CHAPTER 4.
COORDINATE GEOMETRY IN THREE DIMENSIONS

4.1 *Introduction*

Various geometrical figures in three dimensions in three-dimensional space can be described relative to a set of mutually orthogonal axes O*x,* O*y,* O*z,* and a point can be represented by a set of rectangular coordinates (*x, y, z*). The point can also be represented by cylindrical coordinates (ρ , ϕ , *z*) or spherical coordinates (*r* , θ , ϕ), which were described in Chapter 3. In this chapter, we are concerned mostly with (*x, y, z*). The rectangular axes are usually chosen so that when you look down the *z*-axis towards the *xy*-plane, the *y*-axis is 90° counterclockwise from the *x*-axis. Such a set is called a right-handed set. A left-handed set is possible, and may be useful under some circumstances, but, unless stated otherwise, it is assumed that the axes chosen in this chapter are right-handed.

An equation connecting *x, y* and *z,* such as

$$f(x,y,z) = 0 \qquad\qquad 4.1.1$$

or
$$z = z(x,y) \qquad\qquad 4.1.2$$

describes a two-dimensional surface in three-dimensional space. A line (which need be neither straight nor two-dimensional) can be described as the intersection of two surfaces, and hence a line or curve in three-dimensional coordinate geometry is described by two equations, such as

$$f(x,y,z) \;=\; 0 \qquad\qquad 4.1.3$$

and
$$g(x,y,z) \;=\; 0. \qquad\qquad 4.1.4$$

In two-dimensional geometry, a single equation describes some sort of a plane curve. For example,

$$y^2 = 4qx \qquad\qquad 4.1.5$$

describes a parabola. But a plane curve can also be described in <u>parametric</u> form by <u>two</u> equations. Thus, a parabola can also be described by

$$x = qt^2 \qquad\qquad 4.1.6$$

and
$$y = 2qt \qquad\qquad 4.1.7$$

Similarly, in three-dimensional geometry, a line or curve can be described by <u>three</u> equations in parametric form. For example, the three equations

$$x = a \cos t \qquad\qquad 4.1.8$$

$$y = a \sin t \qquad\qquad 4.1.9$$

$$z = ct \qquad\qquad 4.1.10$$

describe a curve in three-space. Think of the parameter t as time, and see if you can imagine what sort of a curve this is.

We shall be concerned in this chapter mainly with six types of surface: the plane, the ellipsoid, the paraboloid, the hyperboloid, the cylinder and the cone.

4.2 *Planes and Straight Lines*

The geometry of the plane and the straight line is, of course, rather simple, so that we can dispose of them in this brief introductory section in a mere 57 equations.

The equation

$$Ax + By + Cz + D = 0 \qquad\qquad 4.2.1$$

represents a plane. If $D \neq 0$ it is often convenient, and saves algebra and computation with no loss of information, to divide the equation through by D and re-write it in the form

$$ax + by + cz = 1. \qquad\qquad 4.2.2$$

The coefficients need not by any means all be positive. If $D = 0$, the plane passes through the origin of coordinates, and it may be convenient to divide the equation 4.2.1 by C and hence to re-write it in the form

$$ax + by + z = 0. \qquad\qquad 4.2.3$$

The plane represented by equation 4.2.2 intersects the yz-, zx- and xy-planes in the straight lines

$$by + cz = 1 \qquad\qquad 4.2.4$$
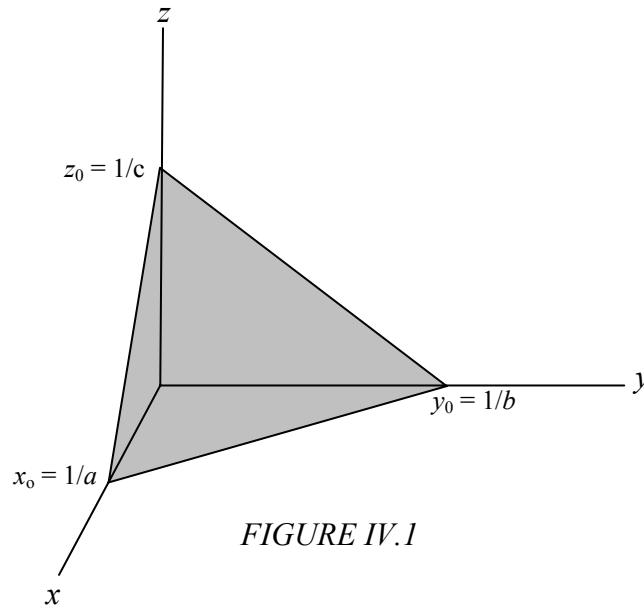
$$cz + ax = 1 \qquad\qquad 4.2.5$$

$$ax + by = 1 \qquad\qquad 4.2.6$$

and it intersects the x-, y- and z-axes at

$$x = x_0 = 1/a \qquad\qquad 4.2.7$$

$$y = y_0 = 1/b \qquad\qquad 4.2.8$$

$$z = z_0 = 1/c \qquad\qquad 4.2.9$$

The geometry can be seen in figure IV.1

FIGURE IV.1

Another way of writing the equation to the plane would be

$$\frac{x}{x_0} + \frac{y}{y_0} + \frac{z}{z_0} = 1. \qquad\qquad 4.2.10$$

In this form, x_0, y_0 and z_0 are the intercepts on the x-, y- and z-axes.

Let P(x,y,z) be a point on the plane 4.2.2. The square S of the distance of that point from the origin is

$$S = x^2 + y^2 + z^2. \qquad\qquad 4.2.11$$

But since (x, y, z) is on the plane, we can write S in terms of x and y alone, by substituting for z from equation 4.2.2:

$$S = x^2 + y^2 + \frac{(1 - ax - by)^2}{c^2}. \qquad\qquad 4.2.12$$

The distance to the origin is least for a point on the plane such that $\dfrac{\partial S}{\partial x}$ and $\dfrac{\partial S}{\partial y}$ are both zero. After some algebra, we find that the point on the plane that is closest to the origin is the point

$$\left(\frac{a}{a^2+b^2+c^2}, \quad \frac{b}{a^2+b^2+c^2}, \quad \frac{c}{a^2+b^2+c^2} \right) \qquad \text{4.2.13}$$

Consequently the perpendicular distance of the origin from the plane is the orthogonal sum of these coordinates, which is

$$\frac{1}{\sqrt{a^2+b^2+c^2}}. \qquad \text{4.2.14}$$

Substitution of x_0 for $1/a$, y_0 for $1/b$ and z_0 for $1/c$ will make these results seem rather obvious.

Further, OP is normal to the plane, and the *direction cosines* (see Chapter 3, especially section 3.3) of OP, i.e. of the normal to the plane, are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad \frac{c}{\sqrt{a^2+b^2+c^2}} \qquad \text{4.2.15}$$

The coefficients a, b, c are direction ratios of the normal to the plane; that is to say, they are numbers that are proportional to the direction cosines.

