Higher-order Voronoi diagrams on triangulated surfaces

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A B S T R A C T

We study the complexity of higher-order Voronoi diagrams on triangulated surfaces under the geodesic distance, when the sites may be polygonal domains of constant complexity. More precisely, we show that on a surface defined by n triangles the sum of the combinatorial complexities of the order-j Voronoi diagrams of m sites, for $j = 1, \ldots, k$, is $O(k^2 n^2 + k^2 m + knm)$, which is asymptotically tight in the worst case.

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1. Introduction

The diverse generalizations of Voronoi diagrams have important applications in several fields and application areas, such as computer graphics, geometric modelling, geographic information systems, visualization of medical data-sets, pattern recognition, robotics and shape analysis. Given a set of m primitives, called *sites*, in an ambient space equipped with a metric, the order-k Voronoi diagram, where k is an integer between 1 and $m-1$, partitions the ambient space into regions such that each point within a fixed region has the same set of k closest sites. Many variants of these diagrams can be found in the literature by considering various ambient spaces, taking sites of different shape or nature, associating weights to the sites, or changing the underlying metrics [3,4,11]. In this paper we concentrate on Voronoi diagrams defined on triangulated surfaces for polygonal sites of constant-description complexity.

Surfaces, sites, and bisectors. Let $\mathcal{P}$ be a triangulated polyhedral surface consisting of n triangles. The surface $\mathcal{P}$ may have nonzero genus. The distance $d(p, q)$ between two points $p, q \in \mathcal{P}$ of the surface is given by their geodesic distance, that is, the length of the shortest path between the two points that is contained in the surface.

In this paper, a *site* s in $\mathcal{P}$ is a polygonal domain contained in $\mathcal{P}$ with constant-description complexity; it may be a point, a segment, a polygonal path with a bounded number of segments, or a domain enclosed by a closed polygonal path with a bounded number of segments.

The distance $d(p, s)$ from a point $p \in \mathcal{P}$ to a site s is defined by the length of the shortest path from p to any point of s, that is, $d(p, s) = \min_{q \in s} d(p, q)$. The *bisector* $\beta(s, s')$ defined by sites s and s' is the locus of points at equal distance from s, s'.

A *terrain* $\mathcal{T}$ is a polyhedral surface embedded in $\mathbb{R}^3$ that is homeomorphic to a topological disk and is xy-monotone—a vertical line intersects $\mathcal{T}$ at most once. The *domain* of a terrain is the projection of $\mathcal{T}$ onto the xy-plane.

Let $S = \{s_1, \ldots, s_m\}$ be a set of m sites in $\mathcal{P}$. We make the following assumptions:

- The sites are pairwise disjoint.
- No vertex of the surface is equidistant from two sites.
- No point of the surface is equidistant from four sites.

When any of the first two assumptions is removed, it may happen that a bisector defined by two sites contains two-dimensional pieces. With the first two assumptions, a bisector does not contain two-dimensional pieces and consists of straight-line, parabolic and hyperbolic arcs [6,8]. A *breakpoint* of a bisector is an intersection point between two adjacent arcs of the bisector. A breakpoint corresponds either to a point where the bisector crosses an edge of $\mathcal{P}$ or to a point where there are two different shortest paths from one of the sites defining the bisector.[3] According to previous results [6,8], a bisector has $O(n^2)$ breakpoints.

Order-k Voronoi diagram. We are interested in the order-k Voronoi diagrams of the sites S. We next recall its definition, together with other related concepts. For a subset S' of the sites, the *Voronoi region* $V(S')$ of S' is defined as

$$\{p \in \mathcal{P} \mid d(p, s') \leqslant d(p, s), \forall s' \in S', \forall s \in S \setminus S'\}.$$

For each integer $1 \leqslant k \leqslant m - 1$, the order-$k$ Voronoi diagram of S, $Vor_k(S)$, is the family of nonempty Voronoi regions $V(S')$, where S' ranges over all subsets of S with k sites. When $k = 1$ and $k = m - 1$ the order-k Voronoi diagrams are called the (*closest-site*) *Voronoi* and the *furthest-site Voronoi* diagram, respectively. A *k-region* is a Voronoi region in the order-k Voronoi diagram. A *k-face* is a maximal connected component of a k-region. A *k-edge* is a connected component in the intersection of two k-faces. A k-edge always lies on the bisector of two sites. A *k-breakpoint* is a breakpoint on a k-edge. A *k-vertex* is the intersection point of three or more k-faces.

