spatial competition can yield insights into political (electoral) competition; his idea was elaborated by Downs [1957], Black [1958], and many others. (Shepsle [1991] and Osborne [1995] survey the field.) Two key respects in which our model departs from Hotelling's are that (1) the set of candidates arises endogenously as the result of citizen entry decisions, and (2) candidates care about the policy that wins the election. Models with each of these features have been studied before.

The simplest variant of Hotelling's model in which the number of candidates arises endogenously posits a set of potential candidates, each of whom has the option of not entering the competition. Unfortunately, this game in general does not possess pure strategy equilibria [Osborne 1993, Propositions 3 and 5].

A further step away from Hotelling's assumptions is taken by Palfrey [1984], who studies a three-candidate model in which the third candidate chooses her position after observing the simultaneous choices of the other two. The third candidate loses in equilibrium (her objective is to maximize the number of votes, not necessarily to win), and her presence affects the other candidates' positions. The appeal of the result is limited by the fact that it no longer holds if each candidate's objective is to win (in which case there is a subgame perfect equilibrium in which one of the first two candidates and the last candidate enter at the median, and the remaining candidate does not enter).

Osborne [1993, Section 4] develops a model of sequential entry in which candidates decide not only whether to enter but also when (in continuous time) to enter; voting occurs only after no more candidates wish to enter. The main result is that, if there are three potential candidates, then only one enters. Feddersen, Sened, and Wright [1990] modify Hotelling's model by allowing candidates to choose whether or not to enter and by having citizens vote strategically. They find that all entering candidates adopt the median position and that the ratio of the spoils of office to the cost of entry provides an upper bound on the number of entrants. While the models of Osborne and Feddersen et al. illuminate some aspects of political competition, their equilibria have features that do not accord well with many actual electoral outcomes, in which there are many candidates with distinct positions.

Several papers study models in which candidates care about the policy carried out, taking one step toward the citizen-candidate formulation that we adopt, among them Wittman [1977, 1983, 1990], Calvert [1985], Alesina [1988], and Roemer [1994]. In these models the candidates, whose number is exogenously fixed to be two, are free to adopt any position. The main question addressed is the degree of similarity in the candidates' positions, which we discuss below.

A formulation that comes close to ours is used by Greenberg and Shepsle [1987], who analyze a situation in which a set of citizens faces the task of electing  $k$  officials. Each citizen votes for her most-preferred candidate from among those who enter the contest, and the  $k$  candidates receiving the most votes are elected, so for  $k = 1$  the system is simple plurality rule. A *k-equilibrium* occurs when  $k$  candidates choose (different) positions such that no additional candidate can choose a position that earns her more votes than any of the original  $k$ . Only in the case  $k = 1$  does an equilibrium generally exist, and in this equilibrium the single candidate chooses the median position. An equilibrium can be interpreted as a situation in which  $k$  citizens enter the election as candidates, each espousing her own most-preferred position. The major respect in which the model differs from ours is the restriction that there be exactly as many candidates as positions. For simple plurality rule, this means that the number of candidates is restricted to one. Consequently, the model cannot address the issues with which we are concerned.

In a recent paper Besley and Coate [1995] independently de-

velop the notion of a citizen-candidate. They formulate a model more general than ours: it differs from ours mainly in that it introduces elements of strategic behavior into the decision to vote. The main point of their paper is to study the efficiency of the outcome of political competition. They study the one-dimensional spatial case under plurality rule (as we do in Section III) under the restrictions that the distribution of ideal points is symmetric and  $b = 0$ . The latter assumption implies that there are never more than two candidates in any such race, in contrast to the predictions of our model (and to reality).

As we noted earlier, the most prominent hypothesis regarding the relationship between the electoral system and the number of candidates is Duverger's Law. Palfrey [1989] (building upon the work of Cox [1987a]) and Feddersen [1992] study models that predict versions of Duverger's Law. Both models assume that voting is strategic. In Palfrey's model there are three candidates with exogenously given positions. The main result captures the idea that supporters of third parties do not want to "waste" their votes. As the number of voters gets large, in any equilibrium in which all three candidates are not tied for first place the share of votes received by one of the candidates goes to zero. In Feddersen's model there are no candidates: citizens may vote for any position in a given finite set. Feddersen gives conditions under which in equilibrium exactly two positions receive votes. Thus, in both papers the prediction is that under plurality rule, two candidates receive (almost) all the votes. Whether or not this is what Duverger himself claimed, it is not the case that plurality-rule elections always feature two candidates, as is clear from both casual observation and the work of Wright and Riker [1989].<sup>8</sup>

Our results contribute also to an understanding of the dispersion observed in candidates' policies. The literature focuses on

8. In more than 25 percent of plurality-rule Democratic gubernatorial primaries in the United States between 1950 and 1982, there were four or more candidates, and in more than 25 percent there was only one candidate. By contrast, in more than 25 percent of primaries under the runoff system there were more than seven candidates, and in fewer than 4 percent there was only one candidate. Even in U. S. presidential elections, in which the two major parties are legally entitled to significant advantages over minor parties (they receive maximal funding for their election campaigns and grants for holding their national conventions, for example), there have been at least eleven candidates in each of the last seven elections, and in three of these elections (1992, 1980, and 1968) a third candidate has received more than 5 percent of the popular vote. As a final example, in the six general elections in Canada between 1962 and 1974, four parties each received at least 5 percent of the popular vote (and, except in 1974, at least 5 percent of the seats in the parliament).

