Using Line and Texture to Visualize Higher-Order Voronoi Diagrams

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Abstract

Higher-order Voronoi diagrams provide a useful tool for studying problems where more than one nearest site is of interest. Understanding and visualizing higher-order Voronoi diagrams is more difficult than ordinary Voronoi diagrams. We propose new techniques using color, line and texture to visualize k-order diagrams. Approaches we develop have several interesting characteristics such as providing useful multi-dimensional information when even a small portion of the figure is available.

1 Introduction

Although Voronoi and Dirichlet provided the first formal definition and analysis of the Voronoi diagram, it's informal usage clearly precedes them. René Descartes, for example, includes diagrams in his works *Principia Philosophiæ* and *Le Monde* which seem to convey the general principle and structure of Voronoi diagrams. Descartes' figures are particularly interesting because they not only illustrate the boundaries of Voronoi cells but also the interior space of the Voronoi cells. See Figure 1.

It is this concept of depicting or rendering a Voronoi diagram that we shall study in this paper. Specifically, we consider approaches for rendering *higher-order* Voronoi diagrams in two dimensions using the Euclidean distance metric.

Each cell in an ordinary Voronoi diagram contains all the points that are closest to a specific generator site, while a higher order Voronoi diagram is one whose cells are formed based on more than one generator site. The order-k Voronoi diagram of n sites in $\mathbb{R}^2$ is a decomposition of the plane into convex polyhedra such that the points in each region have the same k closest sites.

More formally, let $P = \{p_i, \ldots, p_n\}$ be a set of generator sites and let $A^{(k)}(P) = \{\{p_{1,1}, \ldots, p_{1,k}\}, \ldots, \{p_{l,1}, \ldots, p_{l,k}\}$ define a set of size k subsets of P where $p_{i,j} \in P$

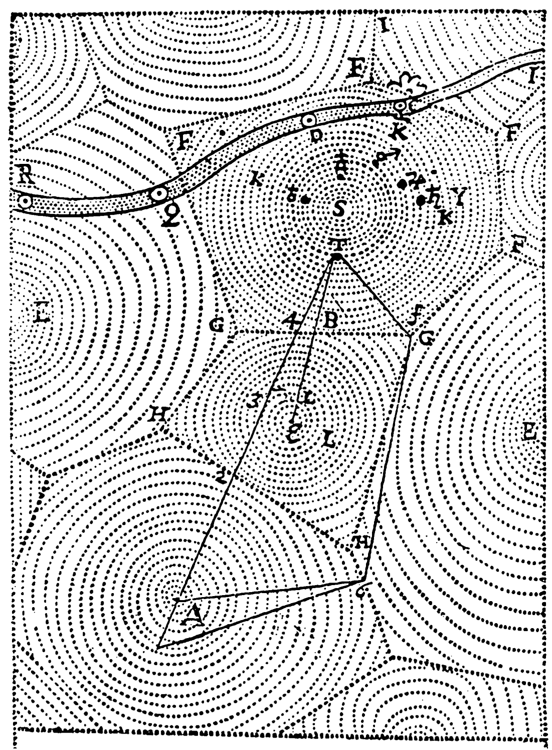

Figure 1. René Descartes uses this figure from *Le Monde* (1664) to represent relationships among stars and comets. Here S is the sun and ϵ is a star.

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and $l = \binom{n}{k}$. Let $P_i^{(k)} = \{p_{i,1}, \ldots, p_{i,k}\}$ denote one such subset. The order-k Voronoi polygon associated with $P_i^{(k)}$ is then described by,

$$V(P_i^{(k)}) = \{p \quad | \quad \max_{p_h}\{d(p,p_h)|p_h \in P_i^{(k)}\} \\ \leq \min_{p_j}\{d(p,p_j)|p_j \in P \backslash P_i^{(k)}\}\}.$$

Or as shown by Miles and Maillardet [9],

$$V(P_i^{(k)}) = \bigcap_{h=1}^{k} V(p_{i,h}|[P\backslash P_i^{(k)}] \cup \{p_{i,h}\}).$$

The order-k Voronoi diagram is a generalization of the classical Voronoi diagram which is obtained if $k = 1$. The *furthest-site* Voronoi diagram is obtained if $k = n - 1$.