The terms faces, edges, and vertices of the order-k Voronoi diagram are chosen because of their resemblance with a graph embedded in the surface $\mathcal{P}$. The removal of all k-edges from the surface leaves a collection of connected sets. The closure of one such connected set is a k-face, and vice versa. Recall the third assumption we have made above: no point of the surface is equidistant from four sites. In this case, no four bisectors intersect at a common point, and hence a k-vertex has precisely three k-edges arriving to it. Finally, let us remark the difference between vertices and breakpoints: while vertices correspond to endpoint of edges, breakpoints can only appear along the relative interior of edges.

The (*combinatorial*) *complexity* of the order-k Voronoi diagram is the total number of k-faces, k-edges, k-vertices, and k-breakpoints.

Our results. Let $Vor_{\leqslant k}(S)$ denote the family of all order-j Voronoi diagrams of S for $j = 1, \dots, k$. Let us define the complexity of $Vor_{\leqslant k}(S)$ as the sum of the complexities of the order-j Voronoi diagrams, $j = 1, \dots, k$. We show that the complexity of $Vor_{\leqslant k}(S)$ is $O(k^2 n^2 + k^2 m + knm)$, and that this bound is asymptotically tight in the worst case. We also prove that the complexity of $Vor_{\leqslant k}(S)$ in so-called realistic terrains (the concept is explained in Section 4) is $O(k^2 n + k\sqrt{n}m + k^2 m)$.

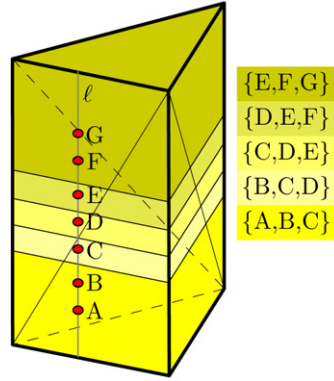[3] This definition of breakpoint is consistent with [1] but different from [2,9].

Fig. 1. Example of an order-3 Voronoi diagram with all the regions without vertices.

Related work. In the Euclidean plane $\mathbb{R}^2$, which can be regarded as a surface of constant complexity, the order-j Voronoi diagram of a set of m points has complexity $O(j(m - j))$, and hence the complexity of $Vor_{\leqslant k}(S)$ is $O(mk^2)$ [7,12]. In the case of a triangulated surface consisting of n triangles and a set of m point sites, the complexity of the closest-site Voronoi diagram is $O(n(n + m))$, while the number of vertices is $O(m)$ [8,10]. For surfaces of genus zero, Aronov et al. [1] show that the furthest-site Voronoi diagram has complexity $\Theta(mn^2)$ in the worst-case. These previous results for surfaces were written for the case where the sites are points; in this case the edges consist of straight-line and hyperbolic arcs. Combining approaches in [6] and [1,8], the results directly extend to segment, polygonal line or polygon sites, where the bisectors may also include parabolic arcs.

2. Structural properties of order-k Voronoi diagrams

2.1. Edges without vertices

Although the definition of edges and vertices resemble those of graphs embedded in $\mathcal{P}$, the situation may be quite different. The following observation is probably folklore.

