Quadratic and Cubic B-Splines by Generalizing Higher-Order Voronoi Diagrams

Extended Abstract

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ABSTRACT

A long-standing problem in spline theory has been to generalize classic B-splines to the multivariate setting, and its full solution will have broad impact. We initiate a study of triangulations that generalize the duals of higher order Voronoi diagrams, and show that these can serve as a foundation for a family of multivariate splines that generalize the classic univariate B-splines. This paper focuses on Voronoi diagrams of orders two and three, which produce families of quadratic and cubic bivariate B-splines. We believe that these families are the most general bivariate B-splines to date and support our belief by demonstrating that a classic quadratic box spline, the Zwart-Powell (ZP) element, is contained in our family. Our work is directly based on that of Neamtu, who established the fascinating connection between splines and higher order Voronoi diagrams.

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Geometrical problems and computations

General Terms

Theory, Algorithms

Keywords

higher order Voronoi diagrams, centroid triangulation, simplex splines, B-splines

1. INTRODUCTION

A *degree k B-spline* is a smooth degree k piecewise polynomial function defined over a set of $k+2$ reals called *knots*. From an arbitrary set of knots $K \subset \mathbb{R}$, we construct the basis for a degree k B-spline space by forming a B-spline for each sequence of $k+2$ consecutive knots in K, as illustrated in Figure 1. The function space spanned by this basis satisfies many mathematical properties—including local support, optimal smoothness, and polynomial reproducibility—that make it attractive for function approximation [6] and for curve modeling in computer-aided geometric design (CAGD).

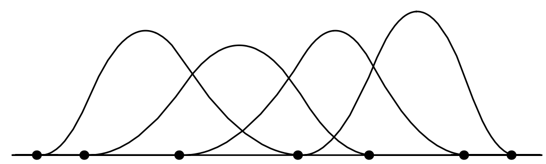

Figure 1: Univariate quadratic B-splines.

Given the success of univariate B-splines, one naturally desires a multivariate analog. In 1976, de Boor [4] introduced a multivariate generalization of the individual B-spline functions: A *degree k simplex-spline* is a smooth, degree k, piecewise-polynomial function defined over a set of $k+s+1$ points $X \subset \mathbb{R}^s$ called *knots*. Geometrically, the simplex spline over X is the "shadow" of a simplex $[Y]$ in $\mathbb{R}^{k+s}$ whose vertices Y project vertically to X; for a point $x \in \mathbb{R}^s$, the function returns the relative measure of the set of points in $[Y]$ that project to x. Figure 2 illustrates simplex splines for $s=1, k=2$ and for $s=2, k=1$.

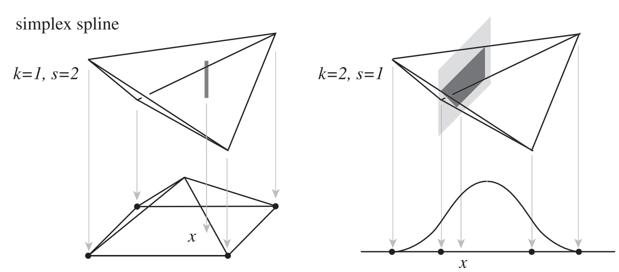

Figure 2: Simplex splines as shadows. The heavy dots are the defining knots. The length of the vertical segment, or the area of the vertical polygon, is the value of the function at the point x.

The generalization of univariate B-splines to simplex splines leaves open, however, the question of how to construct sets of simplex splines to form the "right" basis of a spline space. To be more precise, given a set of knots $K \subset \mathbb{R}^s$, one would like to choose subsets of size $k+s+1$—or *configurations*—

so that the simplex splines over these configurations span a space that has analogous properties to the univariate B-spline space. Two properties are particularly important. First, specializing the construction in one dimension should give the univariate B-splines; second, the constructed spline space should include all polynomials of degree k. The second property, called the *polynomial reproduction*, is the "... *sine qua non* in the consideration of the approximation order of spline spaces" [9]. The problem of choosing the "right" configurations has not been completely solved, although several solutions have been developed. For details of the historic development, we refer to the excellent survey by Neamtu [8]. His own solution [9], using *Delaunay configurations*, seems to be the simplest and most effective. Let us review his solution.

In a knot set $K \subset \mathbb{R}^s$, a $k+s+1$-subset $X \subset K$ is a degree k Delaunay configuration if and only X can be partitioned into two sets, an $s+1$-set t and a k-set I, such that the circumsphere of t has, of all points in K, exactly I inside. (For those familiar with higher order Voronoi diagrams, note that the degree k Delaunay configurations correspond to a subset of the vertices of the order $k+1$ Voronoi diagram of K. The set of simplex splines from the Delaunay configurations satisfies the two properties mentioned earlier. First, in one dimension, the simplex splines become the univariate B-splines, because degree k Delaunay configurations in one dimension are precisely the consecutive runs of $k+2$ knots. Second, for any s and k, the degree k simplex splines reproduce degree k polynomials using the same polar form expression as the univariate B-splines (see Theorem 3.2.)