whether candidates tend to offer the same policies: whether there is "policy convergence." In Hotelling's model there is convergence when there are two candidates. When there are more candidates, then equilibria do not in general exist [Osborne 1993], but when they do, the candidates choose distinct positions [Cox 1987b, 1990]. In Palfrey's [1984] model, in which there is an exogenous set of three candidates, the first two candidates adopt distinct positions in order to minimize the effect of entry by the third. If the assumption of sincere voting is replaced by that of strategic voting in Hotelling's model and entry by candidates is allowed, then in any equilibrium all candidates adopt the same position [Feddersen, Sened, and Wright 1990]. If Hotelling's model is modified by endowing the candidates with (distinct) policy preferences, then in the case that there are two candidates who can commit to the policies they propose there is full convergence when the candidates know the distribution of the citizens' ideal positions, but may not be if the candidates are imperfectly informed [Wittman 1977; Calvert 1985; Roemer 1994]. Alesina [1988] argues that if the candidates are unable to commit to policies, voters will see through any policy announcement, and the only possible outcome is that each candidate carries out her favorite policy. In a model of repeated elections, however, some convergence is possible. The degree depends on discount rates, the difference in the candidates' preferences, and the relative level of support in the population for each of the candidates' preferred positions. What is unique about our analysis is that the possibility of convergence differs fundamentally for the two electoral systems considered. Under plurality rule, the only convergence that is predicted occurs in an equilibrium in which there are at least three candidates. In this case at most two candidates share the same position. Under a runoff system it may be that all candidates share the same position (the median), however many candidates there are.

#### APPENDIX

This appendix contains proofs of all the results stated in the text. Throughout,  $a_i$  denotes the position of candidate  $i$ , and we number the candidates so that  $a_1 \leq a_2 \leq \dots \leq a_n$ . We denote  $q_1 = F^{-1}(\frac{1}{4})$  and  $q_3 = F^{-1}(\frac{3}{4})$ .

*Proof of Lemma 1.* Under the stated conditions the withdrawal of a candidate who loses with certainty either has no ef-

fect on the outcome or causes the set of winners to be the set of candidates whose ideal position is closest to hers (rather than a set of candidates with more distant ideal positions). Since the deviation saves her the cost  $c$ , it is profitable.  $\square$

*Proof of Proposition 1.* In order that no other citizen with the same ideal position wishes to enter, we need  $\frac{1}{2}b \leq c$ . Further, if  $\frac{1}{2}b \leq c$ , then there is an equilibrium in which a single citizen with ideal position  $m$  enters, since any entrant with a different ideal position loses, and the withdrawal of the single candidate yields her  $-\infty$ .

If there is a single candidate with ideal position  $a \neq m$ , then a citizen with ideal position  $d \in (a, 2m - a)$  can win outright by entering, getting a payoff of  $b - c$  rather than  $-|a - d|$ . Thus, a necessary condition for such an equilibrium is  $-|a - d| \geq b - c$  for any such  $d$ , which implies that  $b \leq c$  and  $|m - a| \leq (c - b)/2$ . This condition is also sufficient, since a citizen with ideal position outside  $(a, 2m - a)$  wins with probability at most  $\frac{1}{2}$  if she enters, and the candidate obtains  $-\infty$  if she withdraws.  $\square$

*Proof of Proposition 2.* Part 2(ii) is proved in the text. We now prove part 2(iii). As noted in the text, each candidate's entry is optimal if and only if  $\epsilon \geq c - b/2$ . A noncandidate whose ideal point is outside  $(m - \epsilon, m + \epsilon)$  loses if she enters and does not favorably affect the set of winners, so that it is optimal for her to stay out. Finally, consider a citizen with ideal point  $d \in (m - \epsilon, m + \epsilon)$ . As argued in the text, we need  $\epsilon \leq e_p(F)$  to make it optimal for her to stay out. Now, if  $d \in (m - \epsilon, s(\epsilon, F))$ , then her entry causes the candidate at  $m + \epsilon$  to win, resulting in a payoff for her of  $d - m - \epsilon - c$  rather than  $-\epsilon$ . Thus, we require that  $d - m \leq c$  for all such  $d$ , or  $s(\epsilon, F) - m \leq c$ . Symmetrically, considering  $d \in (s(\epsilon, F), m + \epsilon)$  leads to the requirement  $m - s(\epsilon, F) \leq c$ . If  $d = s(\epsilon, F)$  and  $\epsilon < e_p(F)$ , then the citizen's entry does not affect the outcome. Finally, if  $d = s(\epsilon, F)$  and  $\epsilon = e_p(F)$ , then if the citizen enters she ties for first place with the two existing candidates, obtaining a payoff of  $\frac{1}{3}b - c - \frac{2}{3}\epsilon$  as opposed to  $-\epsilon$  if she stays out, so that for equilibrium we require that  $\epsilon \leq 3c - b$  in this case.

