

The Geometric Stability of Voronoi Diagrams with Respect to Small Changes of the Sites

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ABSTRACT

Voronoi diagrams appear in many areas in science and technology and have numerous applications. They have been the subject of extensive investigation during the last decades. Roughly speaking, they are a certain decomposition of a given space into cells, induced by a distance function and by a tuple of subsets called the generators or the sites. Consider the following question: does a small change of the sites, e.g., of their position or shape, yield a small change in the corresponding Voronoi cells? This question is by all means natural and fundamental, since in practice one approximates the sites either because of inexact information about them, because of inevitable numerical errors in their representation, for simplification purposes and so on, and it is important to know whether the resulting Voronoi cells approximate the real ones well. The traditional approach to Voronoi diagrams, and, in particular, to (variants of) this question, is combinatorial. However, it seems that there has been a very limited discussion in the geometric sense (the shape of the cells), mainly an intuitive one, without proofs, in Euclidean spaces. We formalize this question precisely, and then show that the answer is positive in the case of \mathbb{R}^d , or, more generally, in (possibly infinite dimensional) uniformly convex normed spaces, assuming there is a common positive lower bound on the distance between the sites. Explicit bounds are given, and we allow infinitely many sites of a general form. The relevance of this result is illustrated using several pictures and many real-world and theoretical examples and counterexamples.

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1. INTRODUCTION

1.1 Background

The Voronoi diagram (the Voronoi tessellation, the Voronoi decomposition, the Dirichlet tessellation) is one of the basic structures in computational geometry. Roughly speaking, it is a certain decomposition of a given space X into cells, induced by a distance function and by a tuple of subsets $(P_k)_{k \in K}$, called the generators or the sites. More precisely, the Voronoi cell R_k associated with the site P_k is the set of all the points in X whose distance to P_k is not greater than their distance to the union of the other sites P_j .

Voronoi diagrams appear in a huge number of fields in science and technology and have many applications. They have been the subject of research for at least 160 years, starting formally with L. Dirichlet [17] and G. Voronoi [59], and of extensive research during the last 40 years. For several well written surveys on Voronoi diagrams which contain extensive bibliographies and many applications, see [7], [8], [44], and [23].

Consider the following question:

QUESTION 1.1. *Does a small change of the sites, e.g., of their position or shape, yield a small change in the corresponding Voronoi cells?*

This question is by all means natural, because in practice, no matter which algorithm is being used for the computation of the Voronoi cells, one approximates the sites either because of lack of exact information about them, because of

inevitable numerical errors occurring when a site is represented in an analog or a digital manner, for simplification purposes and so on, and it is important to know whether the resulting Voronoi cells approximate well the real ones.

For instance, consider the Voronoi diagram whose sites are either shops (or large shopping centers), antennas, or other facilities in some city/district such as post offices. See Figures 1-2.

Each Voronoi cell is the domain of influence of its site and it can be used for various purposes, among them estimating the number of potential customers [58] or understanding the spreading patterns of mobile phone viruses [61]. In reality, each site has a somewhat vague shape, and its real location is not known exactly. However, to simplify matters we regard each site as a point (or a finite collection of points if we consider firms of shops) located more or less near the real location. As a result, the resulting cells only approximate the real ones, but we hope that the approximation will be good in the geometric sense, i.e., that the shapes of the corresponding real and approximate cells will be almost the same. (See Section 4 for many additional examples, including ones with infinitely many sites or in higher/infinite dimensional spaces.) As the counterexamples in Section 6 show, it is definitely not obvious that this is the case.

A similar question to Question 1.1 can be asked regarding any geometric structure/algorithm, and, in our opinion, it is a fundamental question which is analogous to the question about the stability of the solution of a differential equation with respect to small changes in the initial conditions.

The traditional approach to Voronoi diagrams, and, in particular, to (variants of) Question 1.1, is combinatorial. For instance, as already mentioned in Aurenhammer [7, p. 366], the combinatorial structure of Voronoi diagrams (in the case of the Euclidean distance with point sites), i.e., the structure of vertices, edges and so on, is not stable under continuous motion of the sites, but it is stable “most of the time”. A more extensive discussion about this issue, still with point sites but possibly in higher dimensions, can be found in Weller [62], Vyalys et al. [60], and Albers et al. [2].

However, it seems that this question, in the geometric sense, has been raised or discussed only rarely in the context of Voronoi diagrams. In fact, after a comprehensive survey of the literature about Voronoi diagrams, we have found only very few places that have a very brief, particular, and intuitive discussion which is somewhat related to this question. The following cases were mentioned: the Euclidean plane with finitely many sites [32], the Euclidean plane with finitely many point sites [7, p. 366], and the d -dimensional Euclidean space with finitely many point sites [2] (see also some of the references therein). It was claimed there without proofs and exact definitions that the Voronoi cells have a continuity property: a small change in the position or the shape of the sites yields a small change in the corresponding Voronoi cells.

