*Robustness of the Multidimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence**

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This analysis demonstrates that important implications of the multidimensional voting model are robust to significant changes in the model's assumptions. (1) If candidates in the model are allowed to be partially or totally interested in the election's policy outcomes, convergence to the median must still occur. (2) If candidates are uncertain about voters' responses, and therefore attempt to maximize the probability of winning, the candidate platforms should still converge in equilibrium under weak assumptions about symmetry of the candidates' situations. (3) If both of these non-standard assumptions are made together, the convergence result no longer holds; but small departures from the classic assumptions lead to only small departures from convergence. In combination with other recent multidimensional voting models that examine behavior in the absence of a median, this study indicates the usefulness of the traditional model for conceptualizing electoral politics.

1. Introduction

The traditional multidimensional, or spatial, model of candidate competition (Downs, 1957; Davis, Hinich, and Ordeshook, 1970) represents the candidate as being concerned solely with winning the election, or, in the same spirit, with maximizing his plurality or total number of votes. In addition, it endows each candidate with complete knowledge about what the election results will be, given any particular choice of platforms by the candidates. One of the most significant conclusions that follows from these assumptions is that, in equilibrium, the candidates must adopt identical platforms. In other words, there is nothing in the basic logic of electoral competition that would lead candidates to offer the voters a choice between different policies.

The argument in this paper demonstrates that candidate convergence at equilibrium still results even if the usual assumptions about candidate motivations and information are relaxed. If candidates are motivated, not by the desire to win office, but by the quality of the policies resulting from an election, they should still compete away their differences and offer identical proposals to the voters. If candidates cannot perfectly predict election outcomes, and instead try to maximize the probability of winning office, then as long as they share the same beliefs about the electorate they will converge in equilibrium. Even if both assumptions are relaxed at the same time, the candidates should diverge only in a limited way. Taken together, these results show that the multidimensional voting model is substantially robust against important changes in its key

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assumptions. In particular, the convergence or near-convergence of candidate platforms in equilibrium is truly a basic property of electoral competition.

The Role of This Analysis

Such robustness against departures from the basic assumptions is vital to the development of any positive theory. As a model of a general process, the basic multidimensional voting model serves mainly as a tractable guide to our thinking about electoral competition (and to more detailed modeling), and not necessarily as a direct generator of hypotheses about real-world elections. It must assume away many features of the real world. If the model's conclusions are robust against complications of that abstract picture, then it has captured the essence of electoral competition; it is a useful abstraction.

For example, a powerful previous result of this type concerns the most drastic "problem" of the basic model: the generic failure of equilibrium to exist (Plott, 1967; Schofield, 1978). If there is almost never any equilibrium, what good are the model's conclusions about the nature of equilibrium? Kramer and McKelvey have demonstrated two ways in which the intuitions generated by the model can withstand the apparent knife-edge character of its equilibrium results. Kramer (1977) has shown that, under a stylized dynamic, two competing candidates should move their platforms toward a "minmax set," centrally located among the voter ideal points. While the platforms may leave the minmax set once having reached it, they will immediately begin to approach it again. Moreover, as the number of voters becomes larger, the minmax set shrinks to a point; that point is the median if one exists (Kramer, 1978).

McKelvey has presented a result in the same spirit that does not depend on the kind of dynamic used by Kramer. In two-candidate competition, either the use of mixed strategies or the successive elimination of dominated strategies would force the candidates to choose platforms in the "uncovered set" (McKelvey, 1983, p. 24). McKelvey has shown that the uncovered set is centrally located among the voter ideal points; is small when the deviation of the voter distribution from symmetry is small (Ferejohn, McKelvey, and Packel, 1984); and shrinks to become the median as the distribution approaches symmetry (McKelvey, 1983, pp. 40–44).

Thus it is helpful to think about electoral competition as taking place in a multidimensional space in which the ideal points of a large number of voters are distributed symmetrically (as in Davis and Hinich, 1968). In particular, there is pressure on the candidates to converge to the central part of the voter distribution, even if that distribution is not perfectly symmetric.

The results presented below serve the same purpose as those of Kramer and McKelvey. Although complete information and a singleminded focus on winning office may not perfectly characterize real elections, it is still useful to think about electoral competition in terms of the (relatively simple) basic model of multidimensional voting. Despite departures from those basic assumptions, there is still an underlying pressure on candidates to converge in their platforms in order to achieve their goals.

Previous Research

This conclusion considerably modifies that drawn by Wittman (1983), who models candidate competition under the same assumptions as here. Observing that "in the real world candidates are both office oriented and issue oriented" (p. 142), Wittman examined a "synthesis model" whose results "differ greatly from the model that assumes that candidates or parties are solely interested in winning" (p. 143). His theoretical results claim that both candidates may differ in the same direction from the center of voter opinion (pp. 143–44) and that candidates are likely to diverge to their own ideal points (pp. 144–45, 155). In other words, once we modify the traditional multidimensional voting model to allow for basic real-world conditions, the major results of that model disappear.

Using identical changes in the traditional assumptions, the present study demonstrates that this disappearing act is not what it seems to be. Relaxing the assumptions results only in a smooth, gradual departure from the traditional results. A strong policy orientation together with such uncertainty will indeed lead to a significant divergence of the candidates as Wittman indicated. But if policy motivation and uncertainty constitute only limited departures or "noise" that occurs when the traditional model is applied, they cause only small perturbations of candidate convergence; and for any degree of policy motivation and uncertainty, divergence is limited by an offsetting convergence tendency. Thus the results below, although related to Wittman's, are shown to reflect the strength, rather than the weakness, of the traditional model as a conceptual device.[1]

Outline of the Paper

In section 2 below, the candidate is motivated by policy concerns rather than by the desire to win office; he has preferences about the policies resulting from the election. All else is just as in the traditional multidimensional voting model. The theorems in this section show that, just as in the traditional model, equilibrium consists of candidate platforms converging at the median. This result about electoral competition also applies, with just a translation of terms, to majority-rule committee processes. In such a committee, previous research (beginning with Black, 1958) has shown that if every member has the power to propose alternatives to the status quo, the ultimate outcome will be the preferred position of the median member, if one exists. The theorems of section 2

[1] The specific theoretical results of Wittman (1983) will be discussed, mainly in footnotes, at the appropriate points below. As will become apparent, I have some quarrels with his theoretical statements themselves as well as with his interpretation of the overall venture.

show that if only two members are empowered to make such proposals then the median position will still be the outcome.

Section 3 returns to the traditional, office-motivated candidate, and asks what happens if the candidates are only probabilistically able to predict election outcomes. Under very weak conditions of symmetry in the contest between the candidates, platform convergence again goes hand-in-hand with equilibrium.

In section 4, the assumptions of the previous sections are combined. It is a surprising fact that convergent platforms can then *never* be in equilibrium. However, small departures from the traditional assumptions yield only small departures from the incentive for candidates to converge to the median. This divergence tendency can in fact be made arbitrarily small. In other words, the conclusions of the traditional model are not knife-edge results, but rather are relaxed continuously as its assumptions are relaxed. Section 5 reexamines the significance of these results.

2. Policy-motivated Candidates with Full Information

In the real world, candidates may be concerned with the policy outcomes of elections for two general reasons. First, a candidate may have genuine preferences about some issues. For such a concern to be relevant, the candidate must be willing at some point to trade off his chances of being elected against the opportunity to achieve a desired policy. Second, previous political trades or connections sometimes make it necessary for the politician to treat the policy concerns of some important supporters as a constraint on his actions. He would then behave as though those concerns were his own, in addition to his desire to hold office.

In this section, the traditional assumption that candidates are interested only in electoral outcomes is replaced with the assumption that they are interested in policy outcomes resulting from elections. Merely *advocating* a desirable platform, however, is assumed to have no intrinsic value to the candidates. They pursue office as a means to achieving desirable policies. As we will see below, the two types of motivation can also be combined without changing the theoretical results.

One might argue, however, that policy-motivated candidates will generally be defeated by office-motivated ones, and that therefore most officeholders can safely be assumed to have been of the latter type. If this is so, why bother to model policy-motivated candidates? To answer this question, consider first that, even accepting this argument, there is no reason to expect that only office-motivated candidates are selected from the "pool" of potential candidates to run for office. Only a few candidates run for a given office at a given time, and they emerge through a vast variety of processes and mechanisms; indeed, the real world is full of amateur candidates, protest candidates, extremist candidates, and other exotic species.

