On 2-Site Voronoi Diagrams under Arithmetic Combinations of Point-to-Point Distances

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***Abstract*—We consider a generalization of Voronoi diagrams, recently introduced by Barequet et al., in which the distance is measured from a *pair* of sites to a point. An easy way to define such distance was proposed together with the concept: it can be the sum-of, the product-of, or (the absolute value of) the difference-between Euclidean distances from either site to the respective point. We explore further the last definition, and analyze the complexity of the nearest- and the furthest-neighbor 2-site Voronoi diagrams for points in the plane with Manhattan or Chebyshev underlying metrics, providing extensions to general Minkowsky metrics and, for the nearest-neighbor case, to higher dimensions. In addition, we point out that the observation made earlier in the literature that 2-point site Voronoi diagrams under the sum-of and the product-of Euclidean distances are identical and almost identical to the second order Voronoi diagrams, respectively, holds in a much more general statement.**

***Keywords*-generalized Voronoi diagrams; distance functions**

I. Introduction

The Voronoi diagram is a fundamental geometric structure, which has found numerous applications in various areas of computer science. Detailed surveys of its history and use were given by Aurenhammer [1] and Sugihara [20]; other excellent sources of information are the books [15] and [11].

For a set S of n points in the Euclidean plane, called *sites*, its Voronoi diagram $V(S)$ partitions the plane into n regions, each corresponding to a distinct site; the Voronoi region $\mathrm{Vor}(s)$ of a site $s \in S$ consists of all the points being closer to s than to any other site from S. Possible generalizations of this concept include consideration of higher dimensions, different types of sites, other metrics or distance functions, and more sophisticated choices of the underlying space.

One of the first generalizations was a Voronoi diagram of order k introduced by Shamos and Hoey [18] and further studied by Lee [13]; any its region is associated with a subset $T \subset S$ of size k, and consists of all the points being closer to any site from T than to any site from $S \setminus T$.

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A few years ago, Barequet et al. [3] introduced so-called *2-point site Voronoi diagrams*, which are obtained as follows. Given a set of point sites in the Euclidean plane, a distance between a *pair* of sites and a point in the plane is defined; then, with each pair of sites, a Voronoi region is associated, consisting of all the points in the plane, for which this pair of sites is the closest one. Recently, this concept has been applied by Dickerson and Goodrich [9] to geographic networks, and further investigated by Barequet and Hanniel [4] and by Dickerson and Eppstein [8].

Yet out of the eight distance functions considered in the works [3], [4], [8], [9], only three lead to Voronoi structures that cannot be interpreted as previously known 1-site Voronoi diagrams—either those for non-point sites or those for points but of higher order. Namely, these are the "difference between distances" distance function examined in [3], the perimeter distance function first defined in [3] and then addressed in [9] (where the respective Voronoi diagram is computed on a graph) and in [4], and an elegant generalization of the latter recently proposed and studied by Dickerson and Eppstein [8]. As a consequence, the mentioned three distance functions look somehow more appealing in the context being discussed.

In this work, we first overview the existing variants of 2-point site Voronoi diagrams and their relationships with the classical types of Voronoi diagrams. In particular, those interrelations immediately suggest the ways for constructing 2-point site Voronoi diagrams under certain distance functions.

After that, we restrict our attention to the 2-point site Voronoi diagrams under arithmetic combinations of point-to-point distances. For the "sum of distances" and "product of distances" distance functions, we point out that the observation made in [3] that the corresponding Voronoi diagrams are identical and almost identical, respectively, to the (1-site) second-order Voronoi diagram for the underlying set of points, holds in a significantly more general statement. Though those properties are not difficult to justify, we believe it is worth to formulate them explicitly, for the

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following two reasons. First, they highlight once again the importance of properly distinguishing between the 2-site distance functions that produce novel Voronoi structures and those that do not. Second, an alternative view at a familiar object can be very fruitful for further investigation of its properties and possible applications; in particular, it would be very interesting if a motivation for studying 2-site Voronoi diagrams under the "sum of distances" distance function were proposed, which would not reduce to the need of finding the two closest sites for a given point (thereby yielding again the notion of the second-order Voronoi diagram), and thus, would affirm the necessity of examining the corresponding geometric structure from the 2-site point of view.

Finally, we consider the 2-point site Voronoi diagrams under the "difference between distances" distance function. For the nearest-neighbor case, we provide the lower bounds on their complexity for Minkowsky metrics (L_p, where $1 \leq p \leq \infty$) in the plane as well as in a higher-dimensional space. Moreover, for Manhattan (L_1) and Chebyshev (L_∞) metrics, we provide an almost-matching upper bound on the complexity of the respective Voronoi diagrams for a set of points in the plane. For the furthest-neighbor case, the corresponding Voronoi structure is given by an overlay of the 1-site nearest- and the furthest-neighbor Voronoi diagrams. Such overlays in the plane endowed with Euclidean (L_2) metric have been considered a few times in the literature in other contexts (see e.g. [10], [17], [2]). We bound their complexity in the plane with L_p metrics, for $1 \leq p \leq \infty$; for the limit cases of L_1 and L_∞, our results also suggest a method for computing the respective Voronoi diagrams in optimal time and space.

The rest of the paper is organized as follows. In the next section, we review and analyze the kinds of 2-point site Voronoi diagrams previously discussed in the literature. In Section III, we address the "sum of distances" and the "product of distances" distance functions, and Section IV is devoted to the "difference between distances" distance function and the corresponding 2-point site Voronoi diagrams. We end in Section V with some concluding remarks.