Another higher order Voronoi diagram we consider is the *ordered* order-k Voronoi diagram. Cells in an ordered order-k Voronoi diagram have the same ordered set of k closest sites. Written more formally [10],

$$V(P_i^{(k)}) = \bigcup_{P_j^{<k>} \in A^{<k>}(P_i^{<k>})} V(P_j^{<k>}),$$

where $P_i^{<k>}$ is an ordered k-tuple of P and $A^{<k>}(P)$ is the set of all ordered k-tuples of P.

A closely related diagram, the k^{th} nearest point diagram is described by [10],

$$V^{[k]}(p_j) = \bigcup_{P_i^{(k-1)} \in A^{(k-1)}(P \backslash \{p_j\})} V(P^{(k-1)} \cup \{p_j\}).$$

2 Related Work

Efficient construction of order-k Voronoi diagrams has been studied by many researchers including Lee [8], Chazelle and Edelsbrunner [4], Aurenhammer [2], Clarkson [5], and Agarwal et al. [1]. The last result uses a randomized approach and finds $O(k(n-k)logn + n\log^3 n)$ expected running time.

Order-k Voronoi diagrams have also been considered in the context of parallel computing [13] and GPU based computation [6]. For a survey of many relevant higher order Voronoi diagram results, see [3] and the book by Okabe et al [10].

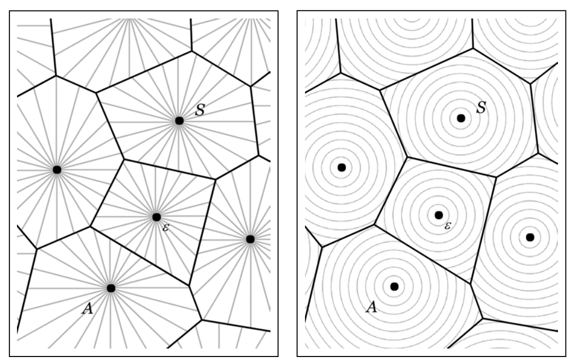
Telea and van Wijk have studied the visualization of order-k Voronoi diagrams with an emphasis on illustrating the relationships between different cells using a series of colored bevels and cushions [12].

In the context of texture, Voronoi diagrams have been used for many different purposes. The *cellular* look that Voronoi diagrams achieve has been exploited by researchers interested in mimicking natural textures in a procedural way. Worley was one of the first to propose a cellular texturing basis function which uses Voronoi cells to emulate naturally occurring cellular phenomena [14]. Worley's method implicitly computes Voronoi diagrams of points assigned x and y coordinates by a pseudo-random noise generator. Worley's function has proven to be useful in defining textures such as waves, stone, metal and leather.

3 Emanations and Contours

The distance fields associated with Voronoi diagrams are often rendered using either emanation lines or contour lines. Emanation lines begin (or emanate) from a generator site and terminate at the edge of the Voronoi cell corresponding to that generator site. See Figure 2 (a). One shortcoming of emanation lines (especially with respect to point generator sites) is that the two dimensional space may be densely represented very close to a generator and very sparsely represented far away from a generator.

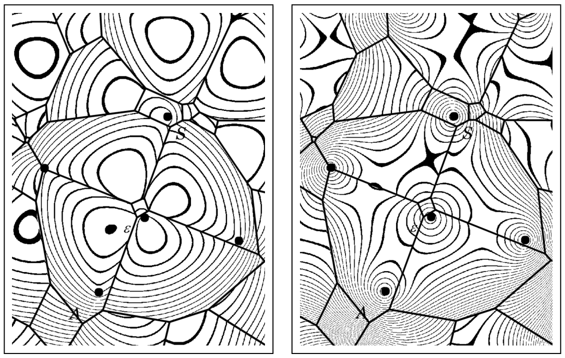
A contour line (or isopleth) is an equipotential curve that represents all points of equal value. Contour plots of the space represented by Voronoi diagrams are extremely useful. Unlike emanation lines, contour lines often provide a better, more uniform, sampling of the space and also provide unique insights into the behavior of the distance function with respect to the generator sites. See Figure 2 (b).