In the election itself, the advantage of the office-motivated candidate would lie in his greater willingness to take a position near his opponent's ideal point. But this means that the presence of even a losing, policy-motivated candidate may have a significant effect on the winning candidate's platform. If a politician's platform at all constrains his actions once in office, then it is still important to learn what happens if the losing candidate is policy motivated. In fact, however, the results below show that, if equilibrium exists, the candidates' motivations don't affect the outcome anyway. And if equilibrium fails to exist, even a policy-oriented candidate could play electoral leapfrog with an office-motivated opponent; again, this is no different from the standard model. There is no reason to expect policy motivation to be "selected out" among candidates; therefore it is valuable to learn that candidate policy motivations don't affect the conclusions of the spatial model.

The other assumptions of simple multidimensional voting models are maintained here. In particular, voter utility functions depend only upon Euclidean distance from the voters' ideal points, and those ideal points are distributed symmetrically in the issue space so that a majority rule equilibrium exists. It is of course well known that this symmetry assumption is highly restrictive. The analysis can still be informative, however. First, with stochastic voting in a large electorate and under numerous concavity assumptions, equilibrium exists just as if the voter distribution were symmetric, and the results of this section hold true. These concavity assumptions seem unlikely, but they are often used in the literature (see for example Hinich, Ledyard, and Ordeshook, 1972; Denzau and Kats, 1977; Hinich, 1978; Coughlin and Nitzan, 1981; and Wittman, 1983, who says "the assumption of concavity is standard," p. 151). Devotees of concavity, at least then, learn something significant from the results presented below (but see Slutsky, 1975, for some problems with such assumptions).

However, there is a second, more important, reason for pursuing this result for a multidimensional issue space and a symmetric voter distribution. As indicated in the introductory section above, many properties of candidate competition in the symmetric distribution case carry over to the more general setting, in particular the pressure on candidates to converge toward the center of the voter distribution. In the absence of direct results concerning policy-motivated candidates in a model like McKelvey's (1983) or Kramer's (1977), the results below should serve as an indication of properties likely to carry over substantially to the case of general, or at least nearly symmetric, voter distributions. More direct results of this type must await further research.[2]

[2] Wittman (1983, p. 155) claims to have shown previously that, in a Kramer-type dynamic setting, platforms will diverge to the candidate ideal points. However, in the previous paper that he cites (Wittman, 1977, pp. 184–85), he did so only for the case in which the "issues" are private distributional shares of a fixed pie, a nonstandard and restrictive setting. Several other restrictive assumptions were also employed.

Definitions and Assumptions

Let S be a compact, convex set of feasible policies in n-dimensional Euclidean space (or any normed linear space). Let f be a symmetric density function describing the distribution of voters' ideal points; let m be the median of f, the point such that any hyperplane containing m divides the space into two closed half-spaces, each containing at least half the voters' ideal points. For the formal analysis here, let us assume that there are infinitely many voters, and that f is continuous on S. Similar results can be derived for a discrete, finite set of voters. Also, for the time being, assume that each voter has a continuous, bounded utility function whose indifference curves are spheres centered at the voter's ideal point, with utility decreasing monotonically with distance away from the ideal point ("Euclidean preferences"). For present purposes we assume that all citizens vote.[3]

Suppose there are two candidates, 1 and 2, each having an ideal point, designated x^* and y^*, respectively. Each candidate i has a utility function u_i which behaves like those of the voters. Later we will consider what happens when the assumption of spherical indifference curves for candidates is relaxed. The goal of candidate i is to maximize $u_i(z)$ where z is the platform of the winning candidate. The candidates' platforms will be denoted x and y, respectively, so $u_1(x)$ represents the utility of candidate 1 if he wins, $u_1(y)$ his utility if candidate 2 wins, and so on.

If we assume that each challenger chooses *the most preferred position* from among those that win by some minimum plurality $\phi_0 > 0$ (otherwise he is maximizing on an open set, which is a bug in Wittman's proof), it is easy to demonstrate that Wittman's conclusion does not hold under standard Euclidean preferences. For the case when a median exists, just use the lemma to Theorem 1 in the Appendix to this paper. Then the platforms converge to the median. When no median exists, the situation is that described by Miller (1980) demonstrating his version of the "competitive solution" in the context of committee decision making. Again, movement to the candidates' (for Miller, the proposers') ideal points does not occur.

[3] Most previous analyses of multidimensional voting models take the same approach, including Plott (1967), Kramer (1977, 1978), McKelvey (1983), and Wittman (1977, 1983). In establishing the pressures upon candidate strategy inherent in voting systems, the object of primary interest is the electoral mechanism. By ignoring nonvoting, these studies address purely the properties of majority rule as such. If it were well known that nonvoting depended predictably on some of the real-world conditions mentioned in the basic multidimensional model, then such behavior would be important to take into account here. But although such theoretical phenomena as abstention due to alienation and indifference are sensible from a modeling standpoint (Hinich and Ordeshook, 1969, 1970), they have not been shown empirically to occur with any consistency. And even theoretically, candidate divergence due to such abstention depends upon special circumstances within the model: vote-maximizing candidates and thick tails in distribution of voter ideal points, for example. Other sources of nonparticipation, such as socializing factors, clearly lie outside the proper scope of this study. Finally, I ignore the free rider problem in voting, assuming that voters vote for whatever reason and that they want to vote for the candidate whose issue positions they most prefer.

Define the *plurality function* $\phi(x, y)$ to be the proportion of citizens voting for candidate 1 minus the proportion voting for candidate 2. Notice that for any fixed x the set

$$\{y \mid \phi(x, y) = 0\}$$

is the set of points at which candidate 2 could achieve a tie. The lemma to Theorem 1 (see Appendix) shows that this set is always a circle centered at the median m when a median exists and preferences are Euclidean.

Simple Results

Consider as an example the case where n (the dimensionality of the issue space) is equal to one. If $x^* < m < y^*$, and if all citizens vote, then the only Nash equilibrium is where at least one candidate j takes the platform m. First, notice that $x > m$ is not satisfactory as a winning platform, since by adopting m candidate 1 could tie or win with a platform he prefers to x. The same reasoning prevents $y < m$ from winning in equilibrium. If, on the other hand, $x < m < y$, then candidate 2 will not be content to have $y \geqslant m + (m - x)$, that is, for y to be further from the median than x, since he would then lose, whereas it would be possible for him to win with a position closer to m but still preferred to x. A similar argument holds for candidate 1's strategy; thus only when $x = m$ or $y = m$ does neither candidate have any incentive to move given that the other will not. (In using Nash equilibrium, of course, we omit any consideration by a candidate of his opponent's reactions.) In the case where x^* and y^* are both on the same side of m, though, this result does not hold; then the ideal which is closest to m becomes the equilibrium point. But the case where the two candidates are opposed to one another in their policy goals, relative to the population, seems the important one. Finally note that, in a setting of dynamic competiton between the candidates (as for example if the candidates were forced to reveal their positions "simultaneously"), the predicted outcome is for *both* candidates to adopt m.

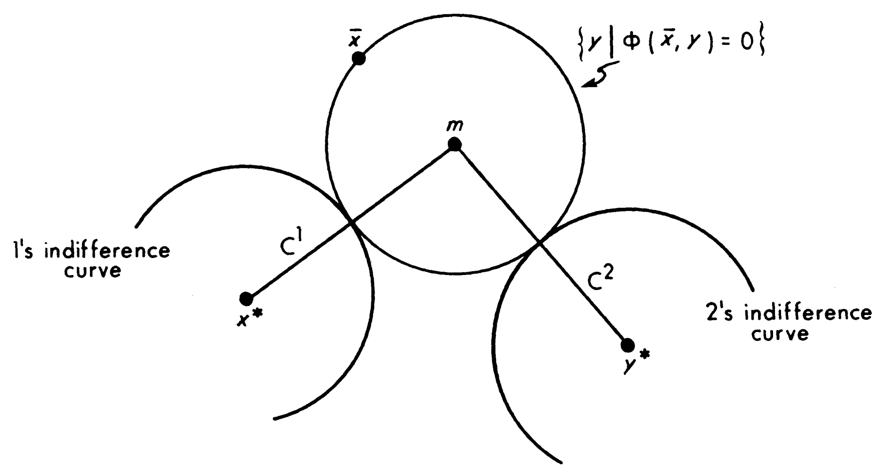
Wittman (1977) generalizes this result[4] to arbitrary values of n for two highly restrictive cases: (1) complete policy opposition between the candidates, in which m lies on a line between x^* and y^*; and (2) complete lack of opposition, where x^* and y^* both lie in the same direction on a line through m. The results correspond to the opposed and nonopposed cases above, respectively. In Theorem 1 below, the convergence result corresponding to case (1) is shown to generalize to *any* situation except that of case (2).