II. Analysis of Previous Results

The notion of 2-point sites Voronoi diagrams was first introduced by Barequet et al. [3]. In particular, they examined six distance functions that give a distance between an unordered pair (q, r) of sites and a point x in the plane:

- sum of distances: $d^+(x, (q, r)) = d(x, q) + d(x, r)$;
- product of distances: $d^*(x, (q, r)) = d(x, q) \cdot d(x, r)$;
- triangle area: $d^{\mathcal{A}}(x, (q, r)) = Area(\triangle xqr)$;
- distance from a line: $d^\ell(x, (q, r)) = d(x, \ell_{qr})$, where ℓ_{qr} denotes the line through q and r;
- distance from a segment: $d^s(x, (q, r)) = d(x, \overline{qr})$, where $\overline{qr}$ denotes the segment with the endpoints q and r;
- difference between distances: $d^-(x, (q, r)) = |d(x, q) - d(x, r)|$.

Barequet et al. [3] also defined a perimeter distance function $d^{\mathcal{P}}(x, (q, r)) = Per(\triangle xqr) = d(x, q) + d(x, r) + d(q, r)$. The 2-site Voronoi diagram under it on a graph was discussed by Disckerson and Goodrich [9], while the one for points in the plane was addressed in [4], and considered as a special case by Dickerson and Eppstein [8], who introduced and studied a generalization of the perimeter distance function of the form $d^c(x, (q, r)) = d(x, q) + d(x, r) + c \cdot d(q, r)$, defined for any constant $c \geq -1$.

Not surprisingly, in the plane with Euclidean metric, 2-point site Voronoi diagrams under the "sum of distances" and the "product of distances" distance functions are very similar to the second-order Voronoi diagrams for points: the former is identical, and the latter is almost identical to it. (The 2-point site Voronoi diagram under the "product of distances" distance function can be obtained from the second-order Voronoi diagram by adding to it all the sites as isolated vertices.) These facts were mentioned in [3].

A 2-point site Voronoi diagram under the triangle area distance function actually represents a special case of weighted Voronoi diagrams for lines in the plane, and a one under the "distance from a line" distance function—the unweighted Voronoi diagram for lines in the plane. The former can be constructed either by means of a general approach based on the computation of the lower envelope of a set of halfplanes, as described in [3] (see also [19]), or by a special-purpose algorithm developed in [5]; the latter can be easily computed in optimal time and space, as suggested in [5].

A 2-point site Voronoi diagram under the "distance from a segment" distance function reduces to a Voronoi algorithm for intersecting segments, and can be computed, for example, with the method described by Karavelas [12].

Finally, we note that 2-point site Voronoi diagrams under the "difference between distances" or under the perimeter distance function, as well as under the generalization of the latter proposed in [8], cannot be represented by 1-(non-point)-site or by higher-order Voronoi diagrams, which makes them particularly attractive. However, the furthest-neighbor 2-point site Voronoi diagram under the "difference between distances" distance function can be obtained as an overlay of the nearest- and the furthest-neighbor 1-point site Voronoi diagrams [3].

III. Sum and Product of Distances

Recall again that in the plane with Euclidean metric, the 2-site Voronoi diagram $V^2_{d^+}(S)$ under the "sum of distances" distance function d^+ is identical to the second-order Voronoi diagram $V^{(2)}(S)$. It was pointed out in [3]; yet we provide below a short and simple reasoning, which explains this fact, and also applies to further cases.

Let S be a set of point sites in the plane. Consider the second-order Voronoi diagram $V^{(2)}(S)$ of S. For any two sites $s', s'' \in S$, their Voronoi region $\mathrm{Vor}^{(2)}(s', s'')$ consists of all points for which s' and s'' are the two nearest neighbors from S. For some pairs of sites, the respective region may be empty; let us choose any two sites q and r, such that $\mathrm{Vor}^{(2)}(q, r) \neq \emptyset$. Observe that for any point $x \in \mathrm{Vor}^{(2)}(q, r)$, the following property holds:

$$d(x, q) + d(x, r) = \min_{u,v \in S, u \neq v} \{d(x, u) + d(x, v)\},$$

where $d(a, b)$ denotes the Euclidean distance between the points a and b in the plane. In other words, for any point $x \in \mathrm{Vor}^{(2)}(q, r)$, the sites q and r minimize the sum of the distances between x and any two distinct sites from S, and thus, $x \in \mathrm{Vor}^2_{d^+}(q, r)$. Conversely, if the sites q and r minimize the sum of the distances between x and any two distinct sites from S, then obviously q and r must be the two nearest neighbors of x in S. We conclude that $V^2_{d^+}(S)$ and $V^{(2)}(S)$ are identical.

Evidently, in the above reasoning, Euclidean metric L_2 can be replaced with any Minkowsky metric L_p, where $1 \leq p \leq \infty$. Moreover, instead of 2-point site Voronoi diagrams, we can consider their k-site analogues, in which a Voronoi region is associated with a k-tuple of sites. Thus, in the plane endowed with any Minkowsky metric, a k-point site Voronoi diagram under the "sum of distances" distance function is identical to the Voronoi diagram of order k. All these observations immediately generalize to higher dimensions.

