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Homework 4

1) Find the radius of convergence and the convergence set

a) $\sum_{n\geq 0} \frac{nx^n}{2^n}$, $a_n = \frac{n}{2^n}$, $c = 0$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{\cancel{2^n}}{n} = \lim_{n\to\infty} \frac{n+1}{n \cdot 2}$

$L = \frac{1}{2} \Rightarrow R = 2 \Rightarrow$ ABS. CONV, $(\forall)\ x \in (-2, 2)$

Check: $x = 2$ and $x = -2$

$x = -2$: $\sum_{n\geq 0} \frac{-2^n \cdot n}{2^n} = (-1)^n \cdot n$ divergent

$x = 2$: $\sum_{n\geq 0} \frac{2^n \cdot n}{2^n} = \sum_{n\geq 0} n = \infty$ divergent

$\Rightarrow$ Convergent set $C = (-2, 2)$

b) $\sum_{n\geq 1} \frac{x^{2n}}{\sqrt{n}}$, $a_n = \frac{1}{\sqrt{n}}$, $c = 0$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \sqrt{n} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1$

$\Rightarrow R = 1 \Rightarrow$ ABS. CONV, $(\forall)\ x \in (-1, 1)$

Check: $x = -1$ and $x = 1$

$x = -1$: $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$, $p = \frac{1}{2} \leq 1$ $\Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges

$x = 1$: $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges

Convergence set $C = (-1, 1)$

c) $\sum_{n\geq 1} (-1)^n \frac{(x-1)^n}{n}$, $a_n = \frac{(-1)^n}{n}$, $c = 1$

$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n}\right|$

$= \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = 1$

ABS CONV: $(\forall)\, x \in (0, 2)$ $\quad (x \in (c-R, c+R))$

Check: $x = 0$, $x = 2$

$x = 0$: $\sum_{n\geq 1} (-1)^n \cdot \frac{(-1)^n}{n} = \sum_{n\geq 1} \frac{1^n}{n} = \sum_{n\geq 1} \frac{1}{n}$ Harmonic S. diverges

$x = 2$: $\sum_{n\geq 1} (-1)^n \cdot \frac{(1)^n}{n} = \sum_{n\geq 1} \frac{(-1)^n}{n}$ Alter series converges

Convergence set: $C = [0, 2]$

2. Study the convergence and compute the sum:

$\sum_{n\geq 2} \frac{x^n}{n(n-1)}$

$\frac{x^n}{n(n-1)} = \int \frac{x^{n-1}}{n-1}\, dx$

$\frac{x^n}{n(n-1)} = \int_0^x \frac{t^{n-1}}{n-1}\, dt = \frac{t^n}{n(n-1)}\Big|_0^x = \frac{x^n}{n(n-1)}$

$\sum_{n\geq 2} \frac{x^n}{n(n-1)} = \sum_{n\geq 2} \left(\int_0^x \frac{t^{n-1}}{n-1}\, dt\right) = \int_0^x \left(\sum_{n\geq 2} \frac{t^{n-1}}{n-1}\right) dt$

$\sum_{n\geq 2} \frac{t^{n-1}}{n-1} = \sum_{n\geq 2} \int_0^t u^{n-2}\, du = \int_0^t \left(\sum_{n\geq 2} u^{n-2}\right) du$

If $|t| < 1 \Rightarrow \int_0^t \left(\sum_{n\geq 2} u^{n-2}\right) du = \int_0^t \frac{1}{1-u}\, du = \ln(1-u)\Big|_0^t$

$= -\ln(1-t)$

$\Rightarrow \int_0^x -\ln(1-t)\, dt$

$$\int_0^x -\ln(1-t)\,dt = -\int_0^x \ln(1-t)\,dt$$

$$f(t) = \ln(1-t) \Rightarrow f'(t) = -\frac{1}{1-t}$$

$$g'(t) = 1 \Rightarrow g(t) = t$$

$$= -\left(f(t)\cdot g(t) - \int_0^x f'(t)\cdot g(t)\,dt\right)$$

$$= -\left(\ln(1-t)\cdot t\Big|_0^x - \int_0^x \frac{-t}{1-t}\,dt\right)$$

$$= -\left(\ln(1-x)\cdot x - \ln(1-0)\cdot 0 - \left(\int_0^x \frac{1-t}{1-t}\,dt - \int_0^x \frac{1}{1-t}\,dt\right)\right)$$

$$= -\left(x\ln(1-x) - \left(t\Big|_0^x + \ln(1-t)\Big|_0^x\right)\right)$$

$$= -\left(x\ln(1-x) - \left(x + \ln(1-x)\right)\right)$$

$$= -x\ln(1-x) - x - \ln(1-x)$$

$$= \ln(1-x)^{-x} - \ln(1-x) - x$$

$$= \ln\frac{1}{(1-x)^x} - \ln(1-x) - x$$

$$= \ln\frac{1}{(1-x)^{x+1}} - x \Rightarrow \text{series converges}$$

If $|t| \geq 1 \Rightarrow \int_0^1 \left(\sum_{n\geq 2} u^{n-2}\right)du$ diverges