(a) (b)

Figure 2. A modern interpretation of Descartes' figure from *Le Monde* using (a) emanation lines and (b) contour lines.

The visual "meaning" of an order-k Voronoi diagram in the context of contour lines leaves room for interpretation. An order-k Voronoi diagram conveys k dimensions of information. What then is the best way to represent this information and these relationships. An additive function that combines the k dimensions of information might seem attractive because the Voronoi diagram of the nearest k sites is also a Voronoi diagram of the minimal summation of any k sites. The rendering while interesting (and similar to what one might find in an illumination study) is difficult to interpret. See Figure 3 (a). A multiplicative model has similar problems. See Figure 3 (b).

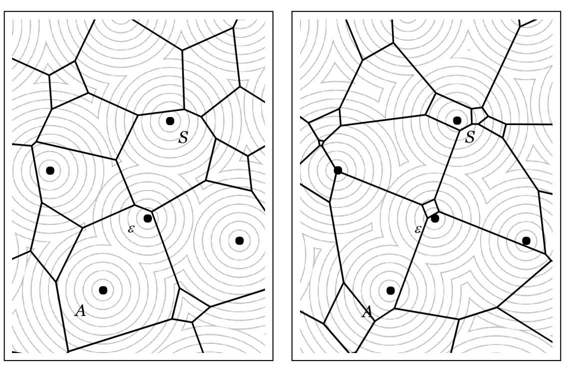


(a) (b)

Figure 3. Contour lines used to illustrate order-3 Voronoi diagrams using and (a) additive and (b) multiplicative model.

3.1 Order-k Plots

Let a k^{th} nearest neighbor plot be a set of equidistant contour lines that represent the distance to the k^{th} nearest generator site. Then, Figure 2 (b) represents a first nearest neighbor contour plot.



(a) (b)

Figure 4. Contour lines used to illustrate (a) order-2 and (b) order-3 Voronoi diagrams.

1^{st} nearest neighbor contour plots provide very intuitive visualizations of both order-1 and order-2 Voronoi diagrams. Abrupt collisions of isocontours associated with different generator points represent boundaries in the underlying distance that are not smooth. Consider the order-2 Voronoi diagram that has been overlaid on top of a 1^{st} nearest neighbor plot in Figure 4 (a). The contours very obviously identify which two generator sites are associated with each Voronoi cell. Furthermore, the contour lines also implicitly identify which of the two sites is closest to any point in the cell. That is, a 1^{st} nearest neighbor plot implicitly provides a visual description of an ordered order-2 Voronoi diagram.

This property does not follow when we consider an order-3 Voronoi diagram. See Figure 4 (b). Using a 1^{st} nearest neighbor plot, we can expect to visually identify (from characteristics of the plot) only the closest generator to an arbitrary point for an order-3 or higher nearest neighbor query. The second and third closest sites may be ambiguous.

We can remove this ambiguity by super-imposing a 1^{st} nearest neighbor plot on top of a 2^{nd} nearest neighbor plot. See Figure 6 (a). The darker 1^{st} nearest neighbor plot represents contours associated with a closest site. The lighter 2^{nd} nearest neighbor plot represents contours associated with the second closest site. The third closest site is then identified implicitly by identifying contours associated with a different generator site in the same Voronoi cell. Let such a plot be called an *order-3 plot*. More generally, let a plot constructed by overlaying nearest neighbor plots for $1, \ldots, k-1$ be called an *order-k plot*.

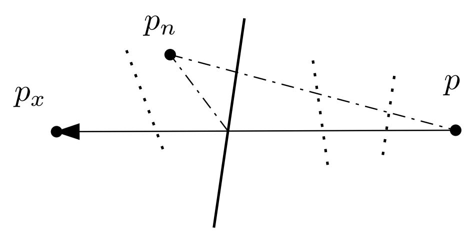


Figure 5. The k^{th} closest neighbor must appear as a $(k-1)^{\text{st}}$ closest neighbor in the same order-k cell.

