Seminar 8

$H_1: (\forall x)(\forall y)(k(x) \wedge os(v) \rightarrow k(y))$

$H_2: (\forall x)(\forall y)(k(x) \wedge d(x, y) \rightarrow k(y))$

$H_3: k(R_3)$

$H_4: d(H_7, R_3)$

$H_5: os(H_7, H_8)$

$C: k(H_8)$

$H_1 \xmapsto[x:R_3]{\text{univ. i.}} (\forall y)(k(R_3) \wedge d(y, R_3) \rightarrow k(y)) = H_6$

$H_6 \xmapsto[y:H_7]{\text{univ. inst.}} k(R_3) \wedge d(H_7, R_3) \rightarrow k(H_7) = H_7$

$$H_3, H_4 \vdash_{\text{conj}} h(R_3) \wedge d(H_7, R_3) = H_8$$

$$H_8, H_7 \vdash_{\text{mp}} h(H_7)$$

$$H_1 \vdash_{\text{univ inst}} (\forall y)(h(H_7) \wedge OS(H_7, y) \to K(y)) = H_9$$

$$H_9 \vdash_{\text{univ. inst.}} (h(H_7) \wedge OS(H_7, H_8) \to K(H_8)) = H_{11}$$

$$H_{10}, H_5 \vdash_{\text{conj}} K(H_8)$$

$$\ldots K(H_8)$$

$U = (\exists x) A(x) \wedge (\exists x) B(x) \to (\forall x)[A(x) \wedge B(x)$

$v'(U) = v'((\exists x) A(x) \wedge (\exists x) B(x)) \to$
$v'((\forall x)\{A(x) \wedge B(x)\})$

$v'((\exists x) A(x)) \wedge v'((\exists x) B(x)) \to v'((\forall x)(A(x)$
$\wedge B(x))$

$= (\exists x)_{x \in D} (x \text{ owns a car}) \wedge (\exists x)(x \text{ has a d.l.}) \to$
$(\forall x)_{x \in D} (x \text{ has both a car and a d.l.})$

$I = \langle D, m \rangle$
D = the set of all ppl from Romania

$m(A) : D \to \{T, F\}$, $m(A)(x)$, "x has a car"

$m(B) : D \to \{T, F\}$, $m(B)(x)$, "x has a driver's license"

$T \wedge T \to F : T \to F = F$

4.1 $(\forall x)(\forall y)(\overset{x:=2}{A(x)} \wedge \overset{y:=2}{A(y)} \to \neg A(\overset{x+y=2}{\mathrm{sum}(x,y)}) \wedge (\forall z) A(\overset{y \cdot z \neq 2}{\mathrm{prod}(y,z)}))$

A : "x is an odd number

5) $(\forall x)(A(x) \to B(x)) \to ((\exists x) A(x) \to (\exists x) B(x))$

$I = \langle \{2, 5\}, m \rangle$, where
$m(A) : \{2, 5\} \to \{T, F\}$, $m(A)(x)$: "$x = 2$"

since this needs to be True, we choose an interpretation that is False

$m(B): \{2, 5\} \to \{T, F\}$, $m(B)(x)$: "$x : 5$"

$$V^I((\forall x)(A(x) \to B(x)) \to (V^I((\exists x) A(x)) \to V^I((\exists x) B(x))) =$$

$$= (2:2 \to 5:2) \wedge (5:2 \to 5:5) \to (2:2 \vee 5:2 \to 2:5 \vee 5:5) =$$

$$= (T \to F) \wedge (F \to T) \to (T \vee F \to F \vee T) =$$

$$= F \wedge T \to (T \to T) =$$

$$= F \to T = T$$

6) $$U = \overbrace{(\exists x)P(x) \wedge (\exists x)Q(x)}^{1^{st}\text{ part}} \longrightarrow \underbrace{(\forall x)(P(x) \wedge Q(x))}_{2^{nd}\text{ part}}$$

$\mathcal{I} = \langle D, m \rangle$, where $D = \{4, 6\}$

$m(P): D \to \{T, F\}$, $m(P)(x) =$ "$x : 2$"

$m(Q): D \to \{T, F\}$, $m(Q)(x) =$ "$x : 3$"

- we need an ~~cntr~~ model $\Rightarrow$ 1^{st} part = True
2^{nd} part = F

$$V^{\mathcal{I}} = V^{\mathcal{I}}((\exists x)P(x)) \wedge V^{\mathcal{I}}((\exists x)Q(x)) \to V^{\mathcal{I}}((\forall x)P(x) \wedge Q(x))$$

$$= \left(\overset{4:2}{P(4)} \vee \overset{6:2}{P(6)}\right) \wedge \left(\overset{4:3}{Q(4)} \vee \overset{6:3}{Q(6)}\right)$$

$$\rightarrow \left(\left(\overset{4:2}{P(4)} \wedge \overset{4:3}{Q(4)}\right) \wedge \left(\overset{6:2}{P(6)} \wedge \overset{6:3}{Q(6)}\right)\right)$$

$$= (T \vee T) \wedge (F \vee T)$$

$$\rightarrow ((T \wedge F) \wedge (T \wedge T))$$

$$= T \wedge T \rightarrow (F \wedge T)$$

$$= T \rightarrow F = F$$

Semantic Tableaux (ST)

Ex 1: Using the ST method decide what kind of formula U_1 is.
If U_1 - consistent find all the models

$$(q \wedge r \to p) \to (p \to r) \wedge q$$

α Rules

$$\begin{array}{ccc} A \wedge B & \neg(A \vee B) & \neg(A \to B) \\ | & | & | \\ A & \neg A & A \\ | & | & | \\ B & \neg P & \neg B \end{array}$$