[4] Wittman actually takes a dynamic approach for his Proposition 5, in which a sequence of elections occurs with a challenger choosing a preferred position which defeats an immobile incumbent. The result is formally identical to the "static" Nash equilibrium result of Theorem 1 here, since such an equilibrium exists.

Candidate Strategies

In order to determine which positions the candidates will favor, we define the following sets: for each candidate i, C^i is the line segment in S whose endpoints are i's ideal point and m. These are points at which candidate i's indifference curves are tangent to the contours of the plurality function (see Figure 1). In other words, for any fixed position of candidate 2, candidate 1 can maximize the utility of his own platform, while achieving a tie vote, only by choosing a position in C^1. As the proof of Theorem 1 shows, candidate i will always choose to be in, or arbitrarily near, the set C^i. Some of the results of this section generalize beyond the case of Euclidean preferences; we will explore this issue later.

FIGURE 1
Elements of Candidate Strategies



In traditional spatial models, a pair of platforms (x, y) is a Nash equilibrium when neither candidate, assuming that the other will remain stationary, sees any way to improve his own plurality. With policy-motivated candidates, the interest in the winning policy must replace the concern with plurality. In order to deal with the indirect influence of winning in this new setting, it is necessary to define the way in which candidates evaluate a tie election. Without loss of generality,[5] we can assume for simplicity that the value to candidate i of a tie election is just

$$t^i(x, y) = (1/2)[u_i(x) + u_i(y)].$$

Nash equilibrium for this model can now be defined as follows: Fix x and y, and suppose u^* is the utility to candidate i of the resulting election outcome—either the utility of the winning position or that of a tie. If, assuming his opponent remains stationary, candidate i cannot move to a winning position with higher utility than u^*, or to a tying position with $t^i > u^*$, then (x, y) is an equilibrium. In particular, a candidate has no incentive to move from a losing position to a winning position if the latter is not strictly preferred to his opponent's position. Further, he has no incentive to move from one position to a preferred position if he would lose in both cases. He might conceivably wish to move from a low-utility, winning platform to a higher-utility, tying platform.

General Results

Two final assumptions are needed to establish Theorem 1. First is the condition of policy disagreement between the candidates. Specifically, assume that x^*, y^*, and m do not all lie on a single line unless m is between x^* and y^*. That is, at the median of the distribution of voters' preferred points, the candidates' utility gradients do not have the same direction. In the case of spherical indifference curves this means that

$$\frac{x^* - m}{||x^* - m||} \neq \frac{y^* - m}{||y^* - m||}$$

Second, there must be a significant number of voters with ideal points arbitrarily near m; if f is a smooth distribution, this is analogous to assuming the presence of a voter at the median in the finite electorate case. This condition takes the form

$$\int_N f(z)\, dz > 0 \text{ for every open set } N \text{ containing } m.$$

We can now state the first theorem. (Lemmas and proofs to all theorems appear in the Appendix.)

[5] All the results in this paper are maintained as long as t^i obeys the following rules:

$$u_i(x) \geqslant t^i(x, y) \geqslant u_i(y),$$

where $i \neq j$, and

$$u_1(x) > u_1(y) \Rightarrow u_1(x) > t^1(x, y),$$

and likewise for candidate 2.

THEOREM 1: Suppose that m is the median of f and that the following hold:

(i) $$\frac{x^* - m}{||x^* - m||} \neq \frac{y^* - m}{||y^* - m||}$$; and

(ii) $\int_N f(z)\, dz > 0$ for every open set N containing m.

Then $(\bar{x}, \bar{y})$ is an equilibrium if and only if $\bar{x} = m$ or $\bar{y} = m$.

Strictly speaking, Theorem 1 is not itself a candidate convergence result since only one candidate need locate at the median to create an equilibrium situation. If we view the election as a standard simultaneous-move, two-player game, however, the only pure strategy either player should choose is the median position.[6]

THEOREM 2: Consider the two-player game in which each player's pure strategy set is the issue space S (the set of possible platforms) and in which the outcomes for each player i are values $u_i(z)$ where z is the strategy of the winning candidate. Then m is a maximin strategy for either player, and it is the only pure strategy that is a maximin strategy.

Theorem 1 can be generalized beyond the case of spherical indifference surfaces for the candidates. Under the original assumption, the sets C^1 and C^2 were just straight-line segments from m to x^* and y^*, respectively. With non-Euclidean preferences, however, the relevant C^i are sets of tangency points between candidate i's indifference curves and the contours of the plurality function ϕ. Elliptical indifference curves for the candidates, for example, would yield C^i sets that are curves instead of straight lines. More general voter preferences might give irregular level curves for ϕ, and the C^i sets could then have branches or even "thick" areas. The same strategic reasoning still applies: a candidate should always choose to be in, or arbitrarily close to, his C^i set. However, these

[6] In his Proposition 1, Wittman (1983) purports to show that "on one or more issues both [candidates' equilibrium] policy positions may be to the left (right) of that point (M^*) that maximizes their probability of winning" (1983, p. 144). There are several problems with this result as Wittman presents it; however, it does suggest a weak result on candidate choice under *determinate*, not probabilistic, outcomes.

First, throughout his proof Wittman compares platforms one issue (dimension) at a time, contrary to the usual approach. Second, Wittman's treatment of the probability of winning here is unusual. He implicitly assumes that there is some point M^* such that moving closer to M^* always increases a candidate's probability of winning. This would give a rather strangely behaved electorate; probability is maximized at the same location, regardless of how extreme the opponent's position is. Wittman offers no further explanation of the underlying assumption.

However, if we give the candidates complete information and let M^* be the median of the voter distribution, reasoning similar to Wittman's does give a result analogous to the one he claims, for the unidimensional case only. It depends directly on the assumption that both candidates' ideals are on the same side of the median. As our subsequent discussion shows, the corresponding multidimensional result holds only when both candidates' ideals lie in exactly the same direction from the median.

sets may now intersect at points other than the median, with the result that new equilibria are created. Suppose candidate 1 takes a position on the intersection of C^1 and C^2 which is not the median; condition (i) of Theorem 1 no longer prevents this. If candidate 2 then occupies a position on C^2 between his own ideal point and candidate 1's position, he has no incentive to move, since he cannot win without taking a position inferior to x. Thus every intersection point of C^1 and C^2 gives rise to a possible equilibrium outcome.

Committee Decision Making

These conclusions about policy-oriented candidates apply equally to models using a finite or infinite electorate.[7] With a finite electorate, a mere change of language yields results about *committee* decision making as well. It does not matter, in the electoral model, whether we treat the candidates themselves as having a vote or not, since we could think of the voting candidate as two separate people for purposes of the model: one just a voter, the other a candidate whose ideal point happens to be identical with that voter's. The voting, policy-oriented candidate for election faces exactly the same problem as a committee member with the power to make proposals, in a committee in which only two members have that power. His task is to choose a proposal which, when matched against an opposing one (perhaps a status quo), will give an outcome closer to his ideal.

Limitations on the power of members to make proposals in committee decision making are an integral feature of recent models exploring the effects of actual legislative institutions, such as party and committee systems, which overlay the basic majority voting rule (Isaac and Plott, 1978; Miller, 1980; Weingast, 1981; Shepsle and Weingast, 1981). Miller in particular asks: "What's it worth to be a party?" in a legislature where full proposal power is held only by the leaders of the parties. The answer given by Theorem 1, if a median voter exists, is "nothing." That is, the outcome of a two-party committee is exactly the same as the outcome when every member has proposal power, because the parties are forced to compete away their advantages in order to win.

