a) $[1,2]=A$

$int(A)=(1,2)$

$cl(A)=[1,2]$

b) $[-3,2)\cup\{3\}=B$

$int(B)=(-3,2)$

$cl(B)=[-3,2]\cup\{3\}$

c) $(-1,1]\cup(2,\infty)=C$

$int(C)=(-1,1)\cup(2,\infty)$

$cl(C)=[-1,1]\cup[2,\infty)$

d) $(-5,5)\cap\mathbb{Z}=\{-4,-3,-2,-1,0,1,2,3,4\}=D$

$\nexists\, int(D)$ or $int(D)=\emptyset$

$cl(D)=\{-4,-3,-2,-1,0,1,2,3,4\}$

H: 5, 8, 10

Homework 1 FARKAS TIBERIA-GIULIA

1) Let $a,b\in\mathbb{R}$ with $a>0$

$S\subseteq\mathbb{R}$ nonempty and bounded above

$\Downarrow$

$\sup\limits_{x\in S}(ax+b)=a\sup(S)+b$

$\Downarrow$

S bounded above $\Rightarrow (\exists)\, ub(S)=[\sup(S),+\infty)\Rightarrow$

$\Rightarrow (\forall)\,\varepsilon>0,\ (\exists)\, x_0\in S$ s.t. $\sup(S)-\varepsilon<x_0\le\sup(S)$ (1)

For $(\forall)\, x\in S\Rightarrow x\le\sup(S)\,|\cdot a$

$ax\le a\sup(S)\,|+b$

$ax+b\le a\sup(S)+b$ (3)

Let $M=\{ax+b \mid x\in S,\ a,b\in\mathbb{R}\}$ $\Bigg|\Rightarrow a\sup(S)+b\in ub(M)$ (2)

From (1), (3) $\Rightarrow (\forall)\,\varepsilon'>0,\ (\exists)\, x_0\in S$ s.t.

$\sup(S)-\varepsilon'<x_0\le\sup(S)\,|\cdot a$

$a\sup(S)-a\varepsilon'<ax_0\le a\sup(S)\,|+b$

$a\sup(S)+b-a\varepsilon'<ax_0+b\le a\sup(S)+b$

Let $\varepsilon'=\dfrac{\varepsilon}{a}\Rightarrow a\sup(S)+b-\varepsilon<ax_0+b\le a\sup(S)+b$

$\overset{(2)}{\Longrightarrow} ax_0+b$ the least upper bound of $M \Rightarrow$

$\Rightarrow \sup(M) = a\sup(S)+b \Rightarrow$

$\Rightarrow \sup(ax_0+b) = a\sup(S)+b$

2) Let $a, b \in \mathbb{R}$

$(\exists)\ U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t.

$$U \cap V = \emptyset$$

$U \in \mathcal{V}(a) \Rightarrow (\exists)\varepsilon_1 > 0$ s.t.

$$U = (a-\varepsilon_1, a+\varepsilon_1) \subseteq \mathcal{V}(a)$$

$V \in \mathcal{V}(b) \Rightarrow (\exists)\varepsilon_2 > 0$ s.t.

$$V = (b-\varepsilon_1, b+\varepsilon_1) \subseteq \mathcal{V}(b)$$

Let $a=-2$ and $b=2$
$\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_2 = \frac{1}{3}$ $\Bigg| \Rightarrow$

$\Rightarrow \begin{cases} U = (-2-\frac{1}{2}, -2+\frac{1}{2}) = (-\frac{5}{2}, -\frac{3}{2}) \\ V = (2-\frac{1}{3}, 2+\frac{1}{3}) = (\frac{5}{3}, \frac{7}{3}) \end{cases} \Bigg| \Rightarrow$

$\Rightarrow U \cap V = \emptyset$

3) Let $A = (0,1) \cap \mathbb{Q}$

$\inf A = 0$, $\sup A = 1$, $\operatorname{int} A = \emptyset$, $\operatorname{cl} A = [0,1]$

$x \in \mathbb{R}$ is the infimum of $A \Leftrightarrow \begin{cases} (\forall) a \in A, x \le a \Rightarrow \\ \Rightarrow x \in lb(A) \\ (\forall) u \in lb(A), x \ge u \end{cases}$

$lb(A) = (-\infty, 0] \Rightarrow (\forall) a \in A, 0 \le a \Rightarrow \inf A = 0$

$x \in \mathbb{R}$ is the supremum of $A = \begin{cases} (\forall) a \in A, x \ge a, x \in ub(A) \\ (\forall) u \in ub(A), x \le u \end{cases}$

$ub(A) = [1, +\infty) \Rightarrow (\forall) a \in A, a \le 1 \Rightarrow \sup(A) = 1$

$\text{int } A = \{x \in \mathbb{R} \mid (\exists) V \in \mathcal{V}(x),\ V \subseteq A\}$

$(\exists) V \in \mathcal{V}(x) \Rightarrow (\exists) \varepsilon > 0$ s.t. $V = (x-\varepsilon, x+\varepsilon) \subseteq A$

How $A = (0,1) \cap \mathbb{Q}$ is a rational set $\Rightarrow$

$\Rightarrow (\nexists) \varepsilon > 0$ s.t. $V \subseteq A \Rightarrow \text{int } A = \emptyset$

$\text{cl } A = \{x \in \mathbb{R} \mid (\exists) V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

How A is a rational set $\Rightarrow (\forall) \varepsilon > 0$ s.t. $V = (x-\varepsilon, x+\varepsilon)$

$V \notin A \Rightarrow \text{cl } A = [0,1]$