More generally, we can consider a distance function $\bar{d}$ in the following form:

$$\begin{aligned}&\bar{d}(x, (s_{i_1}, s_{i_2}, \ldots, s_{i_k})) \\ &\quad = f(\hat{d}(x, s_{i_1}), \hat{d}(x, s_{i_2}), \ldots, \hat{d}(x, s_{i_k})), \qquad (*)\end{aligned}$$

where the function $\hat{d}(x, y)$ defines a distance between any two h-dimensional points x and y, and $f : \mathbb{R}^k_+ \to \mathbb{R}_+$ is a symmetric function; here $\mathbb{R}_+$ denotes the set of all non-negative real numbers. (The function f is required to be symmetric in order to properly define a distance between a point and an *unordered* k-tuple of sites.)

A k-point site Voronoi diagram under the distance function $\bar{d}$ will be identical to the Voronoi diagram of order k, if $\bar{d}$ strictly increases in each argument. In fact, $\hat{d}(x, y)$ even does not have to be a metric: it can also be an asymmetric convex distance function. In this case, the distance between a point and a site is measured from the latter to the former (see e.g. [7]); for consistency, we shall say that the distance $\bar{d}$ between a point and a k-tuple of sites, which is based on a convex distance function $\hat{d}$, is also measured from the latter to the former.

An analogous reasoning applies to the furthest-neighbor case, and finally leads to the conclusion that in h-dimensional space, the furthest k-site Voronoi diagram under the distance function $\bar{d}$ is identical to the k-th order furthest-neighbor Voronoi diagram.

In the context of 2-point site Voronoi diagrams, the only principal difference between the "sum of distances" and the "product of distances" distance functions is that the latter becomes zero if the point x coincides with one of the sites. Thus, each site is equidistant from all the k-tuples of sites, to which it belongs, and therefore, is a vertex of the respective k-site Voronoi diagram. Consequently, in h-dimensional space endowed with a Minkowsky metric, a k-point site Voronoi diagram under the "product of distances" distance function can be obtained from the Voronoi diagram of order k by adding to it all the sites as isolated vertices. An analogous general statement will hold for any distance function $\bar{d}$ of the form (*), if $\bar{d}$ equals zero whenever x coincides with some site, and the function f strictly increases on $\mathbb{R}^k_{>0}$ in each argument, where $\mathbb{R}_{>0}$ denotes the set of all positive real numbers. The furthest k-site Voronoi diagram under such distance function will be identical to the k-th order furthest-neighbor Voronoi diagram.

IV. Difference between Distances

Throughout this section, let $d_p(x, y)$ denote the L_p distance between the points x and y in the plane or in a higher-dimensional space. In the case of Euclidean distance, we shall omit the corresponding subscript '2', and denote the distance between x and y simply by $d(x, y)$.

In this section, we shall consider 2-site Voronoi diagrams in dimension two or higher under the "difference between distances" distance function $d^-_p(x, (s', s'')) = |d_p(x, s') - d_p(x, s'')|$, which is an extension of a similarly-named distance function defined in [3] as $d^-(x, (s', s'')) = |d(x, s') - d(x, s'')|$. We note right away that this distance function cannot be generalized in a straightforward manner to the case of a k-tuple of sites.

A. Nearest-Neighbor Voronoi Diagrams

First, we shall provide a lower bound on the complexity of the 2-site Voronoi diagram $V^2_{d^-_p}(S^h)$ of a set S^h of n sites in h-dimensional space. Let us start with the planar case.

Theorem 1. *The combinatorial complexity of $V^2_{d^-_p}(S^2)$ is $\Omega(n^4)$ in the worst case.*

Proof: For a given $n \geq 1$, consider the point set $S^2 = S_1 \cup S_2$, where $S_1 = \{s^1_i\}^{\lceil \frac{n}{2} \rceil}_{i=1}$ and $S_2 = \{s^2_j\}^{\lfloor \frac{n}{2} \rfloor}_{j=1}$, and
$s^1_i = (\frac{1}{2^{i-1}}, 0)$, for $1 \leq i \leq \lceil \frac{n}{2} \rceil$,
$s^2_j = (0, \frac{1}{2^{j-1}})$, for $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.

For any two points $s_i, s_k \in S_1$, where $1 \leq i < k \leq \lceil \frac{n}{2} \rceil$, their L_p bisector under any metric L_p, $1 \leq p \leq \infty$, coincides with the perpendicular bisector of the horizontal segment $s_i s_k$ under L_2 (Fig. 1). Let us prove that no two of those bisectors are identical.

Let $M = \lceil \frac{n}{2} \rceil$; without loss of generality, assume that $M \geq 2$. Let us add the points $s^1_1, \ldots, s^1_M$ one by one; we claim that after the addition of the m-th point, the desired

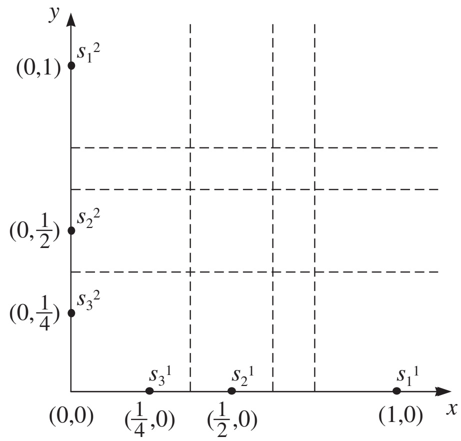

Figure 1. An example of $S^2 = S_1 \cup S_2$ for $n = 6$. The bisectors defined by pairs of points from S_1 are vertical dashed lines; the bisectors defined by pairs of points from S_2 are horizontal dashed lines. The total of 6 bisectors define 9 intersection points.

property holds, where $2 \leq m \leq M$. The proof is by induction.

