J. Appl. Prob. **21**, 548–557 (1984)
Printed in Israel

RADIAL GENERATION OF n-DIMENSIONAL POISSON PROCESSES

M. P. QUINE AND
D. F. WATSON,* *University of Sydney*

Abstract

A simple method is proposed for the generation of successive 'nearest neighbours' to a given origin in an n-dimensional Poisson process. It is shown that the method provides efficient simulation of random Voronoi polytopes. Results are given of simulation studies in two and three dimensions.

GEOMETRICAL PROBABILITY; SIMULATION; VORONOI POLYTOPE

1. Introduction

The joint distribution of ordered distances from the origin in a homogeneous Poisson process is well known (at least in two and three dimensions); see e.g. Kendall and Moran [10]. Conversely, a set of increasing distances with this distribution, together with a sequence of random directions, gives a homogeneous Poisson process. In the next section, we make this simple idea rigorous. In Section 3, we show how the method can be applied to provide an efficient means of simulating random Voronoi polytopes in order to study aspects of their ergodic distribution. Some empirical results are presented in Section 4 for two- and three-dimensional polytopes.

2. Radial generation of Poisson processes

A point $\boldsymbol{x} = (x_1, \cdots, x_n)$ in R^n can be represented in polar coordinates by $(r, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{n-1})$ and

$$\begin{aligned} x_1 &= rc_{n-1}c_{n-2}\cdots c_2c_1 \\ x_2 &= rc_{n-1}c_{n-2}\cdots c_2s_1 \\ x_3 &= rc_{n-1}c_{n-2}\cdots s_2 \\ &\vdots \\ x_n &= rs_{n-1}, \end{aligned}$$

Received 7 December 1981; revision received 21 August 1983.
* Postal address: Department of Mathematical Statistics, The University of Sydney, NSW 2006, Australia.

$c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $0 < \theta_1 \leqq 2\pi$, $-\frac{1}{2}\pi < \theta_j \leqq \frac{1}{2}\pi$, $j = 2, \cdots, n-1$, see e.g. Kendall [9], Deltheil [3].

Suppose that $R_1 < R_2 < \cdots$ is a sequence of random variables such that $\{R_i^n/K_n, i \geqq 1\}$ is a linear Poisson process with parameter λ, where $K_n = \pi^{n/2}/\Gamma(\frac{1}{2}n+1)$ is the volume of the unit sphere in $\boldsymbol{R}^n$. Suppose further that $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots)$ are independent vectors with a common distribution, namely that for each $k \geqq 1$, $(\theta_{k1}, \cdots, \theta_{kn-1})$ are independent, θ_{k1} is uniform on $(0, 2\pi]$ and θ_{kj} has density $(\cos^{j-1}x)/B(\frac{1}{2}, \frac{1}{2}j)$, $-\frac{1}{2}\pi < x \leqq \frac{1}{2}\pi$, $j = 2, \cdots, n-1$. It is straightforward to show that the number of the points $(R_i, \boldsymbol{\theta}_i)$ lying in an annulus $\{(r, \boldsymbol{\theta}) : r_1 \leqq r < r_2\}$ is Poisson with parameter $\lambda K_n(r_2^n - r_1^n)$, and further that, since the Jacobian of the polar transformation is $J = r^{n-1}c_{n-1}^{n-2}c_{n-2}^{n-3}\cdots c_2$, the probability of a small volume round $\boldsymbol{x}$ containing such a point is $\lambda dx_1 \cdots dx_n$. These facts suffice to show that the sequence $\{(R_i, \boldsymbol{\theta}_i), i \geqq 1\}$ is a homogeneous Poisson process in $\boldsymbol{R}^n$ with parameter λ (see Rényi [15], Hammersley [7]).

In practice all these variables can be generated from independent uniform variables U_j, V_{ij} on $(0,1)$: put

$$R_i = \left((-\lambda K_n)^{-1} \sum_{j=1}^{i} \log U_j\right)^{1/n},$$

$$\theta_{ij} = \arcsin\left(N_{i1}/(N_{i1}^2 + \cdots + N_{i,j+1}^2)^{1/2}\right), \qquad j = 2, \cdots, n-1,$$

where $N_{ij} = \Phi^{-1}(V_{ij}) \sim N(0,1)$. In case $n = 3$, it is easier to set $\theta_{i2} = \arcsin(1 - 2V_{i1})$.