To prove part 2(i), first note that  $|m - s(\epsilon, F)| \leq \epsilon$  for all  $\epsilon > 0$ , so that if  $\epsilon \leq c$  then certainly  $|m - s(\epsilon, F)| \leq c$ . Thus, from 2(iii), if  $c - b/2 \leq 0$ , then any  $\epsilon \in (0, \min\{c, e_p(F)\})$  (a nonempty interval) produces an equilibrium. If  $c - b/2 > 0$ , then certainly  $3c - b > 0$ , so that there is an equilibrium if and only if  $c - b/2 \leq e_p(F)$ .  $\square$



*Proof of Lemma 2.* In an equilibrium in which more than two candidates' positions are the same, Lemma 1 implies that each candidate's probability of winning is positive, and hence is the same for all of them. Thus, each of their payoffs is at most  $\frac{1}{3}b - c$ . If one of them withdraws, then she obtains 0 (since then the set of winners is the set of candidates remaining at that position), so we require  $b \geq 3c$ . But then a citizen with ideal point just to either side is better off entering, since she wins outright, obtaining the payoff  $b - c \geq 0$ .  $\square$

*Proof of Proposition 3.* From Lemma 2 there is no equilibrium in which all three candidates have the same ideal point. By Lemma 1 the following two cases remain.

*Each candidate obtains one-third of the votes, and either the three positions are different or exactly two are the same.* Let  $a_2 - a_1 = \epsilon_1$  and  $a_3 - a_2 = \epsilon_2$ . Candidate 1's payoff to  $E$  is  $\frac{1}{3}b - c - \frac{2}{3}\epsilon_1 - \frac{2}{3}(\epsilon_1 + \epsilon_2)$ , while her payoff to  $N$  is  $-2\epsilon_1$ . Thus, for equilibrium we require that  $b \geq 3c + 2(\epsilon_2 - \epsilon_1)$ . Similarly, the optimality of candidate 3's decision implies that  $b \geq 3c + 2(\epsilon_1 - \epsilon_2)$ . Thus, for equilibrium we need that  $b \geq 3c + 2|\epsilon_1 - \epsilon_2|$ . Now, if  $\epsilon_i > 0$  for  $i = 1, 2$ , then the condition that each candidate receive a third of the votes implies that the positions are those stated in part 3(i) of the result. If  $\epsilon_i = 0$  for some  $i$ —say  $\epsilon_2 = 0$ —then the positions are those given in 3(i) since if  $a_2 < t_2$  then any citizen whose ideal point is in  $(a_2, t_2)$  wins outright if she enters, while if  $a_2 > t_2$ , then any citizen whose ideal point is in  $(t_2, a_2)$  wins outright if she enters, and given  $b \geq 3c$  a citizen prefers an outright win to nonentry.

*The three candidates' positions are different, and the middle candidate obtains a smaller fraction of the votes than the other two, who tie.* Let  $a_1 = m_1 - \epsilon_1$ ,  $a_2 = m_1 + \epsilon_1 = m_2 - \epsilon_2$ , and  $a_3 = m_2 + \epsilon_2$  for some  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  and some  $m_1 \in (t_1, m)$  and  $m_2 \in (m, t_2)$  with  $F(m_1) = 1 - F(m_2)$ .

By an argument like that for the previous case, we need that  $b \geq 2c + 2|\epsilon_1 - \epsilon_2|$  for the entry of candidates 1 and 3 (who now each win with probability  $\frac{1}{2}$ ) to be optimal.

Now suppose that candidate 2 withdraws. If  $(a_1 + a_3)/2 = m$ , then the outcome remains the same, so she is better off. Hence  $(a_1 + a_3)/2 \neq m$ . If  $(a_1 + a_3)/2 > m$ , candidate 1 wins, so that the optimality of candidate 2's entry implies that  $-c - \epsilon_1 - \epsilon_2 \geq -2\epsilon_1$ , or  $c \leq \epsilon_1 - \epsilon_2$ . Since  $c > 0$ , this implies that  $\epsilon_1 > \epsilon_2$ . Similarly if  $(a_1 + a_3)/2 < m$ , then we need  $c \leq \epsilon_2 - \epsilon_1$  and hence  $\epsilon_2 > \epsilon_1$ .

Finally, suppose that a citizen with ideal point  $d \in (a_1, a_2)$  enters. Then candidate 3 wins, and the citizen obtains the payoff  $-c - a_3 - d$  rather than  $\frac{1}{2}(a_1 - a_3)$ . Thus, we need  $a_1 - a_3 \geq 2(d - a_3 - c)$  for all  $d \in (a_1, a_2)$ , or  $a_1 - a_3 \geq 2(a_2 - a_3 - c)$ , which is equivalent to  $c \geq \epsilon_1 - \epsilon_2$ . The analogous condition for a citizen with ideal point in  $(a_2, a_3)$  is  $c \geq \epsilon_2 - \epsilon_1$ . Thus, we need  $c \geq |\epsilon_1 - \epsilon_2|$ .

Since  $c \leq \epsilon_1 - \epsilon_2$  if  $(a_1 + a_3)/2 > m$ , and  $c \leq \epsilon_2 - \epsilon_1$  if  $(a_1 + a_3)/2 < m$ , it follows that in either case we have  $c = |\epsilon_1 - \epsilon_2|$ , so that a necessary condition for the existence of this type of equilibrium is  $b \geq 4c$ . Since  $\epsilon_i > 0$  for  $i = 1, 2$ , we have  $\epsilon_i < t_2 - t_1$  for  $i = 1, 2$ . Thus,  $c < t_2 - t_1$ .  $\square$

*Proof of Proposition 4.* We begin by proving the first sentence of the result. Let the number of candidates be  $n$ . By Lemma 1 candidates 1 and  $n$  are winners. First, suppose that  $a_1 = a_2$ . Then  $a_3 > a_1$  by Lemma 2. By Lemma 1 candidates 1 and 2 are winners, so that if 1 withdraws then 2 is the sole winner. Thus, 1's payoff to  $N$  is 0, while her payoff to  $E$  is less than  $(1/k)b - c$ . Hence for an equilibrium of this type we require that  $b > kc$ . A similar argument can be made for an equilibrium in which  $a_{n-1} = a_n$ .