Example: Consider the plane

$$0.50x + 0.25y + 0.20z = 1 \qquad \text{4.2.16}$$

The plane intersects the x-, y- and z-axes at $(2,0,0)$, $0,4,0)$ and $(0,0,5)$.   The point on the plane that is closest to the origin is $(1.4184, 0.7092, 0.5674)$. The perpendicular distance of the origin from the plane is 1.6843. The direction cosines of the normal to the plane are $(0.8422, 0.4211, 0.3369)$.

Now, let $P(x,y,z)$ be a point in the plane 4.2.2, and let P_0 (x_0 , y_0 , z_0) be some point not necessarily in the plane. It is of interest to find the perpendicular distance between P_0 and the plane. Let S be the square of the distance between P_0 and P. Then

$$S = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \qquad \text{4.2.17}$$

We can express this in terms of x and y alone by substitution for z from equation 4.2.2. The distance of P from P_0 is least when $\dfrac{\partial S}{\partial x}$ and $\dfrac{\partial S}{\partial y}$ are simultaneously zero.   These two conditions result in

$$\left(a^2 + c^2\right)x = a + c^2 x_0 - acz_0 - aby \qquad\qquad 4.2.18$$

$$\left(b^2 + c^2\right)y = b + c^2 y_0 - bcz_0 - abx \qquad\qquad 4.2.19$$

These, combined with equation 4.2.2, result in

$$x = \frac{\left(b^2 + c^2\right)x_0 + a\left(1 - by_0 - cz_0\right)}{a^2 + b^2 + c^2} \qquad\qquad 4.2.20$$

$$y = \frac{\left(c^2 + a^2\right)y_0 + b\left(1 - cz_0 - ax_0\right)}{a^2 + b^2 + c^2} \qquad\qquad 4.2.21$$

$$z = \frac{\left(a^2 + b^2\right)z_0 + c\left(1 - ax_0 - by_0\right)}{a^2 + b^2 + c^2} \qquad\qquad 4.2.22$$

These are the coordinates of the point P in the plane that is nearest to P_0. The perpendicular distance between P and P_0 is $p = \sqrt{S}$, or

$$p = \frac{1 - ax_0 - by_0 - cz_0}{\sqrt{a^2 + b^2 + c^2}}. \qquad\qquad 4.2.23$$

This is positive if P_0 is on the same side of the plane as the origin, and negative if it is on the opposite side. If the perpendicular distances of two points from the plane, as calculated from equation 4.4.23, are of opposite signs, they are on opposite sides of the plane. If $p = 0$, or indeed if the numerator of equation 4.4.23 is zero, the point P_0 (x_0, y_0, z_0) is, of course, in the plane.

An equation for the plane containing three specified points can be found as follows. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the three specified points, and let (x, y) be any point in the plane that contains these three points. Each of these points must satisfy an equation of the form 4.2.1. That is,

$$xA + yB + zC + D = 0 \qquad\qquad 4.2.24$$

$$x_1A + y_1B + z_1C + D = 0 \qquad\qquad 4.2.25$$

$$x_2A + y_2B + z_2C + D = 0 \qquad\qquad 4.2.26$$

$$x_3A + y_3B + z_3C + D = 0 \qquad\qquad 4.2.27$$

In these equations, we are treating *A, B, C, D* as unknowns, and the *x, y, z, x_1, y_1* ... as coefficients. We have four linear equations in four unknowns, and no constant term. From the theory of equations, these are consistent only if each is a linear combination of the other three. This is satisfied only if the determinant of the coefficients is zero:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad\qquad 4.2.28$$

and this is the equation to the required plane containing the three points. The reader will notice the similarity of this equation to equation 2.2.4 for a line passing between two points in two-dimensional geometry. The reader might like to repeat the argument, but requiring instead the four points to satisfy an equation of the form 4.2.2. There will then be four linear equations in three unknowns. Otherwise the argument is the same.

We now move on to the question of finding the area of a triangle whose vertices are given. It is straightforward to do this with a numerical example, and the reader is now encouraged to write a computer program, in whatever language is most familiar, to carry out the following tasks. Read as data the *x-y-z* coordinates of three points A, B, C. Calculate the lengths of the sides *a, b, c, a* being opposite to A*, etc. Calculate the three angles at the vertices of the triangle, in degrees and minutes, and check for correctness by verifying that their sum is 180^0 . If an angle is obtuse, make sure that the computer displays its value as a positive angle between 90^0 and 180^0. Finally, calculate the area of the triangle.

The data for several triangles could be written into a data file, which your program reads, and then writes the answers into an output file. Alternatively, you can type the coordinates of the vertices of one triangle and ask the computer to read the data from the monitor screen, and then to write the answers on the screen followed by a message such as "Do you want to try another triangle (1) or quit (2)?". Your program should also be arranged so that it writes an appropriate message if the three points happen to be collinear.

It should be easy to calculate the sides. The angles can then be calculated from equation 3.2.2 and the area from each of the four equations 3.2.3 and 3.2.4. They should all yield the correct answer, of course, but the redundant calculations serve as an important check on the correctness of your programming, as also does your check that the three angles add to 180^0.   Where there are two of more ways of performing a calculation, a careful calculator will do all of them as a check against mistakes, whether the calculation is done by hand or by computer.

Example.   If the coordinates of the vertices are

$$A(7, 4, 3), \quad B(11, 6, 2), \quad C(9, 2, 4)$$

the sides are $\qquad\qquad a = 4.583, \quad b = 3.000, \quad c = 4.899,$

and the angles are $\qquad A = 36^{\circ}\ 42',\ \ B = 77^{\circ}\ 24',\ \ C = 65^{\circ}\ 54'$,

which add up to 180°. The area is 6.708.

Example.   If the coordinates of the vertices are

$$A(6,\ 4,\ 9),\ \ B(2,\ 6,\ 17),\ \ C(8,\ 3,\ 5)$$

the area of the triangle is zero and the points are collinear.