3. Simulation of Voronoi polytopes

Given a realization of a Poisson process in $\boldsymbol{R}^n$, associate with each 'particle' of the process those points in $\boldsymbol{R}^n$ which are closer to the given particle than to any other particle. This results in a partition of $\boldsymbol{R}^n$ into random convex polytopes, whose $(n-1)$-dimensional facets are equidistant from 2 particles, $(n-2)$-dimensional facets equidistant from 3 particles, and so on.

Little is known of the distributions of characteristics such as 'volume' or 'surface area' of these 'Voronoi' polytopes, even in two or three dimensions. However, such a conceptually simple model finds a variety of applications (see e.g. Crain [1]) so that an efficient simulation procedure is desirable. Such a procedure follows easily from the results of Section 2, as we shall see.

The simulation method involves the generation of a sequence of independent 'typical' polytopes. Of course in a single large aggregate, neighbouring polytopes will not be independent, but because of the ergodicity of the underlying point process, the limiting distributions along the sequence and through the aggregate coincide. Hinde and Miles exploit ergodicity in this same way in their simulation study [8], thus avoiding the problem of edge effects that occurs with planar sampling.

For simplicity, we describe the method in R^2. Let p_i denote the 'particle' at (R_i, θ_i), and take the origin to be a particle (p_0, say). First, p_1 is generated, and the perpendicular bisector of (p_0, p_1) is drawn. Then p_2 is generated, the perpendicular bisector of (p_0, p_2) drawn, and so on. After N such lines have been drawn, where $P(N=j)=(j-2)/2^{j-1}$, $j \geqq 3$ and $E(N)=5$ (Wendel [17]; in R^3 the distribution is $(j-2)(j-3)/2^j$, $j \geqq 4$ with mean 7), p_0 will be for the first time surrounded by a convex polygon. However, further particles may produce bisecting lines which reduce the size of this polygon. So the process must be continued until R_j, the distance from p_0 to p_j, exceeds the diameter of the smallest circle with centre p_0 containing the polygon (see Figure 1). Once this happens, the final polygon is mensurated and the procedure starts afresh.

The number of points required to produce a completed polygon is usually around 15 to 20, and each of these points requires only 2 uniform variables. The method of Hinde and Miles [8] employed involved firstly generating a Poisson variable N, and then generating N uniforms. On average, 100 such uniforms were required for each polygon. Our technique is essentially scale invariant so in practice we have taken $\lambda = 1$ throughout. In [8], the parameter had to be chosen with some care.

There are no conceptual problems in extending the method to higher dimensions. There is however a practical problem in constructing the 'bisecting hyperplanes'. This may be dealt with by proceeding via a Delaunay triangulation of the points, which can be performed efficiently in n dimensions (Watson [16]).

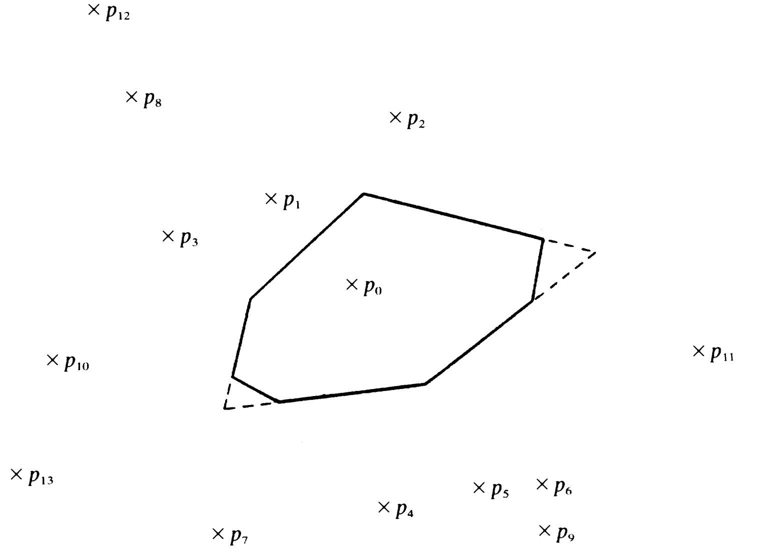

Figure 1. Larger polygon (full + dashed lines) formed by p_5; smaller polygon (full lines only) completed at p_{13}

The algorithm builds a list of Delaunay n-simplices by subdividing an initial very large n-simplex with its circumcentre at the origin. As each point is generated the list is updated to include n-simplices with the new point as a vertex. When the distance from the origin to a new point is greater than twice the maximum circumradius of those n-simplices with a vertex at the origin, the algorithm stops.