Lemma 1. *There exist order-k Voronoi diagrams with k-edges that form closed curves in P and are not adjacent to any k-vertex.*

Proof. Let us consider a prism with triangular base. Consider a line ℓ parallel to an edge of the prism from one base to the other, as in Fig. 1. We place m sites on ℓ, no pair of bisectors intersect and some of them define the boundary of k-regions. Consequently k-regions may have closed curve edges without vertices. $\square$

A key property that we use below is that a k-edge that is not adjacent to any k-vertex must contain a breakpoint, as otherwise it would consist of a single arc contained in a triangle of $\mathcal{P}$, which is not possible.
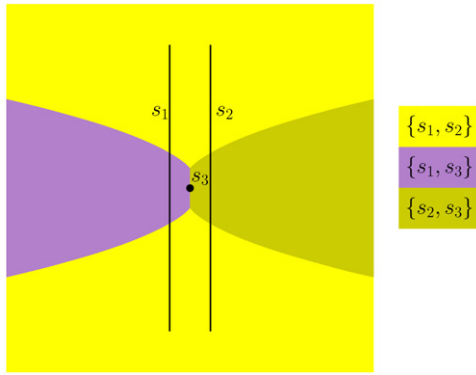
Fig. 2. Example of an order-2 Voronoi diagram in the plane for segments sites with disconnected regions.

2.2. Path-connectivity of the regions

The regions of the closest-site Voronoi diagram are path connected in any surface. The regions of the furthest-site Voronoi diagram are also path-connected in surfaces of genus zero [1]. In the Euclidean plane with point sites, any k-region is also path-connected, since in fact any k-region is convex. The situation is different for the general case, as shown in the following two constructions. The first construction is in the plane but requires using sites that are not points. The second construction is in a terrain, but only requires point sites.

Lemma 2. *For any given $k \geqslant 2$, there exists a set of sites in the* plane *such that the order-k Voronoi diagram has some disconnected regions.*

Proof. In the plane, consider the segment site s_1 between the points $(-1, -7)$ and $(-1, 7)$, the segment site s_2 between the points $(1, -7)$ and $(1, 7)$, and the point site $s_3 = (0, 0)$. See Fig. 2. The 2-region corresponding to the sites s_1, s_2 is disconnected, with a part above and a part below the line $y = 0$. We can extend the construction to any $k \geqslant 2$ by placing $k-2$ new segment sites that are small translations of s_2. ◻

Lemma 3. *For any given $k \geqslant 2$, there exists a polyhedral surface and a set of* point sites *such that the order-k Voronoi diagram has some disconnected regions.*

Proof. We describe a polyhedral surface $\mathcal{P}$ embedded in 3-dimensional Euclidean space. The construction uses three squares: square Q_1 has side length 22 and corners at

$(-11, -11, 0), (-11, 11, 0), (11, 11, 0), (11, -11, 0),$

square Q_2 has side length 20 and corners at

$(-10, -10, 1000), (-10, 10, 1000),$

$(10, 10, 1000), (10, -10, 1000),$

and square Q_3 has side length 18 and corners at

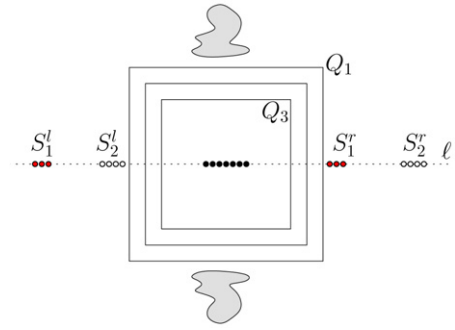$(-9, -9, 0), (-9, 9, 0), (9, 9, 0), (9, -9, 0).$

Fig. 3. Example of an order-7 Voronoi diagram with disconnected regions.

Note that the squares are parallel to the xy-plane, have their centers on the line $x = y = 0$, and have their sides parallel to the x, y coordinate axis. Squares Q_1 and Q_3 are placed at height zero and Q_2 is at height 1000. Let $\mathcal{P}$ be the polyhedral surface obtained by gluing the xy-plane minus the interior of Q_1, the pyramid with bases Q_1 and Q_2 (without the bases), the pyramid with bases Q_2 and Q_3 (without the bases), and the square Q_3. The terrain resembles a volcano. Let ℓ be the curve obtained by intersecting $\mathcal{P}$ with the vertical plane $y = 0$. See Fig. 3 for a projection onto the xy-plane.