Another continuity property was discussed by Groemer [25] in the context of the geometry of numbers. He considered Voronoi diagrams generated by a lattice of points in a d -dimensional Euclidean space, and proved that if a sequence of lattices converges to a certain lattice (meaning that the basis elements which generate the lattices converge with respect to the Euclidean distance to the basis which generates the limit lattice), then the corresponding Voronoi cells of the origin converge, with respect to the Hausdorff

distance, to the cell of the origin of the limit lattice. His result is, in a sense and in a very particular case, a stability result, but it definitely does not answer Question 1.1 (which, actually, was not asked at all in [25]) for several reasons: first, usually the sites or the perturbed ones do not form a (infinite) lattice. Second, in many cases they are not points (singletons). Third, a site is usually different from the perturbed site (in [25] the discussed sites equal $\{0\}$). In this connection, we also note that Groemer’s proof is very restricted to the above setting and it uses arguments based on compactness and no explicit bounds are given.

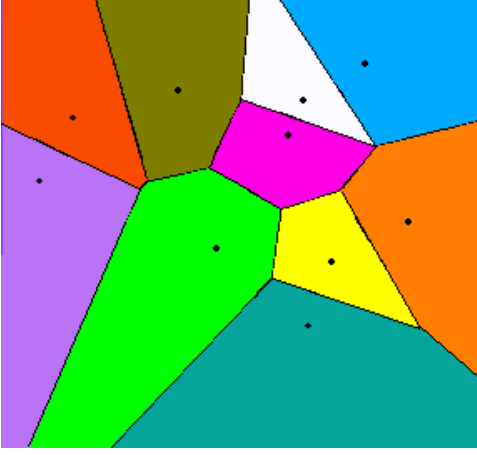
It is quite common in the computational geometry literature to assume “ideal conditions”, say infinite precision in the computation, exact and simple input, and so on. These conditions are somewhat non-realistic. Issues related to the stability of geometric structures under small perturbations of their building blocks (not necessarily the geometric stability) are not so common in the literature, but they can be found in several places, e.g., in [1, 3, 6, 9, 11, 12, 15, 21, 30, 33, 40, 41, 51, 56]. However, in many of the above places the discussion has combinatorial characteristics and there are several restrictive assumptions: for instance, the underlying setting is usually a finite dimensional space (in many cases only \mathbb{R}^2 or \mathbb{R}^3), with the Euclidean distance, and with finitely many objects of a specific form (merely points in many cases). In addition, the methods are restricted to this setting. In contrast, the infinite dimensional case or the case of (possibly infinitely many) general objects or general norms have never been considered.

1.2 Contribution of this paper

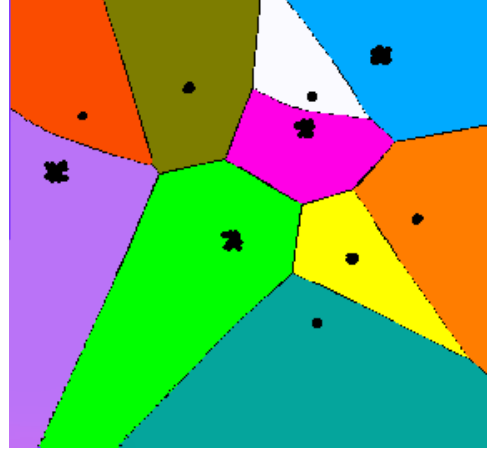
We discuss the question of stability of Voronoi diagrams with respect to small changes of the corresponding sites. We first formalize this question precisely, and then show that the answer is positive in the case of \mathbb{R}^d , or, more generally, in the case of (possibly infinite dimensional) uniformly convex normed spaces, assuming there is a common positive lower bound on the distance between the sites. Explicit bounds are presented, and we allow infinitely many sites of a general form. We also present several counterexamples which show that the assumptions formulated in the main result are necessary. We illustrate the relevance of this result using several pictures and many real-world and theoretical examples and counterexamples. To the best of our knowledge, the main result and the approach used for deriving it are new. Two of our main tools are: a new representation theorem which characterizes the Voronoi cells as a collection of line segments and a new geometric lemma which provides an explicit geometric estimate.

1.3 The structure of the paper

In Section 2 we present the basic definitions and notations. Exact formulation of Question 1.1 and informal description of the main result are given in Section 3. The relevance of the main result is illustrated using many theoretical and real-world examples in Section 4. The main result is presented in Section 5, and we discuss briefly some aspects related to its proof. In Section 6 we present several interesting counterexamples showing that the assumptions imposed in the main result are necessary. We end the paper in Section 7 with several concluding remarks. Since the proof of the main result is quite long and technical, and because the main goal of this paper is to introduce the issue and to discuss it in



1: 10 shopping centers (or post offices) in a flat city. Each shopping center is represented by a point.



2: In reality each shopping center/post office is not a point and its location is approximated. The combinatorial structure is somewhat different and the Voronoi cells are not exactly polygons, but still, their shapes are almost the same as in Figure 1.

a qualitative manner, rather than going deep into technical details, proofs were omitted from this version. Full proofs can be found in [47] and a preliminary version in [48].