It is known that in any random division of space, a special role is played by the origin in that the body B_0 covering it is atypical: it is 'larger' in that large bodies have a proportionately larger chance of covering a given point. There is a simple relation between the distribution of B_0 and the ergodic distribution in the case we have been considering: the joint distribution of n-dimensional content C and any other characteristic Z (e.g. number of vertices) of B_0 satisfies

$$G(dc, dz) = \frac{cF(dc, dz)}{\lambda} \tag{1}$$

where F is the ergodic distribution of C and Z, and λ is the intensity of the Poisson process (see Miles [13], Matheron [11]). So far, we have taken the origin as a Poisson point and hence constructed polytopes with distribution F. However a variation of this procedure will produce polytopes with distribution G: take the perpendicular bisectors of $p_1p_2, p_1p_3, \cdots$, and stop when the distance p_1p_j is greater than the diameter of the smallest hypersphere centred on p_1 containing the polytope covering p_1. These 'larger' polytopes will help give information about the tail of F. For instance with $\lambda = 1$, if Z is any characteristic positively correlated with C, a bivariate histogram $g(c, z)$ of n 'larger' polytopes will give a histogram $\tilde{f}(c, z) = c^{-1}g(c, z)$ which for large c, z is a better estimate of $F(dc, dz)$ than the corresponding histogram $f(c, z)$ of n 'ordinary' polytopes. Crain and Miles [2] resort to similar devices to improve their tail estimates.

More importantly perhaps, (1) continues to hold for another division of space in R^2. The polygon formed by drawing the perpendicular to p_0p_j through p_j, $j = 1, 2, \cdots$ (stopping when T_j exceeds the circumradius of the polygon), will have distribution G corresponding to the ergodic distribution F of polygons formed by an isotropic Poisson line process of intensity $\pi\lambda$.† This tessellation has been studied by planar sampling means in Crain and Miles [2].

4. Some empirical results

In spite of the huge simulation work of Hinde and Miles [8] there are still many aspects of the Voronoi tessellation of a planar Poisson process worthy of attention. We have generated 50 000 such Voronoi polygons from a Poisson process of unit intensity by the methods described in Section 3. In Figures 2 and 3 we give histograms of area (A) and perimeter (S) for polygons with 3–12 sides

† We are indebted to R. Cowan for this observation.

