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PHYS250 - Chpt 4 Reading

**Chapter 4: The Solar System**

- Kepler's Laws
  - Hypothetical solar system: one planet (Earth) which is in orbit around the Sun and the only force in the problem is gravity
  - Newton's law of gravitation of magnitude of this force is:

$$F_G = \frac{G\,M_S\,M_E}{r^2} \,,$$

    - $M_S$ and $M_E$ are the masses of the Sun and Earth
      - Assume mass of the Sun is sufficiently large and its motion can be neglected
    - r is distance between them
    - G is gravitational constant
    - Goal: calculate the position of Earth as a function of time
  - Newton's second law of motion:
    - $F_{G,x}$ and $F_{G,y}$ are the x and y components of the gravitational force

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E}$$

$$\frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_E} \,,$$

$$F_{G,x} = -\frac{G\,M_S\,M_E}{r^2}\cos\theta = -\frac{G\,M_S\,M_E\,x}{r^3} \,,$$

- Write each of the second-order differential equations as two first-order differential equations
  - These can be converted into difference equations that can be solved numerically
    - Useful to consider choice of units and corresponding unit of mass
      - For circular motion we know that the force must be equal to $M_E v^2/r$ which leads to:

$$\frac{dv_x}{dt} = -\frac{G\,M_S\,x}{r^3}$$

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_y}{dt} = -\frac{G\,M_S\,y}{r^3}$$

$$\frac{dy}{dt} = v_y \,.$$

$$\frac{M_E\,v^2}{r} = F_G = \frac{G\,M_S\,M_E}{r^2} \,,$$

$$G\,M_S = v^2\,r = 4\,\pi^2\ \mathrm{AU}^3/\mathrm{yr}^2 \,,$$

- Converting the equations of motion into difference equations in preparation for constructing a computational solution:
    - Delta t is the time step and the factors $4*pi^2 = GM_S$
        - Signals the use of astronomical units
    - Using the Euler-Cromer method will conserve energy exactly over the course of each orbit

$$v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3} \Delta t$$

$$x_{i+1} = x_i + v_{x,i+1} \Delta t$$

$$v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_i^3} \Delta t$$

$$y_{i+1} = y_i + v_{y,i+1} \Delta t \ ,$$

- In analyzing planetary motion, it's useful to visualize the movement of the planet graphically
    - Plotting position of the planet as it becomes available during the calculation
    - The routine planet can be used to simulate the motion of the Earth or any real or hypothetical planet in orbit around the Sun
        - Entry under "eccentricity" refers to the shape of each orbit
- Program can be used to investigate Kepler's Laws for planetary motion
    - All planets move in elliptical orbits, with the Sun at one focus
    - The line joining a planet to the Sun sweeps out equal areas in equal times
    - If T is the period and a the semimajor axis of the orbit, then $T^2/a^3$ is a constant
        - It can be shown analytically that all three of Kepler's Laws are consequences of the fact that the gravitational force follows an inverse-square law
        - It is convenient to modify the program slightly to print out the value of the time when a planet passes a particular point on the orbit
- Inverse-Square Law and Stability of Planetary Orbits: review of analytic results concerning the trajectory of a body in a solar system assumed to contain only the Sun and the body
    - Two-body system, all three of Kepler's Laws are consequences of the fact that the gravitational force follows an inverse-square law
        - Sketch some of the analytic results here to lay the ground for further numerical investigations
        - Interaction force depends only on the separation r
            - Relative motion in this system can be studied as if it were a one-body system
                - Moving body in this equivalent system has a mass equal to the so-called reduced mass
                    - Meu = $m_1m_2/(m_1+m_2)$ where $m_1$ and $m_2$ are the masses of the two original bodies
                - Position of the equivalent body is given by the relative displacement $r = r_2 - r_1$ of the original two bodies
        - Orbital trajectory for a body of reduced mass (meu) is given in polar coordinates:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{L^2} F(r) \ ,$$

- Where L = meu*r^2*theta_dot is the angular momentum and F(r) is the force acting on the body
- For our solar system case, the solution can be expressed as:

$$\frac{1}{r} = \left(\frac{\mu G M_S M_P}{L^2}\right)[1 - e\cos(\theta + \theta_0)] ,$$

or, choosing $\theta_0 = 0$ (which defines the axes of the ellipse),

$$r = \left(\frac{L^2}{\mu G M_S M_P}\right)\frac{1}{1 - e\cos\theta} .$$

  - This result does not give us the position of the planet as a function of time but the shape of the orbital trajectory
  - The max and min of the velocity can be found using:

$$v_{\max} = \sqrt{GM_S}\sqrt{\frac{(1+e)}{a(1-e)}\left(1+\frac{M_P}{M_S}\right)}$$

$$v_{\min} = \sqrt{GM_S}\sqrt{\frac{(1-e)}{a(1+e)}\left(1+\frac{M_P}{M_S}\right)} .$$

