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A theory of policy differentiation in single issue electoral politics

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Abstract. Voter preferences are characterized by a parameter s (say, income) distributed on a set S according to a probability measure F. There is a single issue (say, a tax rate) whose level, b, is to be politically decided. There are two parties, each of which is a perfect agent of some constituency of voters, voters with a given value of s. An equilibrium of the electoral game is a pair of policies, b_1 and b_2, proposed by the two parties, such that b_i maximizes the expected utility of the voters whom party i represents, given the policy proposed by the opposition. Under reasonable assumptions, the unique electoral equilibrium consists in both parties proposing the favorite policy of the median voter. What theory can explain why, historically, we observe electoral equilibria where the 'right' and 'left' parties propose different policies? Uncertainty concerning the distribution of voters is introduced. Let $\{F(t)\}_{t \in T}$ be a class of probability measures on S; all voters and parties share a common prior that the distribution of t is described by a probability measure H on T. If H has finite support, there is in general no electoral equilibrium. However, if H is continuous, then electoral equilibrium generally exists, and in equilibrium the parties propose different policies. Convergence of equilibrium to median voter politics is proved as uncertainty about the distribution of voter traits becomes small.

1. Introduction

In voting models where elections present candidates who are evaluated by voters on a single issue, it is often the case that the unique winning candidate is the one whose position is the best policy for the median voter. Combined with a popular (Downsian) axiom of public choice theory – that candidates propose policies in

* I thank Jacques Drèze, who, in an early discussion, suggested considering a continuum of uncertainty in this problem. I am indebted to Roger Howe, who discovered an error in an earlier version of §6, and made suggestions for how to retrench.

order to win elections – the median voter theorem suggests that when there are only two parties, one should tend to observe both parties proposing the favorite policy of the median voter. More surprisingly, even if candidates are not self-interested in the sense just mentioned, but strive to represent the interests of particular constituencies of voters, and even if there is no 'free entry' for new parties, electoral equilibrium will nevertheless consist in both parties proposing the favorite policy of the median voter (see §2 below, Theorem 2.1).

Yet the real world does not lend clear support to the relevance of this theory: electoral politics in most countries are characterized by diverse political positions. One could explain this by claiming that the aforementioned premises of the median voter theorem fail to hold in reality: elections, for instance, are rarely over single issues. There is now a rich literature on electoral politics where the median voter result does not hold, based either upon the existence of multiple issues, or on some kind of imperfect information. (For a summary of the latter, see Calvert [1986].) Most of these models assume that candidates are not perfect agents of any group of voters: they are either wholly or in part interested in winning elections for their own reasons, and view the policy they advocate as a strategy towards that end.

I think, however, it is imprudent to reject the view that the broad sweep of political competition in many countries and time periods can be represented by a model in which (1) politics are single-issue (left versus right), (2) parties (or candidates) are perfect agents of specific constituencies of voters, and (3) there are essentially two parties. I will call such politics *class politics*, for the prominent example is one in which the single issue is economic policy, the specific constituencies are income groups who will be differentially affected by the policy, and the parties are of the 'left' and the 'right'.

Przeworski and Sprague (1986) argue that European electoral politics during the first half (at least) of the present century were class politics. Alesina and Roubini (1990) show that, in eighteen OECD democracies, the model of 'rational partisan' politics is broadly consistent with the data when there are clearly identifiable shifts from 'left' to 'right', or vice versa. These authors define 'rational partisan' politics as those satisfying (2) above; thus class politics are here defined as single-issue, two-party rational partisan politics.

In what follows, I propose a theory of class politics in which, although politics are single-issue and preferences are single-peaked, electoral equilibrium is not characterized by median voter politics. The tyranny, if you will, of the median voter is vitiated by an assumption that parties (and voters) are uncertain about what the actual distribution of voter preferences is. All voters and parties, however, will have the same priors concerning the distribution F. One might conjecture that this kind of uncertainty would simply cause both parties to advocate those policies preferred by the 'expected median voter', but this turns out not to be so. In the model of Sect. 5 below, parties propose different policies at equilibrium. A further characteristic of equilibrium, perhaps also conforming to reality, is that not both parties expect to win the election at equilibrium, yet this is an equilibrium where each party is representing the interests of its constituents as best it can.

These results are, loosely speaking, the consequence of two postulates: that each party represents the interests of a particular constituency, and that there is a 'continuum of uncertainty' about the distribution of voter types. As I noted in the introductory paragraph, the first assumption alone is insufficient to escape the median voter theorem. In Sect. 2, I present a model of electoral equilibrium where parties represent constituents, but the distribution of voter traits is known to all. In Sect. 3, the case of finite uncertainty (the distribution of voter preferences can be one of a finite number of distributions) is studied, and I show that when an equilibrium exists, it consists in both parties proposing the *ideal* policies of their constituents. In Sect. 4 continuous uncertainty about the distribution of traits is introduced, and it is shown that the payoff functions of the parties are continous (on compact domains), and hence the best-reply correspondence is always non-empty, a property it lacks in the case of finite uncertainty or certainty. Section 5 presents a specific political-economic model, shows how one solves for equilibrium, and shows by simulation how equilibrium politics change with the degree of uncertainty in the political economy. Section 6 studies the behavior of political equilibrium when uncertainty 'converges to zero'. Section 7 concludes.

The models in the literature closest to the present paper are those of Wittman (1983, 1990), Calvert (1985), and Hansson and Stuart (1984). Their models assume 'probabilistic voting', that a given candidate faces an ex ante, exogenously given probability of winning as a function of the policies proposed by both candidates. They also allow for candidates to be 'policy oriented': the assumption of partisan candidates, made in the present paper, is similar in spirit, although neither necessary nor sufficient for 'policy orientedness'. Wittman shows that such candidates do not maximize the probability of winning at equilibrium, and that there is generally differentiation of policies at equilibrium. Calvert reaches similar conclusions, and suggests but does not prove, that if candidates become increasingly policy motivated (and less motivated to win per se), equilibrium might converge to the candidates' ideal points. This suggestion is borne out by Theorem 3.1 of the present paper, although it is generally false in the case of continuous uncertainty. Hansson and Stuart also show the divergence of policies at equilibrium.

These studies, however, are all limited by the assumption of an exogenous function which defines the probability that a particular party wins the election at a given pair of policy proposals. The main conceptual innovation in the present paper is the endogenous determination of the probability of winning elections. The 'state of the world' (the true distribution of voter traits) is exogenously specified as a random variable: but the probability of winning the election is derived directly from voter preferences, and inherits its stochastic nature from the stochastic specification of the state of the world.

Before proceeding, I wish to motivate the usefulness of the present study for those who might view it as 'simply' a rigorous embellishment on the work of Calvert, Wittman, and Hansson and Stuart. In the existing literature in political economy (see, for example, Persson and Tabellini [1990]), political equilibrium is usually modelled as 'median voter' politics. The present paper provides a model which derives, from primitives, a political equilibrium which is a lottery between two policies. When I say 'from primitives', I mean that the uncertainty in the electoral outcome in a political economy derives from preferences, technology, endowments, and uncertainty about the distribution of voter preferences, or more generally, the state of the world. (See the example of Sect. 5.) This model provides, then, a tool for any researcher in political economy for whom uncertainty of the electoral outcome is a salient issue, a tool which is unavailable from the formulations of the earlier studies, because, in those models, the uncertainty in electoral outcomes was not derived from primitives. For example, I have used the present model to study how the uncertain outcome of elections in democracies can affect the actions taken by economic agents, such as concerning the levels of investment they choose (Roemer [in press *a*]). If electoral uncertainty is an important aspect of the political economy of democracies, then one should have a tool to model it.

The work of Palfrey (1984), in which electoral equilibrium also consists in different policies, should also be mentioned. Unlike the previous literature I have cited, and the present paper, Palfrey assumes (à la Downs) that parties are vote maximizers. The issue he raises is the potential entry of a third party. If the two existing parties put forth the same policy, an entrant can always win; this motivates the two parties to put forth different policies, which discourages potential entrants. Unlike the present paper, Palfrey assumes that both parties know for sure the distribution of voter preferences.

Concerning the divergence of electoral equilibrium from the median voter's ideal point. Calvert writes:

...if we treat policy motivation and uncertainty as mere complicating factors in a world where candidates are mostly interested in winning office and are fairly well able to predict voter responses, then this divergence is itself just a complicating factor, small when the other complications are small. Electoral competition still involves an underlying pressure for candidates to converge.[1]

The present study in one way strengthens Calvert's claim, as it shows that even if candidates are completely partisan, as uncertainty diminishes, the convergence to median voter politics occurs (Theorems 2.1 and 6.1). I would, however, take issue with Calvert's conditional clause, as I do not believe that 'policy motivation' and uncertainty are mere noise in a Downsian model with certainty, which, for all practical purposes, accurately describes real political competition.

2. Class politics without uncertainty

Voter preferences are represented by utility functions $v(b,s)$, where $b \in \bar{B}$, $s \in \bar{S}$, $\bar{B}$ and $\bar{S}$ are real intervals, and $\bar{B}$ is compact. 'b' is a policy and 's' is a voter type. Suppose that F is a probability measure on $\bar{S}$, where we interpret, for any measurable subset $S \subset \bar{S}$, $F(S)$ as the fraction of the population with $s \in S$. There are two political parties, representing voters of types $\underline{s}$ and $\bar{s}$. Each party tries to maximize the expected utility of its constituents, facing the policy of the other party. For this to make sense, it is assumed that $v(b,s)$ is, for each s, a von Neumann-Morgenstern utility function on lotteries composed of policies in $\bar{B}$.

We assume that, if there is a tie, a fair coin is tossed to decide the election, and if a voter is indifferent between two policies, he flips a coin to decide how to vote. Let

$$\Omega(b^1,b^2)=\{s \in \bar{S} \mid v(b^1,s) > v(b^2,s)\}$$

and

$$\mathscr{E}(b^1,b^2)=\{s \in \bar{S} \mid v(b^1,s)=v(b^2,s)\} \ .$$

[1] Calvert (1985), p. 86

Then, if one party proposes b^1 and the other party b^2, the first will be supported by fraction $F(\Omega(b^1,b^2))+\frac{1}{2}F(\mathscr{E}(b^1,b^2))$ of the population, and the second by fraction $F(\Omega(b^2,b^1))+\frac{1}{2}F(\mathscr{E}(b^1,b^2))$. I define a *voting game* as a tuple $(v, \bar{B}, \bar{S}, F, \underline{s}, \bar{s})$.