Neamtu's solution, while elegant, is perhaps too restrictive on the types of splines that can be constructed. Let us observe this in the somewhat trivial linear setting: For a fixed knot set K, Neamtu's construction gives the piecewise linear (PL) interpolating functions over the Delaunay triangulation of K, but it seems that a more flexible construction would admit the PL interpolation functions over *any* triangulation of K. This observation in the linear setting is the starting point of our work.

Our generalization are based on two key observations:

1. For the proof of polynomial reproduction, only two properties of Delaunay configurations are needed: First, degree 0 Delaunay configurations form a triangulation; second, for any $i \geq 0$, Delaunay configurations of degree i and $i+1$ *match facets*, as defined in Section 3. This facet-matching property, though perhaps unfamiliar at first glance, has appeared whenever higher order Voronoi diagrams are studied. For example, *old and new vertices* in Lee's algorithm [7], the inclusion and exclusion neighbors in the Delaunay tree [3], and the far and close vertices [2] all correspond to this property.

2. In two dimensions, Lee's algorithm for computing higher order Voronoi diagrams [7] may be dualized and generalized to iteratively compute configurations of successively higher degrees. We call this dualized, generalized Lee's algorithm *link triangulation procedure*. The procedure takes as input a set of configurations of degree i and output a set of configurations of degree $i+1$ that match facets with the input. However, the definition of the procedure does not immediately imply that it can successfully finish.

The consequence of these two observations is that, if the link triangulation procedure can succeed after k iterative applications, then the output degree k configurations give rise to a set of degree k simplex splines that reproduce polynomials in the same way as the Delaunay configurations. We prove that the link triangulation procedure can be iteratively applied up to three times. Although our proof does not yet generalize beyond that, based on computational experiments we conjecture that the link triangulation procedure works generally.

Even without the connection to splines, it is interesting to explore this as a procedure that computes *centroid triangulations*, a term coined by Dominique Schmitt [13]. We discuss this more in Section 5. We believe that the main usefulness of our generalization will come from the ways to specialize the link triangulation procedure to construct splines that suit mathematical or practical needs, such as smooth blending.

We support this belief by demonstrating how to specialize the procedure to reproduce the ZP element—the textbook example of bivariate smooth quadratic *box splines*. Box splines are a family of splines defined on knots from a regular grid. They have a rich set of mathematical properties and powerful computational tools [5], and are often used in computer-aided geometric design applications. Since we can reproduce the ZP element in a simplex spline space on the properly chosen triangulation, we can fillet or blend between two patches of quadratic box splines by completing the triangulations to define the simplex spline between them, and know that we preserve the properties of polynomial reproducibility and optimal smoothness.

2. GEOMETRIC PRELIMINARIES

In $\mathbb{R}^s$, for some $0 \leq i \leq s$, an n dimensional simplex, or n-simplex for short, is the convex hull of a set of $n+1$ affinely independent points. The defining points of a simplex are called *vertices*. For a simplex t, and a vertex v of t, the simplex defined by the set $t \backslash \{v\}$ is the *facet of* t *opposite* v.

It is often convenient to keep track of the relative *orientations* of simplices. For this purpose, an oriented n-simplex is represented by a tuple of $n+1$ vertices $[t_0, \ldots, t_n]$.

Two oriented simplices s and t are *identical*, or $s = t$, if and only if the tuple s can be computed from t by an even number of transpositions, and *opposite*, or $s = -t$, if and only if the tuple s can be computed from t by an odd number number of transpositions. For example, $[a,b] = -[b,a]$, $[a,b,c] = [b,c,a]$, and $[a,b,c] = -[b,a,c]$. For an oriented n-simplex t, and a vertex v with index i in the tuple t, the following "drop" operation ${}_vt$, or ${}_it$, computes the facet of t opposite v:

$${}_it := (-1)^i[t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n].$$

The inverse "insert" operation constructs an n-simplex from an $n-1$-simplex f and a point $v \notin f$:

$${}^vf := [v, f_0, \ldots, f_{n-1}].$$

It is convenient to have the following "swap" operation to go between two simplices sharing a common facet. Let t be a simplex, u be a vertex of t with index i, and v be a point not in t, then, ${}^v_ut$, or ${}^v_it$, is the simplex:

$${}^v_it := [t_0, \ldots, t_{i-1}, v, t_{i+1}, \ldots, t_n].$$

For a set of n-simplices T, summing T, $\sum T$, "cancels" the opposite simplices, and taking the boundary of T, ∂T, gives the facets of T with no opposites: $\partial T := \sum\{_i t \mid t \in T, i = 0..n\}$.

For a point $x \in \mathbb{R}^s$, let $(x_1, \ldots, x_s)$ denote its coordinates. For an s-simplex t, its signed volume, $d(t)$, is the determinant:

$$d(t) := \begin{vmatrix} 1 & (t_0)_1 & \ldots & (t_0)_s \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (t_s)_1 & \ldots & (t_s)_s \end{vmatrix}.$$

When studying s-simpices in $\mathbb{R}^s$, we will always assume that their tuples are ordered to give positive volumes.