The foregoing showed that it was not difficult to calculate numerically the area of the triangle from the coordinates of its vertices. Is it easy to find a simple explicit algebraic formula for the area in terms of $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$ and $(x_3,\ y_3,\ z_3)$? On referring to figure IV.2, we can proceed as follows.
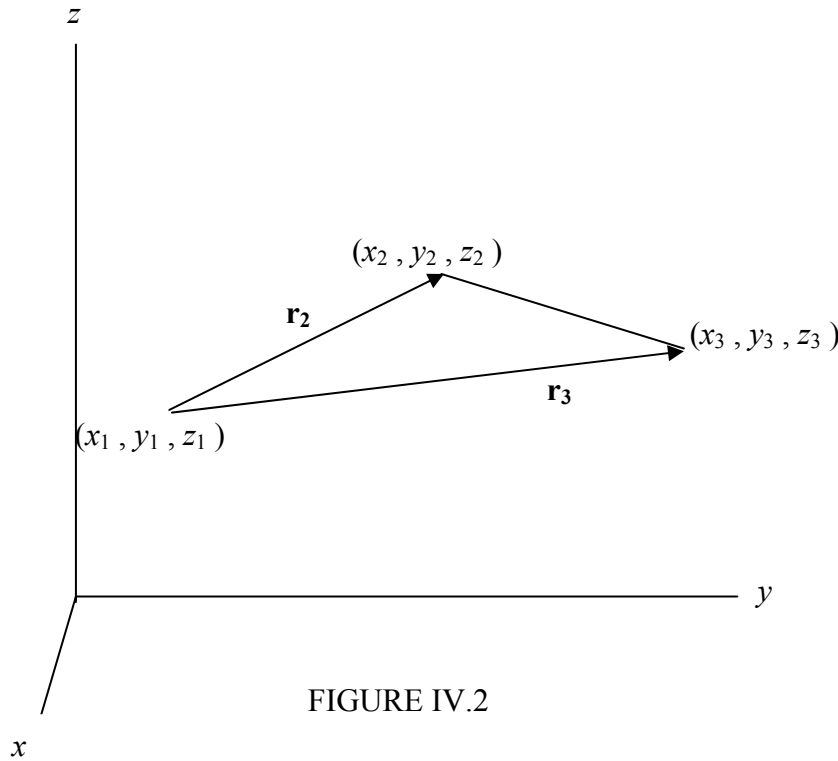
FIGURE IV.2

The vectors $\mathbf{r_2}$ and $\mathbf{r_3}$ can be written

$$\mathbf{r}_2 = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \qquad\qquad 4.2.29$$

$$\mathbf{r}_3 = (x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j} + (z_3 - z_1)\mathbf{k} \qquad\qquad 4.2.30$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors parallel to the x-, y- and z- axes.

The cross product of $\mathbf{r}_2$ and $\mathbf{r}_3$ gives the (vector) area of the parallelogram of which they form two sides. The area $\mathbf{A}$ of the triangle is half of this, so that

$$
\begin{aligned}
2\mathbf{A} = \mathbf{r}_2 \times \mathbf{r}_3 \\
= \quad & [(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)]\mathbf{i} \\
& + [(z_2 - z_1)(x_3 - x_1) - (z_3 - z_1)(x_2 - x_1)]\mathbf{j} \\
& + [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\mathbf{k}
\end{aligned} \qquad 4.2.31
$$

The magnitude of this vector can be found in the usual way, to obtain

$$
\begin{aligned}
4A^2 = \quad & [(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)]^2 \\
& + [(z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)]^2 \\
& + [(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]^2 .
\end{aligned} \qquad 4.2.32
$$

The reader can verify that, if $z_1 = z_2 = z_3$, this reduces to equation 2.2.12 for the area of a triangle the xy-plane. Equation 4.1.32 can also be written

$$
4A^2 = \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ 1 & 1 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}^2 \qquad 4.2.33
$$

This gives the area explicitly in terms of the coordinates of the vertices. If it is zero, the points are collinear.

The volume of a tetrahedron is $\frac{1}{6} \times$ base $\times$ height. By combining equation 4.2.33 for the area of a triangle with equation 4.2.14 for the perpendicular distance of a point from a plane, we can determine that the volume of the tetrahedron whose vertices are

$$(x_0, y_0, z_0), \ (x_1, y_1, z_1), \ (x_2, y_2, z_2), \ (x_3, y_3, z_3)$$

is

$$
\frac{1}{6} \begin{vmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} \qquad 4.2.34
$$

If this determinant is zero, the four points are coplanar.

In three-dimensional coordinate geometry, a straight line is described by two equations, being the intersection of two planes:

$$a_1x + b_1y + c_1z = 1 \qquad\qquad 4.2.35$$

$$a_2x + b_2y + c_2z = 1 \qquad\qquad 4.2.36$$

If $a_1/a_2 = b_1/b_2 = c_1/c_2$, the normals to the two planes have the same direction ratios, so the planes are parallel and do not intersect. Otherwise the normals to the two planes have different direction ratios (a_1, b_1, c_1), (a_2, b_2, c_2), and, since the line of intersection of the planes is at right angles to both normals, the direction ratios of the line are found from the cross product of vectors normal to the planes. The direction ratios of the line of intersection are therefore

$$(b_1c_2 - b_2c_1 ,\ c_1a_2 - c_2a_1 ,\ a_1b_2 - a_2b_1) \qquad\qquad 4.2.37$$

The line crosses the *yz-, zx-* and *xy-* planes at

$$y = \frac{c_2 - c_1}{b_1c_2 - b_2c_1} \qquad\qquad z = \frac{b_1 - b_2}{b_1c_2 - b_2c_1} \qquad\qquad 4.2.38$$

$$z = \frac{a_2 - a_1}{c_1a_2 - c_2a_1} \qquad\qquad x = \frac{c_1 - c_2}{c_1a_2 - c_2a_1} \qquad\qquad 4.2.39$$

$$x = \frac{b_2 - b_1}{a_1b_2 - a_2b_2} \qquad\qquad y = \frac{a_1 - a_2}{a_1b_2 - a_2b_1} \qquad\qquad 4.2.40$$

An example of computing a straight line from the intersection of two planes occurs in meteor astronomy. We can assume a flat Earth, which is tantamount to supposing that the height of a meteor is negligible compared with the radius of Earth, and the height of an observer above sea level is negligible compared with the height of the meteor. Since the heights of meteors are typically a few tens of km, both of these approximations are reasonable, at least for non-instrumental eyewitness accounts.

We suppose that, relative to an arbitrary origin O on the surface of Earth, a witness A is 15 km east and 5 km north of the origin. He sees a fireball start at an angle $\theta = 25^\circ.5$ from his zenith and at an azimuth $\phi = 54^\circ.5$ counterclockwise from his east, and it finishes at $\theta = 36^\circ.7$, $\phi = 16^\circ.7$. (See figure IV.3.)