Define

$$\varphi(b^1,b^2)=F(\Omega(b^1,b^2))+\tfrac{1}{2}F(\mathscr{E}(b^1,b^2)) \ . \tag{2.1}$$

and

$$p(b^1,b^2)=\begin{cases}1 & \text{if} \quad \varphi(b^1,b^2)>\frac{1}{2}\\ \frac{1}{2} & \text{if} \quad \varphi(b^1,b^2)=\frac{1}{2}\\ 0 & \text{if} \quad \varphi(b^1,b^2)<\frac{1}{2} \ .\end{cases} \tag{2.2}$$

A policy which receives a majority vote wins. Hence we can define the *payoff function for voters.* $\pi(b,\bar{b},s)$ as follows, where b and $\bar{b}$ are two policies and s is any voter:

$$\pi(b,\bar{b},s)=v(b,s)\,p(b,\bar{b})+v(\bar{b},s)\,p(\bar{b},b) \ . \tag{2.3}$$

$\pi(b,\bar{b},s)$ is the expected utility of voter s if there is an election contesting policies b and $\bar{b}$.

Definition 2.1. An *electoral equilibrium* of the voting game $(v, \bar{B}, \bar{S}, F, \underline{s}, \bar{s})$ is a pair of policies $(\underline{b},\bar{b})$ such that $\underline{b}$ maximizes $\pi(b,\bar{b},\underline{s})$ on $\bar{B}$ and $\bar{b}$ maximizes $\pi(b,\underline{b},\bar{s})$ on $\bar{B}$.

That is, $(\underline{b},\bar{b})$ is a Nash equilibrium, where $\underline{b}$ is a best response of the $\underline{s}$ party to $\bar{b}$ and $\bar{b}$ is a best response of the $\bar{s}$ party to $\underline{b}$. Each party maximizes the expected utility of its constituents.

Define $\hat{b}(s)\equiv\operatorname*{argmax}_b v(b,s)$.

Lemma 2.1. (Median voter theorem) *Let*

(A1) *$v(b,s)$ be continuous and strictly concave in b,*
(A2) *$v(b,s)$ be continuous in s,*
(A3) *$\hat{b}(s)$ be strictly monotone decreasing, and*
(A4) *F be absolutely continuous w.r.t. Lebesgue measure.*

Then $\hat{b}(s^m)$, where s^m is the median voter type, defeats all other policies by majority vote.

Proof. 1. By (A4), there is a unique s^m such that $F(\{s\le s^m\})=\frac{1}{2}$. The median voter is well-defined.
2. Let $b>\hat{b}(s^m):=b^m$. For ε small enough, $\hat{b}(s^m-\varepsilon)<b$, since continuity of $\hat{b}(s)$ follows from (A1) and (A2). By (A1), it follows that, for small ε, if $s\in[s^m-\varepsilon,s^m]$, then s prefers b^m to b. It also follows immediately from (A1) and (A3) that if $s\ge s^m$ then s prefers b^m to b. Therefore if $s\ge s^m-\varepsilon$, then s prefers b^m to b. This constitutes a majority vote for b^m against b.
3. In like manner, any $b<b^m$ is defeated by b^m. □

We next define:

Definition 2.2. $(v, \bar{B}, \bar{S})$ is said to have the *single crossing property*[2] (SCP) if, for any distinct pair $b^1, b^2 \in \bar{B}$, there is at most one type s such that $v(b^1, s) = v(b^2, s)$.

Theorem 2.1. *Let* (A1)-(A4) *and* SCP *hold, and let* $\underline{s} < s^m < \bar{s}$. *Then* (b^m, b^m) *is the unique electoral equilibrium for the voting game* $(v, \bar{B}, \bar{S}, F, \underline{s}, \bar{s})$.

Proof. 1. First, we note that (b^m, b^m) is an electoral equilibrium. For suppose $b \neq b^m$. Then $p(b, b^m) = 0$, by Lemma 2.1, and so $\pi(b, b^m, s) = v(b^m, s)$, for all s. Thus b^m is itself a best response to b^m, for all s.
2. We next show that no other electoral equilibrium exists. We first observe that if $(\underline{b}, \bar{b})$ is an equilibrium then it must be that $\bar{b} \leq b^m \leq \underline{b}$. Suppose, to the contrary, that both $\underline{b}$ and $\bar{b}$ were located on the same side of b^m - say, $\bar{b} < \underline{b} < b^m$. By (A1) and (A3), $\underline{b}$ defeats $\bar{b}$ (because for small ε, the strict majority coalition $\{s \leq s^m + \varepsilon\}$ all prefer $\underline{b}$). But since $\underline{s} < s^m$, $\underline{s}$ prefers $\underline{b} + \delta$ to $\underline{b}$, for small δ (since $\hat{b}(\underline{s}) > b^m$), and $\underline{b} + \delta$ still attracts a strict majority coalition against $\bar{b}$. Thus $\pi(\underline{b} + \delta, \bar{b}, \underline{s}) > \pi(\underline{b}, \bar{b}, \underline{s})$, and so $(\underline{b}, \bar{b})$ is not an electoral equilibrium.
3. Similar arguments dispose of the various boundary cases, and we finally must consider the possibility of an electoral equilibrium of the form $\bar{b} < b^m < \underline{b}$. If $(\underline{b}, \bar{b})$ is such an equilibrium, then we must have $p(\underline{b}, \bar{b}) = \frac{1}{2}$. For suppose not - say $p(\underline{b}, \bar{b}) = 0$. Then $\pi(\underline{b}, \bar{b}, \underline{s}) = v(\bar{b}, \underline{s})$; but $\pi(b^m, \bar{b}, \underline{s}) = v(b^m, \underline{s}) > v(\bar{b}, \underline{s})$ (since $\underline{s} < s^m$, and invoking (A1)), and so $\underline{b}$ is not a best response of $\underline{s}$ to $\bar{b}$. Therefore $p(\underline{b}, \bar{b}) = \frac{1}{2}$.
4. We note that $v(\bar{b}, \bar{s}) > v(\underline{b}, \bar{s})$ and $v(\underline{b}, \underline{s}) > v(\bar{b}, \underline{s})$. For suppose the first inequality failed. Then

$$v(b^m, \bar{s}) > v(\underline{b}, \bar{s}) \geq v(\bar{b}, \bar{s}) \tag{2.3}$$

and b^m beats $\underline{b}$, so $\bar{b}$ is not, in fact, a best response for $\bar{s}$ to $\underline{b}$.
5. Since some voters prefer $\underline{b}$ to $\bar{b}$ and some vice versa, there must be, by (A2), an s^* such that $v(\underline{b}, s^*) = v(\bar{b}, s^*)$. By SCP, s^* is unique. It follows that $s^* = s^m$. For suppose not, and say $s^* < s^m$. Then, by SCP, either $v(\underline{b}, s) > v(\bar{b}, s)$ for all $s > s^*$ or $v(\bar{b}, s) < v(\underline{b}, s)$ for all $s > s^*$. In the former case, $\varphi(\underline{b}, \bar{b}) > \frac{1}{2}$ and in the latter case, $\varphi(\bar{b}, \underline{b}) > \frac{1}{2}$: either case contradicts the fact that $p(\underline{b}, \bar{b}) = \frac{1}{2}$.
6. Recall that

$$\pi(\underline{b}, \bar{b}, \underline{s}) = \tfrac{1}{2} v(\underline{b}, \underline{s}) + \tfrac{1}{2} v(\bar{b}, \underline{s}) \ . \tag{2.4}$$

Consider, for small positive δ, the policy $\underline{b} - \delta$. (We know that $\underline{b} \neq b_{\min}$ since $\underline{b} > b^m$.) By (A1) and (A2), there exists s^{**} such that $v(\underline{b} - \delta, s^{**}) = v(\bar{b}, s^{**})$. By (A1), it follows that $v(\underline{b} - \delta, s^m) > v(\bar{b}, s^m)$ (draw a picture); it also follows from ¶ 4, and (A1) that $v(\underline{b} - \delta, \underline{s}) > v(\bar{b}, \underline{s})$ and therefore, by the SCP, it follows that

$$v(\underline{b} - \delta, s) > v(\bar{b}, s) \quad \text{for all} \quad s \leq s^m \ , \tag{2.5}$$

[2] v is, in applications, an indirect utility function, generated, for instance, by a direct utility function over resource allocations induced by tax rates 'b'. The present SCP is equivalent to the following (conventional) single crossing property on direct utility functions: that two indifference curves, one from each of two agents, intersect at at most one point.

and indeed, by continuity of v in s, (2.5) holds for all $s \leq s^m + \varepsilon$ for small ε. Hence a majority favor $\underline{b} - \delta$ against $\bar{b}$, and so $p(\underline{b} - \delta, \bar{b}) = 1$. Thus

$$\pi(\underline{b} - \delta, \bar{b}, \underline{s}) = v(\underline{b} - \delta, \underline{s}) \ . \tag{2.6}$$

7. Now invoke ¶ 4, which showed that $v(\underline{b}, \underline{s}) > v(\bar{b}, \underline{s})$. Since δ can be chosen arbitrarily small, it follows from (2.6) and (2.4) that

$$\pi(\underline{b} - \delta, \bar{b}, \underline{s}) > \pi(\underline{b}, \bar{b}, \underline{s}) \ ,$$

contradicting the assumption that $\underline{b}$ is a best response to $\bar{b}$ for $\underline{s}$. □

Theorem 2.1 establishes what was claimed in the introductory section, that even if political parties are not vote maximizers per se, but are sincere agents of voters with particular traits, then, under reasonable assumptions, the unique electoral equilibrium involves undifferentiated, median-voter politics expressed by both parties.

Although it may not be transparent from the proof, the essential reason that other equilibria do not exist in the set-up of Theorem 2.1 is that the best reply correspondence for a party is almost always empty. The intuition is this. Let the $\underline{s}$ party play some strategy $\underline{b}, \underline{b} \neq b^m$. The $\bar{s}$ party wants to respond by playing that strategy that is closest to $\hat{b}(\bar{s})$, yet wins at least 50% of the vote. But there generally is no such strategy. There generally is a sequence of strategies $\bar{b}(\varepsilon_i)$ that the $\bar{s}$ party would choose, where $\varepsilon_i \to 0$, and $\bar{b}(\varepsilon_i)$ beats $\underline{b}$ with a $(50 + \varepsilon_i)$% majority, and $v((\bar{b}(\varepsilon_i), \bar{s}))$ is increasing in i. And, surely, there is a limit policy $\bar{b}$ such that $\bar{b}(\varepsilon_i) \to \bar{b}$. The problem is that at $\bar{b}, \bar{s}$'s utility takes a saltus down, because at $\bar{b}$, which ties $\underline{b}, \underline{s}$ only gets expected utility $\frac{1}{2}v(\bar{b}, \bar{s}) + \frac{1}{2}v(\underline{b}, \bar{s})$. So there is, in fact, no best response by the $\bar{s}$ party to $\underline{b}$. This intuition is important for understanding the central result of section four.