Theorem 1 *An order-k plot constructed from k-1 nearest neighbor plots for* $1, \ldots, k-1$ *can identify the ordered locations of the* k *closest generators of an order-*k *Voronoi diagram.*

Proof: Clearly the k^{th} nearest neighbor plots for $1, \ldots, k$ can be used to identify an ordered set of generator sites (assuming the plots have sufficient detail). The theorem simply implies that with the order-k Voronoi diagram the k^{th} closest site can be implied. Or equivalently, the intersection of an order-k Voronoi diagram with each of the k^{th} nearest

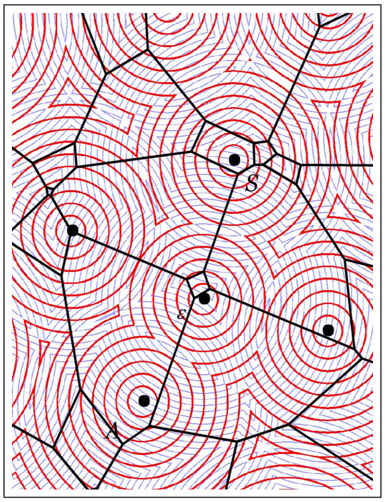


Figure 6. Multidimensional contour lines used to illustrate an order-3 Voronoi diagram.

neighbor Voronoi diagrams for $1, \ldots, k-1$ contains cells associated with all k generator sites.

Consider an arbitrary point p in a cell C associated with an order-k Voronoi diagram, V. We assume the generator sites are in general position. Without loss of generality, assume a line connecting p and and it's k^{th} nearest point p_x does not pass through a vertex of V. Let V' be the ordered order-k Voronoi diagram V' that further subdivides V. We need only show that the k^{th} closest site for p is the $(k-1)^{\text{st}}, \ldots, or 1^{\text{st}}$ closest site somewhere else in V. We prove this by contradiction. Assume the k^{th} closest site, p_x, for C is fixed. Consider the path from p to p_x. Let p_i be a point on this line and let P_i be the set of k nearest points. As we move p_i from p to p_x we will cross edges in V'. As we cross an edge $\in V'$ that is $\notin V$, the order of sites in P_i will change. If we cross an edge $\in V$ the furthest site in P_i is removed and replaced with a new furthest site. Consider the first edge of V to be crossed. Removing p_x from P_i corresponds to a new closest site to p_i, p_n. But since we are moving directly toward p_x from p, if this site is closer, then p_x was not the k^{th} nearest site at p_i. See Figure 5. $\square$

A seemingly minor victory for visualization, Theorem 1 guarantees that $k-1$ nearest neighbor plots are sufficient to completely represent the underlying distance functions in an order-k Voronoi diagram. The difficulty in interpreting figures with even a small k makes this a valuable finding. Consider Figure 6 which uses two sets of contour lines in two different colors and densities to represent an order-3 Voronoi diagram.

Notice that these renderings implicitly or explicitly describe all three higher order Voronoi diagrams we described in Section 1. When a series of nearest neighbor plots are combined as described we have shown the resulting order-k plot accurately describes the ordered order-k Voronoi diagram which is a refinement of the order-k Voronoi diagram which in turn is a generalization of the ordinary Voronoi diagram.

3.2 Efficient Rendering

A naive approach to render an order-k plot might iterate over each value of k and each pixel or scene coordinate and then calculate $\min_k(d(X_i, P_0), \ldots, d(X_i, P_n))$ where X is the set of pixels or sampling points, P is the set of generator sites and $\min_k()$ finds the k^{th} minimal value. The selection of a k^{th} order statistic can be found in linear time and thus such a plot could be calculated in $O(|X|nk)$ running time, where we might expect the number of pixels, $|X|$, to dominate.