Show that the plane containing the witness and the meteor is

$$0.0363x + 0.0911y - 0.0454z = 1 \qquad\qquad 4.2.41$$

A second witness, 30 km east and 15 km north of O, estimates the zenith distance and azimuth of two points on the meteor track to be $\theta = 29^{\circ}.6$, $\phi = 202^{\circ}.9$ and $\theta = 33^{\circ}.6$, $\phi = 242^{\circ}.9$

Show that the plane containing this second witness and the meteor is

$$0.0257x + 0.0153y + 0.0168z = 1 \qquad\qquad\qquad 4.2.42$$

These two equations describe the path of the fireball through the air. Show that, if the meteoroid carries on moving in a straight line, it will strike the ground as a meteorite 42.4 km east and 6.0 km south of the origin O.
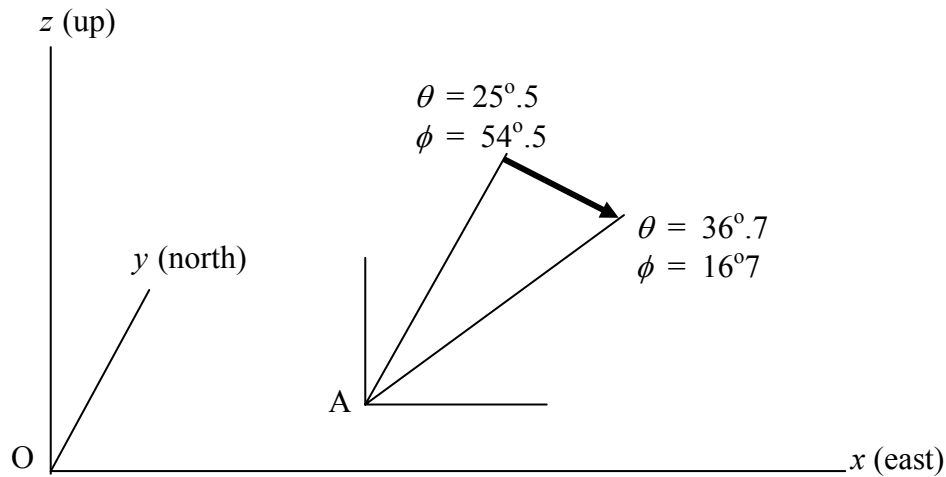

FIGURE IV.3

As we have just discussed, two nonparallel planes intersect in a straight line. Usually, three nonparallel planes intersect at a single unique point; for, if L is a line formed from the intersection of planes P_1 and P_2 , L will usually intersect the plane P_3 at a point.

Example: The planes

$$2x + 3y + 4z - 9 = 0 \qquad\qquad\qquad 4.2.43$$

$$x + y - 8z + 6 = 0 \qquad\qquad\qquad 4.2.44$$

$$5x + 6y - 12z + 1 = 0 \qquad\qquad\qquad 4.2.45$$

intersect at (1, 1, 1).

It will be recalled from the theory of linear equations that three equations

$$A_1x + B_1y + C_1z + D_1 = 0 \qquad\qquad 4.2.46$$

$$A_2x + B_2y + C_2z + D_2 = 0 \qquad\qquad 4.2.47$$

$$A_3x + B_3y + C_3z + D_3 = 0 \qquad\qquad 4.2.48$$

have a unique solution only if

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_2 & C_3 \end{vmatrix} \neq 0 \qquad\qquad 4.2.49$$

and, in the geometrical interpretation, this is the condition that three planes meet in a single point.

Consider, however, the three planes

$$2x + 3y + 4z - 9 = 0 \qquad\qquad 4.2.50$$

$$x + y - 8z + 6 = 0 \qquad\qquad 4.2.51$$

$$5x + 6y - 20z + 12 = 0 \qquad\qquad 4.2.52$$

The direction ratios of the three lines found by combining the planes in pairs (see equation 4.2.37) are

$$(-28, 20, -1) \quad (-84, 60, -3) \quad (28, -20, 1)$$

It will be observed that each is a multiple of either of the others, and the direction cosines of each of the three lines are identical apart from sign: ($\mp$ 0.813, $\pm$0.581, $\mp$ 0.029).

The three lines are, in fact, parallel, and the three planes enclose a prism. A condition for this is that

$$\Delta = 0. \qquad\qquad 4.2.53$$

But consider now the planes

$$2x + 3y + 4z - 9 = 0 \qquad\qquad 4.2.54$$

$$x + y - 8z + 6 = 0 \qquad\qquad 4.2.55$$

$$5x + 6y - 20z + 9 = 0 \qquad\qquad 4.2.56$$

Not only does $\Delta = 0$, but also

$$\Delta' = \begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix} = 0 \qquad\qquad 4.2.57$$

The three lines obtained by combining equations 4.2.54,55,56 in pairs are in fact identical, and the three planes meet in a single line. Each of equations 4.2.54,55,56 is a linear combination of the other two.

In summary, three nonparallel planes meet in a single line if $\Delta \neq 0$. They meet in a single point if $\Delta = \Delta' = 0$. They enclose a prism if $\Delta = 0$, $\Delta' \neq 0$.

4.3 *The Ellipsoid*

Consider the equation

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \qquad\qquad 4.3.1$$

with $a > c$, in the xz-plane. The length of the semi major axis is a and the length of the semi minor axis is c. If this figure is rotated through 360° about its minor (z-) axis, the three- dimensional figure so obtained is called an *oblate spheroid*. The figure of the Earth is not exactly spherical; it approximates to a very slightly oblate spheroid, the ellipticity $(c - a)/a$ being only 0.00335. (The actual figure of the Earth, mean sea level, is often referred to as the *geoid*.)

The equation to the oblate spheroid referred to above is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1. \qquad\qquad 4.3.2$$

If the ellipse 4.3.1 is rotated through 360° about its major (x-) axis, the figure so obtained is called a *prolate* spheroid. A rugby football (or, to a lesser extent, a North American football, which is a bit too pointed) is a good approximation to a prolate spheroid.

The equation to the prolate spheroid just described is

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = 1. \qquad\qquad 4.3.2$$

Either type of spheroid can be referred to as an "ellipsoid of revolution".

The figure described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad\qquad 4.3.3$$

is a *tri-axial ellipsoid*. Unless stated otherwise, I shall adopt the convention $a > b > c$, and choose the coordinate axes such that the major, intermediate and minor axes are along the x-, y- and z-axes respectively. A tri-axial ellipsoid is not an ellipsoid of revolution; it cannot be obtained by rotating an ellipse about an axis.

The special case $a = b = c$:

$$x^2 + y^2 + z^2 = a^2 \qquad\qquad 4.3.4$$

is, of course, a sphere.