In two dimensions, a set of oriented edges P is a *polygon* if $\partial P = \emptyset$ and bounds a finite region. A polygon P is *simple* if each vertex in P is incident on exactly two edges. A simple polygon with only two opposite edges is degenerate and has an empty interior.

A set of triangles T *triangulates the polygon* P if $\partial T = P$ and the vertices of T are the same as the vertices of P. A simple polygon with at least three edges can always be triangulated.

A set of $k+s+1$ knots $X \subset \mathbb{R}^s$ defines a degree k, s-variate simplex spline $M(\cdot \mid X)$ as follows. Let $\pi(\cdot)$ denote the vertical projection map from $\mathbb{R}^{k+s}$ onto $\mathbb{R}^s$. Let Y be any set of point in $\mathbb{R}^{k+s}$ such that $\pi(Y) = X$. Then, the simplex spline is the shadow cast by the simplex $[Y]$, assumed to be uniformly dense:

$$M(x \mid X) := \frac{\mathrm{vol}_k\{y \mid y \in [Y], \pi(y) = x\}}{\mathrm{vol}_{k+s}[Y]}.$$

In the special case $k = 0$, the simplex spline is a normalized membership function over the simplex $[X]$:

$$M(x \mid X) := \begin{cases} 1/|d(X)| & \text{if } x \in [X]; \\ 0 & \text{if } x \notin [X]. \end{cases}$$

Let $K \subset \mathbb{R}^s$ be a set of knots. A subset $X \subset K$ of size $k+s+1$ is a *configuration* of degree k with respect to K. A set of degree k configurations Γ determines a *simplex spline space* of degree k, i.e. $\mathrm{span}\{M(\cdot \mid X) \mid X \in \Gamma\}$. Therefore, the problem of constructing a simplex spline space can be stated in terms of constructing configurations.

3. A FACET-MATCHING PROPERTY

We give a straightforward generalization (Theorem 3.2) of Neamtu's result on the polynomial reproduction property of the simplex splines from Delaunay configurations (Theorem 2 in [9]). The goals are to introduce some notation and to show that the proof hinges on a *facet-matching* property of Delaunay configurations.

In $\mathbb{R}^s$, a degree k *boundary-interior configuration* is a pair (t, I), where t is a tuple of $s+1$ points representing an s-simplex, and I, the *interior set*, is a set k points disjoint from t. A Delaunay configuration (t, I) is a boundary-interior configuration with the property that its interior set I is precisely the set of points in K that fall inside the circumsphere of t.

A boundary-interior configuration (t, I) defines a *normalized simplex spline*:

$$N(\cdot \mid t, I) := d(t) M(\cdot \mid t \cup I).$$

We should note that, in the above definition, the construction of the simplex spline unions the boundary and interior sets. Therefore, this notion of a boundary-interior configuration, as a pair of $s+1$-set and k-set, is consistent with the notion of a configuration as a $k+s+1$-set mentioned earlier. Therefore, without causing any confusion, we will speak of boundary-interior configurations simply as configurations.

Much like the way facets of a simplex are constructed by dropping vertices, the *facet configurations* are constructed by dropping vertices from the boundary simplex, but carrying along the interior set: For a configuration (t, I) and a vertex $v \in t$, the facet configuration of (t, I) opposite v is the pair $(_v t, I)$. For a set of configurations Γ, we use $F(\Gamma)$ to denote all its facet configurations.

We now state a property between two sets of configurations of adjacent degrees, illustrated in Figure 3.

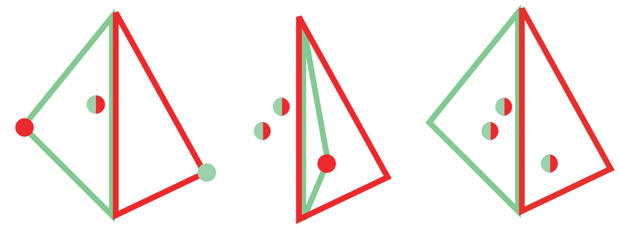

Figure 3: From left to right: case 1a, 1b/2b and 2a of the facet-matching property(3.1) for configurations of degree 2 and 3. For a configuration, its boundary simplex is drawn as a colored triangle, and its interior set is drawn as disks of the same color. Bi-colored disks are interior points shared by two configurations.

Definition 3.1. *Let Γ and Γ' be two sets of configurations of degree k and $k+1$, respectively. Γ and Γ' are said to* match facets *if and only if,*

1. *for each facet configuration $(_u s, I) \in F(\Gamma)$,*
   - *(a) there is either another facet configuration $(_v t, J) \in F(\Gamma)$ such that $_v t = -_u s$ and $I \cup \{v\} = J \cup \{u\}$,*
   - *(b) or there is a facet configuration $(_v t, I') \in F(\Gamma')$ such that $_v t = {_u s}$ and $I \cup \{v\} = I'$*
2. *and, for each facet configuration $(_v t, I') \in F(\Gamma')$,*
   - *(a) there is either another facet configuration $(_u s, J') \in F(\Gamma')$ such that $_v t = -_u s$ and $I' = J'$,*
   - *(b) or there is another facet configuration $(_u s, I) \in F(\Gamma)$ such that $_v t = {_u s}$ and $I' = I \cup \{v\}$.*

The following theorem generalizes Theorem 2 in [9] and states that a set of degree k configurations Γ_k gives rise to a polynomial-reproducing simplex spline space as long as there exist sets of configurations of $\Gamma_0, \ldots, \Gamma_k$ that match facets.