An improved approach begins by computing the ordered order-k Voronoi diagram for the set of generator sites P. Let $T(P)$ be the time to construct such a Voronoi diagram. We then iterate over each Voronoi cell and in turn iterate over each pixel or sampling point captured by that cell. Since the nearest sites and their relative ordering is obtained from the cell, for each pixel we calculate $d(X_i, P_j)$ only for the set of k points in P that are of interest. The resulting running time is then $O(|X|k + T(P))$, where we might expect $|X|k$ to dominate for the more common case where the number of samples in a plot is much larger than the number of generator sites, n.

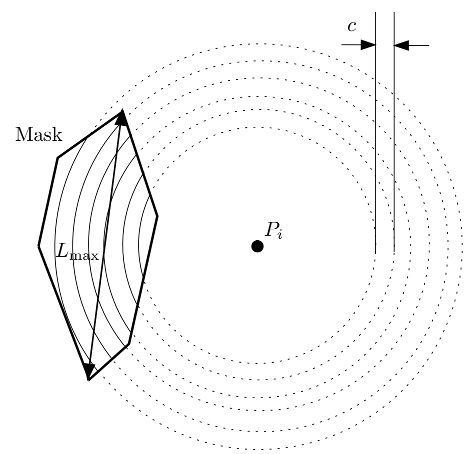


Figure 7. A Voronoi cell can be used as a mask to render contour lines at an interval, c.

If the plot output is vector rather than raster or we assume hardware accelerated circles and masks or arc drawing functions are available, we can consider a third approach that uses the Voronoi diagram cells as a mask and then draws circles into the masked area. We consider each Voronoi cell in turn and draw circles (representing contours) into the mask for values being contoured. See Figure 7. Pixels or sampling points are not used so the running time depends on the number of circles drawn which in turn depends on the contour interval, c, and the maximum distance across a Voronoi cell and it's intersection with the display region, $D_{\max}$. The running time to construct such a plot is then $O(|V|(D_{\max}/c)k + T(P))$ where $|V|$ is the number of Voronoi cells and we assume masking and drawing functions are all constant time operations.

4 Texture

In the previous sections we considered illustration techniques in terms of their ability to express the underlying meaning of higher order Voronoi diagrams. In this section, we consider extending Worley's texture function by considering higher order Voronoi diagrams.

Our first experiment considers order-2 and order-3 plots of several randomly generated points. We shade to the edges of the Voronoi cells based on the distance function and use a $\sin$ based function to determine color. The results Figure 8 and 9 express a number of geometric combinations one could not achieve with an order-1 plot. More specific placement of points can generate many geometric figures reminiscent of Celtic or even Arabic artwork.

In our second series of experiments, we revisit the additive and multiplicative models for combining the k nearest generators. Under the additive model (see Figure 10) the image remains very cellular but under the multiplicative model this quality is lost (see Figure 11).

And finally we consider combining our texture function with Perlin's noise function [11] to provide irregularities in the rendered images. In Figure 12 we use an order-2 plot to generate a wood like grain, while in Figure 13 we combine an order-2 and order-3 plot to create a multilayered texture.

We note that the actual tilings obtained from Order-k Voronoi diagrams are interesting in and of themselves. Using many of the techniques Kaplan has described [7], Higher-Order Voronoi diagrams can be used to construct complex ornamental designs. See Figures 14 and 15.

5 Conclusion

In this paper we have explored several visual aspects of higher order Voronoi diagrams. We have shown how contour lines can be used to effectively visualize higher order Voronoi diagrams and demonstrated certain qualities associated with these diagrams. We then considered the visualization of higher order Voronoi diagrams in the context of procedural textures.

Figure 8. A texture based on an order-2 contour plot.

