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A lower bound on Voronoi diagram complexity

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The Voronoi diagram is a much-studied object in geometry. Informally, one starts with a set of disjoint *sites*, say, points in the plane, and a measure of distance, say, the Euclidean metric. The Voronoi diagram is then a classification of points of the ambient space according to the identity of the closest site or sites. In the above example, it partitions the plane into n open convex *Voronoi cells* each consisting of points that are strictly closer to a specific site than to any other one, *Voronoi edges* consisting of points for which exactly two sites are simultaneously closest, and *Voronoi vertices* which are points from which three or more sites are closest. The number of cells, edges, and vertices is the *combinatorial complexity* of the diagram. In d dimensions, the diagram is defined analogously and in general contains faces of all dimensions, from 0 up to d. Its *complexity* is the total number of (maximal connected) faces of all dimensions.

The Voronoi diagram can similarly be defined in an arbitrary space (such as $\mathbb{R}^d$, for any $d > 0$, d-dimensional sphere $\mathbb{S}^d$, etc.) and for any "reasonable" metric, such as an L_p metric, the metric defined by a centrally symmetric convex "unit ball", or even some "distance functions" which need not be metrics; see, e.g., [9]. One can also allow different types of sites—single points, flats, convex bodies, or more general objects. The interested reader may consult the references [5,9, 1,7,11,8,2] for different variants of Voronoi diagrams that have been considered in the literature.

With very few exceptions, the Voronoi diagram of n pairwise disjoint (possibly non-point) sites in the plane has combinatorial complexity $\Theta(n)$, for a wide variety of choices of the metric and the class of sites. In three dimensions, tight or nearly tight bounds on the worst-case complexity of Voronoi diagrams are known in relatively few special cases, such as for point sites with the Euclidean metric, or for line sites with the distance function defined by a bounded-complexity polyhedral unit ball; see also [5,13,4,3]. The bounds are near-quadratic in n in all these cases. This is conjectured to be true for a much larger class of Voronoi diagrams [4,3,11,12]. Remarkably, no good bounds are known even for Voronoi diagrams of lines in Euclidean 3-space!

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In $\mathbb{R}^d$, the problem has been explored with even less success. The worst-case complexity of the Euclidean Voronoi diagram of points is $\Theta(n^{\lceil d/2 \rceil})$ and the same bounds have been proven for the L_∞ metric and for the distance function defined by a simplex [5,3]. It has been conjectured by Sharir [3,11] that the same or only slightly worse bound holds for more general classes of diagrams, including, at least, those induced by L_p metrics and having arbitrary flats as sites. It has been observed that the general lower-envelope bound of Sharir [10,12] implies that, for fairly general algebraically-defined distance functions and site classes, the complexity of the diagram cannot exceed $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$, with the constant of proportionality depending on ε. This bound is much weaker than the conjectured one for $d > 1$, but applies to a larger class of problems. For example, it holds also when the sites are not disjoint and is easily seen to be close to optimal in this case (in almost any "reasonable" metric the Voronoi diagram of the grid formed by d mutually orthogonal families each consisting of n/d parallel hyperplanes would have complexity $\Omega(n^d)$).

In this note we provide some evidence that the bound derived from envelope analysis is closer to the truth, as the conjecture of Sharir does not hold. Specifically, we construct a collection of n pairwise disjoint $(d-2)$-flats in $\mathbb{R}^d$ that has Voronoi diagram of complexity $\Omega(n^{d-1})$.

Proposition 1. *For $d \geqslant 2$ and sufficiently large n, there is a set of n pairwise disjoint $(d-2)$-flats in $\mathbb{R}^d$ whose Voronoi diagram in the L_∞ metric has complexity $\Omega(n^{d-1})$.*

Proof. The case $d = 2$ is just the Voronoi diagram of n points in $\mathbb{R}^2$, which has complexity $\Theta(n)$ for any set of points, so we assume $d > 2$. Without loss of generality, assume $d-1$ divides n. Put $m = n/(d-1)$. Pick $d-1$ distinct numbers $\varepsilon_1, \ldots, \varepsilon_{d-1} \ll 1$. Define $d-1$ mutually orthogonal families of parallel $(d-2)$-flats, each of size m, as follows. For $i = 1, \ldots, d-1$, the ith family consists of flats f_j^i, $j = 1, \ldots, m$, defined by $x_i = j$ and $x_d = \varepsilon_i$. Informally, in the hyperplane $x_d = 0$ we build an $m \times m \times \cdots \times m$ orthogonal grid of $(d-2)$-flats and then shift each family of parallel flats by a different small amount in $+x_d$-direction; see Fig. 1.