Mixed Motivations

It is plausible, of course, to expect that candidates have preferences about policy *and* prefer to win elections. Indeed, the payoff to winning, itself, is really the only feature that might formally distinguish between the usual models of committee decision making and electoral competition with policy-oriented candidates. If the candidates have chosen platforms x and y, for example,

[7] For a finite electorate, the simplest approach would be to simply count votes, instead of integrating f to get proportions of votes. The plurality function would be bounded by $\pm N$, where N is the number of voters. The C^i sets should now consist of several *points* on the line segment from x^* or y^* to m, since $\phi(\cdot, y)$ is now a step function that increases by 1 every time a new voter is gained. With these changes, all the same results follow as before.

candidate 1's utility might be $u_1(x) + w_1(x)$ if he wins, and $u_1(y)$ if candidate 2 wins, where $w_1(x)$ is an extra utility from winning office with platform x. The only effect of such a change on Theorem 1 would be if $w_1(m) > 0$ and $w_2(m) > 0$, in which case both candidates would be interested in achieving a tie, and (m, m) would be the *only* possible equilibrium outcome. In other words, adding only a small payoff from winning to the policy-motivated model yields a candidate convergence result that is just as strong as in the pure office-seeking model.

3. Office-motivated Candidates with Imperfect Information

Suppose now that the candidates are interested purely in winning office, but that, unlike candidates in the basic model, they are unable to perfectly predict the election's outcome given the candidates' platform choices. Instead, each candidate possesses a subjective probability of winning which connects any pair of platform choices with estimated probabilities that each candidate will win. If the candidates choose platforms x and y respectively, then candidate 1's subjective probability of winning is $P^1(x, y)$; his subjective probability that his opponent will win is $1 - P^1$. (For simplicity, we may regard the probability of a tie as zero.) Candidate 2 believes he will win with probability $P^2(x, y)$. The goal of each candidate now is to maximize his probability of winning.

Agreement, Unbiasedness, and Convergence

Not surprisingly, the candidates in this game may choose very different platforms in equilibrium if they have very different ideas about the probabilities of winning. However, this should properly be regarded not as a feature of electoral institutions themselves, but as a direct result of the candidates' disagreement. In this section, let us assume that the candidates are in agreement about the probable behavior of the electorate, to see whether there are any institutional features driving the candidates to converge or diverge in their platform choices. Accordingly, let us maintain the following:

ASSUMPTION 1 (*Agreement*): $P^1(x, y) = 1 - P^2(x, y)$.

Another noninstitutional reason for the candidates to choose different platforms might be that voters treat the two candidates differently for reasons other than the platforms chosen. For example, consider a one-dimensional issue space in which candidate 1 has the same party label as the retiring incumbent, and suppose that, independent of candidate 1's actual platform, the voters tend to attribute some of the incumbent's positions to him (something like the Humphrey/LBJ problem in 1968). This systematic bias could force candidate 1 to take a more extreme position than his opponent just so that he will be treated as if he had chosen a position identical to candidate 2's. Another way to look at this phenomenon is that if the candidates could freely exchange platforms with one another, they would *not* simply reverse their probabilities of winning. That is, $P^1(x, y)$ is not equal to $P^2(y, x)$.

The goal here is to examine the effects of electoral institutions narrowly defined, excluding such factors as systematic voter biases that may arise. Accordingly, we can apply the following:[8]

ASSUMPTION 2 (*Unbiasedness*): $P^1(x, y) = P^2(y, x)$.

Under these assumptions, the following result holds:

THEOREM 3: Suppose the candidates choose their platforms to maximize their probabilities of winning the election. Then under Assumptions 1 and 2, any equilibrium (x, y) must have $x = y$, or else (x, x) and (y, y) are both equilibria as well.

Convergence to an Estimated Median

Of course, there is no guarantee that any such equilibrium in pure strategies will exist. Theorem 3 simply shows that if there are equilibria, there are convergent equilibria; if there is a unique equilibrium, it must be convergent. A further reasonable condition on the candidates' subjective probability-of-winning functions, however, will guarantee existence. In addition it gives a close analogue to the median voter result of the traditional model, and suggests that there may be generalizations along the lines of Kramer (1977) and McKelvey (1983).

Let us call a position m in S an "estimated median" under P^1 if P^1 obeys the following properties:

(1) $P^1(m, y) \geqslant 1/2$ for all y in S.

(2) Suppose that $P^1(\cdot, y)$ is maximized at x'. There is a continuous path $c(t)$ from x' to m in S, parameterized by $t \in [0, 1]$, such that $c(0) = x'$, $c(1) = m$, and

$$t' > t \text{ implies } P^1(c(t'), y) < P^1(c(t), y).$$

Property (2) just says that there is a way for candidate 1 to move from his maximum to m such that his probability of winning decreases monotonically all along the way. Similar properties define an estimated median for P^2. We can now state the new assumption and its implications:

ASSUMPTION 3: Both P^1 and P^2 have estimated medians.

[8] As it turns out, the convergence of candidate platforms in equilibrium is maintained even if there is bias of the kind described, so long as the bias is of a particular simple form, and its effects are totally independent of the platforms chosen. Thus the following will suffice:

ASSUMPTION 2 *(Positive, Linear Nonpolicy Bias):* There are constants a and b, $b \geqslant 0$, such that

$P^1(x, y) = a + bP^2(y, x)$ for all x and y.

The bias allowed by Assumption 2 might be thought of as a kind of pure public image, the effects of which are independent of the platforms that the candidates choose, and which is not under the control of the candidates during the election campaign.

THEOREM 4: Under Assumptions 1 and 3, there is a unique equilibrium in which each candidate adopts his estimated median.

COROLLARY: Under Assumptions 1, 2, and 3, m is the estimated median for both candidates and (m, m) is the unique equilibrium.

Assumption 3 could be made considerably weaker, at some cost in complicating the model. We could let the "m" corresponding to P^1 be a function m^1 of candidate 2's position, provided that this $m^1(y)$ approaches some m^* as y approaches m^*, and similarly for P^2 and $m^2(x)$. Then m^* serves as the genuine estimated median, and, under some regularity assumptions on the convergence of $m^1(y)$ and $m^2(x)$, both candidates will still wind up at m^*.

To summarize, Theorems 3 and 4 show that there is nothing in the logic of electoral competition under uncertainty that would drive the candidates apart in equilibrium. Indeed, if there are any equilibria, there must be convergent ones. Further, if there is anything resembling the traditional model's median voter, the candidates' platforms will converge there.

4. Policy-motivated Candidates with Uncertainty

Neither the assumption of candidate policy motivations nor the introduction of uncertainty about the probability of winning alters the convergence property of equilibrium in multidimensional candidate competition. In this section, both changes are introduced simultaneously. This gives a surprisingly strong nonconvergence result: no convergent outcome can be an equilibrium. Having established what the candidates will not do in such a game, we can then explore what they will do. At least for the tractable one-dimensional case, the results below indicate that the candidate convergence result is still "robust" in the following sense: if we start, as in section 2 or 3 above, with either certainty or office motivation already relaxed, and then relax the other assumption "slightly," the tendency of the candidates to diverge from the original convergent outcome will also be slight.

Nonconvergence

Assume now that candidates have policy preferences exactly as in section 2 above and that they have subjective probability-of-winning functions as in section 3. Each candidate now wants to maximize the expected utility of the election's outcome. For candidate 1 this is

$$EU_1(x, y) = u_1(x)P^1(x, y) + u_1(y)[1 - P^1(x, y)].$$

Wittman (1983) has given what we might call a "global" divergence result: if the candidates are at identical platforms, candidate 1 can increase expected utility by moving to any point he prefers to that platform. For then if his opponent wins, the outcome is exactly what it would have been under the convergent strategies; if not, the outcome is better.

In what follows, it will be helpful to have a local version of this result.[9] This requires two conditions.[10] Let z be a platform at which the candidates have, hypothetically, converged. First, candidate i's utility is *not flat at* z if every neighborhood of z contains a point that candidate i strictly prefers to z. Second, the probability of winning, P^i, is *nondegenerate at* (z, z) if $P^i(x, y) > 0$ everywhere in some neighborhood of (z, z).

THEOREM 5: Let z be any point in the interior of S, and suppose that $x \neq x^*$ or $y \neq y^*$ (or both). If, for either $i = 1$ or $i = 2$ (or both), u_i is not flat at z and P^i is nondegenerate at (z, z), then (z, z) is not an equilibrium. In particular, either candidate will prefer to move a short distance toward his own ideal point.

