Seminar 11 – Base-change matrices and eigen stuff

V, V' – K.v.s.

$B = (v_1, \dots, v_n)$ basis of V

$B' = (v'_1, \dots, v'_m)$ basis of V'

$f : V \to V'$ linear map

$$[f]_{BB'} = \left([f(v_1)]_{B'} \,\middle|\, [f(v_2)]_{B'} \,\middle|\, \dots \,\middle|\, [f(v_n)]_{B'} \right)$$

$$\forall v \in V : [f(v)]_{B'} = [f]_{BB'} [v]_B$$

V - K. v.s

$B_1 = (v_1, \ldots, v_n)$
$B_2 = (w_1, \ldots, w_n)$ } bases of V

T_{B_1,B_2} = base-change matrix from B_1 to B_2
(transfer V)

! $T_{B_1,B_2} = \underbrace{[\overset{\text{identity map}}{id}]}_{B_2,B_1} =$

$$= \left([w_1]_{B_1} \middle| [w_2]_{B_1} \middle| \ldots \middle| [w_n]_{B_1} \right)$$

$$(T_{B_1,B_2})^{-1} = T_{B_2,B_1}$$

$$[id]^{-1}_{B_2,B_1} = [id]_{B_1,B_2}$$

$\forall\ v \in V$:

$$[v]_{B_1} = [id]_{B_2,B_1} \cdot [v]_{B_2} \quad !$$

$$= T_{B_1 B_2} \cdot [v]_{B_2} \quad !$$

V, V', V'' - K. v.r. mit den Basen

B, B', B'', $f: V \to V'$, $g: V' \to V''$

$f_1, f_2 : V \to V'$

- $[f_1 + f_2]_{B, B'} = [f_1]_{B B'} + [f_2]_{B, B'}$!

- $k \in K$

  $[k f]_{B, B'} = k \cdot [f]_{B, B'}$!

- $[g \circ f]_{B B''} = [g]_{B', B''} \cdot [f]_{B, B'}$!

V, V' - K. v.s

B_1, B_2 - bases of V

B_1', B_2' - bases of V'

$$[f]_{B_1', B_2'} = [id]_{B_2, B_2'} \cdot [f]_{B_1, B_2} \cdot [id]_{B_1', B_1}$$

$B_2 \to B_2'$ $\quad$ $B_1 \to B_2$ $\quad$ $B_1' \to B_1$

$$= T'_{B_2, B_2'} \cdot [f]_{B_1, B_2} \cdot T_{B_1, B_1'}$$

11.2. In $\mathbb{R}^2$ we consider the bases

$B = (v_1, v_2) = ((1,2),(1,3))$

$B' = (v_1', v_2') = ((1,0),(2,1))$

$f, g \in \mathrm{End}(\mathbb{R})$

$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

$[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$

Find $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$

$[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_{B'B}$

$[g]_{B'B} = [id]_{B'B} \cdot [g]_{B'} \cdot [id]_{BB'}$

$=$

$$\{id\}_{B'B} = \left(\{(1,0)\}_B \mid \{(2,1)\}_B \right)$$

$$(1,0) = \alpha(1,2) + \beta(1,3)$$

$$\begin{cases} \alpha + \beta = 1 & \cdot 2 \\ 2\alpha + 3\beta = 0 \end{cases}$$

$$\beta = -2 \Rightarrow \alpha = 3$$

$$\{(1,0)\} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$(2,1) = \alpha(1,2) + \beta(1,3)$$

$$\begin{cases} \alpha + \beta = 2 & \cdot 2 \\ 2\alpha + 3\beta = 1 \end{cases}$$

$$\beta = -3, \ \alpha = 5$$

$$[(2,1)]_{B} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$[id]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[id]_{BB'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$(1,2) = \alpha(1,0) + \beta(2,1)$$

$$\begin{cases} \alpha + 2\beta = 1 \\ \beta = 2 \end{cases} \Rightarrow \alpha = -3$$

$$(1,3) = \alpha(1,0) + \beta(2,1)$$

$$\begin{cases} \alpha + 2\beta = 1 \\ \beta = 3 \end{cases} \Rightarrow \alpha = -5$$

$$\Rightarrow [g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -21+25 & -39+35 \\ 14-15 & 26-21 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & - \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & -30 \\ 12 & 15 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'B'} \cdot [g]_{B'B'}$$

$$= [f]_{B'} \cdot [g]_{B'}$$

$$[f]_{B'B'} = [id]_{BB'} [f]_{B} [id]_{B'B}$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B',B'} \cdot [g]_{B',B'}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \begin{pmatrix} -7 & -13 \\ 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -13 \\ -5 & 7 \end{pmatrix}$$

Eigenstuff

V - K. v.s., $f \in End(V)$

Fixed points $= \{ v \in V \mid f(v) = v \}$

Let $\lambda \in K$

$S(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$

If $S(\lambda) \neq 0 \Rightarrow S(\lambda)$ - eigenspace

λ - eigenvalue

The non-zero element of $S(\lambda)$ - eigenvectors

Th : $f \in End(V)$

The eigenvalues of f are the roots of the characteristic polynomial of the matrix of f on any basis B of V

$$P_f(x) = \det\left([f]_B - x \cdot I_n\right)$$

roots of P_f = eigenvalues

Everything exactly the same for matrices, but for $A \in M_n(K)$ we have:

$$P_A(x) = \det(A - x \cdot I_n)$$

Eigenvectors are found by solving the equation

$$f(v) = \lambda v \iff [f]_B [v]_B = \lambda [v]_B$$

5. Compute the eigenvalues and eigenvectors for the matrix:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$$

__Theorem__ (Rational Root Theorem)

$f = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Z}\{x\}$

If f has an integer root k, then $k \mid a_0$

$$P_A(x) = \det(A - x I_3)$$

$$= \det \begin{pmatrix} 3-x & 1 & 0 \\ -4 & -1-x & 0 \\ -4 & -8 & -2-x \end{pmatrix}$$

$$= (3-x)(-1-x)(-2-x) - (8+4x)$$

$$= (-3 - 3x + x + x^2)(-2 - x) - 8 - 4x$$

$$= (x^2 - 2x - 3)(-x - 2) - 8 - 4x$$

$$= -x^3 + \cancel{2x^2} + 3x - \cancel{2x^2} + \cancel{4x} + \cancel{6 - 8} - \cancel{4x}$$

$$-2$$

$$= -x^3 + 3x - 2$$

For $x_1 = -2 \Rightarrow$

	x^3	x^2	x^1	x^0
	-1	0	3	-2
-2	-1	2	-1	0
1	-1	1	0	
1	-1	0		

$x_1 = -2,\ x_2 = 1,\ x_3 = 1$

$\lambda_1 = -2,\ \lambda_2 = 1$

For $\lambda = -2 \Rightarrow$ we want to find $v = (x, y, z)$ s.t.

$$A\{v\}_E = \lambda \{v\}_E$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

$$\begin{pmatrix} 3x + y \\ -4x - y \\ -4x - 8y - 2z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3x + y = -2x \\ -4x - 4 = -2y \\ -4x - 8y - 2z = -2z \end{cases}$$

... we'll have same values

$x = 0$

$y = 0$

$$S(-2) = \{v = (0,0,z) \mid z \in \mathbb{R}\}$$

$$= \langle (0,0,1) \rangle$$

For $S(1) = \ldots$ some thing

Ex 4: 3.15, 3.14