Now suppose that  $a_1 < a_2$  and  $a_{n-1} < a_n$ . Since  $a_2 \leq a_{n-1}$ , we must have either  $(a_1 + a_n)/2 \geq a_2$  or  $(a_1 + a_n)/2 \leq a_{n-1}$ , so that either  $a_n - a_1 \geq 2(a_2 - a_1)$  or  $a_n - a_1 \geq 2(a_n - a_{n-1})$ . In the former case we claim that candidate 1 can profitably withdraw unless  $b \geq kc$ . If she does so, then only the fraction of the votes received by the candidates at  $a_2$  changes. If these candidates were originally winners, then 1's withdrawal makes them the only winners. If there is only one candidate at  $a_2$  and she originally lost, then since she obtains all of 1's votes, she becomes the outright winner. Thus, in each case 1's withdrawal yields her a payoff of  $-(a_2 - a_1)$  as opposed to at most  $(1/k)b - (1/k)(a_n - a_1) - [(k-2)/k](a_2 - a_1) - c$  when she enters (since she wins with probability  $1/k$ , a candidate at  $a_n$  wins with probability  $1/k$ , and the position of every other candidate is no better for 1 than  $a_2$ ). Since  $a_n - a_1 \geq 2(a_2 - a_1)$ , this latter payoff is at most  $(1/k)b - (a_2 - a_1) - c$ , which is at least  $-(a_2 - a_1)$  only when  $b \geq kc$ .

We now prove the second sentence of the result. By Proposition 3 the condition  $b \geq 3c$  is necessary for a three-candidate equilibrium to exist. We now show that if there are four or more candidates then at least three are winners, so that the second part of the result follows from the first part.

We need to show that there is no equilibrium in which there

are two winners and at least four candidates. By Lemma 1 in any such equilibrium all the candidates take different positions, and the two whose positions are extreme are the winners. The cases of four and of five or more candidates require different arguments, as follows.

First, consider the case of four candidates, with  $a_2 = a_1 + \epsilon_1$ ,  $a_3 = a_2 + \epsilon_2$ , and  $a_4 = a_3 + \epsilon_3$ . If candidate 2's withdrawal leads to a tie for first place between candidates 1 and 3, then certainly 2's withdrawal is beneficial. Thus, 2's withdrawal must lead to an outright victory for either 1 or 3. Similarly candidate 3's withdrawal must lead to an outright victory for either 2 or 4. Suppose that 2's withdrawal leads to a win for 1 and 3's withdrawal leads to a win for 2. The optimality of candidate 2's entry then requires that  $\epsilon_1 \geq \frac{1}{2}\epsilon_1 + \frac{1}{2}(\epsilon_2 + \epsilon_3)$  or  $\epsilon_1 \geq \epsilon_2 + \epsilon_3$  and the optimality of the entry decision of candidate 3 requires  $\epsilon_2 \geq \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}\epsilon_3$  or  $\epsilon_2 \geq \epsilon_1 + \epsilon_3$ . Since these two inequalities are incompatible, this pattern of winners in the event of the withdrawals of candidates 2 and 3 is not possible. Similar arguments eliminate two of the three other possible patterns, leaving the possibility that candidate 3 wins when candidate 2 withdraws and vice versa, which implies that  $\epsilon_2 \geq \epsilon_1 + \epsilon_3$ . Now, we claim that in this case there is a point in  $(a_2, a_3)$  at which an entrant can win for sure. To see this, first note that there is a point in  $(a_2, a_3)$  at which an entrant can receive the votes of all citizens whose ideal points lie in the interval  $(\frac{1}{2}(a_1 + a_3), \frac{1}{2}(a_2 + a_4))$ , since  $\frac{1}{2}(a_2 + a_4) - \frac{1}{2}(a_1 + a_3) = \frac{1}{2}(\epsilon_1 + \epsilon_3) \leq \frac{1}{2}\epsilon_2$ . To complete the argument, we show that the votes of these citizens are enough to win. Let  $\alpha = F(\frac{1}{2}(a_1 + a_2)) = 1 - F(\frac{1}{2}(a_3 + a_4))$ ,  $\beta_1 = F(\frac{1}{2}(a_1 + a_3)) - F(\frac{1}{2}(a_1 + a_2))$ ,  $\beta_2 = F(\frac{1}{2}(a_3 + a_4)) - F(\frac{1}{2}(a_2 + a_4))$ , and  $\gamma = F(\frac{1}{2}(a_2 + a_4)) - F(\frac{1}{2}(a_1 + a_3))$ . The fact that candidate 2 wins when candidate 3 withdraws, and vice versa, means that  $\gamma + \beta_1 > \alpha + \beta_2$  and  $\gamma + \beta_2 > \alpha + \beta_1$ , so that  $\gamma > \alpha$ , completing the argument.