Theorem 3.2. *Let $(\Gamma_0, \ldots, \Gamma_k)$ be sets of configurations of degree 0 to k in $\mathbb{R}^s$. If Γ_0 is a triangulation of the space $\mathbb{R}^s$, and, for $0 \le i \le k-1$, sets Γ_i and Γ_{i+1} match facets, then the normalized simplex splines associated with Γ_k reproduce all degree k polynomials: For a degree k polynomial p with polar form [11] P,*

$$p = \sum_{(t,I)\in\Gamma_k} P(I) N(\cdot \mid t, I).$$

4. THE LINK TRIANGULATION PROCEDURE

Theorem 3.2 begs the question of how to find a sequence of configurations of increasing degrees that match facets. We find that, in two dimensions, the facet-matching property can be established, at least up to degree 3, through a simple construction that we call the *link triangulation procedure*, which, in the dual setting, generalizes Lee's algorithm for computing higher order Voronoi diagrams [7].

Let us first describe some operations to analyze a set of degree k configurations Γ. Let $I(\Gamma)$ denote the set of all interior sets of Γ:

$$I(\Gamma) := \{I \mid (t, I) \in \Gamma\}.$$

Let $V(\Gamma)$ denote a set of $k+1$-sets called *vertices* of Γ:

$$V(\Gamma) := \{I \cup \{v\} \mid v \in t, (t, I) \in \Gamma\}.$$

Note that if Γ is Delaunay, then the vertices are precisely the $k+1$-sets that define regions of an order $k+1$ Voronoi diagram.

For a set of k points I, the *neighborhood* of I in Γ is the set of boundary simplices from the configurations with interior set I:

$$\mathrm{nb}(I, \Gamma) := \{t \mid (t, I) \in \Gamma\}.$$

Note that a neighborhood of I is non-empty if and only if I belongs to the $I(\Gamma)$.

For a set of $k+1$ points J, the *link* of J in Γ is the set of facets:

$$\mathrm{lk}(J, \Gamma) = \{_v t \mid v \in J, (t, J \backslash \{v\}) \in \Gamma\}.$$

We should observe that the link of J is non-empty if and only if J belongs to $V(\Gamma)$. Also, if Γ represents a triangulation, then a link in Γ is just a vertex link defined for triangulations.

It is easy to show that the facet-matching property can be restated in terms of boundaries of neighborhoods and links. Formally:

Lemma 4.1. *Let Γ and Γ' be two sets of configurations of degree k and $k+1$. Then, Γ and Γ' match facets if and only if for any set of $k+1$ points J, the boundary of the neighborhood of J in Γ' is the sum of the edges in the link of J in Γ:*

$$\partial\, \mathrm{nb}(J, \Gamma') = \sum \mathrm{lk}(J, \Gamma).$$

We should note that the summation on the left hand side invoked by the boundary operation and the summation on the right hand side remove opposite facets, leaving facets that match in the way described in the two symmetrical cases (1b and 2b) of Definition 3.1.

Let us now describe the link triangulation procedure, which operate in two dimensions:

Input: a set of degree i configurations Γ
Output: a set of degree $i+1$ configurations Γ'
$\Gamma' \leftarrow \emptyset$
For $J \in V(\Gamma)$ **do**
  $P \leftarrow \sum \mathrm{lk}(J, \Gamma)$
  If $P \neq \emptyset$
    $T \leftarrow \mathrm{triangulate}(P)$
    $\Gamma' \leftarrow \Gamma' \cup \{(t, J)\}_{t \in T}$
Return Γ'.

The above procedure is well-defined if the links in the input configurations are simple polygons. For the moment, let us suppose that this is always true. Then, if we apply this procedure iteratively k times, starting from a triangulation Γ_0, we construct k sets of configurations that clearly satisfy the matching property in Lemma 4.1, hence the facet-matching property, hence the conditions of Theorem 3.2. There are many choices of how to perform the polygon triangulation. In particular, if Γ_0 is a Delaunay triangulation, and we Delaunay-triangulate the link polygons, then the output are Delaunay configurations. For constructing simplex spline spaces, the freedom of polygon triangulations gives local control to the construction.

We now consider the problem of characterizing the output configurations from the link triangulation procedure. In particular, We wish to show that the links are polygons so that the link triangulation procedure can be applied iteratively. We have the following result, which implies that the procedure can be applied three times.

Theorem 4.2. *Let Γ_0 be a planar triangulation and, for $k > 1$, let Γ_k be the set of degree k configurations generated by iteratively applying the link triangulation procedure k times. Then, for $k \leq 2$, for any link P of Γ_k, either P consists of two opposite edges or P is a simple polygon.*

The following lemmas characterize the interior sets of the configurations and support the proof of the above theorem.