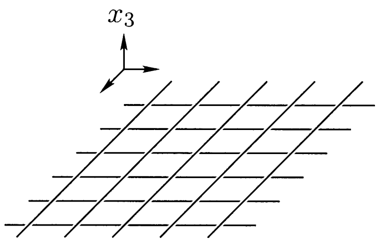

Fig. 1. Construction in $\mathbb{R}^3$.

We claim that the complexity of the Voronoi diagram in the L_∞ metric of the n $(d-2)$-flats is at least m^{d-1}. Indeed, we show that, for any choice of $(j_1, \ldots, j_{d-1}) \in [1, \ldots, m]^{d-1}$, the diagram contains a (two-dimensional) Voronoi face consisting of points equidistant to the $d-1$ flats $\{f_{j_i}^i\}$ and further from the remaining ones. Fix such a tuple of flats and consider the point $x = (j_1 + \frac{1}{4}, \ldots, j_{d-1} + \frac{1}{4}, 0)$. It lies at distance $\frac{1}{4}$ from $f_{j_i}^i$, for all i, while its distance to any other flat of the ith family is at least $\frac{3}{4}$. Hence x lies on the claimed Voronoi face and the proof is complete. □

Proposition 2. *The same lower bound holds for any L_p metric.*

Proof. We use the same set of flats and claim the existence of the same set of at least $\Omega(m^{d-1})$ Voronoi faces. Again, fix the tuple $(j_1, \ldots, j_{d-1}) \in [1, \ldots, m]^{d-1}$. We exhibit a point in the hyperplane $x_d = \frac{1}{4}$ that lies at a distance of $\frac{1}{4}$ from these $d-1$ flats and further than that from the remaining flats. Specifically, the point $x = (x_1, \ldots, x_{d-1}, \frac{1}{4})$ given by

$$x_i = j_i + \left(\left(\tfrac{1}{4}\right)^p - \left(\tfrac{1}{4} - \varepsilon_i\right)^p\right)^{1/p},$$

has the asserted properties. This completes the argument. □

If one views the Voronoi diagram as a minimization diagram of distance functions to the sites [6,10], it is clear that a slight perturbation of the position of the sites does not destroy the above lower bounds, so the flats can be chosen in general position. We also note that the argument of Proposition 2 can be generalized to handle any metric (or even any asymmetric distance function) μ with the following property:

There exist two constants $C, c > 0$, such that for any $x, y \in \mathbb{R}^d$, $c\,d(x, y) \leqslant \mu(x, y) \leqslant C\,d(x, y)$, where $d(\cdot, \cdot)$ is the Euclidean metric.

In particular, any Minkowski distance function defined by a (possibly not centrally symmetric) compact, convex body (the "unit ball") whose reference point lies in its interior satisfies this condition.

We will venture the following conjecture:

Conjecture. *For a "reasonable" class of distance functions and a "reasonable" class of sites, the maximum complexity of a Voronoi diagram of n* disjoint *sites in $\mathbb{R}^d$ is about $\Theta(n^{d-1})$.*

Note that our conjecture, not coincidentally, does not differ from Sharir's in three dimensions. In fact, it is conceivable that in higher dimensions Sharir's conjecture does hold for point sites and even for general "fat" sites.

Our conjecture extends trivially to a lower bound of $\Omega(n^{k+1})$ on the complexity of the Voronoi diagram of n k-flats in $\mathbb{R}^d$, for $k \leqslant d - 2$, but it does not interpolate smoothly between the case of point sites and that of $(d-2)$-flat sites. Obtaining such a smooth interpolations is an interesting open question.

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