2. NOTATION AND BASIC DEFINITIONS

In this section we present our notation and basic definitions. In the main discussion we consider a closed and convex set $X \neq \emptyset$ in some uniformly convex normed space $(\tilde{X}, |\cdot|)$ (see Definition 2.3 below), real or complex, finite or infinite dimensional. The induced metric is $d(x, y) = |x - y|$. We assume that X is not a singleton, for otherwise everything is trivial. We denote by $[p, x]$ and $[p, x)$ the closed and half open line segments connecting p and x , i.e., the sets $\{p + t(x - p) : t \in [0, 1]\}$ and $\{p + t(x - p) : t \in [0, 1)\}$ respectively. The (possibly empty) boundary of X with respect to the affine hull spanned by X is denoted by ∂X . The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$.

DEFINITION 2.1. *Given two nonempty subsets $P, A \subseteq X$, the dominance region $\text{dom}(P, A)$ of P with respect to A is the set of all $x \in X$ whose distance to P is not greater than their distance to A , i.e.,*

$$\text{dom}(P, A) = \{x \in X : d(x, P) \leq d(x, A)\}.$$

Here $d(x, A) = \inf\{d(x, a) : a \in A\}$ and in general we denote $d(A_1, A_2) = \inf\{d(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}$ for any nonempty subsets A_1, A_2 .

DEFINITION 2.2. *Let K be a set of at least 2 elements (indices), possibly infinite. Given a tuple $(P_k)_{k \in K}$ of nonempty subsets $P_k \subseteq X$, called the generators or the sites, the Voronoi diagram induced by this tuple is the tuple $(R_k)_{k \in K}$ of*

non-empty subsets $R_k \subseteq X$, such that for all $k \in K$,

$$\begin{aligned} R_k &= \text{dom}(P_k, \bigcup_{j \neq k} P_j) \\ &= \{x \in X : d(x, P_k) \leq d(x, P_j) \forall j \in K, j \neq k\}. \end{aligned}$$

In other words, the Voronoi cell R_k associated with the site P_k is the set of all $x \in X$ whose distance to P_k is not greater than their distance to the union of the other sites P_j .

In general, the Voronoi diagram induces a decomposition of X into its Voronoi cells and the rest. If K is finite, then the union of the cells is the whole space. However, if K is infinite, then there may be a “neutral cell”: for example, if X is the Euclidean plane, $K = \mathbb{N} = \{1, 2, 3, \dots\}$ and $P_k = \mathbb{R} \times \{1/k\}$, then no point in the lower half-plane $\mathbb{R} \times (-\infty, 0]$ belongs to any Voronoi cell. In the above definition and the rest of the paper we ignore the neutral cell.

We now recall the definition of strictly and uniformly convex spaces.

DEFINITION 2.3. *A normed space $(\tilde{X}, |\cdot|)$ is said to be strictly convex if for all $x, y \in \tilde{X}$ satisfying $|x| = |y| = 1$ and $x \neq y$, the inequality $|(x + y)/2| < 1$ holds. $(\tilde{X}, |\cdot|)$ is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y \in \tilde{X}$, if $|x| = |y| = 1$ and $|x - y| \geq \epsilon$, then $|(x + y)/2| \leq 1 - \delta$.*

Roughly speaking, if the space is uniformly convex, then for any $\epsilon > 0$ there exists a uniform positive lower bound on how deep the midpoint between any two unit vectors must penetrate the unit ball, assuming the distance between them is at least ϵ . In general normed spaces the penetration is not necessarily positive, since the unit sphere may contain line segments. \mathbb{R}^2 with the max norm $|\cdot|_\infty$ is a typical example for this. A uniformly convex space is always strictly convex, and if it is also finite dimensional, then the converse

is true too. The m -dimensional Euclidean space \mathbb{R}^m , or more generally, inner product spaces, the sequence spaces ℓ_p , the Lebesgue spaces $L_p(\Omega)$ ($1 < p < \infty$), and a uniformly convex product of a finite number of uniformly convex spaces, are all examples of uniformly convex spaces. See Clarkson [13] and, for instance, Goebel-Reich [22] and Lindenstrauss-Tzafriri [38] for more information about uniformly convex spaces.

From the definition of uniformly convex spaces we can obtain a function which assigns to the given ϵ a corresponding value $\delta(\epsilon)$. There are several ways to obtain such a function, but for our purpose we only need δ to be increasing, and to satisfy $\delta(0) = 0$ and $\delta(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. One choice, which is not necessarily the most convenient one, is the modulus of convexity, which is the function $\delta : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta(\epsilon) = \inf\{1 - |(x+y)/2| : |x-y| \geq \epsilon, |x| = |y| = 1\}.$$

For specific spaces we can take more convenient functions. For instance, for the spaces $L_p(\Omega)$ or ℓ_p , $1 < p < \infty$, we can take

$$\begin{aligned} \delta(\epsilon) &= 1 - (1 - (\epsilon/2)^p)^{1/p}, \text{ for } p \geq 2, \\ \delta(\epsilon) &= 1 - (1 - (\epsilon/2)^q)^{1/q}, \text{ for } 1 < p \leq 2 \text{ and } \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

We finish this section with the definition of the Hausdorff distance, a definition which is essential for the rest of the paper.