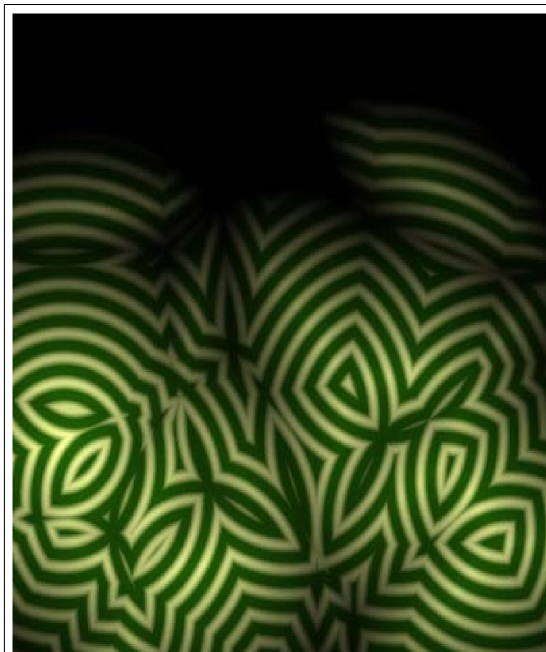


Figure 9. A texture based on an order-3 contour plot.

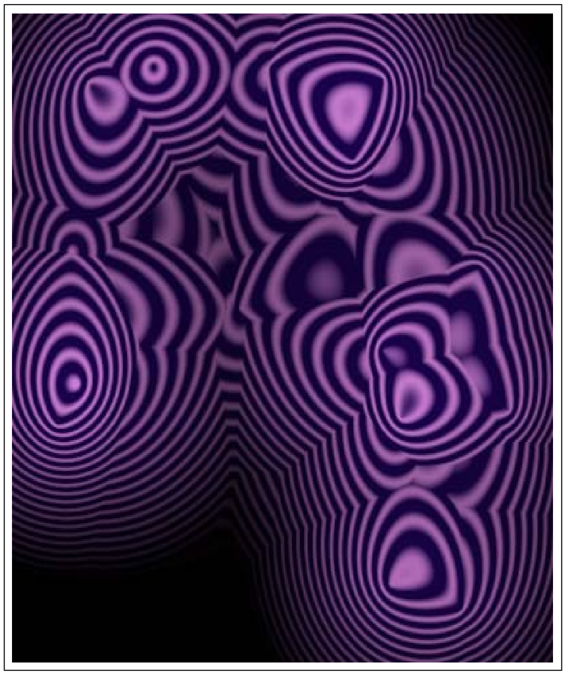


Figure 10. A texture based on an additive model contour plot.

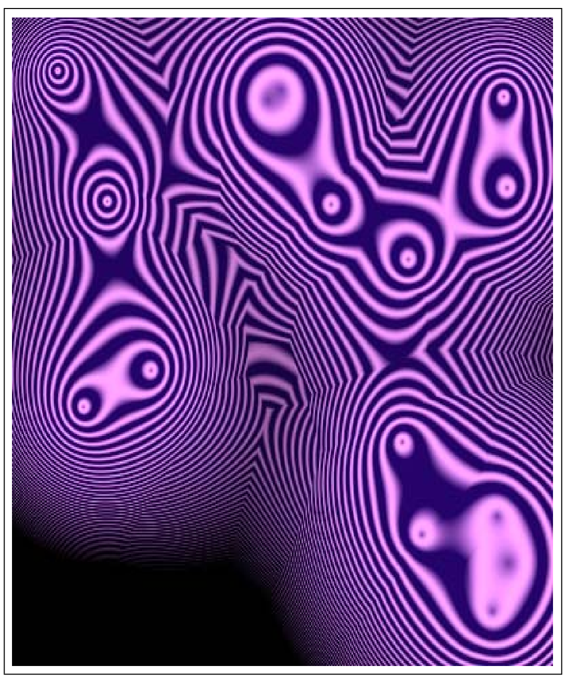


Figure 11. A texture based on a multiplicative model contour plot.

Figure 12. Perlin's noise function is used to break up the regularity of the texture.

