EECE 405/560 —- Homework #1

405 students must hand in problems 1-4, and will receive extra credit for problems 5-6. 560 students must hand in all 6 problems.

1. Let a dart be thrown uniformly at the square dartboard we discussed in class. Let X and Y be the random x- and y-coordinates of the dart's location, and let Z=max(X,Y). Find E[Z] and Var[Z].

   Do so by the following steps:

   1. Draw a picture of the dartboard, and draw the region corresponding to the statement that "$Z \leqslant 0.5$"
   2. Using the picture, write a general formula for the cdf $F(a) = Pr[\max(X,Y) \leq a]$
   3. Differentiate this to get the pdf f(a)
   4. Use f(a) to compute the expected value $E[Z] = E[\max(X,Y)]$
   5. Compute $E[Z^2]$ the same way, and use this with E[Z] to compute Var[Z].

2. Let a dart be thrown uniformly at a circular dartboard of radius R, and let the random variable D denote the distance of the dart from the center. Using the same steps as the last problem, compute E[D] and Var[D].

3. A random variable T takes on values {0, 1, 2, … N} with probability $Pr[T=k] = k/C$, where C is a constant.
   1. What is the value of C?
   2. What is the expected value of T?

4. This pdf is a file consisting of a string of bytes. Determine the distribution of bytes in its file. Provide your answer as a list of all the byte values and their count; which byte value is the most common?

5. A spammer flips a fair coin. If the coin comes up HEADS, the spammer sends 100,000 spam emails to 100,000 randomly chosen individuals. If the coin comes up TAILS, the spammer sends 200,000 spam emails to 200,000 randomly chosen individuals.

   If you are the recipient of one of these emails, you can be taken off the spam list by correctly guessing whether the coin flip came up HEADS or TAILS. Which do you guess, and what is the probability that your guess is right?

6. Using the definition of a random variable and the axioms defining events (next page) prove the following. For step of your proof you should cite the specific axiom that you use.
   1. Prove that axiom 3 follows from axioms 1 and 2.
   2. Prove that axiom 7 follows from axioms 1-6.
   3. Prove that if $E \subseteq F$, then $Pr[E] \leq Pr[F]$.
   4. Prove that the cdf F(a) is nondecreasing: that if $a \leq b$, then $F(a) \leq F(b)$.

Definition of a Probability Space:

A probability space consists of three parts:

A. A set of **outcomes** Ω, with one specific outcome $\omega \in \Omega$ occurring in a random observation;
B. A collection of **events**, subsets of Ω that are assigned probabilities;
C. A **probability function** Pr[E] that assigns a real number to each event $E \subseteq \Omega$.

The events satisfy the following axioms:

1. If E is an event, so is its complement Ω-E (defined as the set $\{\omega \in \Omega$ with $\omega \notin E\}$)
2. If E and F are events, so are $E \cap F$ and $E \cup F$
3. Both Ω and the empty set $\varnothing$ are events

The probability function Pr[] satisfies the following axioms:

4. $0 \leq Pr[E] \leq 1$
5. $Pr[\Omega - E] = 1 - Pr[E]$
6. If $(E \cap F) = \varnothing$ (that is, if E and F are disjoint,) then $Pr[E \cup F] = Pr[E] + Pr[F]$.
7. $Pr[\Omega] = 1$ and $Pr[\varnothing] = 0$

We actually have a slightly stronger version of axioms 2 and 6 to handle infinite sequences of events. We won't discuss them yet.

Definition of a Random Variable:

A random variable X() is a function assigning an outcome to a real number, such that the statements "$X(\omega) \leq a$" and "$X(\omega) < a$" have a well-defined probability: that is, we have axioms

7. The set $E \subseteq \Omega$ defined by $E = \{ \omega \in \Omega$ where $X(\omega) \leq a \}$ is an event.
8. The set $E \subseteq \Omega$ defined by $E = \{ \omega \in \Omega$ where $X(\omega) < a \}$ is an event.