DEFINITION 2.4. Let (X, d) be a metric space. Given two nonempty sets $A_1, A_2 \subseteq X$, the Hausdorff distance between them is defined by

$$D(A_1, A_2) = \max\left\{\sup_{a_1 \in A_1} d(a_1, A_2), \sup_{a_2 \in A_2} d(a_2, A_1)\right\}.$$

Note that the Hausdorff distance $D(A_1, A_2)$ is definitely different from the usual distance $d(A_1, A_2) = \inf\{d(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}$. As a matter of fact, $D(A_1, A_2) \leq \epsilon$ if and only if $d(a_1, A_2) \leq \epsilon$ for any $a_1 \in A_1$, and $d(a_2, A_1) \leq \epsilon$ for any $a_2 \in A_2$. In addition, if $D(A_1, A_2) < \epsilon$, then for any $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $d(a_1, a_2) < \epsilon$, and for any $b_2 \in A_2$ there exists $b_1 \in A_1$ such that $d(b_2, b_1) < \epsilon$. These properties explain why the Hausdorff distance is the natural distance to be used when discussing approximation and stability in the context of sets: suppose that our resolution is at most r , i.e., we are not able to distinguish between two points whose distance is at most some given positive number r . If it is known that $D(A_1, A_2) < r$, then we cannot distinguish between the sets A_1 and A_2 , at least not by inspections based only on distance measurements. As a result of the above discussion, the intuitive phrase “two sets have almost the same shape” can be formulated precisely: the Hausdorff distance between the sets is smaller than some given positive parameter (note that a set and a rigid transformation of it usually have different shapes).

3. EXACT FORMULATION OF THE MAIN QUESTION AND INFORMAL FORMULATION OF THE MAIN RESULT

The exact formulation of Question 1.1 is based on the concept of Hausdorff distance for reasons which were explained at the end of the previous section.

QUESTION 3.1. Suppose that $(P_k)_{k \in K}$ is a tuple of nonempty sets in X . Let $(R_k)_{k \in K}$ be the corresponding Voronoi

diagram. Is it true that a small change of the sites yields a small change in the corresponding Voronoi cells, where both changes are measured with respect to the Hausdorff distance? More precisely, is it true that for any $\epsilon > 0$ there exists $\Delta > 0$ such that for any tuple $(P'_k)_{k \in K}$, the condition $D(P_k, P'_k) < \Delta$ for each $k \in K$ implies that $D(R_k, R'_k) < \epsilon$ for each $k \in K$, where $(R'_k)_{k \in K}$ is the Voronoi diagram of $(P'_k)_{k \in K}$?

The main result (Theorem 5.1) says that the answer is positive. Here is an informal description of it:

ANSWER 3.2. Suppose that the underlying subset X is a closed and convex set of a (possibly infinite dimensional) uniformly convex normed space \tilde{X} . Suppose that a certain boundedness condition on the distance between points in X and the sites holds, e.g., when X is bounded or when the sites form a (distorted) lattice. If there is a common positive lower bound on the distance between the sites, and the distance to each of them is attained, then indeed a small enough change of the (possibly infinitely many) sites yields a small change of the corresponding Voronoi cells, where both changes are measured using the Hausdorff distance; in other words, the shapes of the real cells and the corresponding perturbed ones are almost the same. Moreover, explicit bounds on the changes can be derived and they hold simultaneously for all the cells. There are counterexamples which show that the assumptions imposed above are necessary.

The condition that the distance to a site is attained holds, e.g., when the site is either a closed subset contained in a (translation of a) finite dimensional space, or a compact set, or a convex and closed subset in a uniformly convex Banach space. The “certain boundedness condition on the distance between points in X and the sites” is a somewhat technical condition expressed in (2) (see also Remark 5.2).

4. THE RELEVANCE OF THE MAIN RESULT

In Section 1 we explained why Question 1.1 is natural and fundamental, and mentioned the real-world example of a Voronoi diagram induced by shops/cellular antennas. The goal of this section is to illustrate further the relevance of the main result using a (far from being exhaustive) list of real-world and theoretical examples. In these examples the shape or the position of the real sites are obviously approximated, and the main result (Theorem 5.1) ensures that the approximate Voronoi cells and the real ones have almost the same shape, and no unpleasant phenomenon such as the one described in Figures 5-8 can occur.