Lemma 4.3. *Let Γ_0 be a planar triangulation and, for $k > 1$, let Γ_k denote the degree k configurations generated by applying the link triangulation procedure k times, then,*

1. *$I(\Gamma_1)$ is the vertex set of Γ_0.*
2. *For an edge $\{u, v\}$ of Γ_0, let a and b be the two vertices opposite the facets $[u, v]$ and $[v, u]$ in Γ_0. Then, $\{u, v\}$ belongs to $I(\Gamma_2)$ if and only if $\{a, b\}$ is not chosen as a diagonal in at least one of the polygons $\mathrm{lk}(u, \Gamma_0)$ and $\mathrm{lk}(v, \Gamma_0)$.*
3. *For a set $\{u, v, w\} \in I(\Gamma_3)$, there exist two vertices, say $\{u, v\}$, such that $([u, v], \{w\})$ is a facet configuration of Γ_1.*

For $k \geq 3$, we cannot find counter examples, even with extensive computational experiments: We start with randomly generated sites, randomly flip the edges of the Delaunay triangulation, triangulate the link polygons by randomly cutting ears, and iterate up to five times; in tens of thousands of experiments, all links are simple polygons and star-shaped, just as they are with Delaunay configurations.

5. BOUNDARY-INTERIOR CONFIGS AS CENTROID TRIANGLES

As observed by Aurenhammer [2], an order k Voronoi diagram is dual to a triangulation: For each set of k sites that defines a Voronoi cell, compute the centroid of the lifted sites; the projection of the lower hull of the centroids is a triangulation that is dual to the Voronoi diagram. The dual triangulation can also be constructed by mapping the Delaunay configurations corresponding to the Voronoi vertices to a set of *centroid triangles*. We observe in experiments that, if we apply these maps on the configurations generated from the link triangulation procedure, the resulting centroid

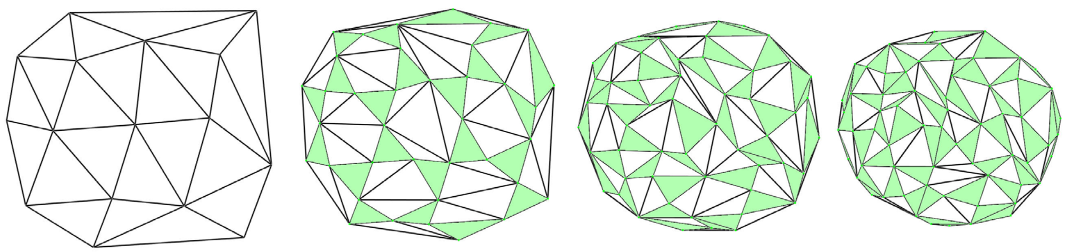

Figure 4: The left most: an order 1 triangulation. The next three: order 2 to 4 centroid triangulations generated by iterative applications of the link triangulation procedure. The type 1 triangles are white; the type 2 triangles are green. Note that, for every two adjacent figures, configurations mapped to the white triangles in the first figure are mapped to the green triangles in the next figure.

triangles also form planar triangulations. Although we are unable to prove the planarity property beyond the first two cases, we still find it worthwhile to discuss this mapping, because, first, it gives us a new way to look at the configurations, and, second, the planarity property is the "missing ingredient" for counting the configurations generated by the link triangulation procedure.

Let us describe centroid triangles and the maps from configurations to centroid triangles. The maps have appeared elsewhere in the Delaunay setting, for example by Andrzejak [1] and Schmitt [12]. Schmitt seems to be the first to coin the term *centroid triangulation*.

For a set of points $V \subset \mathbb{R}^2$, we denote the average, or *centroid*, of V by $\overline{V}$. A triangle is an order k *centroid triangle* if its vertices are centroids of k-sets, and, for any of its edge $\{\overline{U}, \overline{V}\}$, $\#U \cap V = k-1$. The *type* of an order k centroid triangle $[\overline{U}, \overline{V}, \overline{W}]$ is the number $\#(U \cap V \cap W) - k$. It can be proved easily that an order k centroid triangle is either type 1 or 2.

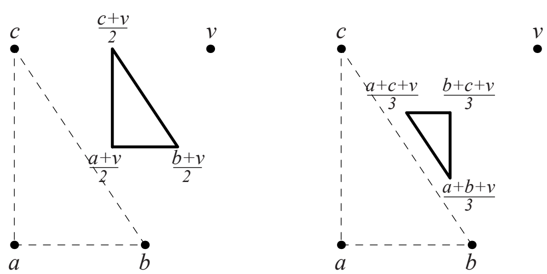

Figure 5: Applying the map A_1, left, and A_2, right, to the configuration $([a, b, c], \{v\})$

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Let K be a set of knots in the plane. There is a bijective correspondence between the set of all possible order k centroid triangles in K and the set of all possible degree $k-1$ and degree $k-2$ configurations in K. The correspondence from the configurations to the centroid triangles is established by the following *average maps* that, for any $i \geq 0$, take a degree i configuration (t, I) to an order $i+1$ or $i+2$ centroid triangle:

$$\begin{array}{l} A_1(t, I) := [\overline{I \cup \{t_0\}}, \overline{I \cup \{t_1\}}, \overline{I \cup \{t_2\}}] \\ A_2(t, I) := [\overline{I \cup {}_0t}, \overline{I \cup {}_1t}, \overline{I \cup {}_2t}] \end{array}.$$