Now suppose that there are  $n \geq 5$  candidates. Let  $a_i = a_{i-1} + \epsilon_{i-1}$  for  $i = 1, \dots, n-1$  and  $\sum_{j=3}^{n-3} \epsilon_j = \delta$ . (If  $n = 5$ , then  $\delta = 0$ .) As in the previous case, if candidate 2 withdraws, then either candidate 1 or candidate 3 must win outright. If candidate 1 wins, then in order for 2's entry to be optimal we require  $\epsilon_1 \geq \frac{1}{2}\epsilon_1 + \frac{1}{2}(\epsilon_2 + \delta + \epsilon_{n-2} + \epsilon_{n-1})$ , or  $\epsilon_1 \geq \epsilon_2 + \delta + \epsilon_{n-2} + \epsilon_{n-1}$ . If candidate 3 wins, then the analogous condition is  $\epsilon_2 \geq \epsilon_1 + \delta + \epsilon_{n-2} + \epsilon_{n-1}$ . Similarly, if candidate  $n$  wins when  $n-1$  withdraws, then in order for the entry of  $n-1$  to be optimal we require that  $\epsilon_{n-1} \geq \epsilon_1 + \epsilon_2 + \delta + \epsilon_{n-2}$ . If candidate  $n-2$  wins, then the analogous

condition is  $\epsilon_{n-2} \geq \epsilon_1 + \epsilon_2 + \delta + \epsilon_{n-1}$ . It is easy to see that no combination of these conditions is possible.  $\square$

*Proof of Proposition 5.* We need to show that no other types of equilibria are possible and to verify the lower bounds on  $b$  for the existence of an equilibrium of types 5(iii) and 5(iv). (The lower bounds in 5(i) and 5(ii) follow from Proposition 4.) Given Lemma 1, there are two types of configuration that remain to be ruled out as equilibria. Throughout we let  $a_2 - a_1 = \epsilon_1$ ,  $a_3 - a_2 = \epsilon_2$ , and  $a_4 - a_3 = \epsilon_3$ .

*The candidates' positions are distinct, and the two middle candidates both lose.* If candidate 2 withdraws, then either candidate 1 wins, candidate 3 wins, or candidates 1 and 3 tie (candidate 4 cannot win, since she definitely receives fewer votes than candidate 1). Similarly, if 3 withdraws, then either 2 wins, 4 wins, or 2 and 4 tie. If either of the ties occurs, then the candidate who withdraws is better off doing so. Thus, in an equilibrium the withdrawal of 2 or 3 leads to certain victory for one of the other candidates.

Suppose that 1 wins if 2 withdraws and 2 wins if 3 withdraws. Then in order for 2's entry to be optimal, we need  $\frac{1}{2}(-\epsilon_1) + \frac{1}{2}(-\epsilon_2 - \epsilon_3) - c \geq -\epsilon_1$ , and in order for 3's entry to be optimal, we need  $\frac{1}{2}(-\epsilon_1 - \epsilon_2) + \frac{1}{2}(-\epsilon_3) - c \geq -\epsilon_2$ . These two inequalities are incompatible with  $\epsilon_3 > 0$ , so this case is impossible. By similar arguments we can rule out equilibria in which 1 wins if 2 withdraws and 4 wins if 3 withdraws, and those in which 3 wins if 2 withdraws and 4 wins if 3 withdraws.

We are left with the case in which 3 wins if 2 withdraws and 2 wins if 3 withdraws, which requires  $\epsilon_2 > \epsilon_1 + \epsilon_3$ . In this case there is a citizen with ideal point in  $(a_2, a_3)$  who obtains the votes of all citizens with ideal points in  $((a_1 + a_3)/2, (a_2 + a_4)/2)$  if she enters, since  $(a_2 + a_4)/2 - (a_1 + a_3)/2 = (\epsilon_1 + \epsilon_3)/2 < \epsilon_2/2$ . Now, the condition that 3 wins if 2 withdraws and 2 wins if 3 withdraws implies that there are enough citizens with ideal points in  $((a_1 + a_3)/2, (a_2 + a_4)/2)$  for the entrant to win. Since she is better off winning outright than staying out, the configuration is not an equilibrium.

*Two candidates share one position, and two candidates share another position. Each candidate wins with probability  $\frac{1}{4}$ .* We need  $\frac{1}{2}(a_1 + a_3) = m$  in order that each candidate win with probability  $\frac{1}{4}$ . If  $a_1 \neq q_1$ , then a citizen with ideal point  $q_1$  can enter and win. Similarly, if  $a_3 \neq q_3$ , then a citizen with ideal point  $q_3$

can enter and win. Thus, for an equilibrium we need that  $a_1 = q_1$  and  $a_3 = q_3$ . For a generic distribution  $F$  these conditions are incompatible with  $\frac{1}{2}(a_1 + a_3) = m$ , so there is no equilibrium of this type.

We now verify the lower bounds on  $b$  for the existence of the equilibria described in 5(iii) and 5(iv).

5(iii). First suppose that  $a_1 < a_2 = a_3 < a_4$ . Candidate 2's payoff is  $\frac{1}{4}b - c - \frac{1}{4}\epsilon_1 - \frac{1}{4}\epsilon_3$ . If she withdraws, her payoff is 0 (since her partner at the position then wins). Thus, a necessary condition for the existence of the equilibrium is  $b \geq 4c + \epsilon_1 + \epsilon_3$ . In order that each candidate receive the same number of votes, we need that  $a_1 < q_1$  and  $a_4 > q_3$ , so that  $\epsilon_1 + \epsilon_3 > q_3 - q_1$ .

Now suppose that  $a_1 = a_2 < a_3 < a_4$ . By an argument like that in the previous case for candidate 2, the optimality of candidate 1's entry requires that  $b \geq 4c + 2\epsilon_2 + \epsilon_3$ . Now, if  $a_1 > q_1$ , a citizen with ideal point  $q_1$  can enter and win outright, so we need  $a_1 < q_1$ . Further, as in the previous case we need  $a_4 > q_3$  in order that candidate 4 receive one-quarter of the votes. Hence  $\epsilon_2 + \epsilon_3 > q_3 - q_1$ .