After the insertion of the second point, our assertion trivially holds. Assume that after the addition of s_m, where $2 \leq m < M$, the statement is true; to demonstrate that it will also hold after s_{m+1} is added, it is sufficient to show that for any i, $1 \leq i \leq m$, the bisector of the points s_i and s_{m+1} will not coincide with any of the previously constructed ones.

Observe that for any j, k, $1 \leq j < k \leq m$, the x-coordinate x_{jk} of the bisector b^1_{jk} of the points s^1_j and s^1_k has a form $a \cdot \frac{1}{2^m}$, where a is a positive integer.

Now consider any point s^1_i, where $1 \leq i \leq m$. The x-coordinate $x_{i,(m+1)}$ of the bisector of s^1_i and s^1_{m+1} equals $c \cdot \frac{1}{2^m} + \frac{1}{2^{m+1}}$, where $c = 2^{m-i}$ (and thus, c is a positive integer). It follows immediately that $x_{i,(m+1)}$ cannot coincide with any x_{jk}, where $1 \leq i, j, k \leq m$ and $j \neq k$.

We conclude that the points from S_1 together define $\Theta(n^2)$ distinct L_p bisectors, which are vertical lines. Similarly, the points from S_2 define $\Theta(n^2)$ distinct bisectors, which are horizontal lines. Moreover, each bisector from the first set intersects all the bisectors from the second set, and each intersection point is a distinct feature of the Voronoi diagram $V^2_{d^-_p}(S^h)$. Therefore, $V^2_{d^-_p}(S^h)$ has $\Omega(n^4)$ features. ∎

Now let us return to the h-dimensional case.

Theorem 2. *The combinatorial complexity of $V^2_{d^-_p}(S^h)$ is $\Omega((\frac{n}{h})^{2h})$ in the worst case.*

Proof: Following the scheme of the proof of Theorem 1, let us consider a set of n points, all of which belong to the coordinate axes; let each of the h axes contain approximately $\frac{n}{h}$ points having the corresponding coordinate 1, $\frac{1}{2}$, $\frac{1}{4}$, …, respectively. For any axis, the L_p bisector of any pair of point belonging to it is a hyperplane, which represents a perpendicular bisector under L_2 of the segment connecting those two points. As in the previous theorem, it can be shown that any two points lying on the same axis define a distinct bisector; thus, a subset of all the points belonging to any axis defines a set of $O((\frac{n}{h})^2)$ bisectors. Any two bisectors defined by two pairs of points lying on different axes intersect. We conclude that all the bisectors under consideration together produce $O((\frac{n}{h})^{2h})$ intersection points, at any of which precisely h bisectors meet (one for each axis). Since each intersection point is a distinct feature of the Voronoi diagram $V^2_{d^-_p}(S^h)$, the claim follows. ∎

For $p = 1$ and $p = \infty$ we also provide an almost-matching upper bound on the complexity of $V^2_{d^-_1}(S)$ and $V^2_{d^-_\infty}(S)$.

Theorem 3. *The combinatorial complexity of $V^2_{d^-_1}(S)$ and $V^2_{d^-_\infty}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).*

Proof: The respective Voronoi surfaces are not algebraic, but every surface can be split into a constant number of planar patches, each of which has a boundary of constant complexity. The collection of these $\Theta(n^2)$ patches fulfills Assumptions 7.1 of [19, p. 188]:

1) Each patch is an algebraic surface of constant maximum degree;
2) The vertical projection of each patch onto the xy-plane is bounded by a constant number of algebraic arcs of constant maximum degree; and
3) Each triple of patches intersects in at most a constant number of points.

Hence, we may apply Theorem 7.7 of [19, p. 191] and obtain the claimed complexity. ∎

B. *Furthest-Neighbor Voronoi Diagrams*

As in the case of the furthest 2-site Voronoi diagrams in the plane under the "difference between distances" distance function with the underlying Euclidean metric, for any h-dimensional point x, the maximum value of $d^-_p(x, (s', s''))$ is attained at the pair of sites from S^h, one of which is the nearest, and the other—the furthest neighbor of x. Thus, the furthest 2-site Voronoi diagram $VF^2_{d^-_p}(S^h)$ in the general case also can be obtained as the overlay of the nearest- and the furthest-neighbor Voronoi diagrams, $V_p(S^h)$ and $VF_p(S^h)$, respectively.

The same holds if instead of $d^-_p(x, (s', s''))$ we consider a distance function

$$\hat{d}^-(x, (s', s'')) = |\hat{d}(x, s') - \hat{d}(x, s'')|,$$

for any convex distance function $\hat{d}$.

In the rest of this section, we discuss the lower bounds on the complexity of the furthest 2-site Voronoi diagrams under the distance function $d^-_p(x, (s', s''))$ in the plane, where $1 \leq p \leq \infty$.

The Planar Case: Let us start with $p = 2$. Consider a set S of sites in the plane and the distance function $d^-(x,(s',s'')) = d_2^-(x,(s',s''))$ defined as the absolute value of the difference of the Euclidean distances from x to s' and to s'', respectively.

Issues of complexity and computation of an overlay of the nearest- and the furthest-neighbor Voronoi diagrams of a set of points in the plane with L_2 metrics were addressed by Ebara et al. [10] and by Roy and Zhang [17]. However, below we develop our own reasoning for this case, in order to further generalize it to L_p metrics, where $1 < p < \infty$.

