Homework-6-Mathematics-Analysis

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Problem Statement

We are tasked with approximating the derivative of a function using finite differences and analyzing the error behavior for two methods: forward difference and centered difference.

We will choose the following function:

$$f(x) = \sin(x)$$

The exact derivative of $f(x)$ is:

$$f'(x) = \cos(x)$$

We will compute the exact value of the derivative at $x = 1$ and approximate it using the two finite difference methods. For the forward difference method, the approximation is given by:

$$\frac{f(x+h) - f(x)}{h}$$

For the centered difference method, the approximation is given by:

$$\frac{f(x+h) - f(x-h)}{2h}$$

We will calculate the errors for different values of h, ranging from 10^{-10} to 10^{-1}, and plot the results.

Python Code Implementation

```
import numpy as np
import matplotlib.pyplot as plt

# Define the function f(x) = sin(x)
def function(x):
    return np.sin(x)
```

```python
# Define the derivative of the function f'(x) = cos(x)
def function_derivative(x):
    return np.cos(x)

# Define the forward difference approximation
def forward_difference(x, h):
    return (function(x + h) - function(x)) / h

# Define the centered difference approximation
def centered_difference(x, h):
    return (function(x + h) - function(x - h)) / (2 * h)

# Set the point x = 1.0
x = 1.0
exact_value = function_derivative(x)

# Generate a range of small values for h between 1e-10 and 1e-1
h_values = np.logspace(-10, -1, 100)

# Lists to store errors for both methods
forward_errors = []
centered_errors = []

# Loop over the values of h and compute the approximations and errors
for h in h_values:
    forward_approx = forward_difference(x, h)
    centered_approx = centered_difference(x, h)

    forward_errors.append(abs(forward_approx - exact_value))
    centered_errors.append(abs(centered_approx - exact_value))

# Plot the errors
plt.loglog(h_values, forward_errors, label=r"Forward Difference Error ($O(h)$)")
plt.loglog(h_values, centered_errors, label=r"Centered Difference Error ($O(h^2)$)")
plt.xlabel("h")
plt.ylabel("Error")
plt.title("Errors in Forward and Centered Difference Approximations")
plt.legend()
plt.grid(True, which="both", linestyle="--")
plt.show()
```

Plot Section

Here is the plot that shows the error behavior for the forward and centered difference approximations as h varies:

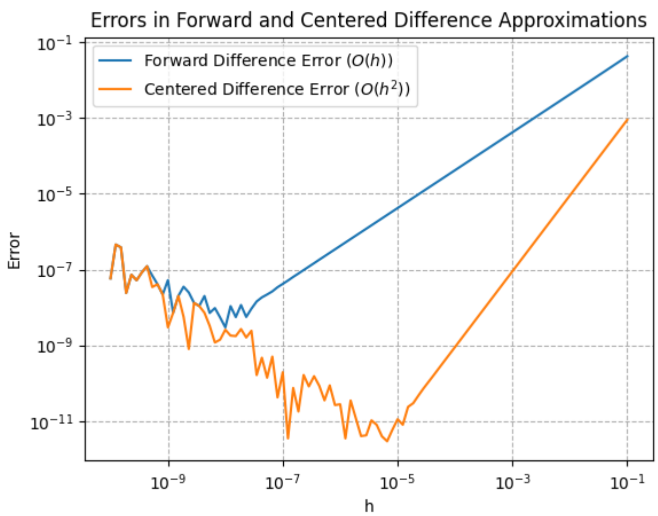

Figure 1: Errors in Forward and Centered Difference Approximations

Explanation of Results

When h becomes very small (e.g., close to 10^{-10}), floating-point precision limitations in computer arithmetic start to affect the calculations. Subtracting very close values $f(x+h)-f(x)$ or $f(x+h)-f(x-h)$ leads to **cancellation errors** (loss of significant digits), causing inaccuracies in the approximation. Thus, after a certain point, the errors start to increase as h gets smaller due to the finite precision of floating-point arithmetic.