3. Finite uncertainty

Uncertainty is 'finite' if the true distribution of the voter trait is known to be in the family $\{F^t\}_{t=1,\ldots,T}$, T finite, with F^t occurring with probability p_t. This is not an empirically realistic case, but I include it because political equilibrium has a very different character, in this case, from both the case of no uncertainty and the case of continuous uncertainty. In other words, modelling uncertainty as discrete gives one poor intuition for what happens in the more realistic case of continuous uncertainty. We have:

Theorem 3.1. *Let $(v, F^1, \ldots, F^T)$ satisfy (A1)–(A4) and* SCP; let $\underline{s} < m^1 < \ldots < m^T < \bar{s}$ *for $i = 1, \ldots, T$, where m^t is the median of F^t. Let $S = [s_1, s_2]$. $B = [b_1, b_2] \supset [\hat{b}(\bar{s}), \hat{b}(\underline{s})]$. Then either the unique equilibrium is $(\underline{b}, \bar{b}) = (\hat{b}(\underline{s}), \hat{b}(\bar{s}))$, or there is no equilibrium.*

Thus, the outcome in the case of certainty is sharply different from the case of finite uncertainty. If an equilibrium exists in the latter case, it consists in both parties playing the ideal policies of their constituents!

For notational simplicity, I will let $T = 2$ and $p_1 = p_2 = \frac{1}{2}$ in the proof of the following lemma and the theorem.

It may be helpful to refer to Fig. 1 during the rest of this section. In the figure, $f^t(s)$ is the density of F^t.

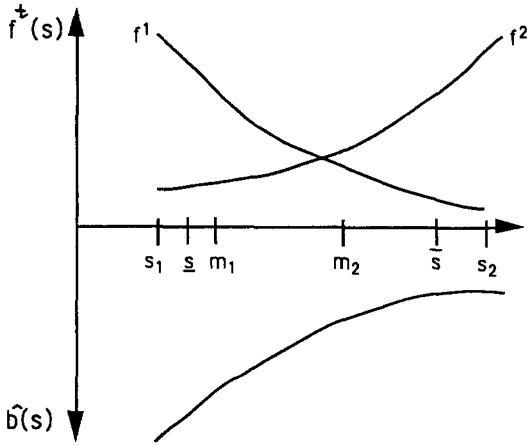

Fig. 1.

Lemma 3.1. *There is no strategy which is a best response to itself for both players.*

Proof. 1. We show no strategy in $[\hat{b}(m_2), \hat{b}(m_1))$ is a best response to itself for $\underline{s}$. Thus, assume $\bar{b} \in [\hat{b}(m_2), \hat{b}(m_1))$ is played by $\bar{s}$. $\underline{s}$ prefers $\hat{b}(m_1)$ to $\bar{b}$. If $\underline{s}$ plays $\hat{b}(m_1)$, he wins iff the state t equals 1; thus his expected utility is

$$\tfrac{1}{2}v(\hat{b}(m_1), \underline{s}) + \tfrac{1}{2}v(\bar{b}, \underline{s}) \ .$$

Thus $\bar{b}$ is not a best reply to itself for $\underline{s}$.
2. Let $\bar{s}$ play $\bar{b} \in [\hat{b}(s_2), \hat{b}(m_2))$. It is again easy to verify that $\hat{b}(m_1)$ is a better reply for $\underline{s}$ than $\bar{b}$ itself.
3. Thus no strategy in $[\hat{b}(s_2), \hat{b}(m_1))$ is a best reply to itself for $\underline{s}$.
4. In like manner, no strategy in $(\hat{b}(m_2), \hat{b}(s_1)]$ is a best reply to itself for $\bar{s}$.
5. It follows that there is no strategy which is a best reply to itself for both players. □

Proof of Theorem 3.1. 1. Let $(\underline{b}, \bar{b})$ be an equilibrium. By Lemma 3.1, $\underline{b} \neq \bar{b}$. Moreover, by SCP both of the next two equations cannot hold:

$$v(\underline{b}, \underline{s}) = v(\bar{b}, \underline{s}) \qquad \text{(i)}$$

$$v(\underline{b}, \bar{s}) = v(\bar{b}, \bar{s}) \ . \qquad \text{(ii)}$$

Let (i) not hold: then $v(\underline{b}, \underline{s}) > v(\bar{b}, \underline{s})$.
2. The probabilities of victory for $\underline{b}$ and $\bar{b}$ are given by $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{4}, \frac{1}{4})$, $(0, 1)$ or $(1, 0)$. Eg., if $\underline{b}$ wins when $t = 1$ and $\bar{b}$ wins when $t = 2$, the probabilities are $(\frac{1}{2}, \frac{1}{2})$; if $\underline{b}$ wins when $t = 1$ and the policies tie when $t = 2$, the probabilities are $(\frac{3}{4}, \frac{1}{4})$.
3. Suppose the probabilities of victory are $(\frac{1}{2}, \frac{1}{2})$. Then $\underline{b}$ wins in one state (say $t = 1$) and $\bar{b}$ in the other. If $\underline{b} \neq \hat{b}(\underline{s})$, then $\underline{s}$ could increase his utility by playing a strategy slightly closer to $\hat{b}(\underline{s})$ than $\underline{b}$ – this would not change the probabilities of victory.[3] In this case, it therefore follows that $\underline{b} = \hat{b}(\underline{s})$, and, in like manner, $\bar{b} = \hat{b}(\bar{s})$.
4. Suppose the probabilities of victory are $(\frac{1}{4}, \frac{3}{4})$. Then $\underline{b}$ loses in one state and ties in the other (say $t = 2$). Since $v(\underline{b}, \underline{s}) > v(\bar{b}, \underline{s})$, $\underline{s}$ could increase his utility by

[3] This step uses the fact that F^t are atomless.

a saltus by playing a strategy arbitrarily close to $\underline{b}$ that defeats $\underline{b}$ in state 2. That is, $\underline{s}$'s utility would jump from

$$\tfrac{1}{4}v(\underline{b},\underline{s})+\tfrac{3}{4}(\bar{b},\underline{s})$$

to

$$\tfrac{1}{2}v(\underline{b}-\varepsilon,\underline{s})+\tfrac{1}{2}v(\bar{b},\underline{s}) \ .$$

Hence, $(\frac{1}{4},\frac{3}{4})$ cannot hold at equilibrium.
5. The same argument shows that $(\frac{3}{4},\frac{1}{4})$ cannot support an equilibrium.
6. Suppose the probabilities of victory are $(0,1)$. It follows that $\bar{b}=\hat{b}(\bar{s})$; for otherwise, $\bar{s}$ could improve his payoff while stilling winning with probability 1. But if $\bar{b}=\hat{b}(\bar{s})$, $\underline{s}$ could play $\hat{b}(m_1)$, increasing his expected utility; so $\underline{b}$ is not a best reply to $\bar{b}$. Therefore $(0,1)$ cannot hold at equilibrium. Similarly, $(1,0)$ cannot hold at equilibrium.
7. To extend the argument to the case of $T>2$, one need only note that, at an equilibrium $(\underline{b},\bar{b})$, the two policies can tie in at most one state t: for $\underline{b}$ ties $\bar{b}$ in t iff $m(t)$ is indifferent between $\underline{b}$ and $\bar{b}$, and this can happen for at most one t by SCP, since $m(t')\neq m(t)$ for all $t\neq t'$. □

I next show by example that both possibilities referred to in Theorem 3.1 can occur.

Example 3.1. An equilibrium fails to exist.

Let $v(b,s)=-\frac{1}{2}b^2+\dfrac{b}{s}$, $S=[1,11]$, $\underline{s}=1$, $\bar{s}=11$, $B=[\frac{1}{11},1]$; let $T=2$, F^t have medians $m_1=\frac{3}{2}$, $m_2=10$, and the probability of each state $t=1,2$ is $\frac{1}{2}$. (We need not specify particular measures F^t.) The reader may verify that v satisfies A1–A3 and SCP.

Given two policies b_1 and b_2, with $b_1>b_2$,

$$\Omega(b_1,b_2)=\{s\,|\,v(b_1,s)>v(b_2,s)\}=\left\{s\,\middle|\,s<\frac{2}{b_1+b_2}\right\} \ .$$

Calculate that $\hat{b}(s)=\dfrac{1}{s}$. Thus we study the pair $(\underline{b},\bar{b})\equiv(\hat{b}(1),\hat{b}(11))=(1,\frac{1}{11})$, which is the only possible equilibrium.

We have

$$\Omega(1,\tfrac{1}{11})=\{s<\tfrac{11}{6}\} \ .$$

It follows that $\underline{b}$ beats $\bar{b}$ if $t=1$ (since $\frac{11}{6}>\frac{3}{2}$) and $\bar{b}$ beats $\underline{b}$ if $t=2$ (since $\frac{11}{6}<10$).

We now check whether $\underline{b}$ is a best response to $\bar{b}$. In state 1, $\underline{s}$ wins with $\underline{b}$, and $\underline{s}$ cannot possibly do better in this state. To increase his expected utility, he would have to play a policy that ties or defeats $\bar{b}$ in state 2. The policy b that ties $\bar{b}$ in state 2 is that policy such that agent '10' is indifferent between b and $\bar{b}$; i.e.:

$$v(b,10)=v(\bar{b},10)$$

or

$$-\tfrac{1}{2}b^2+\frac{b}{10}=-\tfrac{1}{2}(\tfrac{1}{11})^2+\frac{\frac{1}{11}}{10} \ ,$$

or

$$b = 0.228 \ .$$

By playing a policy very close to b, $\underline{s}$ can win in both states and his utility would be approximately $v(b, 1) = 0.225$. At $(\underline{b}, b)$, $\underline{s}$'s utility is

$$\tfrac{1}{2}v(\underline{b}, 1) + \tfrac{1}{2}v(\bar{b}, 1) = 0.295 \ .$$

Thus $\underline{b}$ is a best response to $\bar{b}$.

However, a similar calculation shows that $\bar{b}$ is not a best response to $\underline{b}$. $\bar{s}$'s utility at $(\underline{b}, \bar{b})$ is

$$\tfrac{1}{2}v(\underline{b}, 11) + \tfrac{1}{2}v(\bar{b}, 11) = -0.205 \ .$$

$\bar{s}$ can tie $\underline{s}$ in state 1 by playing $b = \frac{1}{3}$, and thus he can acquire utility of approximately $v(\frac{1}{3}, 11) = -0.02$ by playing a policy close to $\frac{1}{3}$. Hence, by Theorem 3.1, there is no equilibrium for this game.

Example 3.2. An example where $(\hat{b}(\underline{s}), \hat{b}(\bar{s}))$ is an equilibrium.

v, S, and B are as in example 3.1, but $m_1 = 1.05$; all other parameters are as in example 3.1. As before $(\underline{b}, \bar{b}) \equiv (\hat{b}(\underline{s}), \hat{b}(\bar{s})) = (1, \frac{1}{11})$. The argument of example 3.1 still shows that $\underline{b}$ is a best response to $\bar{b}$. We finally calculate that $\bar{b}$ is a best response to $\underline{b}$. To tie $\bar{b}$ in state 1, $\bar{s}$ has to play b such that

$$v(b, \tfrac{21}{20}) = v(1, \tfrac{21}{20}) \quad \text{or} \quad b = 0.91 \ .$$

So $\bar{s}$ can guarantee himself $v(0.91, 11) = -0.327$ by playing b. But his expected utility at $(\underline{b}, \bar{b})$ is -0.205. Therefore $(\underline{b}, \bar{b})$ is Nash.