Again, the logic is simply that candidate i is better off to move slightly, to a nearby, preferred position with a positive probability of winning against his opponent's z. In the remainder of this section, the positive side to this negative result is considered: how far should a candidate diverge from z?

Adding Policy Motivation to Uncertainty

Suppose we begin with a unidimensional version of the type of candidate competition modeled in section 3 above, in which office-motivated candidates maximize their probabilities of winning. If this probability of winning is reasonably well behaved, a candidate can maximize it by adopting a position either identical to that of his opponent or slightly closer to the (estimated) "center" of the distribution of voter ideal points. In particular, let us assume that, for any position y of his opponent, candidate 1's probability of winning is maximized at $y + k(y)$ where $k(y)$ is small (in absolute value) relative to the size of the issue space. Let us assume that the probability of winning declines monotonically with distance away from $y + k(y)$, that $P^1(y + k(y), y) > 0$, and that P^1 is continuously differentiable.

Now consider the case of candidates who are partly office motivated but who also have some concern with the election's policy outcome. Without loss of generality, we can take candidate 1's maximand to be

$$\begin{aligned} v_1(x, y; t) &= t\mathrm{EU}_1(x, y) + P^1(x, y) \\ &= t\{u_1(x)P^1(x, y) + u_1(y)[1 - P^1(x, y)]\} + P^1(x, y) \end{aligned}$$

where t is a nonnegative weight representing the importance candidate 1 places on the policy outcome. Then the following significant result holds:

[9] This theorem also appeared in Calvert (1979, chap. 4).

[10] Corresponding conditions are also required for Wittman's proof, although he leaves them implicit.

THEOREM 6: Fix candidate 2's position y. Then

$$\lim_{t \to 0} \operatorname*{argmax}_x v_1(x, y; t) = y + k(y) \equiv \operatorname*{argmax}_x P^1(x, y)$$

In words, the smaller the importance a candidate places on policy, the closer that candidate will choose to be to the platform that maximizes his probability of winning. If both candidates behave in this way, then any equilibrium pair of platforms will converge in the limit as the candidates' policy motivations approach zero. If there is a point m that functions as an "estimated median" under P^i, as defined in section 3, and such that $y + k(y)$ is always between y and m (along with similar assumptions for candidate y), then the platforms will converge to m in the limit. In a very strong sense, then, the original candidate convergence result is still robust: if we vary the assumptions in section 3 only slightly, the candidates depart only slightly from convergent outcomes.

Adding Uncertainty to Policy Motivation

Starting with policy-motivated candidates and adding a bit of uncertainty to the game is messier to accomplish, but it is possible to establish a result for the one-dimensional case analogous to Theorem 6 above. However, we will need to restrict each candidate's utility function to be linear on each side of the candidate's ideal point. Also, let us assume the existence of an estimated median m similar to that just discussed. Without further loss of generality, we may as well take m to be 0. Finally, without loss of generality we can assume that candidate 1's ideal point x^* is above the median, and let $u_1(x) = x$ for $x < x^*$.

In order to discuss what happens as "uncertainty increases," we need to define that term. This can be done as follows: $\{P^1(x, y; s)\}_{s>0}$ is a *family of probability-of-winning functions parameterized by uncertainty (s)* if it obeys the following properties:[11]

(0) For each $s > 0$, $P^1(\cdot, \cdot; s)$ is a probability-of-winning function.

(1) *Uncertainty:* For each x and y, $P^1(x, y; s)$ approaches .5 in the limit as s approaches infinity.

(2) *Estimated Median $m = 0$:* For any y, $P^1(-y, y; s)$ approaches .5 as s approaches 0.

(3) *Certainty:* For each y, as s approaches zero,
 (a) if $|x| < |y|$ then $P^1(x, y; s)$ approaches 1;
 (b) if $|x| > |y|$ then $P^1(x, y; s)$ approaches 0.
 (In other words, winning is determined solely by who is closest to the median.)

[11] Properties (0), (2), (3), and (5a) are used directly in the proof of Theorem 7. Property (1) just makes it sensible to talk about s as an "uncertainty parameter." Properties (4) and (5b) are added to clarify the situation with regard to some of the possible implications discussed after Theorem 7 is stated.

(4) *Continuity in the Parameter:* For each x and y, $P^1(x,y;s)$ is a continuous function of s.

(5) *Monotonicity:* For each $s>0$, P^1 is continuously differentiable in x and y. P^1 has a maximum in x for every y and a minimum in y for every x such that

(a) if x' maximizes $P^1(x,y;s)$, then

$$\frac{\partial P^1}{\partial x}(x,y;s)>0 \text{ for } x<x' \text{ and}$$

$$\frac{\partial P^1}{\partial x}(x,y;s)<0 \text{ for } x>x';$$

(b) if y' maximizes $P^1(x,y;s)$, then

$$\frac{\partial P^1}{\partial x}(x,y;s)<0 \text{ for } y<y' \text{ and}$$

$$\frac{\partial P^1}{\partial x}(x,y;s)>0 \text{ for } y>y'.$$

The parameterized expected utilities $EU_1(x,y;s)$ are defined according to these probabilities. (A similar definition could of course be applied to P^2 and EU_2, with $y^*<0$ assumed so that the opposition condition of section 2 is satisfied). Then with this setup, the following result holds:

THEOREM 7: Fix any y such that $0<-y<x^*$. Then

$$\lim_{s\to 0} \operatorname*{argmax}_x EU_1(x,y;s) = -y.$$

Furthermore, the convergence will be from below, i.e., from the direction of the estimated median, $m=0$.

In a symmetrical contest in which the candidates' ideal points are equidistant from m and on opposite sides, this means that if candidate 2 adopts his ideal point then under slight uncertainty candidate 1 will also choose to be very close to his own ideal. Should y move closer to m, if uncertainty is small then so will x. Most important, when y is very near m, if uncertainty is small then x will also be very near m. The second part of Theorem 7 is critical: it indicates that the two candidates will indeed respond to initially divergent positions by moving closer to m. Finally, if candidate 2 chooses an extreme y so that $-y\geqslant x^*$, it is obvious from properties (2) and (3) that candidate 1 will react with $x\leqslant x^*$. Thus for any particular "nonmedian" position of the opponent, if uncertainty is small enough, then a candidate will want to be nearer the estimated median than his opponent is (see Lemma 6 to this theorem in the Appendix). Again, the candidate convergence result proves to be robust.

Summary

Theorems 6 and 7 demonstrate facts about the responses of the candidates to one another's positions when one of our two traditional assumptions is dispensed with and the other is then gradually relaxed. In particular, they show that the tendency of the candidates to diverge from the original, convergent outcome is limited by the extent of policy motivation or uncertainty. For any fixed position of the opponent, the divergence tendency can be made arbitrarily small. If equilibrium exists, the theorems imply that it must feature candidate platforms that are close together when office motivation or certainty is only slightly violated. If equilibrium does not exist, which is possible even in the unidimensional case once certainty is dispensed with, the theorems at least hint that the candidates' movements in reaction to one another can be constrained to an arbitrarily small part of the issue space.

However, these results are still limited in the following ways. First, it is not clear whether they still hold under arbitrary candidate utility functions, and whether they have any analogues in multidimensional candidate competition. Second, it is not known how the convergence equation in either theorem compares over different opponent platforms y. That is, suppose that, for a given level of uncertainty, opponent position $y < m$ will cause the candidate to react by choosing an x that is closer to the median than is y. Will a more extreme $y' < y$ also lead to an x' closer to m than is y'? This is a reasonable conjecture since the region between the ideal points is closed and bounded and a good deal of continuity is assumed, but a proof has been elusive.

In any event, Theorems 6 and 7 indicate that the divergence result of Theorem 5 is not the last word on candidate competition in this nontraditional model. If the extent of uncertainty is great and if policy motivation is complete, then indeed the candidates might diverge completely to their ideal points, just as Wittman (1983) indicates. But if we treat policy motivation and uncertainty as mere complicating factors in a world where candidates are mostly interested in winning office and are fairly well able to predict voter responses, then this divergence is itself just a complicating factor, small when the other complications are small. Electoral competition still involves an underlying pressure for candidates to converge.