Figure IV.4 shows the cross-section of a tri-axial ellipse in the yz- plane (a), the xz-plane (b) and (twice - (c), (d)) the xy-plane. If you imagine your eye wandering in the xz-plane from the x-axis (a) to the z-axis (c), you will be convinced that there is a direction in the xz-plane from which the
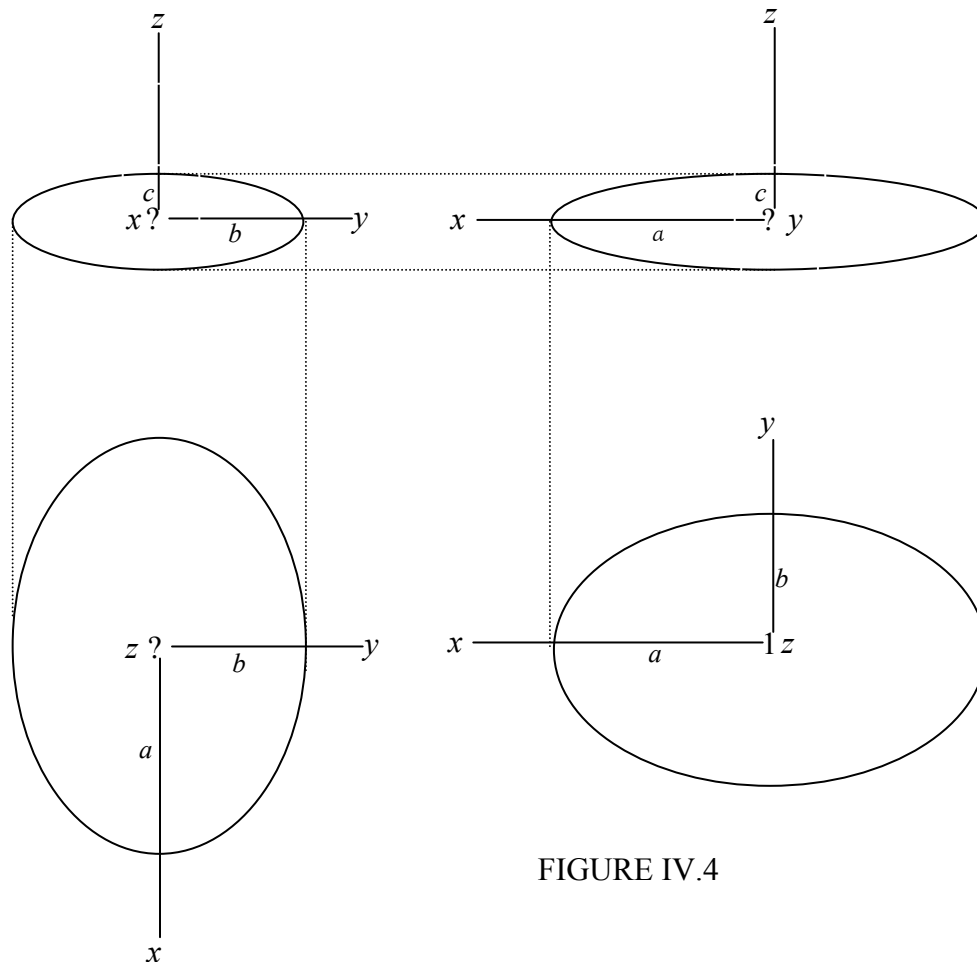

FIGURE IV.4

cross-section of the ellipse is a circle. There are actually two such directions, symmetrically situated on either side of the *z*-axis, but there are no such directions in either the *xy*- or the *yz*-planes from which the cross-section of the ellipsoid appears as a circle. Expressed otherwise, there are two planes that intersect the ellipsoid in a circle. This fact is of some importance in the description of the propagation of light in a bi-axial crystal, in which one of the wavefronts is a tri-axial ellipsoid.

Let us refer the ellipsoid 4.3.3 to a set of axes O*x'y'z'* such that the angles *z*'O*z* and *x*'O*x* are each θ , and the *y*'- and *y*-axes are identical. The equation of the ellipsoid referred to the new axes is (by making use of the usual formulas for the rotation of axes)

$$\frac{\left(z'\sin\theta+x'\cos\theta\right)^2}{a^2}+\frac{y'^2}{b^2}+\frac{\left(z'\cos\theta-x'\sin\theta\right)^2}{c^2}=1. \qquad 4.3.5$$

The cross-section of the ellipsoid in the *x'y'*-plane (i.e. normal to the *z'*-axis) is found by putting *z'* = 0:

$$\frac{(x'\cos\theta)^2}{a^2}+\frac{y'^2}{b^2}+\frac{(x'\sin\theta)^2}{c^2} \quad = \quad 1. \qquad 4.3.6$$

This is a circle if the coefficients of *x'* and *y'* are equal. Thus it is a circle if

$$\cos^2\theta = \frac{a^2\left(b^2-c^2\right)}{b^2\left(a^2-c^2\right)} \quad . \qquad 4.3.7$$

Thus, a plane whose normal is in the *xz*-plane (i.e. between the major and minor axis) and inclined at an angle θ to the minor (*z*-) axis, cuts the tri-axial ellipsoid in a circle. As viewed from either of these directions, the cross-section of the ellipsoid is a circle of radius *b*.

As an asteroid tumbles over and over, its brightness varies, for several reasons, such as its changing phase angle, the directional reflective properties of its regolith, and, of course, the cross-sectional area presented to the observer. The number of factors that affect the light-curve of a rotating asteroid is, in fact, so large that it is doubtful if it is possible, from the light-curve alone, to deduce with much credibility or accuracy the true shape of the asteroid. However, it is obviously of some interest for a start in any such investigation to be able to calculate the cross-sectional area of the ellipsoid 4.3.3 as seen from some direction (θ , ϕ).

Let us erect a set of coordinate axes O*x'y'z'* such that O*z'* is in the direction (θ , ϕ), first by a rotation through ϕ about O*z* to form intermediate axes O*x₁y₁z₁* , followed by a rotation through θ about O*y₁*. The (*x'*, *y'*, *z'*) coordinates are related to the (*x, y, z*) coordinates by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad 4.3.8$$

If we substitute for *x, y, z* in equation 4.3.3 from equation 4.3.8, we obtain the equation to the ellipsoid referred to the O*x'y'z'* coordinate systems. And if we put $z' = 0$, we see the elliptical cross-section of the ellipsoid in the plane normal to O*z'*. This will be of the form

$$Ax'^2 + 2Hx'y' + By'^2 = 1, \qquad 4.3.9$$

where
$$A = \cos^2\theta \left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2} \right) + \frac{\sin^2\theta}{c^2}, \qquad 4.3.10$$

$$2H = 2\cos^2\theta \sin\phi \cos\phi \left(\frac{1}{b^2} - \frac{1}{a^2} \right), \qquad 4.3.11$$

$$B = \frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}. \qquad 4.3.12$$

This is an ellipse whose axes are inclined at an angle ψ from O*x'* given by

$$\tan 2\psi = \frac{2H}{A-B}. \qquad 4.3.13$$

By replacing *x'* and *y'* by *x"* and *y"*. where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x" \\ y" \end{pmatrix} \qquad 4.3.14$$

we shall be able to describe the ellipse in a coordinate system O*x"y"* whose axes are along the axes of the ellipse, and the equation will be of the form

$$\frac{x"^2}{a"^2} + \frac{y"^2}{b"^2} = 1 \qquad 4.3.15$$

and the area of the cross-section is $\pi a"b"$.