    - The orbital period T can be obtained by dividing the area enclosed by the elliptical orbit by the constant rate at which area is swept out according to Kepler's second law
  - Supposed the force law deviates slightly from an inverse-square dependence which says the gravitational force is of the form:

$$F_G = \frac{G\, M_S\, M_E}{r^\beta} .$$

    - If Beta = 2, we have an inverse-square law
      - Behavior of elliptical orbits with this force law can be simulated with the planetary motion program by simply changing the exponent of r in the equations for the velocity

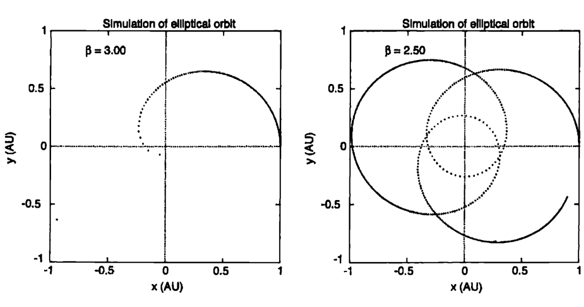
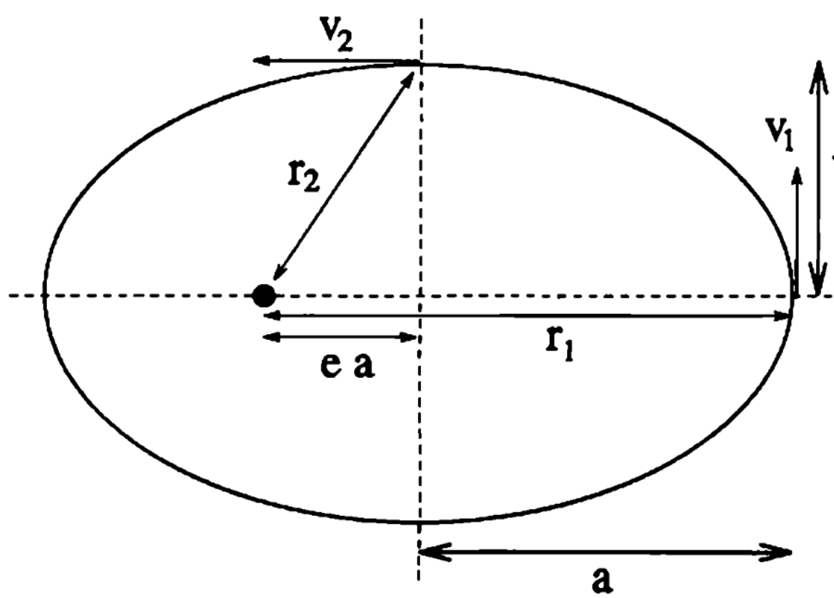



**FIGURE 4.5:** Elliptical orbits calculated for a force law (4 12) with $\beta = 3$ (left) and $\beta = 2.50$ (right).

- Precision of the perihelion of mercury: in a solar system with mor ethan one planet there will be deviations from Kepler's Law
    - Deviations come from a number of sources including the effects of the planets on each other
    - Planets whose orbits deviate the most from circular are Mercury and pLuto
        - Mercury: orientation of the axes of the ellipse that describe its orbit rotate with time
            - Magnitude of this precession is approximately 566 arcseconds per century (one rotation every 230,000 years)
            - Precision of both the experimental measurement and theoretical calculation of precession of Mercury's perihelion are impressive, but did not agree
                - Suggestion that there might be another planet whose orbit was inside that of Mercury or a large amount of dust orbiting near the Sun, whose gravitational attraction was affecting Mercury but neither were confirmed
                - Theory deals with geometry of space and views gravity in a more complicated manner than simple picture of Euclidean space and inverse-square law
                - Sun and Mercury are close enough for these deviations to be significant
                    - Einstein showed that they precisely account for previously unexplained 43 arcseconds of precession
            - Force law predicted by general relativity is:
                - Alpha = $1.1 \times 10^{-8}$ AU$^2$

$$F_G \approx \frac{G\,M_S\,M_M}{r^2}\left(1+\frac{\alpha}{r^2}\right),$$



- Conservation of total energy implies that energies at point 1 and 2 are the same and therefore:

$$-\frac{G\,M_S\,M_M}{r_1} + \frac{1}{2}\,M_M\,v_1^2 = -\frac{G\,M_S\,M_M}{r_2} + \frac{1}{2}\,M_M\,v_2^2\ .$$

- Three-body problem and effect of Jupiter on Earth: without Jupiter, Earth's orbit is stable and unchanging with time
    - Observe effect of gravitational force from Jupiter on Earth's motion

$$F_{E,J} = \frac{G\,M_J\,M_E}{r_{EJ}^2}\ ,$$

    - Magnitude of force between Jupiter and Earth using familiar inverse-square law with Sun replaced by Jupiter