4. Continuous uncertainty about the distribution F

We now suppose, more generally, that the true distribution of the voter traits over $\bar{S}$ is not known with certainty: there is a compact parameter set $\bar{T}$ in some R^n, and all parties and agents believe that the true distribution is given by $F(t)$, a probability measure on $\bar{S}$, where t is distributed according to a probability measure H on the sample space $\bar{T}$. (That is, for T a measurable subset of $\bar{T}$, the probability that the true distribution of the trait on $\bar{S}$ is given by $F(t)$, for some $t \in T$, is $H(T)$.) We may think of $\bar{T}$ as possible states of the world.

A voting game is now a tuple $(v, \bar{B}, \bar{S}, \bar{T}, H, F, \underline{s}, \bar{s})$, where F stands for a family of probability measures $\{F(t) \mid t \in \bar{T}\}$ on $\bar{S}$, and $v: \bar{B} \times \bar{S} \times \bar{T} \to R$. Why might v depend on t? Suppose that the economic problem that lies behind v is as follows: 's' represents income, $F(t)$ is an income distribution, 'b' represents the tax rate on income, and taxes are spent on a public good. Then the value of the public good depends on the income distribution, and so each voter's utility at 'b' depends on t.

We assume:

B1. $v(b, s, t)$ is continuous.

B2. H is absolutely continuous w.r.t. Lebesgue measure on $\bar{T}$.

We define the *expected utility of voter 's' at policy 'b'* as:

$$\int_T v(b, s, t) \, \mathrm{d}H \ , \tag{4.1}$$

and we define the coalition of voters whoe favor b^1 over b^2 as:

$$\Omega(b^1,b^2)=\left\{s\in\bar{S}\,\middle|\,\int_T v(b^1,s,t)\,\mathrm{dH}>\int_T v(b^2,s,t)\,\mathrm{dH}\right\}\ .$$

As in § 2, the set of voters who are indifferent between b^1 and b^2 is

$$\mathscr{E}(b^1,b^2)=\left\{s\in\bar{S}\,\middle|\,\int_T v(b^1,s,t)\,\mathrm{dH}=\int_T v(b^2,s,t)\,\mathrm{dH}\right\}\ .$$

We next define:

$$T'(b^1,b^2)=\{t\in\bar{T}\,|\,F(t)(\Omega(b^1,b^2))>\tfrac{1}{2}\}$$
$$E'(b^1,b^2)=\{t\in\bar{T}\,|\,F(t)(\Omega(b^1,b^2))=\tfrac{1}{2}\}\ .$$

$T'(b^1,b^2)$ is the set of states of the world t under which b^1 beats b^2 by majority vote. $E'(b^1,b^2)$ is the set of states under which b^1 and b^2 tie. We now define the payoff function $\pi(b,\bar{b},s)$ as:

$$\begin{aligned}\pi(b,\bar{b},s)=&\int_{T'(b,\bar{b})} v(b,s,t)\,\mathrm{dH}+\int_{T'(\bar{b},b)} v(\bar{b},s,t)\,\mathrm{dH}\\&+\int_{E'(b,\bar{b})}(\tfrac{1}{2}v(b,s,t)+\tfrac{1}{2}v(\bar{b},s,t))\,\mathrm{dH}\end{aligned}\tag{4.2}$$

The first integral on the r.h.s. of (4.2) is the expected utility of s over those t where b beats $\bar{b}$; the second integral is the expected utility for s for those t where $\bar{b}$ beats b; the third integral is the expected utility for s when b and $\bar{b}$ tie. Thus, the best reply correspondence of a party representing s to a proposal $\bar{b}$ by the opposition is:

$$\beta(\bar{b},s):=\{b\,|\,b\in\operatorname*{argmax}_b\,\pi(b,\bar{b},s)\}\ .$$

We next assume:

B3. *Regularity.*[4] For all $b^1\neq b^2$, $H(E'(b^1,b^2))=0$.

Regularity is a reasonable assumption if H is atomless (see B2), but it always fails if H has finite support. The case of Sect. 2, complete certainty, is, of course, a special case of H having finite support (namely, H puts all the mass on one point of $\bar{T}$), and the argument of the last paragraph of Sect. 2 capitalized on the fact that, for that case, there are many pairs (b^1,b^2) such that $H(E'(b^1,b^2))=1$.

Definition 4.1. An *electoral equilibrium* for the electoral game $(v,\bar{B},\bar{S},\bar{T},H,F,\underline{s},\bar{s})$ is a pair of policies $(\underline{b},\bar{b})$ such that $\underline{b}\in\beta(\bar{b},\underline{s})$ and $\bar{b}\in\beta(\underline{b},\bar{s})$.

We next assume:

B4. For any measurable set $S\subset\bar{S}$, $F(t)(S)$ is continuous in t.

Proposition 4.1. *If B1, B2, B3, and B4 hold then* $\pi(b,\bar{b},s)$ *is continuous in b and* $\bar{b}$, *for any s.*

Proof. 1. We shall prove continuity in b; continuity in $\bar{b}$ follows from a similar

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[4] In Sect. 6, we will deduce regularity from more primitive assumptions.

argument. By regularity, the third term in (4.2) is zero. The proof goes by showing that for b^1 near b^2, the sets $T'(b^1, \bar{b})$ and $T'(b^2, \bar{b})$ are very close, and so the payoffs $\pi(b^1, \bar{b}, s)$ and $\pi(b^2, \bar{b}, s)$ are close (see (4.2)).
2. Until step 6 below, we study the case where $b \neq \bar{b}$. Recall that $\Omega(b, \bar{b}) = \left\{ s \mid \int_T v(b,s,t)\,\mathrm{dH} > \int_T v(\bar{b},s,t)\,\mathrm{dH} \right\}$.
Hence, we may write $\Omega(b, \bar{b})$ as a disjoint union

$$\Omega(b,\bar{b}) = \sum_{n=0}^{\infty} S^n(b,\bar{b}) \tag{4.4}$$

where

$$S^n(b,\bar{b}) = \left\{ s \mid \frac{1}{n} \geq \int_T (v(b,s,t) - v(\bar{b},s,t))\,\mathrm{dH} > \frac{1}{n+1} \right\} .$$

By continuity of v and compactness of $\bar{T}$, for any integer N, we can choose δ_N so that $|b^1 - b^2| < \delta_N$ implies that

$$s \in \sum_{n=0}^{N} S^n(b^1, \bar{b}) \Rightarrow s \in \Omega(b^2, \bar{b}) .$$

Thus $s \in \Omega(b^1, \bar{b}) \setminus \Omega(b^2, \bar{b})$ implies $s \in \sum_{N+1}^{\infty} S^n(b^1, \bar{b})$. Similarly, we can choose ε_N so that $|b^1 - b^2| < \varepsilon_N$ implies that

$$s \in \sum_{n=0}^{N} S^n(b^2, \bar{b}) \Rightarrow s \in \Omega(b^1, \bar{b})$$

and so

$$s \in \Omega(b^2, \bar{b}) \setminus \Omega(b^1, \bar{b}) \quad \text{implies} \quad s \in \sum_{N+1}^{\infty} S^n(b^2, \bar{b}) .$$

Thus $s \in \Omega(b^1, \bar{b}) \Delta \Omega(b^2, \bar{b})$, for $|b^1 - b^2| < \min(\varepsilon_N, \delta_N)$, implies

$$s \in \sum_{N+1}^{\infty} S^n(b^1, \bar{b}) \cup \sum_{N+1}^{\infty} S^n(b^2, \bar{b}) := \hat{S}^N .$$

Now for any t, $F(t)$ is a measure on $\bar{S}$, and by σ-additivity of $F(t)$, it follows that $F(t)(\hat{S}^N)$ approaches zero as N gets large. This is seen by evaluating $F(t)(\Omega(b, \bar{b}))$ according to (4.4).
3. We next consider the function $g_N(t) = F(t)(\hat{S}^N)$. By B4, g_N is continuous on $\bar{T}$, and so achieves a maximum value c_N on $\bar{T}$; say $g_N(t^N) = c_N$. $\{c^N\}$ is a decreasing sequence, since

$$\begin{aligned} c_{N+1} &= g_{N+1}(t^{N+1}) = F(t^{N+1})(\hat{S}^{N+1}) \\ &\leq F(t^{N+1})(\hat{S}^N) \leq F(t^N)(\hat{S}^N) = c_N . \end{aligned}$$

I claim $\lim_{N\to\infty} c_N = 0$. Let t^* be the limit point of $\{t^N\}$. We have

$$\lim c_N = \lim_{N\to\infty} F(t^N)(\hat{S}^N) = \lim_{N\to\infty} F(t^*)(\hat{S}^N) ;$$

but by ¶ 2 we know $\lim_{N\to\infty} F(t^*)(\hat{S}^N)=0$, proving the claim.

4. Next, recall that $T'(b,\bar{b})=\{t \mid F(t)(\Omega(b,\bar{b}))>\frac{1}{2}\}$, and write $T'(b,\bar{b}) = \sum_{m=0}^{\infty} T^m(b,\bar{b})$ where

$$T^m(b,\bar{b})=\left\{t \,\Big|\, \tfrac{1}{2}+\frac{1}{m}\geq F(t)(\Omega(b,\bar{b}))>\tfrac{1}{2}+\frac{1}{m+1}\right\} .$$

We have shown in ¶ 2 that if b^1 and b^2 are sufficiently close, then $\Omega(b^1,\bar{b})$ and $\Omega(b^2,\bar{b})$ differ by a set contained in $\hat{S}^N$, and in ¶ 3 we have shown that $F(t)(\hat{S}^N)$ approaches zero uniformly in t (i.e., for any $\varepsilon>0$ there is an N such that $\forall\, t\in\bar{T}$, $F(t)(\hat{S}^N)<\varepsilon$). We now apply the technique of ¶ 2 again. Given any integer M, by ¶ 3, we can choose δ_M so that

$$|b^1-b^2|<\delta_M \Rightarrow |F(t)(\Omega(b^1,\bar{b}))-F(t)(\Omega(b^2,\bar{b}))|<\frac{1}{M+1} ,$$

for all t. Now $t\in \sum_{m=0}^{M} T^m(b^1,\bar{b}) \Rightarrow F(t)(\Omega(b^1,\bar{b}))>\frac{1}{2}+\frac{1}{M+1}$ and so

$$t\in \sum_{m=0}^{M} T^m(b^1,\bar{b}) \Rightarrow F(t)(\Omega(b^2,\bar{b}))>\tfrac{1}{2} \Rightarrow t\in T'(b^2,\bar{b}) .$$

Thus $t\in T'(b^1,\bar{b})\backslash T'(b^2,\bar{b}) \Rightarrow t\in \sum_{M+1}^{\infty} T^m(b^1,\bar{b})$. In like manner, we can choose ε_M so that if $|b^1-b^2|<\varepsilon_M$, then

$$t\in T'(b^2,\bar{b})\backslash T'(b^1,\bar{b}) \Rightarrow t\in \sum_{M+1}^{\infty} T^m(b^2,\bar{b}) .$$

Thus if $|b^1-b^2|<\min(\delta_M,\varepsilon_M)$ and $t\in T'(b^1,\bar{b})\Delta T'(b^2,\bar{b})$, then $t\in\hat{T}^M$, where

$$\hat{T}^M:=\sum_{M+1}^{\infty} T^m(b^1,\bar{b})\cup \sum_{M+1}^{\infty} T^m(b^2,\bar{b}) .$$

By σ-additivity of H, we have $\lim_{M\to\infty} H(\hat{T}^M)=0$.