Clearly, applying the map A_1 to degree $k-1$ configurations and A_2 to degree $k-2$ configurations both give order k centroid triangles. In the other direction, choose an order k triangle $[\overline{U}, \overline{V}, \overline{W}]$. Let $I := U \cap V \cap W$, and $X := U \cup V \cup W$. If the triangle is type 1, it maps to a degree $k-1$ configuration $([U \backslash I, V \backslash I, W \backslash I], I)$; if it is type 2, it maps to a degree $k-2$ configuration $([X \backslash U \backslash I, X \backslash V \backslash I, X \backslash W \backslash I], I)$.

In general, two triangles s and t are called *neighbors* if there are vertices $u \in s$, $v \in t$ such that the facets ${}_us$ and ${}_vt$ are opposite. The following lemma shows that facet matching property between degree $k-1$ and $k-2$ configurations correspond to the neighbor relations between order k centroid triangles.

LEMMA 5.1. *Consider two neighboring order-k centroid triangles with edges* $\pm[\overline{J \cup \{a\}}, \overline{J \cup \{b\}}]$. *The pair of configurations mapped to the two triangles can be characterized as follows.*

1. *If both triangles are of type 1, then there must be points $u \notin J$ and $v \notin J$ such that the configurations have the form $([a, b, u], J)$ and $([b, a, v], J)$.*
2. *If both triangles are of type 2, then there must be points $u \in J$ and $v \in J$ such that the configurations have the form $([a, b, u], J \backslash \{u\})$ and $([b, a, v], J \backslash \{v\})$.*
3. *If the two triangles are of opposite types, then there must be points $u \notin J$ and $v \in J$ such that the configurations have the form $([a, b, u], J)$ and $([a, b, v], J \backslash \{v\})$.*

By the above lemma, for two sets of configurations Γ and Γ' of degree $k-1$ and $k-2$, if the centroid triangles $A_1(\Gamma)$ and $A_2(\Gamma')$ form a planar triangulation, then Γ and Γ' match facets. Therefore, the planarity property for Γ and Γ' is stronger than the facet matching property, in essentially the same way as the property that a set of triangles T form a planar triangulation is stronger than the property that each edge in T is shared by two neighboring triangles.

For $k = 2$ and 3, we can easily show that this stronger property is satisfied by the input and output configuration sets from the link triangulation procedure, although we are not able to do this for larger k. Suppose that this property is indeed satisfied. Then, we have the following simple description of the link triangulation procedure: Given a set of configurations Γ, it triangulates the holes left after placing the triangles $A_2(\Gamma)$ on the plane (See Figure 4.) This description makes it straightforward to generalize Lee's bound on the size of higher order Voronoi diagrams [7], as stated below. All that is needed to generalize the proof is to observe that the triangles in a polygon triangulation form a binary tree, whether they are Delaunay or not.

THEOREM 5.2. *Let Γ_0 be a planar triangulation and, for $k \geq 1$, let Γ_k be the set of degree k configurations generated by iteratively applying the link triangulation procedure k times. Then, $\#\Gamma_k = \Theta(nk)$.*

6. B-SPLINES: COLLECTING SIMPLEX SPLINES BY NEIGHBORHOODS

We describe how to collect simplex splines to form a coarser basis, as Neamtu does in [10]. There are certain computational benefits to using the coarser basis.

Let Γ be a set of degree k configurations. The simplex splines associated with Γ can be partitioned based on neighborhoods and summed to form a coarser basis called (multivariate) *B-splines*:

$$\Big\{ B_I(\cdot) := \sum_{t \in \mathrm{nb}(I,\Gamma)} N(\cdot \mid t, I) \Big\}_{I \in I(\Gamma)}.$$

Collecting simplex splines into the coarser B-splines does not break the two key properties of the spline space. First, in one dimension, the multivariate B-splines become the classic univariate B-splines, since, in $\mathbb{R}$, each neighborhood has exactly one configuration. Second, the polynomial reproduction property is preserved, since we can easily derive the the polynomial reproduction formula for the B-splines by collecting terms in the equation of Theorem 3.2 that belong to the same neighborhoods and get $p = \sum_{I \in I(\Gamma)} P(I) B_I$.

For applications, these B-splines should always be preferred over the simplex splines. First, the B-splines are more natural: In the linear case, the B-splines are precisely the PL interpolation functions over a triangulation, while the simplex splines form a finer linear spline basis. In the higher order cases, a B-spline tends to be more symmetrically shaped around the centroid of the interior knot set, while a simplex spline is usually asymmetrical (even for knots on a grid). Second, the B-splines are easier to compute with, due to a facet-form of B-splines that we introduce next.