We place $3k$ point-sites along ℓ, as follows. First, we place k sites in Q_3, symmetrically respect to its center. We then place k sites along ℓ in each side outside Q_1, distributed in four groups: S_1^l, S_2^l, S_1^r and S_2^r. Groups S_1^l, S_1^r contain $\lfloor k/2 \rfloor$ points, while groups S_2^l, S_2^r contain $\lceil k/2 \rceil$ points. Within each group, the points are regularly spaced at distance ϵ, where $0 < \epsilon \ll 1/k$. The points S_2^l are placed to the left of Q_1, along ℓ, at distance ϵ from Q_1, and the points S_1^l at distance $1/2$ from S_2^l. We do the same on the right of Q_1 but placing S_1^r closer to Q_1 than S_2^r (see Fig. 3). The construction is symmetric with respect to the plane $y = 0$. (It is also symmetric with respect to the plane $x = 0$ when k is even.)

Consider the Voronoi region R having $S_1^r \cup S_2^l$ as closest sites. First, note that R is nonempty because it contains the points $(0, 12, 0)$ and $(0, -12, 0)$ in its interior, and hence R is a k-region. Note that no point on ℓ is in R, while the points $(0, 12, 0), (0, -12, 0)$ are on different sides of ℓ. This means that R has at least two connected components. The construction for $k = 7$ is shown in Fig. 3 (xy-projection). ◻

3. Complexity of order-($\leqslant k$) Voronoi diagrams

3.1. Upper bounds

In this section we will give bounds for the complexity of $V_{\leqslant k}(S)$. Our approach is similar to the approach used by Clarkson and Shor [5] or Sharir [13]. Let B_j be the total number of j-breakpoints, and let $B_{\leqslant k}$ be the number of j-breakpoints for $j = 1, \dots, k$. Similarly, let V_j be the number of j-vertices and let $V_{\leqslant k}$ be the total number of j-vertices for $j = 1, \dots, k$.

Lemma 4. *The complexity of the order-k Voronoi diagram is $O(B_k + V_k)$.*

Proof. Let E_k and F_k be the number of k-edges and k-faces, respectively.

Each k-edge that is not adjacent to a k-vertex (cf. Lemma 1) contains some k-breakpoints. Therefore, the number of k-edges that are not adjacent to any k-vertex is bounded by B_k. The number of k-edges adjacent to some k-vertex is bounded by $3V_k$ because each k-vertex is adjacent to at most three k-edges. Therefore, we conclude that $E_k = O(B_k + V_k)$.

To bound F_k, we can use Euler's formula to obtain the bound $F_k \leqslant E_k - V_k + 2 - g$, where $g \geqslant 0$ is the genus of the surface.[4] Therefore $F_k \leqslant E_k + 2 = O(E_k)$. We conclude that the complexity of the order-k Voronoi diagram, defined as $F_k + E_k + V_k + B_k$, is $O(B_k + V_k)$. □

Lemma 5. *We have $B_{\leqslant k} = O(kn(kn + m))$ for any $1 \leqslant k \leqslant m - 1$.*

Proof. Draw a random sample R of S, by independently drawing each element of S with probability p. We will set below p to an appropriate value. Let $B(R)$ be the number of breakpoints of the order-1 Voronoi diagram of R in $\mathcal{P}$. From the bounds on the complexity of order-1 Voronoi diagrams, discussed in Section 1, we know that $B(R) = O(n^2 + |R|n)$. Since $|R|$ follows a binomial distribution, we have

$$\mathbb{E}[|R|] = mp,$$

and therefore

$$\mathbb{E}[B(R)] = O(n^2 + nmp). \tag{1}$$

Consider a breakpoint b in some j-edge of the order-j Voronoi diagram of S. Let s, s' be the two sites defining the bisector that contains b. Consider the set S_b of $j - 1$ sites distinct from s, s' that are at distance at most $d(b, s)$ from b. The point b is a breakpoint in the order-1 Voronoi diagram of the random sample R if and only if s, s' are in R and none of the $j - 1$ sites S_b is in R. Therefore, the probability that b is a breakpoint in the order-1 Voronoi diagram of R is precisely $p^2(1 - p)^{j-1}$. By linearity of expectation we then have