Lemma 1. *The combinatorial complexity of $VF^2_{d^-}(S)$ is $O(kn)$, where k is the number of points from S lying on the boundary of the convex hull of S.*

Proof: It is well-known that in the Euclidean plane, the nearest- and the furthest-neighbor Voronoi diagrams $V(S)$ and $VF(S)$, respectively, have the complexity $O(n)$ and $O(k)$, respectively (see e.g. [16]). Taking this into account, the desired bounds can be easily obtained by analyzing, in which ways the vertices, the edges and the faces of the overlay can be produced. ■

Theorem 4. *The combinatorial complexity of $VF^2_{d^-}(S)$ is $\Theta(n^2)$ in the worst case.*

Proof: Consider a unit circle $\mathcal{C}$ centered at the origin O. For a given $n \geq 2$, consider two point sets $S_1 = \{s_1, \ldots, s_{\lceil \frac{n}{2} \rceil}\}$ and $S_2 = \{s_{\lceil \frac{n}{2} \rceil + 1}, \ldots, s_n\}$ having approximately the same size, each consisting of distinct points, and such that all the points from S_1 lie in the open segment $((-\frac{\sqrt{2}}{2}, 0), O)$, and all the points from S_2 lie in the (smaller) open arc of $\mathcal{C}$ between the points (1,0) and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ (see Fig. 2a). Let $S = S_1 \cup S_2$.

For any point $q \in S_1$, let $r_v(q)$ denote the vertical ray directed downwards having the origin at q. For any point $t \in S_2$, let $r_O(t)$ denote the ray passing through O with the origin at t.

Consider any two points $q \in S_1$ and $t \in S_2$; let $u_{qt} = r_v(q) \cap r_O(t)$. Observe that for u_{qt}, q and t are the nearest and the furthest neighbors from S, respectively. Therefore, u_{qt} belongs to the Voronoi region $\mathrm{VorF}_{d^-}(q,t)$ of the pair of sites (q,t) in $VF^2_{d^-}(S)$.

Thus, for any two points q and t, being from S_1 and S_2, respectively, we can indicate a point in the plane contained in the Voronoi region $\mathrm{VorF}_{d^-}(q,t)$ (see Fig. 2b). Since there are $\Theta(n^2)$ such pairs of points, we conclude that $VF^2_{d^-}(S)$ contains $\Theta(n^2)$ non-empty regions. Together with Lemma 1, this implies the claim. ■

The reasoning developed in the proof of Theorem 4 extends to the case of $d_p^-(x,(s',s''))$, where $1 < p < \infty$, as demonstrated below.

Theorem 5. *The combinatorial complexity of $VF^2_{d_p^-}(S)$ is $\Omega(n^2)$ in the worst case.*

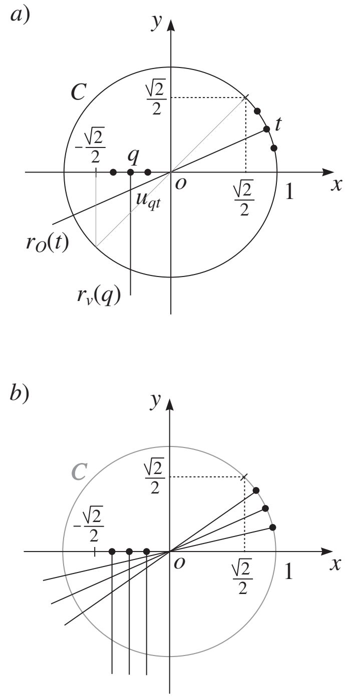

Figure 2. a) The unit circle $\mathcal{C}$ is centered at the origin O. The sets S_1 and S_2 consist each of three points. For the points $q \in S_1$ and $t \in S_2$, the rays $r_v(q)$ and $r_O(t)$ are depicted, intersecting at the point u_{qt}. b) For all the points from $S = S_1 \cup S_2$, their associated rays are shown; those six rays together define nine intersection points, each belonging to a distinct Voronoi region of $VF^2_{d^-}(S)$.

Proof: Consider a unit L_p-sphere $\mathcal{C}_p$ centered at the origin O. For a given $n \geq 2$, consider two point sets S_1 and S_2 defined as in the proof of Theorem 4, with the only difference that the points from S_1 lie in the open segment $((-2^{-1/p}, 0), O)$, and those from S_2—in the (smaller) open arc of $\mathcal{C}_p$ between the points (1,0) and $(2^{-1/p}, 2^{-1/p})$. Let $S = S_1 \cup S_2$. For any point $q \in S_1$, and for any point $t \in S_2$, define the rays $r_v(q)$ and $r_O(t)$, respectively, again, as in the proof of Theorem 4.

Consider any two points $q \in S_1$ and $t \in S_2$; for the point $u_{qt} = r_v(q) \cap r_O(t)$, let x_u and y_u denote its respective coordinates. For any point $v \in S$ different from q, $|x_v - x_u| > 0 = |x_q - x_u|$ and $|y_v - y_u| \geq |y_q - y_u|$. It follows immediately that q is the nearest neighbor of u_{qt} in S.

Next, observe that for any point $w \in S_1$, $|x_w - x_u| < 2^{-1/p}$ and $|y_w - y_u| < 2^{-1/p}$, and thus, $d_p(w, u_{qt}) < 1$. On the other hand, since x_t and y_t are both positive, and x_u and y_u are both negative, $d_p(t, u_{qt}) = (|x_t - x_u|^p + |y_t - y_u|^p)^{1/p} > (|x_t|^p + |y_t|^p)^{1/p} = 1$. We conclude that u_{qt} is further from t than from any point from S_1; therefore, the furthest neighbor of u_{qt} in S necessarily belongs to S_2.