5. Now let γ be an upper bound for $v(b,s,t)$, for given s, on $\bar{B}\times\{s\}\times\bar{T}$. Let $\varepsilon>0$ be given. By ¶ 4, we can choose δ so that

$$|b^1-b^2|<\delta \Rightarrow H(T'(b^1,\bar{b})\Delta T'(b^2,\bar{b}))<\frac{\varepsilon}{3\gamma+1} \quad \text{and}$$

$$H(T'(\bar{b},b^1)\Delta T'(\bar{b},b^2))<\frac{\varepsilon}{3\gamma+1} .$$

We can also choose δ' so that $|b^1-b^2|<\delta'$ implies

$$\operatorname*{Max}_t |v(b^1,s,t)-v(b^2,s,t)|<\frac{\varepsilon}{3\gamma+1}$$

by the continuity of v and compactness of $\bar{B}\times\bar{T}$. We can then write, for $|b^1-b^2| < \min(\delta,\delta')$, from (4.2)*:

* This step assumes that v is non-negative. But this is no restriction as v is bounded on its compact domain, being continuous, and is a von Neumann-Morgenstern utility. Therefore, w.l.o.g., we may choose v to be non-negative.

$$|\pi(b^1,\bar{b},s)-\pi(b^2,\bar{b},s)|$$

$$\leq \left| \int_{T'(b^1,\bar{b})} v(b^1,s,t)\,\mathrm{dH} - \int_{T'(b^2,\bar{b})} v(b^2,s,t)\,\mathrm{dH} + \int_{T'(\bar{b},b^1)\Delta T'(\bar{b},b^2)} v(\bar{b},s,t)\,\mathrm{dH} \right|$$

$$\leq \left| \int_{T'(b^1,\bar{b})\cap T'(b^2,\bar{b})} (v(b^1,s,t)-v(b^2,s,t))\,\mathrm{dH} \right.$$

$$\left. + \int_{T'(b^1,\bar{b})\Delta T'(b^2,\bar{b})} v(b^1,s,t)\,\mathrm{dH} + \frac{\gamma\varepsilon}{3\gamma+1} \right|$$

$$\leq \frac{\varepsilon}{3\gamma+1} H(T'(b^1,\bar{b})\cap T'(b^2,\bar{b})) + \frac{2\gamma\varepsilon}{3\gamma+1}$$

$$\leq \frac{\varepsilon(2\gamma+1)}{3\gamma+1} < \varepsilon .$$

This proves the continuity of $\pi(b,\bar{b},s)$ in b when $b \neq \bar{b}$.
6. Finally, we prove continuity of π at the point $b=\bar{b}$. Note that $E'(\bar{b},\bar{b})=\bar{T}$, so $\pi(\bar{b},\bar{b},s)=\int_{\bar{T}} v(\bar{b},s,t)\,\mathrm{dH}$. Now

$$\lim_{b\to\bar{b}} \pi(b,\bar{b},s) = \lim_{b\to\bar{b}} \left(\int_{T'(b,\bar{b})} v(b,s,t)\,\mathrm{dH} + \int_{T'(\bar{b},b)} v(\bar{b},s,t)\,\mathrm{dH} \right)$$

$$= \int_{\bar{T}} v(\bar{b},s,t)\,\mathrm{dH} ,$$

since $v(b,s,t)$ approaches $v(\bar{b},s,t)$ uniformly in t as b approaches $\bar{b}$. Hence $\pi(b,\bar{b},s)$ is continuous at $b=\bar{b}$. □

Remark. In Proposition 4.1, $\bar{B}$ can be a compact set of any dimension.

As a consequence, we have:

Corollary 4.1. *Assume B1–B4 for the voting game $(v,\bar{B},\bar{S},\bar{T},H,F,\underline{s},\bar{s})$. Suppose, for each $\bar{b}$ and $s\in\{\underline{s},\bar{s}\}$ the function $\pi(b,\bar{b},s)$ achieves a maximum on $\bar{B}$ at at most one policy b. Then an electoral equilibrium exists.*

Proof. Define the best reply correspondence $\hat{\beta}: \bar{B}\times\bar{B}\to\bar{B}\times\bar{B}$ by

$$\hat{\beta}(\bar{b},\underline{b}) = (\beta(\underline{b},\bar{s}), \beta(\bar{b},\underline{s})) .$$

By Proposition 4.1, $\pi(\cdot,\bar{b},s)$ is continuous on the compact set $\bar{B}$, and so achieves a maximum. The best reply correspondence is therefore non-empty everywhere. By the theorem's premise, the best reply correspondence is therefore single-valued. It also follows from Proposition 4.1 that the best reply function is a continuous map of $\bar{B}\times\bar{B}$ into itself, and so by Brouwer's fixed point theorem, an electoral equilibrium exists. □

Remark. Corollary 4.1 may not be very useful, because the premise of a single-valued best-reply correspondence may not often hold in applications. I have no proof of existence of equilibrium in the general case of this article, although Roemer (1993) proves existence for a special class of cases. The existence proofs of analogous models in the previous literature, in Wittman (1983, 1990) and Hansson and Stuart (1984), are incorrect, as I show in the just mentioned paper. Proposition 4.1 at least establishes that in the case of continuous uncertainty the payoff functions are continuous, and so the best-reply correspondence is everywhere defined, which is not true in the case of finite uncertainty of Sect. 3.

A comment comparing Theorem 2.1 and Corollary 4.1 is in order. As pointed out in the last paragraph of §2, the best-reply correspondence is almost always empty in the voting games of §2, but assumptions on v guaranteeing 'single-peakedness' imply that b^m is a best reply to itself – so despite the generic emptiness of the best-reply correspondence, a single Nash equilibrium, (b^m, b^m), does exist. In contrast, in the model of this section, single-peakedness is not assumed. Even if it were, there is no clearly defined median voter, as the distribution of voters is uncertain. Regularity, however, guarantees that the best-reply correspondence is everywhere non-empty.[5]

There is no reason to expect the equilibrium policies of the two parties to be the same with continuous uncertainty. Rather than attempt a genericity result to this effect here, I refer the reader to Roemer (1993, Theorem 4.1), which proves, in a sub-class of models, that equilibria always consist in different policies, and each party always wins with positive probability, as long as $\underline{s}$ and $\bar{s}$ are sufficiently far apart. The next section will illustrate that in general equilibria in the continuous case consist of parties playing different policies.

5. Economic-political equilibrium: an example

In this section, I present a model of an economy where parties propose different tax policies, and I calculate how the equilibrium pair of policies depends upon the degree of uncertainty concerning the distribution of voter traits.

There is a population of agents with different skill levels s, where s is distributed in the interval $\bar{S}=[c, \infty)$, and c is a positive constant. There is a family of probability distributions, $\{F(t), t \in [0, \bar{T}]\}$, on $\bar{S}$. There is a probability measure H on $\bar{T}:=[0, \bar{T}]$ (forgive the notational abuse). Thus, if $T \subset \bar{T}$ is a measurable subset, then everyone believes that the correct distribution of skills on $\bar{S}$ is given by $F(t)$ for some $t \in T$, with probability $H(T)$.

All agents have the same utility function for income and labor, $u(y, L)$, where y is income and L is labor performed. The government will levy a tax of rate b on income, for some $b \in [0,1]:=\bar{B}$. Taxes are used to purchase a public good which is valued like income by all consumers. Thus, if consumer i earns y^i income with labor L^i and per capita taxes are I, the value of the public good consumed by the agent, then his utility (at tax rate b) will be $u((1-b)y^i+I, L^i)$.[6]

Skill is defined as follows: an agent of type s has a personal production function $y=sL$. It is assumed that there is no restriction on the amount of labor an individual can perform.

We proceed to define an economic equilibrium for the *economic model* $(u, \bar{B}, \bar{S}, \bar{T}, H, F)$.

Definition 5.1. An *economic equilibrium at tax rate b* for the model $(u, \bar{S}, \bar{B}, \bar{T}, H, F)$ is a function $\bar{L}: \bar{S} \to \mathscr{R}_+$, and a function $I: \bar{T} \to \mathscr{R}_+$ such that:

[5] In fact, non-emptiness of the best reply correspondence follows from B1–B3 alone. I have chosen not to prove this proposition here in order to save space, and have instead deduced non-emptiness from the continuity of the pay-off function which requires B4 as well.

[6] Alternatively, taxes are spent to provide each citizen with a private good, such as health services, in equal amounts.

(i) for all $s \in \bar{S}$, $\bar{L}(s)$ solves

$$\operatorname*{Max}_{L} \int_{\bar{T}} u((1-b)sL+I(t),L)\,\mathrm{d}H$$

(ii) For all $t \in \bar{T}, I(t) = \int_{\bar{S}} bs\bar{L}(s)\,\mathrm{d}F(t)$.

Statement (i) says: if the agents believe that $I(t)$ will be the value of the public good when the skill distribution is given by $F(t)$, then they will supply labor in amount $\bar{L}(s)$. Statement (ii) says: if agents supply labor $\bar{L}(s)$, then the value of the public good when the skill distribution is $F(t)$ will indeed be $I(t)$.

Suppose an equilibrium exists and is unique for each b: call it $\{\bar{L}(s,b), I(t,b)\}$. Then we define the *indirect utility function*

$$v(b,s,t) = u((1-b)\bar{L}(s,b) + I(t,b), \bar{L}(s,b)) \; . \tag{5.1}$$

The function $v(b,s,t)$ enables us to define a *voting game* $(v, \bar{B}, \bar{S}, \bar{T}, H, F, \underline{s}, \bar{s})$ associated with the economic model $(u, \bar{B}, \bar{S}, \bar{T}, H, F)$. Of course, $\underline{s}$ and $\bar{s}$ are given in $\bar{S}$. We are interested in studying what kind of politics are associated with given economies $(u, \bar{B}, \bar{S}, \bar{T}, H, F)$, and given political parties $\underline{s}$ and $\bar{s}$. To this end, we study the electoral equilibrium of the associated voting game to an economic model.[7]

We now specialize to an example for computational purposes.