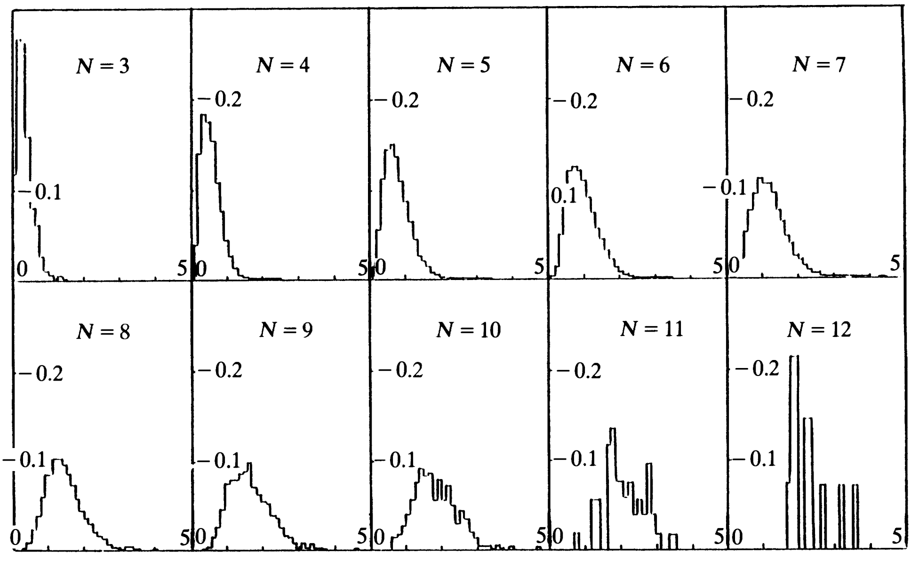

Figure 2. Histograms of areas of polygons classified by number of sides

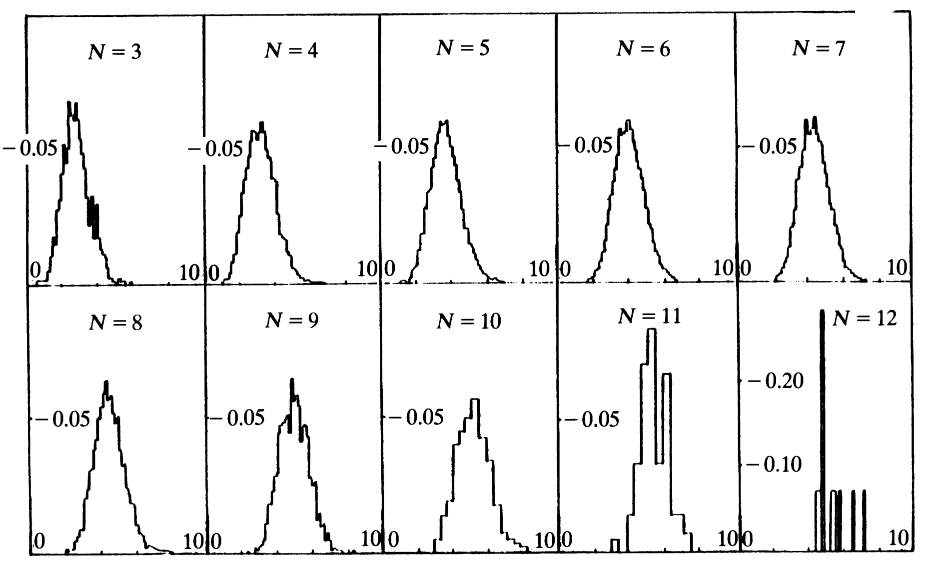

Figure 3. Histograms of perimeters of polygons classified by number of sides

(we encountered no polygons with more than 12 sides). Tables 1 and 2 give the first four conditional moments of A and S given N and also the unconditional moments; for comparison note $E(A) = 1$, $V(A) = 0.529^2$, $E(S) = 4$. In addition, Table 1 gives the observed frequency distribution of N itself which should be compared with the more accurate estimates in [8].

TABLE 1
Area of polygons

N	3	4	5	6	7	8	9	10	11	12	overall
mean	0.3593	0.5579	0.7778	0.9950	1.2233	1.4474	1.6609	1.8681	2.0626	1.9900	0.9973
Δ_N^2 (mean)	—	0.0213	−0.0027	0.0111	−0.0042	−0.0106	−0.0063	−0.0127	−0.2671	—	—
s.d.	0.2200	0.3043	0.3793	0.4361	0.4940	0.5322	0.5750	0.6349	0.5778	0.5358	0.5268
skewness	1.4112	1.2428	1.0981	1.0492	0.9157	0.7913	0.6679	0.7656	0.2660	1.1904	1.0271
kurtosis (−3)	2.8590	2.5808	1.9923	2.7785	1.5493	1.0586	0.6088	1.0552	−0.1242	0.4308	1.6917
f_N	0.0114	0.1090	0.2603	0.2922	0.1985	0.0909	0.0289	0.0076	0.0010	0.0003	1.0001