5(iv). Suppose that  $a_1 < a_2 < a_3 = a_4$ . Consider the optimality of candidate 2's entry. Her payoff is  $-\frac{1}{3}\epsilon_1 - \frac{2}{3}\epsilon_2 - c$ . If when she withdraws the remaining candidates still tie for first place, then her entry is clearly not optimal. If when she withdraws 3 and 4 tie for first place, then her payoff is  $-\epsilon_2$ , so her entry is optimal only if  $c < \frac{1}{3}(\epsilon_2 - \epsilon_1)$ . But in order for a citizen with ideal point slightly greater than  $a_2$  to stay out, we need  $-\frac{1}{3}\epsilon_1 - \frac{2}{3}\epsilon_2 > -\epsilon_1 - c$ , since her entry causes 3 or 4 to win, or  $c \geq \frac{2}{3}(\epsilon_2 - \epsilon_1)$ , a contradiction. The final possibility is that if 2 withdraws then 1 wins, in which case her entry is optimal if and only if  $c \leq \frac{2}{3}(\epsilon_1 - \epsilon_2)$ , or  $\epsilon_1 \geq \frac{3}{2}c + \epsilon_2$ . Now, for candidate 3's entry to be optimal, we need that  $\frac{1}{3}b - c - \frac{1}{3}(\epsilon_1 + \epsilon_2) \geq 0$ , or  $b \geq 3c + \epsilon_1 + \epsilon_2$ . It follows that  $b > \frac{9}{2}c$ .  $\square$

*Proof of Proposition 6.* Suppose that the candidates' common ideal position is  $m$ . Then entry by a citizen with a different ideal position results in certain defeat: if either  $k = 2$ , or  $k \geq 3$  and the position of the entrant is far from  $m$ , then the entrant fails to reach the runoff; otherwise, she loses in the runoff. Thus, it is optimal for such a citizen not to enter. It is optimal for another citizen with ideal position  $m$  not to enter since  $b/(k + 1) \geq c$ , and it is optimal for the  $k$  candidates to enter since  $b/k \leq c$ .  $\square$

*Proof of Proposition 7.* It is immediate that in any equilibrium the candidates' positions are  $m - \epsilon$  and  $m + \epsilon$  for some

$\epsilon \geq 0$ . The case  $\epsilon = 0$  is covered in Proposition 6. The line of argument for  $\epsilon > 0$  closely follows that in the proof of Proposition 2. Differences arise only because (1) a citizen whose ideal point is  $m - \epsilon$  or  $m + \epsilon$  has a positive probability under a runoff system of winning if she enters, and  $b \leq 4c$  is necessary for her nonentry to be optimal; (2) the upper bound on  $\epsilon$  under which no citizen with an ideal point in  $(m - \epsilon, m + \epsilon)$  can win outright is  $e_r(F)$  rather than  $e_p(F)$ ; (3) a citizen with ideal point in  $(m - \epsilon, m + \epsilon)$  who enters does not affect the identity of the winner (so that no condition like  $c \geq |m - s(\epsilon, F)|$  is needed); and (4) if  $\epsilon = e_r(F)$ , then there is a citizen with ideal point in  $(m - \epsilon, m + \epsilon)$  who is the ultimate winner with probability  $\frac{1}{2}$  (rather than  $\frac{1}{3}$  if  $\epsilon = e_p(F)$  under plurality rule), yielding the condition  $\epsilon \leq 2c - b$  rather than  $\epsilon \leq 3c - b$ .  $\square$

*Proof of Proposition 8.* First, consider the possibility of an equilibrium in which  $a_1 < a_2 = a_3$ . Candidate 1 must receive at most half of the votes, otherwise she wins outright on the first ballot, and candidates 2 and 3 are better off withdrawing. Further, her probability of ultimately winning must be positive (otherwise she prefers to withdraw), so that she must receive exactly half of the votes in the first round. Thus,  $a_1 = m - \epsilon$  and  $a_2 = m + \epsilon$  for some  $\epsilon > 0$ . Now, in order for candidates 2 and 3 to prefer  $E$  to  $N$ , we need  $\frac{1}{4}b - \frac{1}{2} \cdot 2\epsilon - c \geq \frac{1}{2} \cdot (-2\epsilon)$ , or  $b \geq 4c$ . But if another citizen with ideal position  $a_1$  enters, then each of the four wins with probability  $\frac{1}{4}$ , so to deter such entry we need  $b \leq 4c$ . Hence for such an equilibrium we need  $b = 4c$ .

Now consider the possibility of an equilibrium in which  $a_1 < a_2 < a_3$ . Each candidate must obtain one-third of the votes in the first round, or else one of them has no chance of winning, and hence prefers to withdraw. Hence the positions satisfy the condition in Proposition 3(i). Now, candidates 1 and 3 each prefer to withdraw unless she has a positive probability of winning in a runoff. Since she can win only if she faces the other extreme candidate, we must have  $m - a_1 = a_3 - m$ , which implies that the positions are those given in the result.