For example, suppose the semi axes of the ellipsoid are $a = 3$, $b = 2$, $y = 1$, and we look at it from the direction $\theta = 60°$, $\phi = 45°$. Following equations 4.4.9,10,11,12, we obtain for the equation of the elliptical cross-section referred to the system $Ox'y'z'$

$$0.795138x'^2 + 0.0694x'y' + 0.1805y'^2 = 1. \qquad\qquad 4.3.16$$

From equation 4.4.13 we find $\psi = 3°.22338$. Equation 4.4.14 then transforms equation 4.4.16 to

$$0.797094x''^2 + 0.178600y''^2 = 1 \qquad\qquad 4.3.17$$

or
$$\frac{x''^2}{(1.1201)^2} + \frac{y''^2}{(2.3662)^2} = 1. \qquad\qquad 4.3.18$$

The area is $\qquad\qquad \pi \times 1.1202 \times 2.3663 = 8.362.$

It is suggested here that the reader could write a computer program in the language of his or her choice for calculating the cross-sectional area of an ellipsoid as seen from any direction. As an example, I reproduce below a Fortran program for an ellipse with $(a, b, c) = (3, 2, 1)$. It is by no means the fastest and most efficient Fortran program that could be written, but is sufficiently straightforward that anyone familiar with Fortran and probably many who are not should be able to follow the steps.

```
A=3.
B=2.
C=1.
A2=A*A
B2=B*B
C2=C*C
READ(5,*)TH,PH
TH=TH/57.29578
PH=PH/57.29578
STH=SIN(TH)
CTH=COS(TH)
SPH=SIN(PH)
CPH=COS(PH)
STH2=STH*STH
CTH2=CTH*CTH
SPH2=SPH*SPH
CPH2=CPH*CPH
AA=CTH2*(CPH2/A2+SPH2/B2)+STH2/C2
TWOHH=2.*CTH*STH*CPH*(1./B2−1./A2)
BB=SPH2/A2+CPH2/B2
PS=.5*ATAN2(TWOHH,AA−BB)
SPS=SIN(PS)
```

```
CPS=COS(PS)
AAA=CPS*(AA*CPS+TWOHH*SPS)+BB*SPS*SPS
BBB=SPS*(AA*SPS–TWOHH*CPS)+BB*CPS*CPS
SEMAX1=1./SQRT(AAA)
SEMAX2=1./SQRT(BBB)
AREA=3.1415927*SEMAX1*SEMAX2
WRITE(6,1)AREA
1 FORMAT(' Area = ',F7.3)
STOP
END
```

4.4 *The Paraboloid*

The equation $x^2 = 4qz = 2lz$ is a parabola in the *xz*-plane. The distance between vertex and focus is q, and the length of the semi latus rectum $l = 2q$. The equation can also be written

$$\frac{x^2}{a^2} = \frac{z}{h} \qquad\qquad 4.4.1$$

Here a and h are distances such that $x = a$ when $z = h$, and the length of the semi latus rectum is $l = a^2/(2h)$.

If this parabola is rotated through 360° about the *z*-axis, the figure swept out is a *paraboloid of revolution*, or *circular paraboloid*. Many telescope mirrors are of this shape. The equation to the circular paraboloid is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z}{h}. \qquad\qquad 4.4.2$$

The cross-section at $z = h$ is a circle of radius a.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{h}, \qquad\qquad 4.4.3$$

in which we shall choose the *x*- and *y*-axes such that $a > b$, is an elliptic paraboloid and, if $a \neq b$, is not formed by rotation of a parabola. At $z = h$, the cross section is an ellipse of semi major and minor axes equal to a and b respectively. The section in the plane $y = 0$ is a parabola of semi latus rectum $a^2/(2h)$. The section in the plane $x = 0$ is a parabola of semi latus rectum $b^2/(2h)$. The elliptic paraboloid lies entirely above the *xy*-plane.

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{h}$$

4.4.4

is a hyperbolic paraboloid, and its shape is not quite so easily visualized. Unlike the elliptic paraboloid, it extends above and below the plane. It is a saddle-shaped surface, with the saddle point at the origin. The section in the plane $y = 0$ is the "nose down" parabola $x^2 = a^2 z / h$ extending above the xy-plane. The section in the plane $x = 0$ is the "nose up" parabola $y^2 = -b^2 z / h$ extending below the xy-plane. The section in the plane $z = h$ is the hyperbola
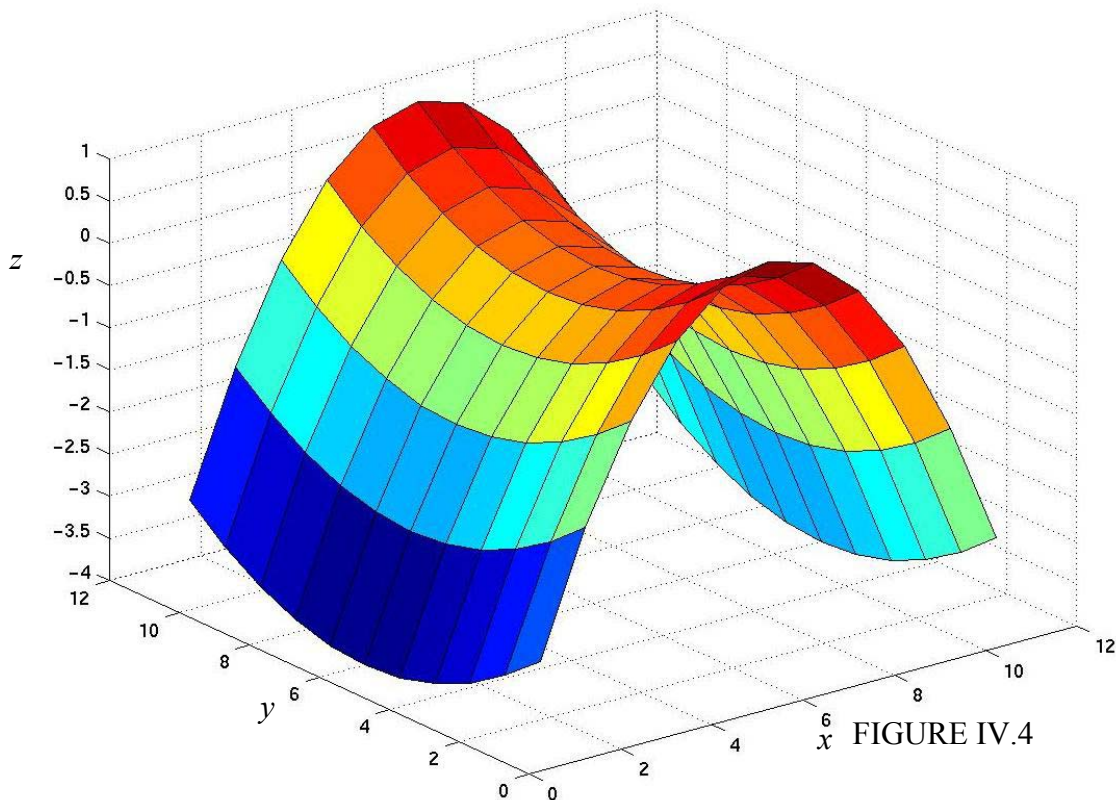
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

4.4.5

The section with the plane $z = -h$ is the conjugate hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

4.4.6

The section with the plane $z = 0$ is the asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

4.4.7

The surface for $a = 3$, $b = 2$, $h = 1$ is drawn in figure IV.4.