TABLE 2
Perimeter of polygons

N	3	4	5	6	7	8	9	10	11	12	overall
mean	2.8026	3.2174	3.6500	4.0295	4.3820	4.7005	4.9805	5.2444	5.4749	5.4147	3.9978
Δ_N^2 (mean)	—	0.0178	−0.0531	−0.0270	−0.0340	−0.0385	−0.0161	−0.0334	−0.2907	—	—
s.d.	0.8232	0.8345	0.8475	0.8487	0.8586	0.8442	0.8333	0.8778	0.7403	0.7624	0.9746
skewness	0.4254	0.3192	0.3066	0.3551	0.4120	0.2170	0.1486	0.1572	−0.2671	1.0702	0.2350
kurtosis (−3)	0.3293	0.1320	0.2063	0.9870	2.1016	0.1019	−0.1106	−0.0827	0.9709	0.2888	0.4193

It is of interest to notice the nice relationship of mean area (and to a lesser extent mean perimeter) to N. We conjecture that for a process with intensity λ

$$E(A \mid N) = \frac{2N-3}{9\lambda} \tag{2}$$

(which is consistent with $E(A) = 1/\lambda$), but are unable to give a proof of even a linear relationship. Miles and Maillardet [14], p. 101, suggest that for large N, the conditional distribution of A is approximately $\Gamma(N, 4\lambda)$ which implies $E(A \mid N) \sim \frac{1}{4}N/\lambda$.

Finally, we give the results of a study of 2500 Voronoi polyhedra generated from a unit Poisson process in R^3. Figure 4 gives histograms of number of faces (N) per polyhedron, number of full neighbours (F) (i.e. faces which in the aggregate would be intersected by the line through the associated nuclei — see Meijering [12], p. 283) per polyhedron, number of edges (E) per face, and volume (V), surface area (A) and total edge length (L) per polyhedron. Table 3 gives corresponding numerical data. Tables 4–6 give a numerical breakdown of V, A and L by N. The corresponding histograms, which we do not show, exhibit roughly the same shapes as their polygonal counterparts (i.e. V skewed to the right as in Figure 2, etc.).

The only other related simulation study in R^3 that we are aware of is that of Finney [4], [5]. This is concerned with a Voronoi tessellation based on the centres of randomly packed equal spheres. Comparison of Figure 4 with Figures 3 and 4 of [4] show the considerable disparities between the two models. Finney

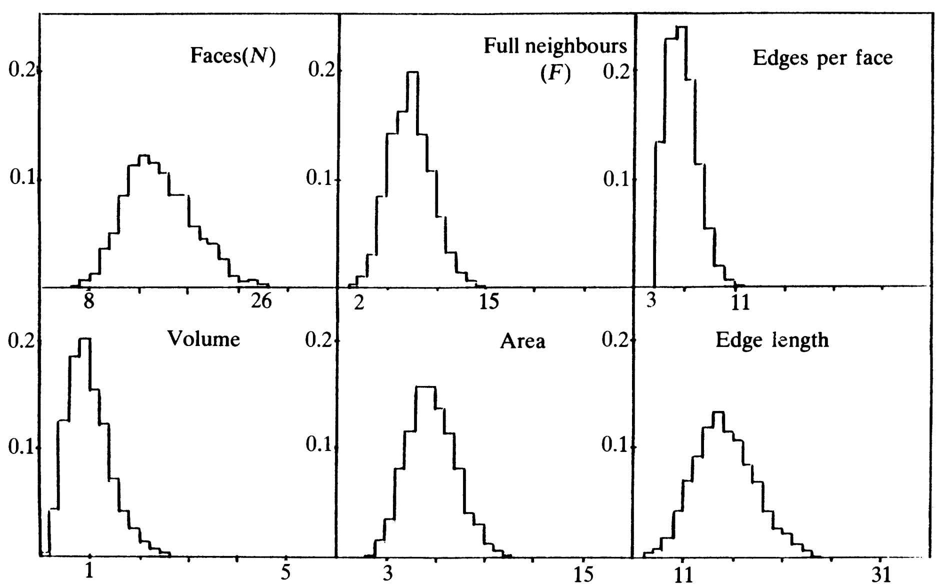

Figure 4. Histograms of characteristics of 2500 polyhedra

TABLE 3
Summary data for 2500 polyhedra

	N	F	V	A	L	E/face
mean	15.5052	7.9620	0.9982	5.8188	17.4810	5.2261
Expected:	15.5355	8	1	5.8209	17.4956	5.228
maximum	31	17	3.5933	12.8305	31.7164	13
minimum	6	2	0.1026	1.5966	6.1077	3
s.d.	3.3916	2.1822	0.4262*	1.4857	3.7199	1.5734
skewness	0.3762	0.3624	0.8109	0.3193	0.2951	0.6250
kurtosis (-3)	-0.0020	0.1800	1.0336	0.0951	-0.0359	0.1817

* Expected value is 0.424 (Gilbert [6]).