In this configuration, with probability  $\frac{1}{3}$ , candidate 1 is in a runoff with candidate 2, which candidate 2 certainly wins. With probability  $\frac{1}{3}$  she is in a runoff with candidate 3, which she wins with probability  $\frac{1}{2}$ . With probability  $\frac{1}{3}$  she does not make it to a runoff, which candidate 2 wins. Thus, her payoff is  $\frac{1}{3}(-2\epsilon_1) + \frac{1}{3}[\frac{1}{2}b - \frac{1}{2}(2\epsilon_1 + 2\epsilon_2)] + \frac{1}{3}(-2\epsilon_1) - c$ , where  $\epsilon_i = a_{i+1} - a_i$ . If she withdraws, then she obtains  $-2\epsilon_1$  (since candidate 2 then cer-

tainly wins). Thus, her entry is optimal if and only if  $b \geq 6c + 2(\epsilon_2 - \epsilon_1)$ . A similar calculation for candidate 3 yields the condition  $b \geq 6c + 2(\epsilon_1 - \epsilon_2)$ , so that we need  $b \geq 6c + 2|\epsilon_2 - \epsilon_1|$ .  $\square$

*Proof of Proposition 9.* Each candidate wins with probability  $1/k$ , so her expected payoff is  $b/k - c - \epsilon$ . If she withdraws, then the ideal position of the eventual winner is  $m - \epsilon$  with probability  $\frac{1}{2}$  and  $m + \epsilon$  with probability  $\frac{1}{2}$  if  $k = 4$  and is the same as hers if  $k \geq 6$ . Thus, her entry is optimal if and only if  $b > 4c$  if  $k = 4$  and  $b \geq k(c + \epsilon)$  if  $k \geq 6$ .

The entry of a noncandidate whose ideal point is outside  $(m - \epsilon, m + \epsilon)$  is not optimal, since it causes the ideal point of the eventual winner to be more remote.

Finally, consider citizens with ideal points in  $(m - \epsilon, m + \epsilon)$ . If  $\epsilon > e_r^k(F)$ , then there is such a citizen who can win outright if she enters, and hence prefers to do so. If  $\epsilon = e_r^k(F)$ , then there is such a citizen who gets into a runoff with probability  $2/(k + 1)$ , and wins if she does so, obtaining the expected payoff  $2b/(k + 1) - c - (1 - 2/(k + 1))\epsilon$ . Since  $b \geq kc$ , this payoff exceeds  $-\epsilon$ , so that she is better off entering. Finally, if  $\epsilon < e_r^k(F)$ , then the entry of a citizen with ideal point  $d \in (m - \epsilon, s(\epsilon, F))$  causes the ideal point of the winner to become  $m + \epsilon$ . To deter such entry, it is necessary and sufficient that  $-\epsilon \geq -c - (m + \epsilon - d)$  for all such  $d$ , or  $s(\epsilon, F) - m \leq c$ . Similarly, the optimality of the action of a citizen with ideal point in  $(s(\epsilon, F), m + \epsilon)$  requires that  $m - s(\epsilon, F) \leq c$ .  $\square$

*Proof of Proposition 10.* We first show that for a generic distribution  $F$  the configuration in parts (ii) and (iii) of Proposition 5 are not possible under a runoff system.

$a_1 < a_2 \leq a_3 < a_4$ , and each candidate obtains the same number of first-round votes. Since each candidate obtains one-quarter of the first-round votes, we have  $a_2 = 2q_1 - a_1$ ,  $a_3 = 2m - a_2$ , and  $a_4 = 2q_3 - a_3$ . Hence  $a_4 = 2q_3 - 2m + 2q_1 - a_1$ . Now, if 1 and 4 tie in a runoff against each other, then  $a_4 - m = m - a_1$ . Combined with  $a_4 - m = m - a_1$ , we deduce that we need  $m = \frac{1}{2}(q_1 + q_3)$ , which generically is not satisfied. Thus, for a generic distribution either 1 loses against 4 or vice versa.

Suppose that 1 loses against 4, so that she loses any runoff. The ultimate winner is 2 with probability  $\frac{5}{6}$ , 3 with probability  $\frac{5}{12}$ , and 4 with probability  $\frac{1}{6}$ . If 1 withdraws, then the ultimate winner is 2 with probability  $\frac{3}{4}$  and 3 with probability  $\frac{1}{4}$  if  $a_2 < a_3$  and 2 and 3 each with probability  $\frac{1}{2}$  if  $a_2 = a_3$ . Hence in either case 1 is better off withdrawing.

$a_1 < a_2 < a_3 = a_4$ , and each candidate obtains the same number of first-round votes. The ultimate winner is 2 with probability  $\frac{2}{3}$ , 3 with probability  $\frac{1}{6}$ , and 4 with probability  $\frac{1}{6}$ . If 1 withdraws, then the ultimate winner is 2 with probability 1. Hence 1 is better off withdrawing.

We now show that the configuration in part (iv) of Proposition 5 is not possible under a runoff system.

$a_1 = a_2 < a_3 < a_4$ , and candidate 3 receives fewer votes than the other three candidates. The ultimate winner is 1 with probability  $\frac{1}{2}$  and 2 with probability  $\frac{1}{2}$  (1 and 2 both beat 4 in a one-on-one contest). If 4 withdraws, then the ultimate outcome is the same, so she is better off withdrawing.

It remains to show that the configuration in part 5(i) is possible in equilibrium only if  $b > 6c$ .