One example is in molecular biology for modeling the proteins structure (Richards [49], Kim et al. [34], Bourquard et al. [10]), where the sites are either the atoms of a protein or special selected points in the amino acids and they are approximated by spheres/points. Another example is related to collision detection and robot motion (Goralski-Gold [24], Schwartz et al. [52]), where the sites are the (static or dynamic) obstacles located in an environment in which a vehicle/airplane/ship/robot/satellite should move. A third example is in solid state physics (Ashcroft-Mermin [5]; here the common terms are “the first Brillouin zone” or “the Wigner-Seitz cell” instead of “the Voronoi cell”), where the sites are infinitely many point atoms in a (roughly) periodic structure which represents a crystal. A fourth example

is in material engineering (Li-Ghosh [36]), where the sites are cracks in a material. A fifth example is in numerical simulations of various dynamical phenomena, e.g., gas, fluid or urban dynamics (Slotterback et al. [54], Mostafavi et al. [43]). Here the sites are certain points/shapes taken from the sampled data of the simulated phenomena, and the cells help to cluster and analyze the data continuously. A sixth example is in astrophysics (Springel et al. [55]) where the (point) sites are actually huge regions in the universe (of diameter equals to hundreds of light years) used in simulations performed for understanding the behavior of (dark) matter in the universe. A seventh example is in image processing and computer graphics, where the sites are either certain important features/parts in an image (Tagare et al. [57], Dobashi et al. [18], Sabha-Dutr  [50]) used for processing/analyzing it, or they are points form a useful configuration such as (approximate) centroidal Voronoi diagram (CVD) which induces cells having good shapes (Du et al. [19], Liu et al. [39], Faustino-Figueiredo [20]).

An eighth example is in computational geometry, and it is actually a large collection of familiar problems in this field where Voronoi cells appear and being used, possibly indirectly: (approximate) nearest neighbor searching/the post office problem, cluster analysis, (approximate) closest pairs, motion planning, finding (approximate) minimum spanning trees, finding good triangulations, and so on. See, e.g., Aurenhammer [7], Aurenhammer-Klein [8], Clarkson [14], Indyk [31], and Okabe et al. [44]. Here the sites are either points or other shapes, and the space is usually \mathbb{R}^n with some norm. In some of the above problems our stability result is clearly related because of the analysis being used (e.g., cluster analysis) or because the position/shapes of the sites are approximated (e.g., motion planning, the post office problem). However, it may be related also because in many of the previous problems the difficulty/complexity of finding an exact solution is too high, so one is forced to use approximate algorithms, to impose a general position assumption, and so on. Now, by perturbing slightly the sites (moving points, replacing a non-smooth curve by an approximating smooth one, etc.), one may obtain much simpler and tractable configurations and now, using the geometric stability of the Voronoi cells, one may estimate how well the obtained solution approximates the best/real one.

As for a theoretical example of a different nature, we mention Kopeck  et al. [35] in which the stability results described here have been used, in particular, for proving the existence of a zone diagram (a concept which was first introduced by Asano et al. [4] in the Euclidean plane with point sites) of finitely many compact sites which are strictly contained in a (large) compact and convex subset of a uniformly convex space, and also for proving interesting properties of Voronoi cells there.

Another example is for the infinite dimensional Hilbert space $L_2(I)$ for some I (perhaps an interval or a finite dimensional space): functions in it are used in signal processing and in many other areas in science and technology. In practice the signals (functions) can be distorted, e.g., because of noise, and in addition, they are approximated by finite series (e.g., finite Fourier series) or integrals (e.g., Fourier transform). Given several signals, the (approximate) Voronoi cell of a given signal may help, at least in theory, to cluster or analyze data related to the sites. Such an analysis can be done also when the signal is considered as a point in a finite

dimensional space of a high dimension. See, for instance, Conway-Sloane [16, pp. 66-69, 451-477] (coding) and Shannon [53] (communication) for a closely related discussion (in [53] Voronoi diagrams are definitely used in various places, but without their explicit name).

We mention several additional examples related to our stability result, sometimes in a somewhat unexpected way. For instance, Voronoi diagrams of infinitely many sites generated by a Poisson process (Okabe et al. [44, pp. 39, 291-410]), Voronoi diagrams of atom nuclei used for the mathematical analysis of stability phenomena in matter (Lieb-Yau [37]), Voronoi diagrams of infinitely many lattice points in a multi-dimensional Euclidean space which appear in the original works of Dirichlet [17] and Voronoi [59] (see also Groemer [25] and Gruber-Lekkerkerker [26] regarding the geometry of numbers and quadratic forms; Groemer used his stability result for deriving the Mahler compactness theorem [42]), and packing problems such as the Kepler conjecture and the Dodecahedral conjecture (Hales [27],[28], Hales-McLaughlin [29]; because of continuity arguments needed in the proof) or those described in Conway-Sloane [16].

5. THE MAIN RESULT AND SOME ASPECTS RELATED TO ITS PROOF

In this section we formulate the main result and discuss briefly issues related to its proof. See also the remarks after Theorem 5.1 for several relevant clarifications.