$$\mathbb{E}[B(R)] = \sum_{j=1}^{m-1} B_j p^2 (1 - p)^{j-1}$$

$$\geqslant p^2 \sum_{j=1}^{k} B_j (1 - p)^{j-1}$$

$$\geqslant p^2 (1 - p)^{k-1} \sum_{j=1}^{k} B_j$$

$$= p^2 (1 - p)^{k-1} B_{\leqslant k}.$$

Rearranging and substituting Eq. (1) we see

$$B_{\leqslant k} \leqslant \frac{\mathbb{E}[B(R)]}{p^2(1 - p)^{k-1}}$$

$$= \frac{O(n^2 + nmp)}{p^2(1 - p)^{k-1}}.$$

Finally, setting $p = 1/k$ we obtain

$$B_{\leqslant k} = O\left(\frac{n^2 + nm/k}{(1/k)^2(1 - 1/k)^{k-1}}\right)$$

$$= O(k^2 n^2 + knm),$$

where we have used that

$$\frac{1}{(1 - 1/k)^{k-1}} = \left(\frac{k}{k-1}\right)^{k-1} \leqslant e. \qquad □$$

Lemma 6. *We have $V_{\leqslant k} = O(k^2 m)$ for any $1 \leqslant k \leqslant m - 1$.*

Proof. This proof is very similar to the previous one. Draw a random sample R from S, by independently drawing each element of S with probability p. Let $V(R)$ be the number of vertices of the order-1 Voronoi diagram of R in $\mathcal{P}$. From the known bounds, as discussed in Section 1, we have $V(R) = O(|R|) = O(mp)$.

Consider a j-vertex v, and let s_1, s_2, s_3 be the three sites defining v: v is in the intersection of the bisectors $\beta(s_1, s_2)$, $\beta(s_1, s_3)$, $\beta(s_2, s_3)$. Consider the set S_v of sites distinct from s_1, s_2, s_3 that are at distance at most $d(v, s_1)$ from v. The cardinality of S_v is either $j - 2$ (when v is also a $(j-1)$-vertex) or $j - 1$ (when v is also a $(j+1)$-vertex). The point v is a vertex in the order-1 Voronoi diagram of the random sample R if and only if s_1, s_2, s_3 are in R and none of the sites S_v are in R. Therefore, the probability that v is a vertex in the order-1 Voronoi diagram of R is precisely $p^3(1 - p)^{|S_v|} \geqslant p^3(1 - p)^{j-1}$. Since a j-vertex is also a $(j-1)$-vertex or a $(j+1)$-vertex, each vertex is counted twice in the sum $V_1 + V_2 + \cdots + V_{m-1}$. By linearity of expectation we then have

$$\mathbb{E}[V(R)] \geqslant \frac{1}{2} \sum_{j=1}^{m-1} V_j p^3 (1 - p)^{j-1}$$

$$\geqslant \frac{1}{2} p^3 \sum_{j=1}^{k} V_j (1 - p)^{j-1}$$

$$= \frac{1}{2} p^3 (1 - p)^{k-1} V_{\leqslant k}.$$

As in the previous proof, we rearrange, substitute $V(R) = O(mp)$, and set $p = 1/k$ to get

$$V_{\leqslant k} \leqslant \frac{2 \cdot \mathbb{E}[V(R)]}{p^3(1 - p)^{k-1}}$$

$$= \frac{2m/k}{(1/k)^3(1 - 1/k)^{k-1}}$$

$$= O(k^2 m). \qquad □$$

Combining Lemmas 4–6 we can conclude our main result:

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[4] There is equality only for nonorientable surfaces without boundary where each k-face is a topological disk. For orientable surfaces we can replace the term g by $2g$.

Theorem 1. *Let $\mathcal{P}$ be a polyhedral surface with n triangles, let S be a set of m sites on $\mathcal{P}$, and let k be an integer between 1 and $m - 1$. The complexity of $Vor_{\leqslant k}(S)$ is $O(k^2n^2 + k^2m + knm)$.*

3.2. Lower bounds

We will show that the bound of Theorem 1 is asymptotically optimal in the worst case. For this, we give three constructions, each showing the need of one summand in the bound $O(k^2n^2 + k^2m + knm)$. Our constructions only use terrains, and hence the upper bounds in Theorem 1 are tight even for terrains.