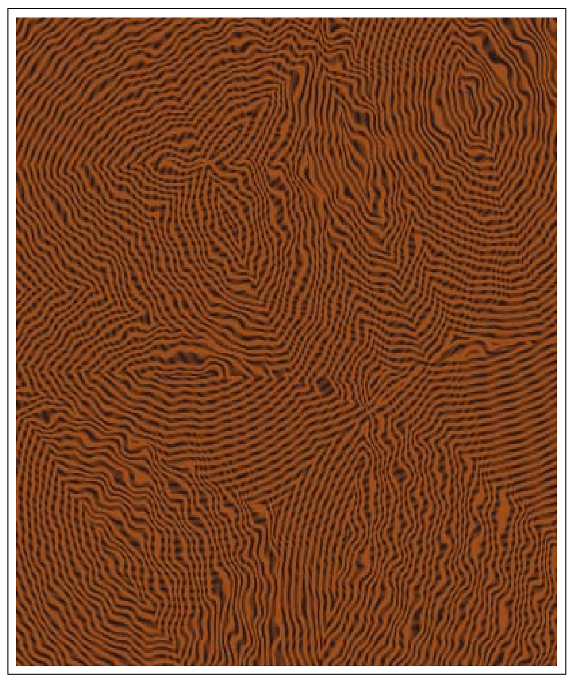


Figure 13. Two textures based on order-1 and order-2 contour plots have been combined.

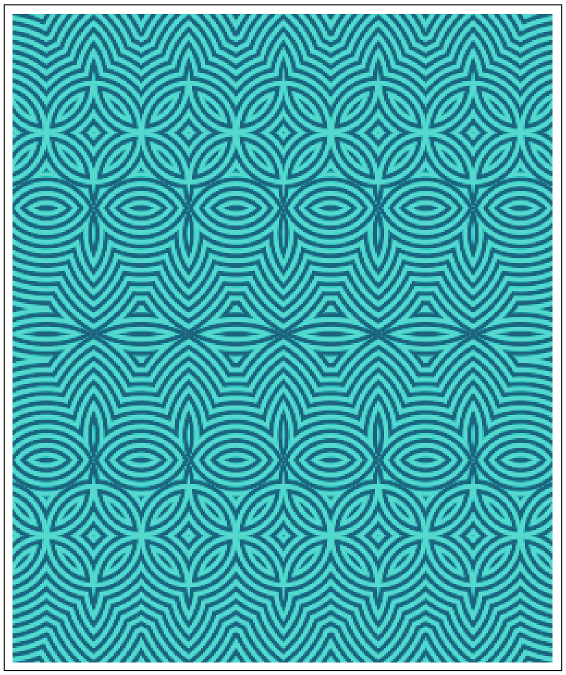


Figure 14. A pattern constructed from an interesting Voronoi diagram based tiling of the plane.

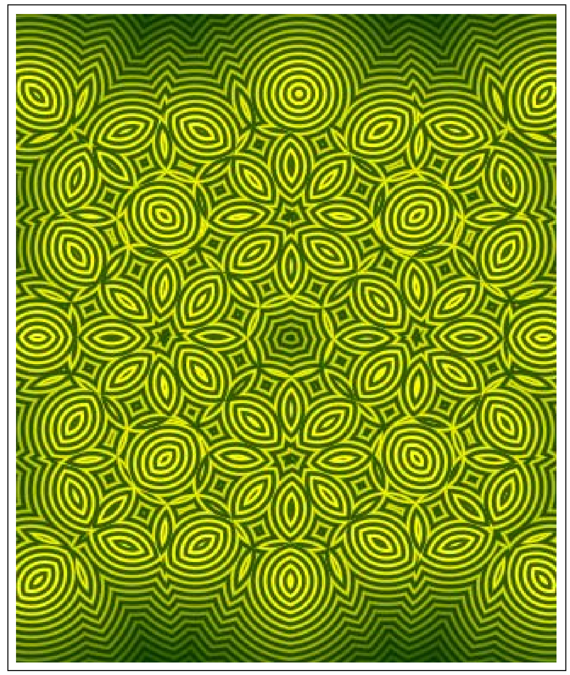


Figure 15. A pattern constructed by processing points with a symmetry generator.

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