To demonstrate that t is the furthest neighbor of u_{qt} in S_2, let us restrict our attention to the open arc $\mathcal{C}_p^{\mathrm{I}}$ of $\mathcal{C}_p$ lying in the first quadrant, and consider the L_p distance from u_{qt} to a point $(x,y) \in \mathcal{C}_p^{\mathrm{I}}$ as a function of x. Thus, we obtain a function $U : (0,1) \to \mathbb{R}$ defined by the formula

$$\begin{aligned} U(x) &= d_p(u_{qt}, (x, y(x))) \\ &= (|x - x_u|^p + |y(x) - y_u|^p)^{1/p} \\ &= ((x - x_u)^p + ((1 - x^p)^{1/p} - y_u)^p)^{1/p}. \end{aligned}$$

We start by finding extrema of $U(x)$ on (0,1); to this end, we shall first determine the roots of $U'(x)$ on (0,1).

Let $U_1(x) = (x - x_u)^p + ((1 - x^p)^{1/p} - y_u)^p$; observe that the roots of $U'(x)$ are the same as those of

$$U_1'(x) = p(x - x_u)^{p-1} - p((1 - \frac{y_u}{(1 - x^p)^{1/p}}) \cdot x)^{p-1}.$$

Letting $U_1'(x) = 0$, we obtain immediately $\frac{x_u}{y_u} = \frac{x}{(1-x^p)^{1/p}} = \frac{x}{y(x)}$. Consequently, the only root of $U_1'(x)$ on $(0,1)$ is x_t, and the same holds for $U'(x)$. This implies that an extremum of $U(x)$ on $(0,1)$ can be attained only at x_t. To assure that it is indeed so, and that the respective extremum is a strict local maximum, it is sufficient to show that $U''(x_t) < 0$, which is straightforward to demonstrate by elementary calculus. Moreover, we thereby derive that $U'(x)$ is positive on $(0, x_t)$ and negative on $(x_t, 1)$; it follows that $d(u_{qt}, t) > d(u_{qt}, z)$, $\forall z \in \mathcal{C}_p^{\mathrm{I}} \setminus \{t\}$, and, in particular, t is the furthest neighbor of u_{qt} in S_2.

We conclude that for any $q \in S_1$ and any $t \in S_2$, the point u_{qt} belongs to the Voronoi region $\mathrm{VorF}_{d_p^-}(q,t)$; therefore, $VF^2_{d_p^-}(S)$ contains $\Theta(n^2)$ non-empty regions. The claim follows. ■

Now let us consider the two limit cases of L_p: the metrics L_1 and L_∞. The proofs of Theorems 4 and 5 rely on the fact that for any point t lying inside an L_p-disk D_p centered at a point $c \neq t$, there exists a unique furthest point from t on the boundary of D_p, where $1 < p < \infty$. However, for L_1- and L_∞-disks, this property does not hold: in either case, for any inner point t of the respective disk other than the center of the latter, the set of furthest points from t on the disk boundary is infinite. More precisely, such a point set represents either a single or two adjacent edges of the square being the disk under consideration. As a consequence, the lower bounds on the complexity of $VF^2_{d_1^-}(S)$ and $VF^2_{d_\infty^-}(S)$ differ from that on the complexity of $VF^2_{d_p^-}(S)$, where $1 < p < \infty$, as will be shown in the remainder of this section.

For a set S of sites in the plane, let R_1^S and R_∞^S denote the minimum bounding rectangles for S, the sides of which have a slope of ± 1, and are axis-parallel, respectively (see Fig. 3a,b).

Lemma 2. *For any point t in the plane, any its furthest neighbor in S under the L_1 metric lies on the boundary of R_1^S.*

Proof: Let $f \in S$ denote the furthest neighbor of t in S under the L_1 metric. Suppose for contradiction that f lies inside R_1^S (see Fig. 4).

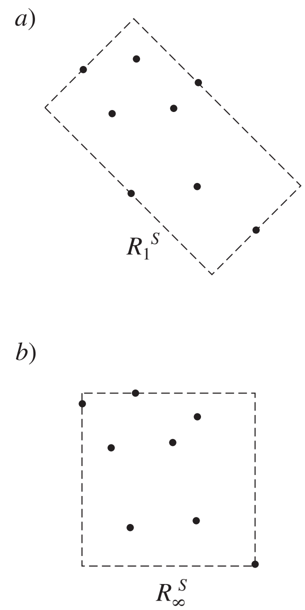

Figure 3. For the set S consisting of eight sites (black), the minimum bounding rectangles a) R_1^S and b) R_∞^S are depicted dashed.

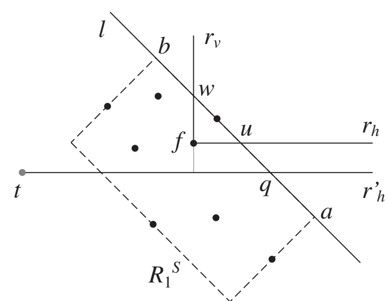

Figure 4. The set S consists of eight sites (black). The rectangle R_1^S is shown dashed; the line ℓ passes through its rightmost and topmost vertices a and b, respectively. For a point t (gray), the site $f \in S$ is conjectured to be the furthest neighbor in S. The horizontal rays r_h and r'_h and the vertical ray r_v (solid black) intersect ℓ at the points u, q and w, respectively.