THEOREM 5.1. *Let $(\tilde{X}, |\cdot|)$ be a uniformly convex normed space. Let $X \subseteq \tilde{X}$ be closed and convex. Let $(P_k)_{k \in K}$, $(P'_k)_{k \in K}$ be two given tuples of nonempty subsets of X with the property that the distance between each $x \in X$ and each P_k, P'_k is attained. For each $k \in K$ let $A_k = \bigcup_{j \neq k} P_j$, $A'_k = \bigcup_{j \neq k} P'_j$. Suppose that the following conditions hold:*

$$\eta := \inf\{d(P_k, P_j) : j, k \in K, j \neq k\} > 0, \quad (1)$$

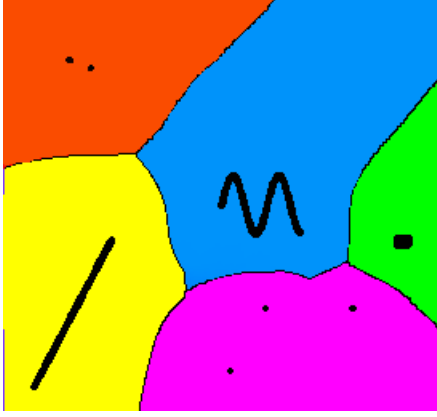
$\exists \rho \in (0, \infty)$ such that for all $k \in K$ and for all $x \in X$

the open ball $B(x, \rho)$ intersects A_k . (2)

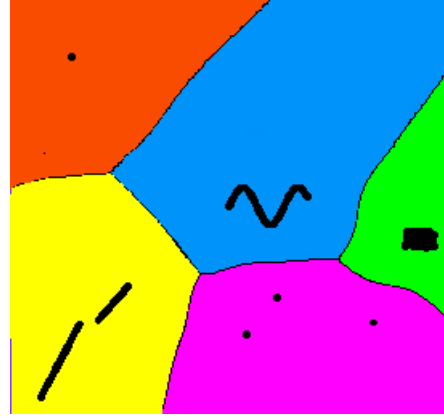
For each $k \in K$ let $R_k = \text{dom}(P_k, A_k)$, $R'_k = \text{dom}(P'_k, A'_k)$ be, respectively, the Voronoi cells associated with the original site P_k and the perturbed one P'_k . Then for each $\epsilon \in (0, \eta/6)$ there exists $\Delta > 0$ such that if $D(P_k, P'_k) < \Delta$ for each $k \in K$, then $D(R_k, R'_k) < \epsilon$ for each $k \in K$.

See Figures 3, 4 for an illustration. The pictures were produced using the algorithm described in [46].

REMARK 5.2. Condition (2) may seem somewhat complicated at a first glance, but it actually expresses a certain uniform boundedness condition on the distance between any point in X to its neighbor sites. No matter which point $x \in X$ and which site P_k are chosen, the distance between x and the collection of other sites P_j , $j \neq k$ cannot be arbitrary large. A sufficient condition for it to hold is when a uniform bound on the diameter of the cells (including the neutral one, if it is nonempty) is known in advance, e.g., when X is bounded or when the sites form a (distorted) lattice. But (2) can hold even if the cells are not bounded, e.g., when the setting is the Euclidean plane and $P_k = \mathbb{R} \times \{k\}$ where k runs over all integers.



3: Illustration of Theorem 5.1: five sites in a square in (\mathbb{R}^2, ℓ_p) where the parameter is $p = 3.14159$.



4: The sites have been slightly perturbed: the two points have merged, the “sin” has shrunk, and so on. The cells have been slightly perturbed.

REMARK 5.3. In general, we have $\Delta = O(\epsilon^2)$. However, if there is a positive lower bound on the distance between the sites and the boundary of X (relative to the affine hull spanned by X), i.e., if the sites are strictly contained in the (relative) interior of X , then actually the better estimate $\Delta = O(\epsilon)$ holds. The constants inside the big O can be described explicitly: when $\Delta = C\epsilon^2$ we can take

$$C = \frac{1}{16(\rho + 5\eta/12)} \cdot \delta \left(\frac{\eta}{12\rho + 5\eta} \right),$$

and when $\Delta = C\epsilon$ we can take

$$C = \min \left\{ \frac{1}{16} \delta \left(\frac{\eta}{12\rho + 5\eta} \right), \frac{d(\bigcup_{k \in K} P_k, \partial X)}{8(\rho + \eta/6)} \right\}.$$

In the second case, in addition to the inequality $\epsilon < \eta/6$, the inequality $\epsilon \leq 8 \cdot d(\bigcup_{k \in K} P_k, \partial X)$ should be satisfied too.

The proof of Theorem 5.1 is quite long and technical, and hence we have decided to omit it from this version. Despite this, we want to say a few words about the proof and some of the difficulties which arise along the way. First, as the counterexamples mentioned in Section 6 show, one must take into account all the assumptions mentioned in the formulation of the theorem.