5. Conclusion

The traditional multidimensional model of electoral competition, as used, for example, by Davis and Hinich (1968), concludes that candidate platforms will converge to the median[12] voter in equilibrium. Among its simplifying

[12] Davis and Hinich called it the "mean," which of course is the median in a symmetric distribution. Their approach deals more with the median properties of that position, however. Hinich (1978) has since obtained results indicating that there is a separate significance to the mean as such under certain assumptions.

assumptions are symmetry of the distribution of voter ideal points, a single-minded devotion of the candidates to winning office, and full information about how the electorate will respond to the platforms. Previous work by Kramer (1977, 1978) and McKelvey (1983) indicates that the traditional model's conclusions are altered only in a gradual way as the model departs from the symmetry assumption. The present study provides similar results concerning the other two assumptions. If candidates are policy motivated and if they do not have full advance information about the electoral results of their platforms, then candidate convergence still occurs, either fully or "almost," depending on the extent to which the assumptions are relaxed. Further, the median voter, or a probabilistic analogue to it, carries a significance similar to that in the traditional model.

The traditional model thus meets an important test for a useful abstraction: its conclusions are robust to small departures from its simplifying assumptions. It serves as a useful, that is, reasonably tractable and arguably accurate, conceptual device for thinking about electoral competition in general. It provides a good starting point for more detailed modeling of particular electoral institutions, which themselves may seriously violate the traditional model's assumptions and conclusions, because it reveals the properties that underlie all electoral competition, even though these properties may be counteracted by the particular conditions of a given real world situation. Thus, although convergence to the center of voter opinion is a pressure that generally underlies democratic electoral systems, we cannot necessarily conclude that the policies of a given real government will tend toward that center.

Finally, to look at it in a slightly different way, this study implies that the traditional model is a true prototype for more specialized models of electoral competition, rather than a misleading special case. One might think about the alternative versions as a (two-dimensional) continuum: at one extreme is the traditional model, in which candidates are interested only in winning, electoral outcomes are perfectly predictable, and platforms converge to the median. Completely weakening the assumptions gives another extreme: if candidates are interested solely in policy outcomes, and if probabilities of winning are independent of the candidates' positions, then the candidates will diverge completely to their respective ideal points. In between, variations in the assumptions yield corresponding gradations in the divergence of candidates. We could as well include in this scheme results based on symmetric versus asymmetric voter distributions, adding another dimension of the same kind (although some regions of the resulting three-dimensional continuum have not yet been explored). Short of the policy-motivated, zero-information extreme, one implication of the prototype model always remains: any electoral system has a built-in pressure favoring convergence of competing alternatives toward the center of the voter distribution.

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APPENDIX

Proofs of Theorems

To establish Theorem 1, we need the following technical lemma, whose proof is illustrated in Figure A1.

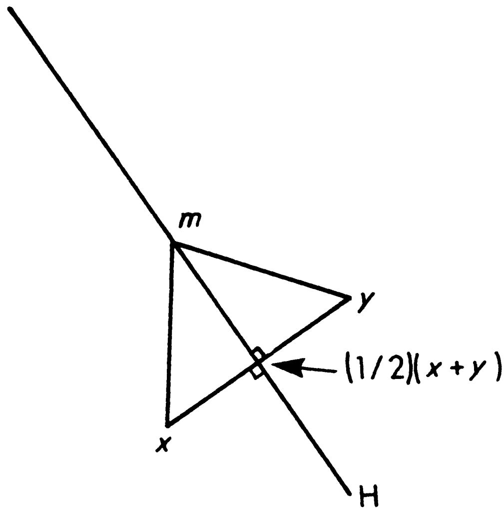


FIGURE A1

Lemma to Theorem 1

LEMMA: Suppose f is symmetric, with median m. Then $\phi(x, y) = 0$ if and only if

$||x - m|| = ||y - m||$.

PROOF: $\phi(x, y) = 0$ if and only if there is some vector h such that

(1) $\int_{\{z \mid z \cdot h > 0\}} f(z)\, dz = \int_{\{z \mid z \cdot h < 0\}} f(z)\, dz,$

since we assumed spherical indifference curves. This h is normal to a hyperplane H satisfying

(2) $\frac{1}{2}(x + y) \in H$; and

$(x - y) \cdot (z - w) = 0$ for all z and $w \in H$.

That is, H is the set of perpendicular bisectors of the line segment from x to y. Because h satisfies (1), and because m is a median, m must be in the hyperplane H. By (2), then,

$$(x - y) \cdot \left[\frac{x + y}{2} - m\right] = 0.$$

Multiplying by ½ and rearranging terms in the first factor,

$$\left[x - \frac{x+y}{2}\right] \cdot \left[\frac{x+y}{2} - m\right] = 0.$$

The Pythagorean theorem therefore applies (see Figure A1):

$$(3) \quad ||x - m||^2 = ||x - \tfrac{1}{2}(x + y)||^2 + ||\tfrac{1}{2}(x + y) - m||^2,$$

and likewise

$$(4) \quad ||y - m||^2 = ||y - \tfrac{1}{2}(x + y)||^2 + ||\tfrac{1}{2}(x + y) - m||^2.$$

But

$$(5) \quad x - \tfrac{1}{2}(x + y) = \tfrac{1}{2}(x - y) = \tfrac{1}{2}(x + y) - y;$$

the norms of the left- and right-hand sides of (5) are equal; so combining (3) and (4) gives

$$||x - m||^2 = ||y - m||^2$$

as required.

THEOREM 1: Suppose that m is the median of f and that the following hold:

(i) $\dfrac{x^* - m}{||x^* - m||} \neq \dfrac{y^* - m}{||y^* - m||}$; and

(ii) $\int_N f(z)\, dz > 0$ for every open set N containing m.

Then $(\bar{x}, \bar{y})$ is an equilibrium if and only if $\bar{x} = m$ or $\bar{y} = m$.

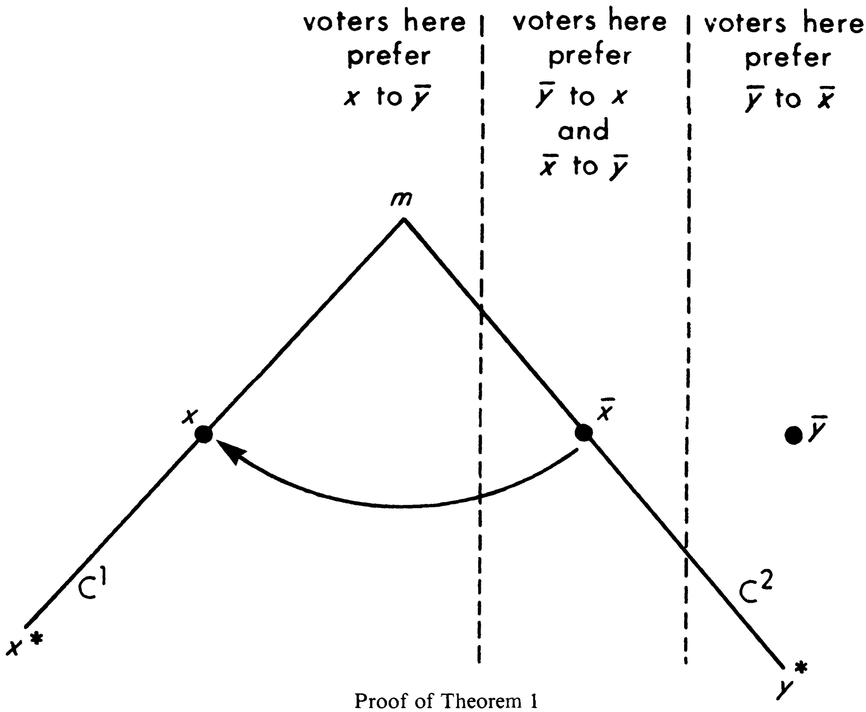
PROOF: Suppose $\bar{x} \neq m$ and $\bar{y} \neq m$, and consider the case in which $\phi(\bar{x}, \bar{y}) > 0$. By moving to the point y such that $||\bar{x} - m|| = ||y - m||$ and such that distance from y^* is minimized (so he is in C^2), candidate 2 could achieve a tie, according to the lemma. He will prefer to do so, that is, $u_2(y)$ will be greater than $u_2(\bar{x})$, unless that y is equal to $\bar{x}$. Thus unless $\bar{x} \in C^2$, $(\bar{x}, \bar{y})$ cannot be a Nash equilibrium. But now suppose $\bar{x} \in C^2$. Then candidate 1 could maintain his winning outcome and gain utility by moving to the point x such that $||x - m|| = ||\bar{x} - m||$ and $x = \alpha m + (1 - \alpha)x^*$ for some $0 < \alpha < 1$, that is, to $x \in C^1$ (see Figure A2). Candidate 1 still wins since he is closer to m than is $\bar{y}$ (an immediate consequence of the lemma). Thus $(\bar{x}, \bar{y})$ is not a Nash equilibrium. A similar proof holds when $\phi(\bar{x}, \bar{y}) < 0$.