C1. $u(y,L) = y - \frac{d}{2}L^2$ (quasilinear utility with quadratic cost)

C2. $F(t)$ is the Pareto distribution with density function

$$f(s,t) = \frac{t+3}{c}\left(\frac{c}{s}\right)^{t+4} \quad \text{on} \quad \bar{S} = [c, \infty) \; .$$

The given Pareto distribution has mean $\mu(t) = \frac{(t+3)c}{t+2}$ and median $m(t) = 2^{\left(\frac{1}{t+3}\right)} c$. It is also convenient to calculate the second moment, which I call $\tilde{\mu}(t)$:

$$\tilde{\mu}(t) := \int_{c}^{\infty} s^2 f(s,t)\,\mathrm{d}s = \frac{t+3}{t+1}c^2 \; . \tag{5.2}$$

Thus as t increases, skill becomes more concentrated near its minimum value c; higher t means a less egalitarian distribution of skill.

Lemma 5.1. *Assume C1. For any I and b, the solution of*

$$\operatorname*{Max}_{L} u((1-b)sL+I,L)$$

is given by $L(s,b) = \frac{(1-b)s}{d}$.

[7] This paper does not propose an endogenous theory of the emergence of parties, which is, of course, an important question.

Proof. Easy computation.

Quasi-linearity assures us, as the lemma says, that labor supply is independent of the value of total taxes. It is this feature that simplifies enormously the calculation of economic equilibrium.

Define

$$I(t)=\int_{\bar{S}} sL(s,b)\,\mathrm{dF}(t)=\left(\frac{1-b}{d}\right)\tilde{\mu}(t)$$

and

$$\int_{\bar{T}} I(t)\,\mathrm{dH}=\frac{(1-b)}{\mathrm{d}}\int_{\bar{T}}\tilde{\mu}(t)\,\mathrm{dH}\ .$$

We have:

Proposition 5.1. *Assume C1, C2. Then for each* $b\in[0,1]$ *there is a unique economic equilibrium for the model* $(u,\bar{B},\bar{S},\bar{T},F,H)$ *given by:*

$$\bar{L}(s,b)=\frac{(1-b)s}{d}\ ,\qquad I(t)=\left(\frac{1-b}{d}\right)\tilde{\mu}(t)\ .$$

Proof. This follows immediately from Lemma 5.1 and the above. □

Assume now:

C3. H is the uniform measure on $[0,\bar{T}]$.

Under C1, C2 and C3, we can compute

$$\int_{\bar{T}}\tilde{\mu}(t)\,\mathrm{dH}=c^2\left[1+2\frac{\log(1+\bar{T})}{\bar{T}}\right]:=\tilde{M}(\bar{T},c)\ . \tag{5.3}$$

From (5.1), and Proposition 5.1 we have the indirect utility is given by:

$$\begin{aligned} v(b,s,t)&=(1-b)sL(s)+bI(t)-\frac{d}{2}(L(s))^2\\ &=\frac{(1-b)^2s^2}{2d}+\frac{b(1-b)}{d}\tilde{\mu}(t)\ , \end{aligned} \tag{5.4}$$

and so expected utility for voter s at tax rate b is given by

$$\begin{aligned} \tilde{v}(b,s)&=\int_{\bar{T}} v(b,s,t)\,\mathrm{dH}\\ &=\frac{(1-b)^2s^2}{2d}+\frac{b(1-b)}{d}\tilde{M}(\bar{T},c)\ . \end{aligned} \tag{5.5}$$

Now the set of voters who prefer policy b^1 to policy b^2 is given by (see § 2):

$$\Omega(b^1,b^2)=\{s\,|\,\tilde{v}(b^1,s)>\tilde{v}(b^2,s)\}\ . \tag{5.6}$$

From (5.5) and (5.6), we can compute that

$$b^1<b^2\quad\text{implies}\quad\Omega(b^1,b^2)=\{s\,|\,s^2>2\tilde{M}(\bar{T},c)\,\Psi(b^1,b^2)\} \tag{5.7}$$

where

$$\Psi(b^1,b^2) := \frac{b^2(1-b^2)-b^1(1-b^1)}{(1-b^1)^2-(1-b^2)^2} \; . \tag{5.8}$$

We can also compute from (5.5) that the favorite policy of voter s is:

$$\hat{b}(s) := \begin{cases} \dfrac{\tilde{M}(\bar{T},c)-s^2}{2\tilde{M}(\bar{T},c)-s^2} & \text{for} \quad s^2<\tilde{M}(\bar{T},c) \\ 0 & \text{for} \quad s^2 \geq \tilde{M}(\bar{T},c) \; . \end{cases} \tag{5.9}$$

We must compute $T'(b^1,b^2)$, the subset of $\bar{T}$ for which $\Omega(b^1,b^2)$ is a majority coalition. According to (5.7), $\Omega(b^1,b^2)$ comprises exactly one-half the population when $F(t)$ is such that the median of $F(t)$ is s such that

$$s^2 = 2\tilde{M}(\bar{T},c)\,\Psi(b^1,b^2) \; . \tag{5.10}$$

Call this value of t (if there is one) $\bar{t}(b^1,b^2)$. Since $m(t)=2^{\left(\frac{1}{t+3}\right)}c$, we have $\sqrt{2\tilde{M}\Psi}=c\,2^{\frac{1}{\bar{t}+3}}$, or

$$\bar{t}(b^1,b^2) = \frac{2\log 2}{\log \dfrac{2\tilde{M}\Psi(b^1,b^2)}{c^2}} - 3 \; , \tag{5.11}$$

if the value on the r.h.s. of (5.11) is in $[0,\bar{T}]$. Under $F(\bar{t})$, $\Omega(b^1,b^2)$ has measure $\frac{1}{2}$. As t falls, the median of $F(t)$ increases. Thus for $t<\bar{t}(b^1,b^2)$, $F(t)(\Omega(b^1,b^2)>\frac{1}{2}$. Hence, the probability that b^1 defeats b^2 in an election is given by $H([0,\hat{t}(b^1,b^2)])$, where we take

$$\hat{t}(b^1,b^2) = \begin{cases} 0 & \text{if} \quad \bar{t}(b^1,b^2)\leq 0 \\ \bar{t}(b^1,b^2) & \text{if} \quad \bar{t}(b^1,b^2)\in[0,\bar{T}] \\ \bar{T} & \text{if} \quad \bar{t}(b^1,b^2)>\bar{T} \; . \end{cases} \tag{5.12}$$

We can verify that regularity holds given our assumptions C1–C3, and finally we can write down the payoff function in the electoral game induced by the economic model for a party representing $\bar{s}$ (see (5.4) and (5.2)) as:

$$\begin{aligned} \pi(b,\underline{b},\bar{s}) = {} & \frac{(1-b)^2\bar{s}^2}{2d}\,\frac{\hat{t}(b,\underline{b})}{\bar{T}} + \int_0^{\hat{t}} \frac{b(1-b)}{d}\,\tilde{\mu}(t)\,\frac{1}{\bar{T}}\,\mathrm{dt} \\ & + \frac{(1-\bar{b})^2\bar{s}^2}{2d}\left(\frac{\bar{T}-\hat{t}(b,\underline{b})}{\bar{T}}\right) \\ & + \int_{\hat{t}}^{\bar{T}} \frac{\underline{b}(1-\underline{b})}{d}\,\tilde{\mu}(t)\,\frac{1}{\bar{T}}\,\mathrm{dt} \; , \qquad \text{for } b<\underline{b} \; , \end{aligned}$$

with an analogous expression for $b>\underline{b}$. There is an analogous payoff function for $\underline{s}$.

We can now search for an interior electoral equilibrium, when the two parties represent $\underline{s}$ and $\bar{s}$, by looking for a solution $(\underline{b},\bar{b})\in[0.1]^2$ to the following two equations:

Table 1

$\bar{T}$	$\bar{b}$	$\underline{b}$	$\frac{t(\underline{b},\bar{b})}{\bar{T}}$	$\hat{b}(\bar{s})$	$\hat{b}(\underline{s})$	$\hat{b}(E(m(t)))$
0.001	0.3200	0.3201	N.A.	0	0.400	0.3201
0.01	0.3193	0.3197	0.49	0	0.400	0.3200
0.3	0.2974	0.3089	0.48	0	0.389	0.303
0.5	0.2839	0.3031	0.49	0	0.382	0.293
1	0.2547	0.2922	0.49	0	0.367	0.273
2	0.2102	0.2786	0.48	0	0.344	0.265
4	0.1545	0.2622	0.47	0	0.308	0.204
6	0.1225	0.2496	0.46	0	0.282	0.178
9	0.0959	0.2326	0.46	0	0.253	0.152
20	0.0646	0.1842	0.47	0	0.1892	0.103
30	0.0558	0.1551	0.47	0	0.1570	0.082

$$\begin{aligned} &D_1\pi(\bar{b},\underline{b},\bar{s})=0 \\ &D_2\pi(\bar{b},\underline{b},\underline{s})=0 \ , \end{aligned} \tag{5.13}$$

where D_i stands for the derivative of π w.r.t. its i^{th} co-ordinate.

Equations (5.13) were solved by computer, for the model with the following parameters: $c=20$, $d=5$, $\underline{s}=20$, $\bar{s}=1{,}000$ and for various values of $\bar{T}$. Note $\underline{s}$ is the least skilled type, and $\bar{s}$ is in the top 1% of skill types for all the Pareto measures $\{F(t), t\in\bar{T}\}$ on $\bar{S}$, for $\bar{T}$ taking on the values in Table 1. The results are presented in Table 1. Recall that the parameter t is distributed according to Lebesgue measure on $\bar{T}=[0,\bar{T}]$, so when $\bar{T}$ is small, there is very little uncertainty concerning the distribution of skills. The fourth column in Table 1 is the probability that the $\bar{s}$-party wins the election at the electoral equilibrium: as remarked in § 1, it differs from one-half. Recall $\hat{b}(\underline{s})$ and $\hat{b}(\bar{s})$ are the favorite policies of the two types. The last column computes the favorite policy of the 'expected median voter', where

$$E(m(t)) := \int_{\bar{T}} m(t)\,\mathrm{dH} \ ,$$

and $m(t)$ is the median voter under $F(t)$.

Note that when uncertainty is very small ($\bar{T}=0.001$), the electoral equilibrium is approximaately at the favorite policy of the median voter. (In this case $E(m(t))$ is virtually the median voter under the measure $F(0)$.) But for large $\bar{T}$, clearly neither party plays the ideal policy of the expected median voter.

6. Convergence of equilibrium policies as uncertainty becomes small

In this section, we study the convergence of equilibria as uncertainty varies. To this end, we fix an interval $T:=[0,\bar{T}]$, a family $\{v(\cdot,\cdot,t)\}$, compact[8] intervals $\bar{B}=[b_1,b_2]$, $\bar{S}=[s_1,s_2]$, voters (or parties) $\underline{s}$ and $\bar{s}$, and a family $\{F(t), t\in\bar{T}\}$ of measures on $\bar{s}$. We shall consider sequences of measures $\{H^i\}$ on $\bar{T}$; each H^i defines a voting game $\mathscr{G}^i=(v,\bar{B},\bar{S},\bar{T},H^i,F,\underline{s},\bar{s})$, where uncertainty of the distribution of voters is described by H^i. In particular, recall from § 5 that

[8] Compactness of $\bar{S}$ can be dispensed with.