TABLE 4
Volume of polyhedra

N	mean	s.d.	skewness	kurtosis (-3)
6	0.1792	0	0	0
7	0.2182	0.0986	0.4052	-1.1423
8	0.3556	0.1373	0.6974	-0.0520
9	0.3997	0.1441	0.2287	-0.7723
10	0.5065	0.1763	0.1137	-0.6277
11	0.5828	0.2021	1.3031	2.0635
12	0.6974	0.2311	0.7194	0.3110
13	0.7541	0.2156	0.5224	0.3364
14	0.8339	0.2493	0.5960	0.6800
15	0.9398	0.2769	0.3110	0.6542
16	1.0402	0.2910	0.3664	-0.3984
17	1.1405	0.2931	0.3504	-0.2228
18	1.2175	0.3194	0.8739	1.4594
19	1.2728	0.3134	0.7035	0.7823
20	1.4631	0.3943	0.6529	0.5700
21	1.5820	0.3625	0.4279	-0.5475
22	1.6028	0.3398	0.3176	-0.1948
23	1.7935	0.3650	0.1515	-0.8828
24	1.8605	0.3762	0.1905	-1.3545
25	1.9262	0.5981	1.7276	2.1873
26	1.8854	0.2884	1.0791	-0.1165
29	1.8778	0	0	0
31	2.0712	0	0	0

considers the present model as a limiting 'zero-density' version of his (see e.g. [4], p. 488), although we can see no theoretical justification. In this context, his claim ([4], p. 485) that lower density gives greater symmetry in the histogram of V seems hard to reconcile with our Figure 4. The fundamental differences between the models are perhaps best illustrated by comparing his 'typical polyhedra' ([4], Figure 2a) with Figure 5 which shows two aspects of a typical member of our 2500 polyhedra.

TABLE 5
Surface area of polyhedra

N	mean	s.d.	skewness	kurtosis (−3)
6	2.8005	0	0	0
7	2.5438	0.6623	−0.1140	−1.3490
8	3.3115	0.6830	0.0290	−0.7457
9	3.5641	0.8710	0.3158	−0.3928
10	3.9699	0.8774	−0.1562	−0.5226
11	4.3360	0.9085	0.9649	0.9175
12	4.7899	0.9761	0.3978	0.0932
13	5.0200	0.8841	0.1561	−0.0644
14	5.3041	0.9855	0.3779	0.3597
15	5.6898	1.0708	−0.0938	0.2424
16	6.0216	1.0546	0.0984	−0.4930
17	6.3872	1.0316	0.0554	−0.4950
18	6.5986	1.0533	0.6302	0.8011
19	6.7807	1.0546	0.4391	0.0450
20	7.3224	1.2000	0.2638	−0.0659
21	7.6826	1.1520	0.2516	−0.6805
22	7.7856	1.0368	0.0167	−0.5089
23	8.1985	1.0957	−0.0890	−0.8253
24	8.4749	1.1834	0.2816	−1.3032
25	8.4903	1.6134	1.5759	1.7113
26	8.4771	0.7653	0.7103	−1.1617
29	8.8453	0	0	0
31	9.0523	0	0	0