Second, in order to arrive to the generality described in the theorem, one is forced to avoid many familiar arguments used in computational geometry and elsewhere, such as Euclidean arguments (standard angles, trigonometric functions, normals, etc.), arguments based on lower envelopes and algebraic arguments (since the intersections between the graphs generating the lower envelope may be complicated and since the boundaries of the cells may not be algebraic), arguments based on continuous motion of points, arguments based on finite dimensional properties such as compactness (since in infinite dimensional spaces closed and bounded balls are not compact), arguments based on finiteness (since we allow infinitely many sites and sites consist of infinitely many points) and so on. Our way to overcome these difficulties is to use a new representation theorem for dominance regions as a collection of line segments (Theorem 5.4 below) and a new geometric lemma (Lemma 5.5 below) which enables us to

arrive to the explicit bounds mentioned in the theorem. As a matter of fact, we are not aware of any other way to obtain these explicit bounds even in a square in the Euclidean plane with point sites. These tools also allow us to overcome the difficulty of a potential infinite accumulated error due to the possibility of infinitely many sites/sites with infinitely many points/infinite dimension.

THEOREM 5.4. *Let X be a closed and convex subset of a normed space $(\tilde{X}, |\cdot|)$, and let $P, A \subseteq X$ be nonempty. Suppose that for all $x \in X$ the distance between x and P is attained. Then $\text{dom}(P, A)$ is a union of line segments starting at the points of P . More precisely, given $p \in P$ and a unit vector θ , let*

$$T(\theta, p) = \sup\{t \in [0, \infty) : p + t\theta \in X \text{ and } d(p + t\theta, p) \leq d(p + t\theta, A)\}.$$

Then

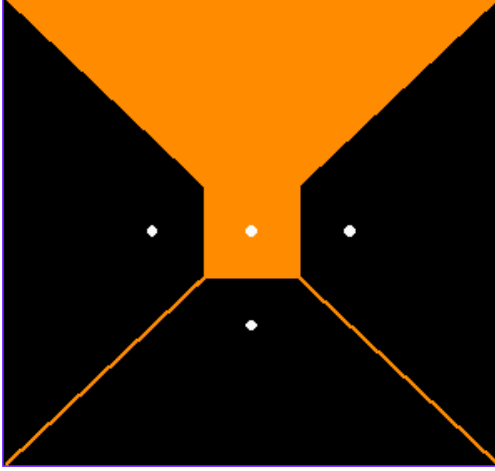
$$\text{dom}(P, A) = \bigcup_{p \in P} \bigcup_{|\theta|=1} [p, p + T(\theta, p)\theta].$$

When $T(\theta, p) = \infty$, the notation $[p, p + T(\theta, p)\theta]$ means the ray $\{p + t\theta : t \in [0, \infty)\}$.

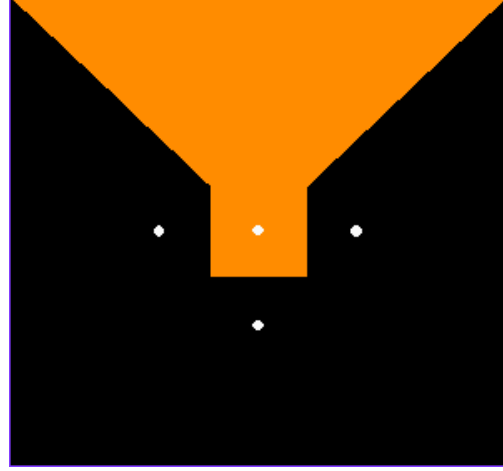
LEMMA 5.5. *Let $(\tilde{X}, |\cdot|)$ be a uniformly convex normed space and let $A \subseteq \tilde{X}$ be nonempty. Suppose that $y, p \in \tilde{X}$ satisfy $d(y, p) \leq d(y, A)$ and $d(p, A) > 0$. Let $x \in [p, y)$. Let $\sigma \in (0, \infty)$ be arbitrary. Then $d(x, p) < d(x, A) - r$ for any $r > 0$ satisfying*

$$r \leq \min \left\{ \sigma, \frac{4d(p, A)}{10}, d(y, x) \delta \left(\frac{d(p, A)}{10(d(x, A) + \sigma + d(y, x))} \right) \right\}.$$

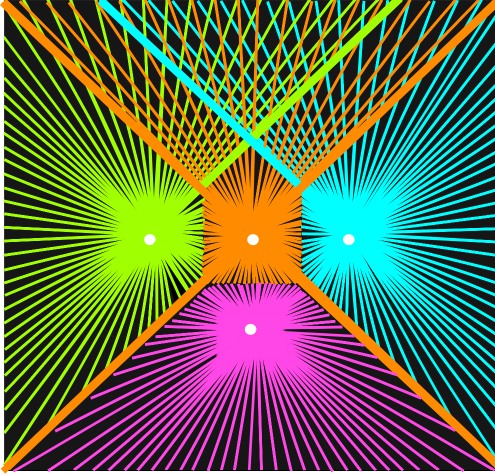
The proof of Lemma 5.5 is based on the strong triangle inequality of Clarkson [13, Theorem 3]. It is interesting to note that although this inequality was formulated in the very famous paper [13] of Clarkson, it seems that it has been almost totally forgotten, and in fact, despite a comprehensive search we have made in the literature, we have found evidences to its existence only in [13] and later in [45].