FIGURE IV.4

4.5 *The Hyperboloid*

The equation

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad\qquad\qquad 4.5.1$$

is a hyperbola, and *a* is the semi transverse axis. (As described in Chapter 2, *c* is the semi transverse axis of the conjugate hyperbola.)

If this figure is rotated about the *z*-axis through 360°, the surface swept out is a *circular hyperboloid* (or *hyperboloid of revolution*) of one sheet. Its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1. \qquad\qquad\qquad 4.5.2$$

Imagine two horizontal rings, one underneath the other. The upper one is fixed. The lower one is suspended from the upper one by a large number of vertical strings attached to points equally spaced around the circumference of each ring. Now twist the lower one through a few degrees about a vertical axis, so that the strings are no longer quite vertical, and the lower ring rises slightly. These strings are generators of a circular hyperboloid of one sheet.

If this figure is rotated about the *x*-axis through 360°, the surface swept out is a *circular hyperboloid* (or *hyperboloid of revolution*) *of two sheets*. Its equation is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2} - \frac{z^2}{c^2} = 1. \qquad\qquad\qquad 4.5.3$$

The equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad\qquad\qquad 4.5.4$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad\qquad\qquad 4.5.5$$

represent hyperbolas of one and two sheets respectively, but are not hyperbolas of revolution, since their cross sections in the planes $z = $ constant and $x = $ constant $> a$ respectively are ellipses rather than circles. The reader should imagine what the cross- sections of all four hyperboloids are like in the planes $x = 0,\ y = 0$ and $z = 0$.

4.6 *The Cylinder*

In three-dimensional solid geometry the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad\qquad 4.6.1$$

represents a cylinder of elliptical cross-section, whose axis coincides with the *z*-axis. The equation to a cylinder with an axis in another position and with another orientation can be obtained by the usual processes of translation and rotation of axes (see Chapter 3, section 3.7).

4.7 *The Cone*

The equation

$$x^2 + y^2 = a^2z^2 \qquad\qquad 4.7.1$$

represents a circular cone whose vertex in at the origin and whose axis coincides with the *z*-axis. The semi-vertical angle α of the cone is given by

$$\alpha = \tan^{-1} a. \qquad\qquad 4.7.2$$

In this context, the word "vertical" has nothing to do with "upright"; it merely means "of or at the vertex". A little knowledge of Latin is useful even today!

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2 \qquad\qquad 4.7.3$$

is a cone with elliptical cross-section.

If the vertex of the cone remains at the origin, but the vertex is in some arbitrary direction (described, for example, by the direction cosines, or by spherical angles θ and ϕ) the equation can be derived by a rotation of coordinate axes. This will introduce terms in *yz, zx* and *xy*, but it will not produce any terms in *x, y* or z, nor will it introduce a constant. Therefore the equation to such a cone will have only second degree terms in it. The equation will be of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad\qquad 4.7.4$$

With one proviso, the converse is usually true, namely that equation 4.7.4 represents a cone with vertex at the origin. The one proviso is that if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \qquad\qquad 4.7.5$$

equation 4.7.4 will factorize into two linear expressions, and will represent two planes, which intersect in a line that contains the origin - and this could be regarded as a special case of a cone, with zero vertical angle. If, however, the two linear factors are identical, the two planes are coincident.

The equation to a cone of semi vertical angle α whose vertex is at the origin and whose axis has direction cosines (l, m, n) can be found as follows. The equation

$$lx + my + nz + h = 0 \qquad\qquad 4.7.6$$

represents a plane that is perpendicular to the axis of the cone and is at a distance h from the origin (i.e. from the vertex of the cone). Let P (x,y,z) be a point in the plane and also on the surface of the cone, at a distance r from the origin. The semi vertical angle of the cone is then given by

$$\cos\alpha = h / r. \qquad\qquad 4.7.7$$

But

$$r^2 = x^2 + y^2 + z^2 \qquad\qquad 4.7.8$$

and, from equation 4.7.6,

$$h^2 = (lx + my + nz)^2 \qquad\qquad 4.7.9$$

Thus

$$\left(x^2+y^2+z^2\right)\cos^2\alpha = \left(lx+my+nz\right)^2 \qquad\qquad 4.7.10$$

is the required equation to the cone.

It might be thought that the equation to an inclined cone is unlikely to find much application in astronomy. Here is an application.

A fireball (i.e. a bright meteor, potentially capable of depositing a meteorite) moves down through the atmosphere with speed V along a straight line trajectory with direction cosines (l, m, n) referred to some coordinate system whose xy-plane is on the surface of Earth (assumed flat). If v is the speed of sound, and $V > v$, the meteoroid will generate a conical shock front of semi vertical angle α given by

$$\sin\alpha = v / V. \qquad\qquad 4.7.11$$

At a time t before impact, the coordinates of the vertex will be (lVt, mVt, nVt), and the equation to the conical shock front will then be

$$[(x+lVt)^2 +(y+mVt)^2 + (z+nVt)^2]\cos^2\alpha = [l(x+lVt)+m(y+mVt)+n(z+nVt)]^2 \qquad 4.7.12$$

Part of this shock front (at time t before impact) has already reached ground level, and it intersects the ground in a conic section given by putting $z = 0$ in equation 4.7.12:

$$\left(\cos^2 \alpha - l^2\right)x^2 + \left(\cos^2\alpha - m^2\right)y^2 - 2lmxy - 2Vt\sin^2 \alpha \left(lx + my\right) - V^2t^2 \sin^2 \alpha = 0 \qquad 4.7.13$$

and witnesses on the ground on any point on this conic section will hear the shock front at the same time. Further details on this can be found in Tatum, J.B., *Meteoritics and Planetary Science*, **34**, 571 (1999) and Tatum, J.B., Parker, L. C. and Stumpf, L. L., *Planetary and Space Science* **48**, 921 (2000).

4.8 *The General Second Degree Equation in Three Dimensions*

The general second degree equation in three dimensions is

$$ax^2 + by^2 + cz + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad 4.8.1$$

This may represent a plane or pair of planes (which, if not parallel, define a straight line), or an ellipsoid, paraboloid, hyperboloid, cylinder or cone. The equation could, if convenient, be divided through by d (or any of the other constants), and there are in reality only nine independent constants. Therefore nine points in space are sufficient to determine the second degree surface on this they lie.

If d is zero, the surface contains the origin. If u, v and w are all zero, and the surface is an ellipsoid, hyperbolic paraboloid or a hyperboloid, the origin is at the centre of the figure. If the figure is an elliptic paraboloid, the origin is at the vertex. If u, v, w and d are all zero, the surface is a cone with the proviso mentioned in section 4.7. If a, b, c, f, g, h are all zero, the surface is a plane.