Let Γ be a set of configurations of degree $k-1$. The facet-form B-spline basis associated with Γ is the set of degree k splines defined over the links of Γ:

$$\Big\{ B_J(x) := \sum_{f \in \mathrm{lk}(J,\Gamma)} d({}^{x}f) M(x \mid f \cup J) \Big\}_{J \in V(\Gamma)}.$$

By simple algebraic manipulation, it is easy to show that the facet-form B-splines, defined over the links of the degree $k-1$ configurations Γ, are the same as the B-splines defined over the neighborhoods of any set of degree k configurations Γ' that match facets with Γ. Therefore, for construction of degree k B-splines by the link triangulation procedure, it suffices to compute configurations up to degree $k-1$.

B-splines can also be described in a way analogous to the way PL functions over a triangulation are commonly described: Given an order k centroid triangulation, for each vertex $\overline{V}$ of the triangulation, there is basis function constructed over V and the link facets of V. See Figure 6 for an illustration. This description also makes it clear that the edge graph of the centroid triangulation gives a representation of the adjacency relations between the B-splines, which can be useful for computation.

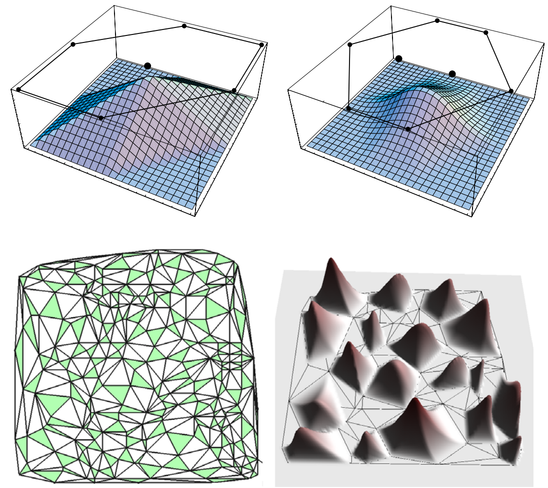

Figure 6: First row: Linear and quadratic B-splines with their facet forms. The points from the interior set are drawn slightly enlarged; the facets are drawn as polygons. Second row: An order 2 centroid triangulation and some of its associated quadratic B-splines.

7. REPRODUCTION OF ZP-ELEMENT

We demonstrate the usefulness of the link-triangulation procedure by reproducing the Zwart-Powell element [6], a textbook example of bivariate smooth quadratic *box spline*, depicted in Figure 7. The formal statement can be found in Theorem 7.1.

Reproducing box splines has a practical application for blending surface patches. In CAD, surfaces are often constructed by joining together a collection of box spline patches. Controlling the quality of the surfaces around the patch boundaries is a difficult problem. Two of the most common approaches to handling the patch boundaries have drawbacks: In engineering, surface patches are often stitched together by a delicate process of parameter tweaking with humans in the loop, with conflicting goals of satisfying the boundary smoothness constraints while controlling the geometric shape around the boundaries. In computer graphics, subdivision surfaces (which generalize box splines via the subdivision algorithm) handle patch boundaries by using special subdivision rules for the extraordinary vertices, but the portions of the surface around extraordinary vertices are not polynomials and have lower smoothness. Our B-splines provide an alternative: Given a collection of box spline patches, we simply triangulate the underlying mesh, then construct a B-spline space over the entire mesh that will reproduce the box spline basis within the patches. The mathematics guarantees the same smoothness everywhere; the surface around patch boundaries are no longer special. A designer can then concentrate on controlling the shape around the patch boundaries by changing basis coefficients or changing link triangulations, without worrying about smoothness.

Our proof the B-splines reproduce the ZP element is geometric. A B-spline is the sum of shadows of 4-simplices; a box spline is the shadow of a 4-cube. This naturally leads us to find a suitable tessellation of the 4-cube into simplices.

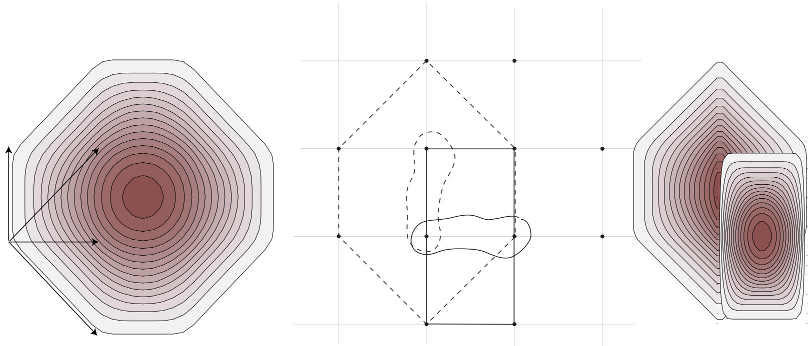

Figure 7: Left: ZP element with its defining vector set Ξ. Right: the two types of B-splines in (1) and their facet forms, distinguished by dotted or solid lines(center).

7.1 Polyhedron splines

Recall that s-variate simplex splines are shadows cast by simplices of uniform density to $\mathbb{R}^s$. Box splines are shadows cast by hypercubes. In general, polyhedron splines are shadows cast by polyhedron. We formalize the notation for projection and shadow functions, which will then let us define box splines and simplex splines very concisely.