$a_1 < a_2 < a_3 < a_4$ , and candidate 2 receives fewer votes than the other three candidates. If 1 wins a runoff against 4, then the ultimate winner is 3 with probability  $\frac{2}{3}$  and 1 with probability  $\frac{1}{3}$ . If 4 withdraws, then 3 is the certain winner (since the runoff is then between 1 and 3), so she is better off doing so. Similarly, if 4 wins a runoff against 1, then 2 is better off withdrawing. The remaining possibility is that 1 and 4 tie in a runoff, in which case 1 wins probability  $\frac{1}{6}$ , 3 wins with probability  $\frac{2}{3}$ , and 4 wins with probability  $\frac{1}{6}$ . If 4 withdraws, then 3 is the certain winner, so the optimality of 4's entry implies that  $\frac{1}{6}b - c - \frac{2}{3}\epsilon_3 - \frac{1}{6}(\epsilon_1 + \epsilon_2 + \epsilon_3) \geq -\epsilon_3$ , or  $b \geq 6c + \epsilon_1 + \epsilon_2 - \epsilon_3$ , where  $\epsilon_i = a_{i+1} - a_i$  for  $i = 1, 2, 3$ . If 2 withdraws then 3 is also the certain winner, so the optimality of 2's entry implies that  $-c - \frac{1}{6}\epsilon_1 - \frac{2}{3}\epsilon_2 - \frac{1}{6}(\epsilon_2 + \epsilon_3) \geq -\epsilon_2$ , or  $6c \leq -\epsilon_1 + \epsilon_2 - \epsilon_3$ , or, in particular,  $\epsilon_2 > \epsilon_1 + \epsilon_3$ . Thus, for an equilibrium we need that  $b \geq 6c + \epsilon_1 + \epsilon_1 + \epsilon_3 - \epsilon_3 > 6c$ .

DEPARTMENT OF ECONOMICS, MCMASTER UNIVERSITY  
DEPARTMENT OF ECONOMICS, UNIVERSITY OF WESTERN ONTARIO

## REFERENCES

- Alesina, Alberto, "Credibility and Policy Convergence in a Two-Party System with Rational Voters," *American Economic Review*, LXXVIII (1988), 796-805.  
Besley, Timothy, and Stephen Coate, "An Economic Model of Representative Democracy," Working Paper 95-02, Center for Analytic Research in Economics and the Social Sciences, University of Pennsylvania, 1995.  
Black, Duncan, *The Theory of Committees and Elections* (Cambridge, England: Cambridge University Press, 1958).  
Calvert, Randall L., "Robustness of the Multidimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence," *American Journal of Political Science*, XXIX (1985), 69-95.  
Cox, Gary W., "Duverger's Law and Strategic Voting," unpublished paper, Department of Government, University of Texas, Austin, 1987a.



- , "Electoral Equilibrium under Alternative Voting Institutions," *American Journal of Political Science*, XXXI (1987b), 82–108.
- , "Multicandidate Spatial Competition," *Advances in the Spatial Theory of Voting*, J. M. Enelow and M. J. Hinich, eds. (Cambridge, England: Cambridge University Press, 1990).
- Downs, Anthony, *An Economic Theory of Democracy* (New York: Harper and Row, 1957).
- Duverger, Maurice, *Political Parties* (London, England: Methuen, 1954).
- Feddersen, Timothy J., "A Voting Model Implying Duverger's Law and Positive Turnout," *American Journal of Political Science*, XXXVI (1992), 938–62.
- Feddersen, Timothy J., Itai Sened, and Stephen G. Wright, "Rational Voting and Candidate Entry under Plurality Rule," *American Journal of Political Science*, XXXIV (1990), 1005–16.
- Greenberg, Joseph, and Kenneth A. Shepsle, "The Effect of Electoral Rewards in Multiparty Competition with Entry," *American Political Science Review*, LXXXI (1987), 525–37.
- Hotelling, Harold, "Stability in Competition," *Economic Journal*, XXXIX (1929), 41–57.
- Osborne, Martin J., "Candidate Positioning and Entry in a Political Competition," *Games and Economic Behavior*, V (1993), 133–51.
- , "Spatial Models of Political Competition under Plurality Rule: A Survey of Some Explanations of the Number of Candidates and the Positions They Take," *Canadian Journal of Economics*, XXVIII (1995), 261–301.
- Palfrey, Thomas R., "Spatial Equilibrium with Entry," *Review of Economic Studies*, LI (1984), 139–56.
- , "A Mathematical Proof of Duverger's Law," *Models of Strategic Choice in Politics*, Peter C. Ordeshook, ed. (Ann Arbor: University of Michigan Press, 1989).
- Roemer, John E., "A Theory of Policy Differentiation in Single Issue Electoral Politics," *Social Choice and Welfare*, XI (1994), 355–80.
- Rogoff, Kenneth, "Equilibrium Political Budget Cycles," *American Economic Review*, LXXX (1990), 21–36.
- Shepsle, Kenneth A., *Models of Multiparty Electoral Competition* (Chur, Switzerland: Harwood Academic Publishers, 1991).
- Tabellini, Guido, and Alberto Alesina, "Voting on the Budget Deficit," *American Economic Review*, LXXX (1990), 37–49.
- Wittman, Donald, "Candidates with Policy Preferences: A Dynamic Model," *Journal of Economic Theory*, XIV (1977), 180–89.
- , "Candidate Motivation: A Synthesis of Alternative Theories," *American Political Science Review*, LXXVII (1983), 142–57.
- , "Spatial Strategies When Candidates Have Policy Preferences," *Advances in the Spatial Theory of Voting*, James M. Enelow and Melvin J. Hinich, eds. (Cambridge, England: Cambridge University Press, 1990).
- Wright, Stephen G., and William H. Riker, "Plurality and Runoff Systems and Numbers of Candidates," *Public Choice*, LX (1989), 155–75.