Without loss of generality, assume that $x_t \leq x_f$ and $y_t \leq y_f$; the other three possible cases are symmetric.

Let a and b denote the rightmost and the topmost vertices of R_1^S, respectively, and let ℓ denote the line supporting the segment ab. Observe that the slope of ℓ is -1, and both t and f lie below ℓ. Consider a horizontal and a vertical rays r_h and r_v, respectively, having their origins at f and directed to the right and upwards, respectively. Let $u = r_h \cap \ell$ and $w = r_v \cap \ell$. Note that the t is L_1-equidistant from any point of uw, and this distance is obviously greater than $d_1(t, f)$.

Recall that by the definition of R_1^S, at least one point $g \in S$ belongs to ab. If $ab \subset uw$, we conclude immediately that $d_1(t, g) > d_1(t, f)$, contradicting our assumption that f is the furthest neighbor of t in S.

Otherwise, at least one of u and w must lie inside ab; first, suppose that u does. Let r'_h denote the horizontal ray with the origin at t directed to the right, and let $q = r'_h \cap \ell$. Observe that the L_1 distance between t and any point of qu equals $d_1(t,u)$, and thus, is greater than $d_1(t,f)$. Moreover, if $q \in ab$, then for any point $z \in aq$, $d_1(t,z) > d_1(t,q) = d_1(t,u) > d_1(t,f)$. Together with the assumption that f is the furthest neighbor of t in S, this implies that no point from S falls inside au. Similarly, in case w lies inside ab, no point from S can fall inside wb. We conclude that $g \in uw$. As mentioned above, this implies that $d_1(t,g) > d_1(t,f)$, which again contradicts our assumption. ∎

Corollary 1. *Any site $s \in S$ having a non-empty Voronoi region in the furthest site Voronoi diagram $VF_{d_1}(S)$ of S under the metric L_1, lies on the boundary of R_1^S.*

Lemma 3. *For any set S of n sites in the plane, the furthest site Voronoi diagram $VF_{d_1}(S)$ has a constant complexity.*

Proof: By Corollary 1, any site $s \in S$ having a non-empty Voronoi region in $VF_{d_1}(S)$ must lie on the boundary of R_1^S. Therefore, $VF_{d_1}(S)$ is identical to $VF_{d_1}(S_B)$, where $S_B \subset S$ consists of all the sites from S lying on the boundary of R_1^S.

Let a and b denote the rightmost and the topmost vertices of R_1^S, respectively; denote the line supporting ab by ℓ. Let $s_{i_1}, \ldots, s_{i_k}$ denote the sites from S that lie inside the segment ab, in the order in which they are encountered when walking from a to b, where $1 \le k \le n$ and $1 \le i_m \le n$, for $1 \le m \le k$. Consider any point t in the plane; let ℓ_h and ℓ_v denote the horizontal and vertical lines through t, and let $u = \ell \cap \ell_h$ and $w = \ell \cap \ell_v$.

If $s_{i_1}s_{i_k} \subset uw$, then all the sites $s_{i_1}, \ldots, s_{i_k}$ are equidistant from t. If $s_{i_1} \notin uw$ but $s_{i_k} \in uw$, then s_{i_1} is the unique furthest neighbor of t in the subset of sites $S_{ab} = \{s_{i_1}, \ldots, s_{i_k}\}$. Similarly, if $s_{i_1} \in uw$ but $s_{i_k} \notin uw$, then s_{i_k} is the unique furthest neighbor of t in S_{ab}. Finally, in case both s_{i_1} and s_{i_k} fall outside uw, either one of those sites is the unique furthest neighbor of t in S_{ab}, or s_{i_1} and s_{i_k} are the only two furthest neighbors of t in S_{ab}. It follows that $VF_{d_1}(S_B)$ is identical to $VF_{d_1}(S_B \setminus S_{ab} \cup \{s_{i_1}, s_{i_k}\})$.

Having applied a similar reasoning to each of the three other sides of R_1^S, we conclude that at most eight sites from S fully determine the structure of $VF_{d_1}(S)$. This implies the claim. ∎

Corollary 2. *The combinatorial complexity of $VF^2_{d_1^-}(S)$ is asymptotically the same as that of the Voronoi diagram $V_{d_1}(S)$ of S under the metric L_1.*

Together with the known bounds on the complexity of the Voronoi diagram under the metric L_1 of a set of points in the plane (see e.g. [6]), this implies the following theorem.

Theorem 6. *For a set S of n points in the plane in L_1-general position, the combinatorial complexity of $VF^2_{d_1^-}(S)$ is $O(n)$. If the points from S are not in L_1-general position, the complexity of $VF^2_{d_1^-}(S)$ can be as high as $\Omega(n^2)$.*

The bounds on the complexity of $VF^2_{d_\infty^-}(S)$ are derived analogously.

Lemma 4. *For any point t in the plane, any of its furthest neighbors in S under the L_∞ metric lies on the boundary of R_∞^S.*

Proof: Let $f \in S$ denote the furthest neighbor of t in S under the L_∞ metric. Suppose for contradiction that f lies inside R_1^S.