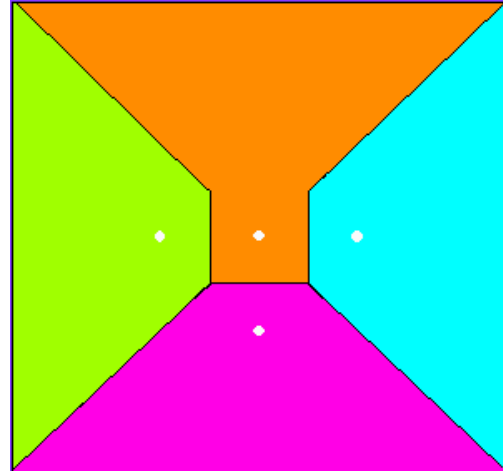
5: Four sites in a square in $(\mathbb{R}^2, \ell_\infty)$. The cell of $P_1 = \{(0, 0)\}$ is displayed. The other sites are $P_2 = \{(2, 0)\}$, $P_3 = \{(-2, 0)\}$, $P_4 = \{(0, -2)\}$.



6: Now either $P_1 = \{(\beta, \beta)\}$ or $P_4 = [-\beta, \beta] \times [-2 - \beta, -2 + \beta]$, $\beta > 0$ arbitrary small. The two lower rays have disappeared. No stability.



7: The full diagram of Figure 5. Note the large intersection between cells 1,2,3. Each cell is represented as a union of rays (see Theorem 5.4) and some rays were emphasized.



8: The full diagram of Figure 6 when $P_1 = \{(\beta, \beta)\}$. Cells 2,3 have been (significantly) changed too.

6. COUNTEREXAMPLES

In this section we mention several counterexamples which show that the assumptions in Theorem 5.1 are essential.

If the space is not uniformly convex, then the Voronoi cells may not be stable as shown in Figures 5-8. Here the setting is point sites in a square in \mathbb{R}^2 with the max norm.

The positive common lower bound expressed in (1) is necessary even in a square in the Euclidean plane. Consider $X = [-10, 10]^2$, $P_{1,\beta} = \{(0, \beta)\}$ and $P_{2,\beta} = \{(0, -\beta)\}$, where $\beta \in [0, 1]$. If $\beta = 0$, then the cell $\text{dom}(P_{1,\beta}, P_{2,\beta})$ is X . However, if $\beta > 0$, then $\text{dom}(P_{1,\beta}, P_{2,\beta})$ is the upper half of X . A more interesting example occurs when considering in X the rectangle $P_{1,\beta} = [-a, a] \times [-10, -\beta]$ and the line segment $P_2 = [-10, 10] \times \{0\}$, where $a, \beta \in [0, 1]$. If $\beta = 0$, then $d(P_{1,\beta}, P_2) = 0$, and the cell $\text{dom}(P_{1,\beta}, P_2)$ contains the rectangle $[-a, a] \times [0, 10]$. However, if $\beta > 0$, then this cell does not contain this rectangle.

The assumption expressed in (2) is essential even in the Euclidean plane with two points. Indeed, given $\beta \geq 0$, let $P_{1,\beta} = \{(\beta, 1)\}$ and $P_2 = \{(0, -1)\}$. Then $\text{dom}(P_{1,0}, P_2)$ is the upper half space. However, if $\beta > 0$, then the half space $\text{dom}(P_{1,\beta}, P_2)$ contains points (x, y) with $y \rightarrow -\infty$. Therefore the Hausdorff distance between the cell $\text{dom}(P_{1,0}, P_2)$ and the perturbed cell $\text{dom}(P_{1,\beta}, P_2)$ is infinity, so there can be no stability.

7. CONCLUDING REMARKS

We conclude the paper with the following remarks. First, despite the counterexamples mentioned above, some of the assumptions can be weakened, with some caution. For instance, under a certain compactness assumption and a certain geometric condition that the sites should satisfy, the main result can be generalized to normed spaces which are not uniformly convex. A particular case of these assumptions is when the space is m -dimensional with the $|\cdot|_\infty$ norm, the sites are positively separated, and no two points of different sites are on a hyperplane perpendicular to one of the standard axes.

Second, an interesting (but not immediate) consequence of the main result is that it implies the stability of the (multi-dimensional) volume, namely a small enough change in the sites yields a small change in the volumes of the corresponding Voronoi cells.

Third, it can be shown that the function T defined in Theorem 5.4 also has a certain continuity property if the space is uniformly convex and this expresses a certain continuity property of the boundary of the cells.

Fourth, the estimate for Δ from Remark 5.3 is definitely not optimal and it can be improved, but, as simple examples show, Δ cannot be too much larger and its estimate should be taken into account when performing a relevant analysis. There is nothing strange in this and the situation is analogous to the familiar case of real valued functions. For instance, consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 0$, $x < 0$, $f(x) = (1/\beta)x$, $x \in [0, \beta]$, $f(x) = 1$, $x > \beta$, where $\beta > 0$ is given. Although f is continuous, a “large” change near $x = 0$ (of more than β) will cause a large change to f .

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