We describe a construction attaining complexity $\Omega(mk^2)$. Consider the Euclidean plane and observe that the complexity of $Vor_{\leqslant k}(S)$ for a set S of m points is $\Omega(mk^2)$ [7]. Clipping the plane with a large enough rectangle, so that $n = O(1)$, we obtain the desired bound.

We describe a construction attaining complexity $\Omega(knm)$. Consider the Euclidean plane, and take the point sites $s_i = (i, 0)$, $i = 1, \ldots, m$. Sites with consecutive indices, like $s_i, s_{i+1}, \ldots, s_{i+j-1}$, define a j-region, which is a vertical strip. Therefore, Vor_j consists of $m - j + 1$ regions, and $Vor_{\leqslant k}$ has $\Omega(km)$ regions. Clip the plane with a sufficiently large axis-parallel rectangle, and triangulate it with n 'long' edges that connect the left boundary to the right boundary. Each such 'long' edge intersects each region in $Vor_{\leqslant k}$, and hence the complexity of $Vor_{\leqslant k}$ is $\Omega(kmn)$.

We describe a construction attaining complexity $\Omega(k^2n^2)$. Let us assume for simplicity that m is even. Consider the Euclidean plane and define the $m/2$ point sites $s_i = (2 + \varepsilon \cdot i, 0)$, for $i = 1, \ldots, m/2$, where $\varepsilon > 0$ is a sufficiently small value to be set below. Define also the sites s_i' by $s_i' = -s_i$, for $i = 1, \ldots, m/2$. Note that the bisector between s_i and s_i' is the line $x = 0$. Let us place n small rectangular obstacles along the line $x = 1/2$. For concreteness, let R_t be the axis-aligned square with center $(1/2, t)$ and side length $1/n$. We erect an almost vertical wall over each of the edges of each R_t. This is like effectively considering each R_t as an obstacle. Note that there is a line through s_i that separates the rectangle R_t from R_{t+1}, for any $i = 1, \ldots, m/2$ and $t = 1, \ldots, n$. See Fig. 4.

Let β denote the bisector between sites s_1 and s_1'. The bisector β is not a line anymore, but contains $\Theta(n)$ straight-line arcs—contained in the line $x = 0$—and $\Theta(n)$ hyperbolic arcs—contained in the halfplane $x \geqslant 0$. Consider a value $\delta > 0$ sufficiently small such that the vertical line $x = \delta$ intersects each hyperbolic arc in β. Let ℓ denote the vertical line $x = \delta/2$. We can take ε to be a sufficiently small value such that the line ℓ intersects all the hyperbolic arcs in any of the bisectors between a site s_i and a site s_j', for any $i, j \in \{1, \ldots, m/2\}$. This is possible because, looking at a bounded portion of the plane, the bisector $\beta(s_1, s_1')$ changes continuously with a small perturbation of the sites s_1, s_1'. Therefore, the line ℓ intersects $\Theta(n)$ times the bisector $\beta(s_i, s_j')$, for any $i, j \in \{1, \ldots, m/2\}$. We clip the surface with a large enough rectangle, and triangulate it adding n edges that are parallel to ℓ and very close together; each of the added edges should intersect $\Theta(n)$ times any bisector $\beta(s_i, s_j')$.
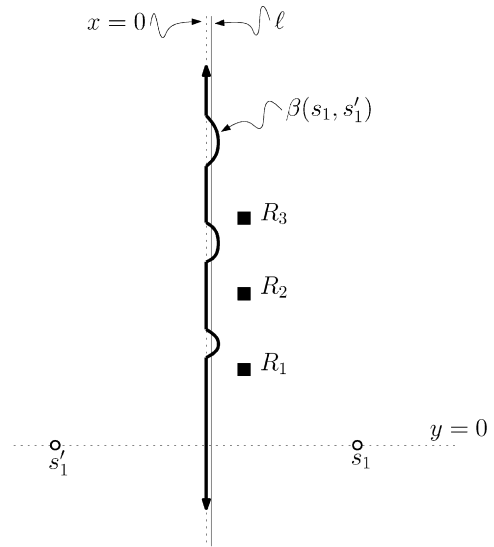

Fig. 4. The lower bound for $\Omega(k^2n^2)$ when $n = 3$; the hyperbolic arcs are symbolic and only the sites s_1, s_1' are shown.