Let us consider a particular example:

$$3x^2 - 4y^2 + 6z^2 + 8yz - 2zx + 4xy + 14x - 10y - 4z + 5 = 0 \quad 4.8.2$$

What sort of a surface is this?

We need to do two things. First we need to rotate the coordinate axes so that they are parallel to the figure axes. The equation referred to the figure axes will have no terms in yz, zx or xy. Then we need to translate the axes so that the origin is at the centre of the figure (or at the vertex, if it is an elliptical paraboloid).

Mathematically, we need to find the eigenvectors of the matrix

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -4 & 4 \\ 1 & 4 & 6 \end{vmatrix} \qquad\qquad 4.8.3$$

Some readers will readily know how to do this. Others may not, and may not even be quite certain what an eigenvector is. Section 4.9 may be of interest to either group of readers. In any case, the eigenvectors are found to be

$$\begin{pmatrix} l_{11} \\ l_{12} \\ l_{13} \end{pmatrix} = \begin{pmatrix} -0.069\,5481 \\ +0.318\,8310 \\ +0.945\,2565 \end{pmatrix} \qquad \begin{pmatrix} l_{12} \\ l_{22} \\ l_{32} \end{pmatrix} = \begin{pmatrix} -0.240\,6405 \\ +0.914\,2071 \\ -0.326\,0635 \end{pmatrix} \qquad \begin{pmatrix} l_{13} \\ l_{23} \\ l_{33} \end{pmatrix} = \begin{pmatrix} -0.968\,1194 \\ -0.250\,1441 \\ +0.013\,1423 \end{pmatrix}$$

with corresponding eigenvalues 7.422 7590, 5.953 0969, 3.530 3380.

The elements of the eigenvectors are the direction cosines of the present coordinate axis with respect to the figure axes. To express the equation to the surface relative to coordinate axes that are parallel to the figure axes, we replace

$$x \text{ by } l_{11}x \ + l_{12}y + l_{13}z$$

$$y \text{ by } l_{21}x \ + \ l_{22}y \ + l_{23}z$$

$$z \text{ by } l_{31}x \ + \ l_{32}y \ + l_{33}z$$

This will make the terms in *yz, zx* and *xy* vanish; this should be checked numerically, particularly as it is easy to rotate the axes in the wrong sense. When the substitutions are made, the equation is found to be

$$7.422\,7590x^2 - 5.953\,0969y^2 + 3.530\,3380z^2 - 7.943\,0994x - 11.206\,7840y - 11.104\,7998z + 5 = 0.$$
$$4.8.4$$

Notice that there are now no terms in *yz, zx* or *xy*.

Now we need to translate the origin of coordinates to the centre of the figure (or to the vertex if it is an elliptic paraboloid). It will readily be seen that this can be done by substituting

$$x \ - \ \alpha \qquad \text{for } x$$

$$y - \ \beta \qquad \text{for } y$$

$$z - \gamma \quad \text{for } z$$

where α = (coefficient of x)/(twice the coefficient of x^2) = $-0.535\,050\,336$

β = (coefficient of y)/(twice the coefficient of y^2) = $+0.941\,256\,643$

γ = (coefficient of z)/(twice the coefficient of z^2) = $-1.572\,767\,224$.

The equation then becomes

$$7.422\,7590\,x^2 - 5.953\,0969\,y^2 + 3.530\,3380\,z^2 - 0.583\,3816 = 0 \qquad \text{4.8.5}$$

or
$$\frac{x^2}{(0.280\,346)^2} - \frac{y^2}{(0.313\,044)^2} + \frac{z^2}{(0.406\,507)^2} = 1. \qquad \text{4.8.6}$$

The surface is a hyperboloid of one sheet, elliptical in any y = constant cross-section.

The surfaces described by second-degree equations in three dimensions - ellipsoids, paraboloids, hyperboloids, cones and cylinders - are generally called quadric surfaces. The surface described by the equation

$$\left[(x^2 + y^2)^{\frac{1}{2}} - a \right]^2 + z^2 = b^2, \quad b < a \qquad \text{4.8.7}$$

is not one of the quadric surfaces. If the square root is isolated and squared, the resulting equation will contain terms of degree four. The surface is a fairly familiar one, and the reader should try to imagine what it is. Failing that, if your computer skills are up to it, you might try to draw the surface in three-dimensional space. The only hint I give is to suggest that you put $y = 0$ in equation 4.8.7 to see what the section is in the z-x plane.

4.9 *Matrices*

This book has assumed a knowledge of matrices, which may or may not be justified. In this section, we do not attempt a thorough treatment of the subject, which must be sought elsewhere, but the remarks may be of use both to novices and to the more experienced.

However you may have the opportunity of learning about the manipulation of matrices, it is suggested that you should aim to understand and know how to carry out at least the following operations on matrices whose elements are real numbers:

Multiply a vector by a matrix
Multiply a matrix by a matrix
Calculate the determinant of a square matrix
Invert a square matrix
Diagonalize a symmetric matrix
Test a matrix for orthogonality

Numerous other aspects of matrix manipulation are possible, and the subject expands greatly if we allow the elements to be complex numbers. These six, however, are particularly useful for many applications. It might be mentioned, however, that solving simultaneous linear equations by Kramers' Rule or by inverting a matrix are very inefficient ways of solving such equations, and that is not the main purpose of acquiring the above skills.

Most, or doubtless all, of the above operations are available in many modern mathematical computer packages. This is not what I mean, however, by "understand and know how to carry out". The student should carry out at least once by hand calculator, step by step, each of the above operations, and, at each step, try to understand not only the algebraic and arithmetic steps, but also try to visualize the geometric interpretation, particularly when rotating axes and calculating eigenvectors. After doing a hand calculation, you should then write a series of short computer programs (rather than one vast, all-encompassing matrix package) for each operation, so that when, in future, you need to do any of these things, you can instantly obtain the answer without having to go through tedious calculations. For example, in the previous section, when I needed the eigenvectors of the matrix, I was able to generate them with a single word "EIGEN" on a computer; a considerable amount of arithmetic was actually performed by the computer.

On the question of testing a matrix for orthogonality, the usual application in mechanical and geometrical problems is to test a matrix of direction cosines. The tests can not only detect mistakes, but it can locate them and even suggest what the correct element should be. Tests for orthogonality are as follows. The student should try to think of the geometric interpretation of each.

The sum of the squares of the elements in any row or any column is unity. (This test does not guard against mistakes in signs of the elements.)

The sum of the products of corresponding elements in any two rows or in any two columns is zero.

Every element is equal to plus or minus its cofactor.

The determinant of the matrix is plus or minus one.

A minus sign in the last two tests indicates that the two sets of axes differ in chirality (handedness). This usually does not matter, and can easily be dealt with by reversing the signs of the elements of one eigenvector and of its corresponding eigenvalue.