MATH214 Homework 1

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1 Exercise 1

1.1 Question 1

Let $(S, *)$ and $(S', *')$ and $(S'', *'')$ be three algebraic structures. Suppose $f : S \to S'$, $g : S' \to S''$ and $h = f \circ g : S \to S''$.

$$\text{Then we have} \qquad \begin{cases} f(x * y) = f(x) *' f(y) \\ g(f(x) *' f(y)) = g \circ f(x) *'' g \circ f(y) \end{cases}$$

Thus we have $g \circ f(x * y) = g \circ f(x) *'' g \circ f(y)$, namely $h(x * y) = h(x) *'' h(y)$. So the composition of f and g is a homomorphism.

1.2 Question 2

According to the definition of isomorphisim, an isomorphisim is a bijection.
Let f be an isomorphisim from S to S' and $a \in S$, $a' \in S'$. Since f is a bijection, we have $f(a) = a'$ and $f^{-1}(a') = a$. We can easily observe that f^{-1} is also bijective becuase f is the inverse function of f^{-1}.
Next we need to show that f^{-1} is a homomorphism. $\forall x', y' \in S'$,

$$\begin{aligned} f^{-1}(x' *' y') &= f^{-1}(f(x) *' f(y)) \\ &= f^{-1}(f(x * y)) \\ &= x * y \\ &= f^{-1}(x') * f^{-1}(y) \end{aligned}$$

1.3 Question 3

a) homomorphism
b) endomorphism
c) automorphism
d) automorphism
e) homomorphism

2 Exercise 2

Set 3, 4, 6 are vector subspaces meeting the requirments.

3 Exercise 3

3.1 Question 1

$\forall \alpha \in V$ and $\forall k \in \mathbb{R}$, we have $k \cdot \alpha = k \cdot (x, y, z) = (kx, ky, kz)$. It's obvious that $kx - 2ky + 3kz = k(x - 2y + 3z) = 0$. So V is a subspace of $\mathbb{R}^3$.

3.2 Question 2

A set is a collection of distinct objects.
A vector space is a mathematical structure that consists of a set of vectors, together with operations of addition and scalar multiplication.

3.3 Question 3

Since $B \subset C$, we only need to prove $C \subset B$, namely $\forall c \in C$, $c \in B$.
Since $A + B = A + C$, $\forall a \in A$ and $b \in B$, we have

$$\begin{cases} a + b \in A + C \\ a + c \in A + B \end{cases}$$

A is a subspace of V, so $0 \in A$. If $a = 0$, then $c \in A + B$. Since $c \in C$, $c \in B$.
That is, $C \subset b$, $B = C$.

3.4 Question 4

First, show that $A \cap (B + (A \cap C)) \subset (A \cap B) + (A \cap C)$.
$\forall x \in A \cap (B + (A \cap C))$, we have

$$\begin{cases} x \in A \\ x \in (B + (A \cap C)) \end{cases}$$

3.5 Question 5

I'm afraid this is beyond me.

4 Exercise 4

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