Next suppose $\phi(\bar{x}, \bar{y}) = 0$, still assuming $\bar{x} \neq m$ and $\bar{y} \neq m$. By the lemma, $||\bar{x} - m|| = ||\bar{y} - m||$. By the same type of argument as above, if $x \notin C^1$ or if $y \notin C^2$ one candidate can maintain a tie while moving to a preferred position, and hence $(\bar{x}, \bar{y})$ cannot be an equilibrium. So suppose $\bar{x} \in C^1$ and $\bar{y} \in C^2$. Let $x = \varepsilon m + (1 - \varepsilon)\bar{x}$ for some small $\varepsilon > 0$. By continuity of u_1, ε can be small enough that

$$u_1(x) = u_1(\bar{x}) - \delta$$

for arbitrarily small $\delta > 0$. Since $\bar{x} \neq \bar{y}$ by the opposition assumption, we can choose ε small enough so that

$$u_1(x) > \frac{u_1(\bar{x}) + u_1(\bar{y})}{2}.$$



Proof of Theorem 1

Finally, since there is a positive proportion of voters near m, moving slightly closer to m picks up those voters for candidate 1; the closed half-space of voter ideal points at least as close to x as to $\bar{y}$ now contains more than half the ideal points, so $\phi(x, \bar{y}) > 0$. In words, this means candidate 1 can move slightly closer to m along C^1 so as to change the tie to a win and have a position he prefers to the tie he started with. (The same could of course be said for candidate 2.)

Thus $(\bar{x}, \bar{y})$ is never a Nash equilibrium when neither platform is equal to m.

Conversely, suppose that $\bar{x} = m$. Because m is a majority-rule equilibrium, the best candidate 2 could do would be to tie $\bar{x}$ by choosing $y = m$. But if $y = m$, it is *not* true that

$$\frac{u_2(\bar{x}) + u_2(y)}{2} > u_2(\bar{x});$$

in fact, the two values are equal. Hence in the Nash equilibrium sense candidate 2 has no incentive to change from $\bar{y}$. A similar argument holds for $\bar{y} = m$. Thus $\bar{x} = m$ or $\bar{y} = m$ implies that $(\bar{x}, \bar{y})$ is a Nash equilibrium.

Theorem 2: Consider the two-player game in which each player's pure strategy set is the issue space S (the set of possible platforms) and in which the outcomes for each player i are values $u_i(z)$ where z is the strategy of the winning candidate. Then m is a maximin strategy for either player, and it is the only pure strategy that is a maximin strategy.

Proof: Suppose candidate 1 chooses $x = m$. Since he will then either tie with $y = m$ or win, he is guaranteed a payoff of $u_1(m)$. Any other choice of a pure strategy leaves open the possibility that candidate 2 could win with $u_1(y) < u_1(m)$. Hence $u_1(m)$ is candidate 1's security level, achievable in pure strategies only through $x = m$.

Suppose candidate 1 chooses a mixed strategy given by the probability measure p on S. Again, his security level is no higher than $u_1(m)$, since $y = m$ would ensure that outcome against p. Thus no mixed strategy has a security level higher than $u_1(m)$.

Similar arguments, of course, hold for candidate 2.

THEOREM 3: Suppose the candidates choose their platforms to maximize their probabilities of winning the election. Then under Assumptions 1 and 2, any equilibrium (x, y) must have $x = y$, or else (x, x) and (y, y) are both equilibria as well.

PROOF: This is a simple consequence of the assumptions. They require that the candidate competition is a two-player, constant-sum (because $P^1 + P^2 = 1$), symmetric (because $P^1(x, y) = P^2(y, x)$) game. For any such game, the equilibria must be as given by the theorem.

THEOREM 4: Under Assumptions 1 and 3, there is a unique equilibrium in which each candidate adopts his estimated median.

PROOF: First, suppose each candidate adopts his estimated median, m_i. Then each estimates that he has a probability of winning of .5, by Assumption 1 and property (1). Hence (m_1, m_2) is an equilibrium, also by property (1).

To see that this equilibrium is unique, suppose that (x, y) is an equilibrium. By Assumption 1, one candidate must have a probability of winning no greater than .5. Suppose $P^1(x, y) \geqslant .5$. By the definition of equilibrium, x maximizes P^1 against y. By property (2), if candidate 1 moves along the curve $c(t)$ toward m_1, his probability of winning strictly decreases, so $P^1(m_1, y) < ½$, contradicting property (1). The only way to avoid this contradiction is if $x = m_1$. But then by property (1) it must be candidate 2 who has $P^2 \leqslant .5$, and the same argument applies; hence $y = m_2$.

THEOREM 5: Let z be any point in the interior of S, and suppose that $x \neq x^*$ or $y \neq y^*$ (or both). If, for either $i = 1$ or $i = 2$, u_i is not flat at z and P^i is nondegenerate at (z, z), then (z, z) is not an equilibrium.

PROOF: Suppose $i = 1$, and fix candidate 2's platform at z. Let

$$d = \frac{\nabla u_i(z)}{||\nabla u_i(z)||}$$

be the direction of 1's utility gradient. Since u_i is not flat, there is a value ε_0 such that for all $\varepsilon \geqslant \varepsilon_0$,

$$u_i(z + \varepsilon d) > v_i(z).$$

For some sufficiently small $\delta_0 > 0$, nondegeneracy implies

$$P^1(z + \delta d, z) > 0$$

for all $\delta \leqslant \delta_0$. Let γ be the minimum of δ_0 and ε_0. At z, 1's expected utility is

$$P^1(z, z)u_1(z) + [1 - P^1(z, z)]u_1(z),$$

while at $z + \gamma d$ his expected utility is

$$P^1(z + \gamma d, z)\, u_1(z + \gamma d) + [1 - P^1(z + \gamma d, z)]\, u_1(z).$$

Since $P^1(z + \gamma d, z) > 0$ and $u_1(z + \gamma d) > u_1(z)$, position $z + \gamma d$ is preferred by candidate 1. Thus (z, z) is not an equilibrium.

THEOREM 6: Fix candidate 2's position y. Then

$$\lim_{t \to 0} \operatorname*{argmax}_x v_1(x, y; t) = y + k(y) \equiv \operatorname*{argmax}_x P^1(x, y).$$

PROOF: Let $EU^* = \max_x EU_1(x, y)$. In what follows, it is assumed that $EU^* \geqslant 0$. Straightforward changes will accomplish the proof when $EU^* < 0$. Furthermore, this proof assumes that $y < y + k(y) < x^*$; similar proofs will handle permutations of this ordering. Notice that for

$x < y + k(y)$, both $P^1(x, y)$ and $u_1(x)$, and thus $v_1(x, y; t)$, are lower than they would be at $y + k(y)$. Hence we can ignore the interval $(-\infty, y + k(y)]$.

Choose any k_0 such that $y + k(y) < y + k_0$. Let

$$t_0 = \frac{P^1(y + k(y), y) - P^1(y + k_0, y)}{EU^* + |u_1(y)|}$$

Then for all $x > y + k_0$,

$$t < t_0 \Rightarrow tEU^* + P^1(y + k_0, y) < P^1(y + k(y), y) - t|u_1(y)|.$$

But we know $EU_1(x, y) \leqslant EU^*, EU_1(y + k(y), y) \geqslant u_1(y)$, and $P^1(x, y) \leqslant P^1(y + k_0, y)$, so this gives

$$v_1(x, y; t) < v_1(y + k(y), y; t).$$

In other words, for an arbitrarily small interval above $y + k(y)$, we can find t sufficiently small that $v_1(x, y; t)$ is maximized in that interval. That is,

$$\lim_{t \to 0} \operatorname*{argmax}_{x} v_1(x, y; t) = y + k(y)$$

as required.