Let $S_{i,j}$ denote the subset of sites

$$\{s_1, s_2, \ldots, s_i\} \cup \{s_1', s_2', \ldots, s_j'\}.$$

Note that a point p in the plane whose x coordinate is between -1 and 1 is closer to s_i than to s_{i+1} for any $i \in \{1, \ldots, m/2 - 1\}$. Similarly, such point is closer to s_j' than to s_{j+1}'. This implies that the $i + j$ sites $S_{i,j}$ define a nonempty $(i + j)$-region in $Vor_{i+j}(S)$ whose boundary is determined by the bisectors $\beta(s_i, s_{j+1}')$ and $\beta(s_{i+1}, s_j')$. Therefore, the boundary of $V(S_{i,j})$ has complexity $\Omega(n^2)$ because bisector $\beta(s_i, s_{j+1}')$ is intersected n times by each of the n edges parallel to ℓ. We conclude that any set $S_{i,j}$ defines a $(i + j)$-region of complexity $\Omega(n^2)$. Since $k \leqslant m$, there are $\Theta(k^2)$ pairs of values $i, j \in \{1, \ldots, m/2\}$ adding to at most k. This means that $Vor_{\leqslant k}$ has at least $\Omega(k^2)$ regions, each of complexity $\Omega(n^2)$. Therefore in this construction the complexity of $Vor_{\leqslant k}$ is $\Omega(k^2n^2)$.

4. Complexity of order-k Voronoi diagrams on realistic terrains

We next restrict ourselves to so-called realistic terrains [9]. Let $\alpha, \beta, \gamma, \delta$ be constants independent of n, m. A *realistic terrain* is a terrain where:

 (i) the domain is a rectangle whose aspect ratio is upper bounded by α;
 (ii) the angle of any segment contained in $\mathcal{T}$ with the horizontal plane is upper bounded by $\beta < \frac{\pi}{2}$;
 (iii) the ratio between the longest and the shortest edge in the xy-projection of $\mathcal{T}$ is upper bounded by γ;
 (iv) the triangulation of the domain induced by the xy-projection is a γ-low-density triangulation: any square Q intersects at most γ edges of length greater than the side length of Q.

Aronov et al. [2] have recently showed that in a realistic terrain the closest-site Voronoi diagram of m point sites has complexity $O(n + m\sqrt{n})$; this improves the previous bound by Moet et al. [9]. Using this bound in Lemma 5 we obtain the following result.

Lemma 7. *In a realistic terrain $B_{\leqslant k} = O(k^2 n + k\sqrt{n}m)$ for any $1 \leqslant k \leqslant m - 1$.*

Proof. Consider the proof of Lemma 5 for a realistic terrain. Using the bound

$$B(R) = O\left(n + |R|\sqrt{n}\right) = O\left(n + \sqrt{n}mp\right),$$

instead of Eq. (1), one readily obtains

$$B_{\leqslant k} = O\left(k^2 n + k\sqrt{n}m\right), \qquad \square$$

Combining Lemmas 4, 6, and 7 we can conclude the following.

Theorem 2. *Let $\mathcal{T}$ be a realistic terrain with n triangles, let S be a set of m point sites, and let k be an integer between 1 and $m - 1$. The complexity of $Vor_{\leqslant k}(S)$ is $O(k^2 n + k\sqrt{n}m + k^2 m)$.*

5. Conclusions

We have given tight bounds for the combinatorial complexity of order-$(\leqslant k)$ Voronoi diagrams in triangulated surfaces. Our upper bound of course also holds for the combinatorial complexity of the order-k Voronoi diagram. Obtaining better upper bounds for the order-k Voronoi diagram seems a much more challenging problem, even if restricted only to terrains or realistic terrains.

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