Consider the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, where $m \leq n$. Let Ξ denote an $m \times n$ matrix, whose columns span $\mathbb{R}^m$, that represents a projection map from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., a point $y \in \mathbb{R}^n$ is taken to $\Xi y \in \mathbb{R}^m$.

Let $\mathcal{P}$ denote a polyhedron in $\mathbb{R}^n$. Then, the polyhedron shadow function, $M_\Xi(\cdot \mid \mathcal{P}) : \mathbb{R}^m \to \mathbb{R}$, is the measure of P that projects onto a point $x \in \mathbb{R}^m$:

$$M_\Xi(x \mid \mathcal{P}) := \frac{\mathrm{vol}_{n-m}\{y \mid y \in \mathcal{P}, \Xi y = x\}}{\mathrm{vol}_n \mathcal{P}}.$$

Box splines. The box spline $M_\Xi : \mathbb{R}^m \to \mathbb{R}$ is the shadow of an n-cube $[0,1]^n$: $M_\Xi(x) := M_\Xi(x \mid [0,1]^n)$.

Simplex splines. Let $X \subset \mathbb{R}^m$ denote a set of $n+1$ points. Let $\Xi : \mathbb{R}^n \to \mathbb{R}^m$ be the "vertical" projection that simply omits the last $n-m$ coordinates. Let Y denote a set of points such that $\Xi Y = X$. Then, the simplex spline with respect to X, $M(\cdot \mid X)$, is the shadow of an n-simplex with vertical projection: $M(x \mid X) := M_\Xi(x \mid [Y])$.

7.2 ZP element and B-splines

We define two sets of quadratic splines over the integer grid $\mathbb{N} \times \mathbb{N}$. The first is the integer translates of the ZP element. The second is the set of B-splines defined by applying the link triangulation procedure once in a certain way in order to reproduce the ZP element.

The *Zwart-Powell element* [6], or ZP element for short, is the quadratic box spline: $M_{\left[\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{smallmatrix}\right]}$, as shown on the left of figure 7. To form the basis of a spline space, all integer translates of a single ZP element are taken, so that, for each grid cell, there is a ZP element centered at the cell center.

We now construct a B-spline space over grid points by applying the link triangulation procedure once as shown in Figure 8. Let E denote the vertical and horizontal segments in the grid $\mathbb{N} \times \mathbb{N}$, i.e. a pair of grid points $\{u, v\}$ belongs to E if and only if exactly one of their coordinates differ by one. Then, the quadratic B-splines defined over Γ_1 are the facet-form B-splines:

$$\{B_{\{u,v\}}\}_{\{u,v\} \in E}. \tag{1}$$

The B-splines in (1) are translated copies of two types of B-splines as shown in Figure 7. One type of B-spline, which we shall call the P type, comes from vertical segments in E; the other, the Q type, comes from horizontal segments in E.

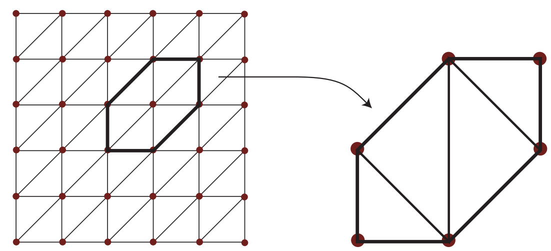

Figure 8: Left: grid triangulation Γ_0. Right: the triangulation of a single vertex link.

7.3 Reproducing the ZP element

We prove the following main result:

THEOREM 7.1. *The space of quadratic B-splines as defined in (1) contains the space spanned by integer translates of the ZP element.*

For the proof, we show that the ZP element, whose support is centered around a grid square, equals the sum of four B-splines, of which two are type P from the vertical segments of the square and two are type Q from the horizontal segments of the square. See Figure 7 for an illustration.

The equality is shown by first decomposing the B-splines and the ZP element as sums of shadows of 4-simplices and then matching the sums. For the B-splines decomposition, we triangulate their defining polygons; for the ZP element decomposition, we first tessellate the 4-cube to four pieces to match the four P, Q type B-splines, then triangulate each piece to match the decompositions of individual B-splines.

8. DISCUSSION

We have initiated a study of the connection of order k centroid triangulations (or, in the dual, generalizing higher order Voronoi diagrams) to bivariate B-splines on arbitrary knot sets. The solution we have provided here is based a simple procedure. For lower orders, we have shown the procedure to be correct and immediately used it as a tool to study a certain spline theory problem. However, the analysis of the procedure becomes difficult as the order increases.

We believe that the solution lies in some appropriate geometric characterization of the centroid triangulations, which we do not yet have.

There are many other interesting geometric problems on centroid triangulations whose solutions have applications to splines. For example, how can we create a hierarchical of nested centroid triangulations? Solutions to this problem might lead to subdivision algorithms for displaying B-spline surfaces.

Acknowledgments

We are indebted to Dominique Schmitt, who has several results on centroid triangulations as geometric and combinatorial objects in their own right, for many helpful discussions.

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