$\bar{t}^i(b^1, b^2)$ is that value of t, if there is one, such that the median voter of $F(t)$ is indifferent between b^1 and b^2, i.e.,

$$\int_{\bar{T}} v(b^1, m(\bar{t}^i), t)\, \mathrm{d}H^i = \int_{\bar{T}} v(b^2, m(\bar{t}^i), t)\, \mathrm{d}H^i \ .$$

The expected utility of voter s in game $\mathscr{E}^i$ at b is

$$\tilde{v}^i(b, s) := \int_{\bar{T}} v(b, s, t)\, \mathrm{d}H^i \ .$$

Definition 6.1. We say (v, B, S, T, F, H) has the *single crossing property* (SCP*) if $(\tilde{v}, B, S)$ has the SCP (see Definition 2.2).

The favorite policy of voter s in game $\mathscr{E}^i$ is

$$\hat{b}^i(s) = \operatorname*{argmax}_b \int_{\bar{T}} v(b, s, t)\, \mathrm{d}H^i \ .$$

We shall assume:

D.1. $v(b, s, t)$ is C^1 in b and s, concave in b, and for any s, bounded on $\bar{B} \times \{s\} \times \bar{T}$. The derivative v_{12} exists, is continuous, and has the same sign for all (b, s, t). A3 holds for each t. SCP holds for $v(\cdot, \cdot, t)$, for all t.
D.2. H^i is absolutely continous w.r.t. Lebesgue measure, and its density h^i is continuous of $\bar{T}$.
D.3. $F(t)$ is equivalent to Lebesgue measure on $\bar{S}$. The median function, $m(t)$, is C^1 and $m'(t) < 0$ on $\bar{T}$. The density functions $f(t, s)$ of $F(t)$ are continuous and strictly positive on $\bar{T} \times \bar{S}$, and are differentiable in t.

Lemma 6.1. *Under D3,* $\int_{s_1}^{m(t)} f_1(t, s)\, \mathrm{d}s$ *is bounded away from*

$$0: \quad (\exists\, \varepsilon > 0)(\forall\, t) \left(\int_{s_1}^{m(T)} f_1(t, s)\, \mathrm{d}s > \varepsilon \right) .$$

Proof. By D3, it follows that $m'(t)$ and $f(t, s)$ are bounded away from zero on $\bar{T}$ and $\bar{T} \times \bar{S}$, respectively. By definition of $m(t)$:

$$\int_{s_1}^{m(t)} f(t, s)\, \mathrm{d}s = \tfrac{1}{2} \ .$$

Differentiating gives:

$$m'(t) = \frac{-\int_{s_1}^{m(t)} f_1(t, s)\, \mathrm{d}s}{f(t, m(t))} \ . \tag{6.1}$$

Since $m'(t)$ and $f(t, m(t))$ are bounded away from zero, the lemma follows. □

Lemma 6.2. *Assume D1, D3, and* SCP*. *Let* $\mathscr{E} = (v, \bar{B}, \tilde{S}, \bar{T}, H, F, \underline{s}, \bar{s})$ *be a voting game with an equilibrium* $(\underline{b}, \bar{b})$, $\underline{b} \neq \bar{b}$, *where both parties have a positive probability of victory. Then the function* $\bar{t}(b^1, b^2)$ *is continuously differentiable in a neighborhood of* $(\underline{b}, \bar{b})$.

Proof. 1. Since some voters prefer $\underline{b}$ and some prefer $\bar{b}$, there is a unique voter $\tilde{s}(\underline{b},\bar{b})$ who is indifferent between $\underline{b}$ and $\bar{b}$ (by SCP*). By definition $\bar{t}(\underline{b},\bar{b})$ is that value of t such that $m(t)=\tilde{s}(\underline{b},\bar{b})$. Such a t exists and is interior in $\bar{T}$ because both parties have a positive probability of victory. By definition:

$$\int_{s_1}^{\tilde{s}(\underline{b},\bar{b})} f(\bar{t}(\underline{b},\bar{b}),s)\,\mathrm{d}s=\tfrac{1}{2}\ , \tag{6.2}$$

and (6.2) holds as we vary b (or $\bar{b}$) slightly.
2. If the first derivative $\bar{t}_1(\underline{b},\bar{b})$ exists, then by implicit differentiation of (6.2), we have

$$\bar{t}_1(\underline{b},\bar{b})=-f(\bar{t},\tilde{s})\,\frac{\partial\tilde{s}}{\partial b}\Big/\int_{s_1}^{\tilde{s}} f_1(\bar{t},s)\,\mathrm{d}s\ , \tag{6.3}$$

where $\tilde{s}(b,\bar{b})$ is defined by

$$\tilde{v}(b,\tilde{s}(b,\bar{b}))-\tilde{v}(\bar{b},\tilde{s}(b,\bar{b}))=0\ , \tag{6.4}$$

and so, by implicit differentiation,

$$\frac{\partial\tilde{s}}{\partial b}=\frac{\tilde{v}_1(\underline{b},\tilde{s})}{\tilde{v}_2(\bar{b},\tilde{s})-\tilde{v}_2(\underline{b},\tilde{s})}\ . \tag{6.5}$$

By D1, $\tilde{v}_{21}$ inherits the common (non-zero) sign of v_{21}, and so the denominator of (6.5) is non-zero (noting $\underline{b}\neq\bar{b}$). By the implicit function theorem, $\tilde{s}$ is therefore indeed C^1. Indeed, by (6.3), $\bar{t}_1$ will have been shown to be C^1 by the implicit function theorem, if the denominator on the r.h.s. of (6.3) is non-zero. But this follows from Lemma 6.1. □

Lemma 6.3. *Assume D3 and* SCP*. *Then H is regular (see B3).*

Proof. Given $b^1\neq b^2$; recall $E'(b^1,b^2)=\{t\in\bar{T}\,|\,F(t)(\Omega(b^1,b^2))=\tfrac{1}{2}\}$. Suppose $E'(b^1,b^2)$ is not empty. By SCP*, $\Omega(b^1,b^2)$ must be of the form $[s_1,\tilde{s})$ or $(\tilde{s},s_2]$, and $t\in E'$ iff $m(t)=\tilde{s}$. Since $m'(t)<0$, it follows that E' consists of only one element, and so $H(E'(b^1,b^2))=0$ (H is atomless). □

The following condition postulates a sequence of measure H^i which 'converge to certainty'.

Definition 6.2. We say that a sequence of absolutely continuous (w.r.t. Lebesgue measure) measures $\{H^i\}$ on $\bar{T}$ *converges to certainty at* t^* (has property CC at T^*) if h^i are quasiconcave and $(\forall\varepsilon,\delta>0)\ (\exists M)\ (i>M\Rightarrow H^i(N_\delta(t^*))>1-\varepsilon)$ where $N_\delta(t^*)$ is a δ-neighborhood of t^* in $\bar{T}$.

In our application we think of the games $\mathscr{E}^i$ converging to a game where the distribution of voters is $F(t^*)$ for sure.

Lemma 6.4. *Let* $\{H^i\}$ *have property* CC *at* t^* *and suppose there exist positive numbers* δ_1 *and* δ_2 *and a sequence* $\{t_i\}$ *such that:*

$$\forall i\ \delta_1<\int_{t_i}^{\bar{T}} h^i(t)\,\mathrm{d}t<1-\delta_2\ . \tag{+}$$

Then $h^i(t^i)\to\infty$.

Proof. 1. Let $T_i \subset \bar{T}$ be the set of points at which h^i achieves its maximum value. By quasiconcavity, T_i is either a point or an interval. Generally, $T_i = [r_i, r_i']$, where possibly $r_i' = r_i$. $\{t_i\}$ can be written as the union of two subsequences $\{t_i^1\}$ and $\{t_i^2\}$, where, for all i, $t_i^1 \leq r_i'$ and $t_i^2 > r_i'$.

Possibly one of these subsequences is finite or empty. We show that if $\{t_i^j\}_i$ for $j = 1, 2$ is infinite then $h^i(t_i^j) \to \infty$.

2. Suppose $\{t_i^1\}$ is infinite, and suppose that $h_i(t_i^1) \leq K$ for all i, some number K. By quasiconcavity, h^i is monotone increasing on $[0, t_i^1]$. Hence h^i is bounded by K on $[0, t_i^1]$. By CC, for any ε, $\delta > 0$, we can choose i sufficiently large that:

$$\int_0^{t_i^1} h^i(t)\,\mathrm{d}t < K\delta + \varepsilon \ .$$

Now choose ε and δ such that $\varepsilon + K\delta < \delta_2$. It follows that

$$\int_{t_i}^{\bar{T}} h^i(t)\,\mathrm{d}t > 1 - (\varepsilon + K\delta) > 1 - \delta_2 \ ,$$

contradicting hypothesis $(+)$. It follows, by contradiction, that $h^i(t_i^1) \to \infty$.

3. Suppose that $\{t_i^2\}$ is infinite, and suppose that $\{t_i^2\} \leq K$, for all i, some number K. By quasiconcavity, h^i is monotone decreasing on $[t_i^2, \bar{T}]$, and it follows, as in step 2, that for any ε, $\delta > 0$, and for i sufficiently large:

$$\int_{t_i^2}^{\bar{T}} h^i(t)\,\mathrm{d}t < \varepsilon + K\delta \ .$$

As in step 2, this implies, by contradiction, that $h^i(t_i^2) \to \infty$. □

Theorem 6.1. *Assume D1, D2, D3, and* SCP* *and $\underline{s} < m(t) < \bar{s}$ for all $t \in \bar{T}$. Let $\{H^i\}$ be a sequence of measures with property CC at t^*. Let $(\underline{b}^i, \bar{b}^i)$ be an interior equilibrium in the voting game $\mathscr{G}^i$, and let both parties have probabilities of victory that are bounded away from zero. Then b^i and $\bar{b}^i$ converge to the same policy.*

Proof. 1. By Lemma 6.3, regularity holds for each i. Hence we can write the payoff function for $\underline{s}$ in game i for b near $\underline{b}$ as

$$\pi^i(b, \underline{b}; \underline{s}) = \int_{T'(b,\bar{b})} v(b, \underline{s}, t)\,\mathrm{dH}^i + \int_{T'(\bar{b},b)} v(\bar{b}, \underline{s}, t)\,\mathrm{dH}^i \tag{6.6}$$

(recall (4.2)).

2. Suppose the conclusion false. We may, w.l.o.g., assume that $\underline{b}^i \to \underline{b}$ and $\bar{b}^i \to \bar{b}$ where $\underline{b} \neq \bar{b}$ (i.e., choose a convergent subsequence, by compactness of $\bar{B}$). Hence $\underline{b}^i \neq \bar{b}^i$ for large i, and so $\bar{t}^i(\underline{b}^i, \bar{b}^i)$ is well defined and C^1 by Lemma 6.2.