The proof of Theorem 7 is best accomplished via a series of lemmas. As mentioned in the text, Lemma 6 itself has a substantive interpretation. Throughout these proofs, let us assume $y < 0$, so $-y$ is the position opposite the median from y and closer to x^* (see Figure A3). Furthermore, assume, as in the statement of Theorem 7, that $-y < x^*$.

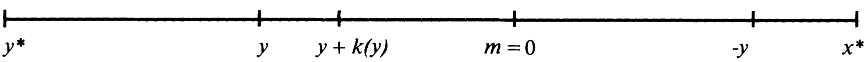


FIGURE A3

Basic Situation for Theorem 7 and Its Lemmas

LEMMA 1: Fix $\varepsilon > 0$. As $s \to 0$, $EU_1(-y + \varepsilon, y; s) \to y$.

PROOF: $EU_1(-y + \varepsilon, y; s) = (-y + \varepsilon)\, P^1(-y + \varepsilon, y; s) + y[1 - P^1(-y + \varepsilon, y; s)]$. Since $\varepsilon > 0$, by property (3b)

$$\lim_{s \to 0} P^1(-y + \varepsilon, y; s) = 0.$$

Thus

$$\lim_{s \to 0} EU_1(-y + \varepsilon, y; s) = y.$$

LEMMA 2: $EU_1(-y, y; s) \to 0$ as $s \to 0$

PROOF: As $s \to 0$, property (2) gives $P^1(-y, y; s) \to .5$. Thus

$$-yP^1(-y, y; s) + y[1 - P^1(-y, y; s)] \to \frac{-y}{2} + \frac{y}{2} = 0.$$

LEMMA 3: Fix any positive $\varepsilon < y/(y - x^*)$. There is an $s_0 > 0$ such that

$$s < s_0 \Rightarrow \operatorname*{argmax}_{x \geqslant -y} EU_1(x, y; s) < -y + \varepsilon.$$

PROOF: By property (5a), $P^1(x, y; s) < P^1(-y + \varepsilon, y; s)$ for all $x > -y + \varepsilon$. By Lemma 1, there is an s_1 such that

$$s < s_1 \Rightarrow P^1(-y + \varepsilon, y; s) < \varepsilon.$$

Thus $P^1(x, y; s) < \varepsilon$ for all $x > -y + \varepsilon$, and we have

$$\begin{aligned} EU_1(x, y; s) &< x\varepsilon + y(1 - \varepsilon) \\ &\leqslant x^*\varepsilon + y(1 - \varepsilon) \text{ for all } x \geqslant -y + \varepsilon. \end{aligned}$$

Now by Lemma 2, for any $\delta > 0$ there is an s_2 such that

$$s < s_2 \Rightarrow EU_1(-y, y; s) > -\delta$$

If we choose this $\delta < -x^*\varepsilon - y(1 - \varepsilon)$, which is possible since $\varepsilon < y/(y - x^*)$, then

$$-\delta > x^*\varepsilon + y(1 - \varepsilon)$$

and we have, for $s < \min\{s_1, s_2\}$,

$$EU_1(-y, y; s) > EU_1(x, y; s) \text{ for all } x \geqslant -y + \varepsilon.$$

Letting $s_0 = \min\{s_1, s_2\}$, the result is obtained.

LEMMA 4: Fix any positive $\varepsilon < -2y$. As $s \to 0$,
$EU_1(-y - \varepsilon, y; s) \to -y - \varepsilon$.

PROOF: Since $|-y - \varepsilon| < |y|$, by property (3a)

$$P^1(-y - \varepsilon, y; s) \to 1 \text{ as } s \to 0.$$

Thus,

$$(-y - \varepsilon) P^1(-y - \varepsilon, y; s) + y[1 - P^1(-y - \varepsilon, y; s)] \to -y - \varepsilon.$$

LEMMA 5: Fix any positive $\varepsilon < \min\{.5, -y/4\}$. There is an $s_0 > 0$ such that

$$(x \geqslant -y \text{ and } s < s_0) \Rightarrow EU_1(x, y; s) < EU_1(-y - \varepsilon, y; s).$$

PROOF: For $x \geqslant -y + \varepsilon$, by Lemma 3 there is an s_1 such that

$$s < s_1 = EU_1(x, y; s) < \varepsilon.$$

For x in the interval $[-y, -y + \varepsilon)$, we get

$$\begin{aligned} (6) \quad EU_1(x, y; s) &= xP^1(x, y; s) + y[1 - P^1(x, y; s)] \\ &\leqslant xP^1(-y, y; s) + y[1 - P^1(-y, y; s)] \\ &= (x + y)P^1(-y, y; s) + (-y)P^1(-y, y; s) + y[1 - P^1(-y, y; s)]. \end{aligned}$$

Now by property (2) there is an s_2 such that

$$s < s_2 \Rightarrow P^1(-y, y; s) < .5 + \varepsilon.$$

Since we also have $x + y < \varepsilon$, $s < s_2$ gives, using inequality (6),

$$EU_1(x, y; s) < \varepsilon(.5 + \varepsilon) + EU_1(-y, y; s).$$

Since we chose $\varepsilon < .5$, this is

$$EU_1(x, y; s) < \varepsilon + EU_1(-y, y; s).$$

By Lemma 2, there is an s_3 such that

$$s < s_3 \Rightarrow EU_1(-y, y; s) < \varepsilon.$$

Hence for $s < \min\{s_2, s_3\}$, we have $EU_1(x, y; s) < 2\varepsilon$ whenever x is in the interval $[-y, -y + \varepsilon)$.

Finally, by Lemma 4 there is an s_4 such that

$$s < s_4 \Rightarrow EU_1(-y - \varepsilon, y; s) > -y - \varepsilon - \varepsilon = -y - 2\varepsilon > 2\varepsilon,$$

the latter since $\varepsilon < -y/4$ gives $-y > 4\varepsilon$. Thus for $s < \min\{s_2, s_3, s_4\}$ we have

$$EU_1(x, y; s) < 2\varepsilon < EU_1(-y - \varepsilon, y; s) \text{ for } x \text{ in } [-y, -y + \varepsilon),$$

while for $x \geqslant -y + \varepsilon$, $s < s_1$ gave us

$$EU_1(x, y; s) < \varepsilon < 2\varepsilon < EU_1(-y - \varepsilon, y; s).$$

Letting $s = \min\{s_1, s_2, s_3, s_4\}$ gives the desired result.

LEMMA 6: For any $y < 0$, there is an $s_0 > 0$ such that

$$s < s_0 \Rightarrow \underset{x}{\operatorname{argmax}} EU_1(x, y; s) < -y.$$

PROOF: Immediate from Lemma 5.

THEOREM 7: Fix any y such that $0 < -y < x^*$. Then

$$\lim_{s \to 0} \underset{x}{\operatorname{argmax}} EU_1(x, y; s) = -y.$$

Furthermore, the convergence will be from below, i.e., from the direction of the estimated median, $m = 0$.

PROOF: By reasoning similar to that in Theorem 5, $EU_1(x, y; s)$ will never be maximized by $x < y$. Furthermore, if s is small enough, $EU_1(-y, y; s) > 0$ by property (2), so continuity of P^1 means that Theorem 5 applies, and EU_1 is not maximized at $x = y$ either. By Lemma 6, $EU_1(x, y; s)$ is maximized at $x < -y$ for sufficiently small s. Hence we can find an s_0 such that

$$s \leqslant s_0 \Rightarrow y < \underset{x}{\operatorname{argmax}}\, EU_1(x, y; s) < -y.$$

Now let x_0 be any position strictly between y and $-y$, and choose x_1 such that $x_0 < x_1 < -y$. By Lemma 4 there is an $s_1 < s_0$ such that

$$s < s_1 \Rightarrow |EU_1(x_0, y; s) - x_0| < \frac{x_1 - x_0}{2}$$

$$\text{and} \qquad |EU_1(x_1, y; s) - x_1| < \frac{x_1 - x_0}{2}.$$

Hence

$$EU_1(x_1, y; s) > EU_1(x_0, y; s).$$

That is, for any $x_0 < -y$ we can find s sufficiently small that $EU_1(x, y; s)$ is maximized *closer* to $-y$. This proves the theorem.

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