Homework 6

1) a) $\lim\limits_{x \to 0} \dfrac{\sin x - x + \frac{x^3}{6}}{x^5} =$

$= \lim\limits_{x \to 0} \dfrac{\sum\limits_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - x + \frac{x^3}{6}}{x^5}$

$= \lim\limits_{x \to 0} \dfrac{(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \dots) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5}$

$= \lim\limits_{x \to 0} \frac{1}{5!} - \underbrace{\frac{x^2}{7!}}_{\to 0} + \underbrace{\frac{x^4}{9!}}_{\to 0} \underbrace{- \dots}_{\to 0}$

$= \dfrac{1}{120}$

b) $\lim\limits_{x \to 0} \dfrac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4}$

Taylor series: $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(x_0)}{n!} (x - x_0)^n$

$f(x) = e^{x^2} \to f'(x) = 2xe^{x^2}, \quad f'(0) = 0$

$f''(x) = 2e^{x^2} + 4xe^{x^2}, \quad f'(0) = 2$

$f'''(x) = 4xe^{x^2} + 4(2xe^{x^2} + 4x\overset{3}{e}x^2), \quad f'(0) = 0$

$f^{(4)}(x) = 4e^{x^2} + 8x\overset{2}{e}^{x^2} + 4(2e^{x^2} + 24x^4e^{x^2}); \quad f'(0) = 12$

$\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n = 1 + 0 + \dfrac{2 \cdot x^2}{2!} + 0 + \dfrac{12 \cdot x^4}{4!} + \dots$

$= 1 + x^2 + \dfrac{x^4}{2} + \dfrac{x^6}{6} \dots$

$= 1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dots$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^{x^2} - \cos x = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$= \cancel{1} + \overset{2/}{x^2} + \overset{12}{\frac{x^4}{2!}} + \overset{120}{\frac{x^6}{3!}} + \dots - \cancel{1} + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

$$= \frac{3x^2}{2} + \frac{11x^4}{4!} + \frac{127x^6}{6!} + \dots$$

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x - \frac{3x^2}{2}}{x^4} = \frac{\cancel{\frac{3x^2}{2}} + \frac{11x^4}{4!} + \frac{127x^6}{6!} + \dots - \cancel{\frac{3x^2}{2}}}{x^4} =$$

$$= \lim_{x \to 0} \frac{11}{4!} + \underbrace{\frac{127x^2}{6!} + \dots}_{\to 0} = \frac{11}{24}$$

2. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

$$f(x) = \ln(1+x) \Rightarrow f'(x) = \frac{1}{1+x}, \; f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2}, \; f''(0) = -1$$

$$f^{(3)}(x) = \frac{2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3}, \; f^{(3)}(0) = 2$$

$$f^{(4)}(x) = -\frac{2 \cdot 3\,(1+x)^2}{(1+x)^6} = -\frac{3!}{(1+x)^4}, \; f^{(4)}(0) = -3!$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}, \; f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

Taylor series: $\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n =$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot x^n$$

3. prove that: $\frac{f(x+h)-f(x)}{h}$ approximates $f'(x)$ with error of order h (first order approximation)

$$f'(x) = \frac{f(x+h)-f(x)}{h} + O(h)$$

Oh means $e_1(h) = O(e_2(h))$

$$\lim_{h \to 0} \frac{e_1(h)}{e_2(h)} \in (0, \infty)$$

$$f(x) = T_m(x) + \frac{f^{(m+1)}(c)}{(m+1)!}(x-x_0)^{m+1}$$

$$f(x+h) = f(x) + f'(x)(x-x_0) + \frac{f''(c)}{2!}(x-x_0)^2$$

h - error of order $\Rightarrow x - x_0 = h$

$$f(x+h) = f(x) + h f'(x) + h^2 \frac{f''(c)}{2}$$

$$f(x+h) - f(x) - h f'(x) = h^2 \frac{f''(c)}{2}$$

$$h\left(\frac{f(x+h)-f(x)}{h} - f'(x)\right) = h^2 \frac{f''(c)}{2}$$

$$\frac{f(x+h)-f(x)}{h} - f'(x) = h \frac{f''(c)}{2} \Rightarrow$$

$$\Rightarrow f'(x) = \frac{f(x+h)-f(x)}{h} - \underbrace{h \frac{f''(c)}{2}}_{\to O(h)}$$

$\Rightarrow$ error is proportional to $h^1 \Rightarrow \frac{f(x+h)-f(x)}{h}$ is a first order approximation

$\frac{f(x+h)-f(x-h)}{2}$ approximates $f'(x)$ with an error of order h^2 (second order approximation)

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} + O(h^2)$$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2 f''(\cancel{x})}{2!} + \frac{h^3 f^{(3)}(c_1)}{3!}$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2 f''(x)}{2!} - \frac{h^3 f^{(3)}(c_2)}{3!}$$

$$f(x+h) - f(x-h) = \cancel{f(x)} + h f'(x) + \cancel{\frac{h^2 f''(x)}{2!}} + \frac{h^3 f^{(3)}(c_1)}{3!} -$$

$$- \cancel{f(x)} + h f'(x) - \cancel{\frac{h^2 f''(x)}{2!}} + \frac{h^3 f^{(3)}(c_2)}{3!}$$

$$f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3 f^{(3)}(c_1)}{3!} + \frac{h^3 f^{(3)}(c_2)}{3!}$$

$$2h f'(x) = f(x+h) - f(x-h) - \frac{h^3 f^{(3)}(c_1)}{3!} + \frac{h^3 f^{(3)}(c_2)}{3!} \quad \Big| : 2h$$

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{(h^2 f^{(3)}(c_1) + h^2 f^{(3)}(c_2))}{2 \cdot 3!}$$

$$= O(h^2)$$

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} + O(h^2)$$

$\Rightarrow$ error is proportional to $h^2 \Rightarrow$ $\frac{f(x+h)-f(x-h)}{2h}$ is the second order approximation