Without loss of generality, assume that $x_t \le x_f$ and $d_\infty(t,f) = |x_f - x_t| = x_f - x_t$; the other possible cases are symmetric. By definition of R_∞^S, any side of it contains a site from S. Let $g \in S$ denote the site lying on the rightmost side of R_∞^S. Then, $d_\infty(t,g) \ge |x_g - x_t| = x_g - x_t > x_f - x_t$, which contradicts our assumption that f is the furthest neighbor of t in S. ∎

Corollary 3. *Any site $s \in S$, having a non-empty Voronoi region in the furthest site Voronoi diagram $VF_{d_\infty}(S)$ of S under the metric L_∞, lies on the boundary of R_∞^S.*

Lemma 5. *For any set S of n sites in the plane, the furthest site Voronoi diagram $VF_{d_\infty}(S)$ has a constant complexity.*

Proof: By Corollary 3, any site $s \in S$ having a non-empty Voronoi region in $VF_{d_\infty}(S)$, must lie on the boundary of R_∞^S. Therefore, $VF_{d_\infty}(S)$ is identical to $VF_{d_\infty}(S_B)$, where $S_B \subset S$ consists of all the sites from S lying on the boundary of R_∞^S.

Suppose that a site $g \in S_B$ interior to a horizontal side of R_∞^S is the furthest neighbor in S_B for some point t in the plane. Then $d_\infty(t,g) = |y_g - y_t|$ (otherwise, arguing as in the proof of Lemma 4, we would indicate a site being further from t than g). Similarly, if a site g' interior to a vertical side of R_∞^S is the furthest neighbor in S_B of a point t', then $d_\infty(t',g') = |x_{g'} - x_{t'}|$.

Consider a set $S_T \subset S_B$ of all the sites interior to the top side T of R_∞^S; suppose that $S_T \neq \emptyset$. The above observation implies that if for a point t in the plane, a site $f \in S_T$ is its furthest neighbor in S_B, then any other site $g \in S_T$ is its furthest neighbor in S_B as well. Thus, $VF_{d_\infty}(S_B)$ is identical to $VF_{d_\infty}(S_B \setminus S_T \cup \{s\})$, where s is any site from S_T.

Arguing in a similar way, we conclude that $VF_{d_\infty}(S_B)$ is identical to the Voronoi diagram $VF_{d_\infty}(S')$, where the set S' of sites is obtained from S_B by considering each side of R_∞^S and removing from S_B all the sites interior to that side except one. Since at most four sites may reside at the vertices of R_∞^S, we conclude immediately that $|S'| \le 8$. This implies the claim. ∎

Corollary 4. *The combinatorial complexity of $VF^2_{d_\infty^-}(S)$ is asymptotically the same as that of the Voronoi diagram $V_{d_\infty}(S)$ of S under the metric L_∞.*

Again, applying the known bounds on the complexity of the Voronoi diagram under the metric L_∞ of a set of points in the plane (see e.g. [6]), we derive the following theorem.

Theorem 7. *For a set S of n points in the plane in L_∞-general position, the combinatorial complexity of $VF^2_{d^-_\infty}(S)$ is $O(n)$. If the points from S are not in L_∞-general position, the complexity of $VF^2_{d^-_\infty}(S)$ can be as high as $\Omega(n^2)$.*

It was pointed out in [3] that $VF^2(S)$ can be obtained in $O(n^2)$ time using the existing algorithms for computation of the nearest- and the furthest-neighbor Voronoi diagrams in the Euclidean plane, and of their overlay. The Voronoi diagrams $VF^2_{d^-_1}(S)$ and $VF^2_{d^-_\infty}(S)$ can be efficiently constructed in a similar fashion.

Theorem 8. *Let $\mathcal{R}$ be a plane endowed with the L_1 (L_∞) metric. Let $S \subset \mathcal{R}$ be a set of points in L_1- (L_∞-) general position. The furthest-neighbor 2-site Voronoi diagram $VF^2_{d^-_1}(S)$ ($VF^2_{d^-_\infty}(S)$) under the "difference of distances" distance function can be computed in optimal $\Theta(n \log n)$ time and linear space.*

Proof: Let us consider the case of L_1 metrics; the reasoning for the case of L_∞ will be fully analogous.

By Lemma 2, $VF_{d^-_1}(S)$ has constant complexity. Moreover, it follows from the proof of that Lemma that the points from S having non-empty cells in $VF_{d^-_1}(S)$ can be found in linear time. Therefore, $VF_{d^-_1}(S)$ can be constructed in total linear time. Moreover, $V_{d^-_1}(S)$ has linear complexity, and can be obtained in optimal $\Theta(n \log n)$ time [14]. In particular, we derive immediately that the overlay of $V_{d^-_1}(S)$ and $VF_{d^-_1}(S)$ has linear complexity, and can be computed in linear time.

Thus, the most time-consuming task is the computation of $V_{d^-_1}(S)$, and we never use more than linear storage. The claim follows. ■

V. Conclusion

Among the distance functions, under which 2-point site Voronoi diagrams have been considered in the literature, only the "difference between distances" and the "perimeter", along with its generalization proposed in [8], produce new Voronoi structures that cannot be interpreted as any kind of 1-site Voronoi diagrams. As a consequence, the mentioned distance functions are of particular interest in the respective context.

Our results presented in this paper extend the knowledge on the 2-point site Voronoi diagrams under the "difference between distances" distance function. In addition, we point out that an earlier observation by Barequet et al. [3], which established similarity between 2-point site Voronoi diagrams under the "sum of distances" or the "product of distances" distance functions and the classical second-order Voronoi diagram, generalizes to much broader classes of distance functions. This derivation should further strengthen the interest towards distance functions, which give rise to Voronoi structures of novel types.

Future research directions include introducing and studying distance functions of principally different forms and consideration of non-point objects as sites.

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