TABLE 6
Edge length of polyhedra

N	mean	s.d.	skewness	kurtosis (−3)
6	8.3709	0	0	0
7	8.2628	1.4532	−0.3445	−1.5061
8	9.9569	1.2381	0.4064	−0.1070
9	10.8845	1.3099	0.1236	−0.7956
10	12.0746	1.5781	−0.3234	−0.0948
11	13.0190	1.5282	0.4100	−0.0361
12	14.2727	1.6545	0.1264	−0.5229
13	15.0322	1.5328	0.0261	−0.3305
14	15.9908	1.6667	−0.1489	0.1208
15	16.9609	1.8392	−0.2958	0.2973
16	18.0298	1.7455	0.0945	−0.5916
17	19.0526	1.7390	−0.0148	−0.0252
18	19.8660	1.7949	0.1264	0.1730
19	20.6327	1.8340	0.3695	0.3293
20	22.0126	2.0524	0.1996	−0.3851
21	23.0832	1.9480	−0.0241	−0.1711
22	23.4927	1.9285	−0.3035	0.3840
23	24.8104	1.9565	−0.0041	−0.7994
24	25.7486	2.1333	−0.2257	−1.1336
25	26.5108	2.5135	−0.7660	−0.3417
26	26.3243	1.0949	0.8823	−0.4306
29	27.6197	0	0	0
31	29.7026	0	0	0

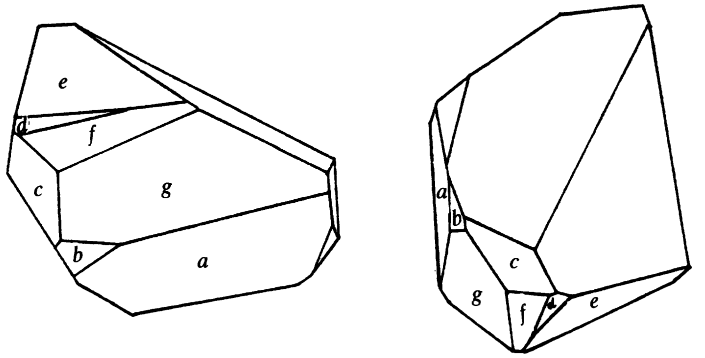

Figure 5. A typical Voronoi polyhedron

References

[1] Crain, I. K. (1972) Monte-Carlo simulation of the random Voronoi polygons: preliminary results. *Search* **3**, 220–221.

[2] Crain, I. K. and Miles, R. E. (1976) Monte-Carlo estimates of the distributions of the random polygons determined by random lines in a plane. *J. Statist. Comput. Simul.* **4**, 293–325.

[3] Deltheil, R. (1926) *Probabilités géométriques. Traité du calcul des probabilités et de ses applications.* Gauthier-Villars, Paris.

[4] Finney, J. L. (1970) Random packings and the structure of simple liquids I. The geometry of random close packing. *Proc. R. Soc. London* A **319**, 479–493.

[5] Finney, J. L. (1970) Random packings and the structure of simple liquids II. The molecular geometry of simple liquids. *Proc. R. Soc. London* A **319**, 495–507.

[6] Gilbert, E. N. (1962) Random subdivisions of space into crystals. *Ann. Math. Statist.* **33**, 958–972.

[7] Hammersley, J. M. (1972) Stochastic models for the distribution of particles in space. *Suppl. Adv. Appl. Prob.*, 47–65.

[8] Hinde, A. L. and Miles, R. E. (1980) Monte-Carlo estimates of the distributions of the random polygons of the Voronoi tessellation with respect to a Poisson process. *J. Statist. Comput. Simul.* **10**, 205–223.

[9] Kendall, M. G. (1961) *A Course in the Geometry of n Dimensions.* Griffin, London.

[10] Kendall, M. G. and Moran, P. A. P. (1963) *Geometrical Probability.* Griffin, London.

[11] Matheron, G. (1975) *Random Sets and Integral Geometry.* Wiley, New York.

[12] Meijering, J. L. (1953) Interface area, edge length and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.* **8**, 270–290.

[13] Miles, R. E. (1961) Random Polytopes: The Generalisation to *n* Dimensions of the Intervals of a Poisson Process. Ph.D. Thesis, University of Cambridge.

[14] Miles, R. E. and Maillardet, R. J. (1982) The basic structures of Voronoi and generalized Voronoi polygons. *J. Appl. Prob.* **19**A, 97–111.

[15] Rényi, A. (1967) Remarks on the Poisson process. In *Lecture Notes in Mathematics* **31**, Springer-Verlag, Berlin, 280–286.

[16] Watson, D. F. (1981) Computing the *n*-dimensional Delaunay tessellation with application to Voronoi polytopes. *Computer J.* **24**, 167–172.

[17] Wendel, J. G. (1962) A problem in geometric probability. *Math. Scand.* **11**, 109–111.