3. By definition of $\bar{t}^i$, $F(\bar{t}^i(\underline{b}^i, \bar{b}^i))(\Omega(\underline{b}^i, \bar{b}^i)) = \frac{1}{2}$. *Suppose* s prefers $\underline{b}^i$ to $\bar{b}^i$ (this isn't obvious from our assumptions). Since $\underline{s} < m(\bar{t}^i)$, it follows by SCP* that all s in $[s_1, m(\bar{t}^i))$ prefer $\underline{b}^i$ to $\bar{b}^i$, and so $\Omega(\underline{b}^i, \bar{b}^i) = [s_1, m(\bar{t}^i))$. For any $\varepsilon > 0$:

$$\begin{aligned} F(\bar{t}^i + \varepsilon)(\Omega(\underline{b}^i, \bar{b}^i)) &= F(\bar{t}^i + \varepsilon)([s_1, m(\bar{t}^i))) \\ &> F(\bar{t}^i + \varepsilon)[s_1, m(\bar{t}^i + \varepsilon))) = \tfrac{1}{2} \ , \end{aligned}$$

where the inequality follows since $m'(t) < 0$ and $F(t)$ is equivalent to Lebesgue measure on $\bar{S}$. This shows that $(\bar{t}^i+\varepsilon)\in T'(\underline{b}^i,\bar{b}^i)$ for all ε, and so $T'(\underline{b}^i,\bar{b}^i)=(\bar{t}^i,\bar{T}]$. Hence we can write (6.6) as:

$$\pi^i(b,\underline{b}^i;\underline{s})=\int_{\bar{t}^i}^{\bar{T}} v(b,\underline{s},t)\,\mathrm{d}H^i+\int_0^{\bar{t}^i} v(\bar{b}^i,\underline{s},t)\,\mathrm{d}H^i \ . \tag{6.7}$$

(3′. If the supposition in step 3 is false then it would follow that $\Omega(\underline{b}^i,\bar{b}^i)=(m(\bar{t}^i),s_2]$, with the consequence that the limits on the first integral would interchange with the limits on the second integral. The argument that follows would still hold.)

4. By Lemma 6.2, $\dfrac{\partial\bar{t}^i}{\partial b}$ exists, and so we may differentiate expression (6.7) w.r.t. b:

$$0=\frac{\partial\pi^i}{\partial b}(\underline{b}^i,\bar{b}^i,\underline{s})=\int_{\bar{t}^i}^{\bar{T}} v_1(b^i,\underline{s},t)\,h^i(t)\,\mathrm{d}t$$

$$+\frac{\partial\bar{t}^i}{\partial b}\cdot h^i(\bar{t}^i)[v(\bar{b}^i,\underline{s},\bar{t}^i)-v(\bar{b}^i,\underline{s},\bar{t}^i)] \ , \tag{6.8}$$

where $h^i(t)$ is the density of H^i. The derivative has been set equal to zero since $(\underline{b}^i,\bar{b}^i)$ is an interior point of $\bar{B}\times\bar{B}$. (6.8) is the fundamental equation of the analysis.

5. v_1 is continuous on the compact set $\bar{B}\times\{\underline{s}\}\times\bar{T}$, and so is bounded. Therefore the first term on the r.h.s. of (6.8) is bounded. It follows that the second term is also bounded in i.

6. We analyze this second term. By hypothesis, each party wins with probability bounded away from zero. That is, there exist positive numbers δ_1 and δ_2 such that

$$\forall i\ \delta_1<\int_{\bar{t}^i(\underline{b}^i,\bar{b}^i)}^{\bar{T}} h^i(t)\,\mathrm{d}t<1-\delta_2 \ . \tag{6.9}$$

By Lemma 6.4, it follows that $h^i(\bar{t}^i)\to\infty$.

7. Next, we observe that $\dfrac{\partial\bar{t}^i}{\partial b}(\underline{b}^i,\mathrm{\bar{b}}^i)$ is bounded away from zero. Examine (6.3) and (6.5). Let $\tilde{s}^i(\underline{b}^i,\bar{b}^i)\to\tilde{s}^*$ (pick a subsequence if necessary). By continuity, $\tilde{s}^*$ is indifferent between $\underline{b}$ and $\bar{b}$ when the distribution of voter traits is $F(t^*)$ for sure so that $v_1(\underline{b},\tilde{s}^*,t^*)\neq 0$, by strict concavity of $v(\cdot,\tilde{s}^*,t^*)$. But, by CC, it follows that $\tilde{v}_1^i(\underline{b},\tilde{s}^i)\to v_1(\underline{b},\tilde{s}^*,t^*)$, and so $\tilde{v}_1^i(\underline{b}^i,\tilde{s}^i)$ are bounded away from zero for large i. Now (6.5) implies $\dfrac{\partial\tilde{s}^i}{\partial b}$ are bounded away from zero, and (6.3) implies that $\dfrac{\partial\bar{t}}{\partial b}(\underline{b}^i,\bar{b}^i)$ are bounded away from zero (using the fact that $f(t,s)$ are bounded away from zero as well).

8. It follows, by the previous three steps, and considering the second term in (6.8), that

$$\lim_i[v(\bar{b}^i,\underline{s},\bar{t}^i)-v(\underline{b}^i,\underline{s},\bar{t}^i)]=v(\bar{b},\underline{s},t^*)-(v(\underline{b},\underline{s},t^*)=0 \ .$$

9. In like manner, we can deduce that

$$v(\bar{b},\bar{s},t^*)-v(\underline{b},\bar{s},t^*)=0 \ .$$

10. But the conclusions of the previous two steps contradict SCP for $v(\cdot, \cdot, t^*)$. This contradiction establishes the theorem. □

Indeed, it is not surprising that we have:

Corollary 6.1. *Let the hypotheses of Theorem 6.1 be satisfied. Then* $\{\underline{b}^i\}$ *and* $\{\bar{b}^i\}$ *converge to* $\hat{b}(m(t^*))$.

Proof. 1. By Theorem 6.1, $\{\underline{b}^i\}$ and $\{\bar{b}^i\}$ converge to a policy b^{**}. Suppose $b^{**} \neq \hat{b}(m(t^*))$. W.l.o.g., suppose $b^{**} > \hat{b}(m(t^*))$. Then, by A3, and since $\bar{s} > m(t^*)$:

$$v(b(m(t^*)), \bar{s}, t^*) > v(b^{**}, \bar{s}, t^*) \ . \tag{6.10}$$

2. Since $\{H^i\}$ has property CC at t^*, $\tilde{v}^i(b,s) = \int_{\bar{T}} v(b,s,t)\,\mathrm{d}H^i \to v(b,s,t^*)$, for all b and s. For large i, the payoff to $\bar{s}$ at $(\underline{b}^i, \bar{b}^i)$ becomes arbitrarily close to $\tilde{v}^i(b^{**}, \bar{s})$, and therefore arbitrarily close to $v(b^{**}, \bar{s}, t^*)$. But I claim that were $\bar{s}$ to play $b^* \equiv \hat{b}(m(t^*))$ against $\underline{b}^i$ for games with large i, his payoff would become arbitrarily close to $v(b^*, \bar{s}, t^*)$. By (6.10), this will show that, for large i, $\bar{b}^i$ was indeed not a best response of $\bar{s}$ to $\underline{b}^i$ in game $\mathscr{G}^i$, a contradiction that will establish the corollary.
3. I establish the stated claim. Since $b^{**} > b^*$, a strict majority of voters prefer b^* to b^{**} when the distribution of voters is $F(t^*)$ for sure. By D3, it follows that, for δ sufficiently small, a strict majority of voters prefer b^* to b^{**} and hence to $\underline{b}^i$ when the distribution of voters is $F(t)$ for sure, when $t \in N_\delta(t^*)$. By CC, for any small ε, and for i sufficiently large, the probability that the true distribution of voters is $F(t)$, for $t \in N_\delta(t^*)$, is at least $1-\varepsilon$. But if $t \in N_\delta(t^*)$, then we have established b^* defeats $\underline{b}^i$, for large i, with probability at least $1-\varepsilon$, and so

$$\pi^i(b^*, \underline{b}^i; \bar{s}) > (1-\varepsilon)\tilde{v}^i(b^*, \bar{s}) + \varepsilon\tilde{v}^i(\underline{b}^i, \bar{s})$$

and so

$$\liminf \pi^i(b^*, \underline{b}^i, \bar{s}) \geq (1-\varepsilon) v(b^*, \bar{s}, t^*) + \varepsilon v(b^{**}, \bar{s}, t^*) \ . \tag{6.11}$$

Since this argument holds for any ε, we have

$$\liminf \pi^i(b^*, \underline{b}^i, \bar{s}) \geq v(b^*, \bar{s}, t^*) \ . \tag{6.12}$$

Indeed, (6.12) is not exactly the stated claim, but it is sufficient to establish the contradiction of step 2. □

7. Conclusion

In many countries and time periods, electoral equilibrium between 'left' and 'right' parties seems to have been characterized by parties proposing highly differentiated platforms. It is not intuitively appealing to explain this observed differentiation by noting that politics are not single-issue, thus providing an escape clause from the median voter theorem: for it seems approximately accurate to say that, in many historical periods and countries, political competition has been single-issue. The escape clauses provided by claiming that political competition is a multiperiod game or that parties are Downsian but are forestalling the entry of new parties by differentiating their policies do not strike me as cogent moves. The analysis provided here offers an alternative explanation for differentiated politics, in a single-issue, single-period framework, based on uncertainty about the distribution of voter preferences.

That uncertainty may exist in reality for numerous reasons: (1) there may, indeed, be poor information on the distribution of voter preferences; (2) there may be uncertainty about which citizens will actually vote, although the distribution of preferences in the population is known (see Roemer 1993); (3) some voters may not vote rationally, that is, may not maximize their expected utility;[9] (4) it may be possible to persuade voters of different theories of how the economy works, thus influencing voter preferences over policies, but with uncertain success. (This theme is developed in Roemer [in press b].) Only the first reason has been precisely modelled here, although differentiated politics will also characterize electoral equilibrium in models of the other phenomena as well.

The issue of credibility has not been touched upon here: that partisan parties may have an incentive to campaign on one policy but to implement another if elected. Alesina (1988) discusses this problem in a model where elections are a repeated game, and shows that repetition tends to push partisan parties to converge in their policy announcements.

Theorem 6.1 suggests that we should observe decreasing polarization of political equilibrium due to an improved technology of opinion polling. The reliability of such polls has increased dramatically in the past fifty years (recall the polls predicted that Dewey would win the U.S. presidential race in 1948), and that should decrease the uncertainty concerning the relevant distribution of voter traits that parties face. *Ceteris paribus*, political equilibrium should move towards median voter politics. Thus what may appear as a deradicalization of parties of the Left and Right may not reflect changes in the politics of their constituents, but may simply be a change in political strategies implied by technological change in polling procedures. It would be interesting to test this hypothesis against the data.

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[9] Voters may have 'false consciousness', voting as if they have different values of s than they actually do; or, more charitably, they may vote for the policy that is optimal for a value of s they conjecture they will have in the future.

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