β rules

$$A \lor B \;\Rightarrow\; A \;\big|\; B$$

$$\neg(A \land B) \;\Rightarrow\; \neg A \;\big|\; \neg B$$

$$A \to B \;\Rightarrow\; \neg A \;\big|\; B$$

$$(q \land r \to p) \to (p \to r) \land q \quad (1)$$

β for 1

$\neg(q \land r \to p)$ (2)	$(p \to r) \land q$ (3)
α for (2)	
$q \land r$ (4)	$p \to r$
$\neg p$ (5)	q
α for 4	$\neg p$ | r
q	$\odot$ | $\odot$
r	
$\odot$	

ABSORPTION $(U \vee (U \wedge V) = U)$

$$\Rightarrow DNF(V_7) = (\neg p \wedge q \wedge r) \vee (q \wedge \neg p) \vee (q \wedge r)$$

$$\equiv \underbrace{(q \wedge \neg p)}_{T} \vee \underbrace{(q \vee r)}_{T}$$

	P	Q	R
i_1	F	T	T
i_2	F	T	F
i_3	T	T	T
i_4	F	T	T

the same

Ex 2: Prove V_7 is a tautology using the semantic tableaux method

$$V_7 = (p \to (q \to r)) \to (q \to (p \to r))$$

$$\neg V_7 = \neg((p \to (q \to r)) \to (q \to (p \to r)))$$

$$= \neg(p \to (q \to r)) \to \neg(q \to (p \to r))$$

| α rule

$p \to (q \to r)$ (1)

$\neg(q \to (p \to r))$ (2)

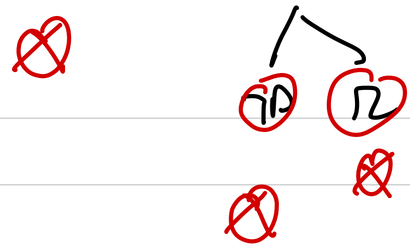
q

$\neg(p \to r)$

p

$\neg r$

$\neg p$ $\qquad$ $p \to r$



⇒ it is a closed table ⇒

- $\neg U$ · closed ⇒ U - tautology

Ex 3: Decide whether the logical consequence holds

$$p \wedge (q \to r),\ q \vee r \models p \to (q \to r)$$

$$(p \wedge (q \to r)) \wedge (q \vee r) \wedge \neg(p \to (q \to r))$$

$$U_1, U_2, U_3, \dots, U_n \models V$$

iff

$U_1 \wedge U_2 \wedge \dots \wedge U_n \wedge \neg V$ has a closed semantic t.

$(p \land (q \to r)) \land (q \lor r) \land \neg(p \to (q \to r))$ (1)

α für 1

$p \land (q \to r)$ (2)

$q \lor r$ (3)

p (4)

$\neg(q \to r)$ (5)

α für 2

p

$q \to r$

α für 5

q

$\neg r \qquad r$

$q \qquad r$

... all branches closed $\Rightarrow$ logical consequence

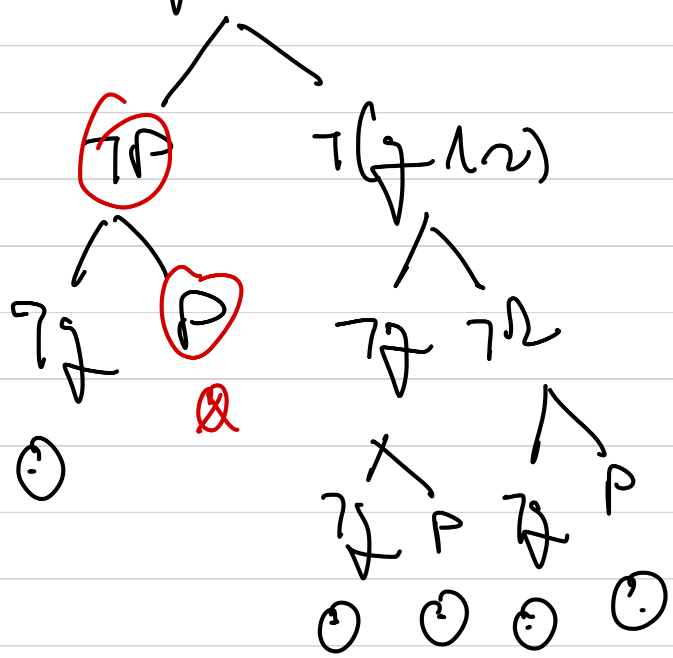
Find the anti-models

$U_7: \neg p \vee \neg(q \wedge p) \to q \wedge \neg p$

$\neg U_7: \neg(\neg p \vee \neg(q \wedge p) \to q \wedge \neg p))$ (1)

$\neg p \vee \neg(q \wedge p)$ (2)

$\neg(q \wedge \neg p)$ (3)



$$DNF(\neg U_7) : (\neg q \wedge \neg p) \vee \overset{F}{(p \wedge \neg p)} \vee (\neg q \wedge \neg q) \vee (p \wedge \neg q) \vee (\neg q \wedge \neg r) \vee (p \wedge \neg r)$$

$$\equiv \cancel{(\neg q \wedge \neg p)} \vee (\neg q) \vee \cancel{(p \wedge \neg q)} \vee \underbrace{\cancel{(\neg q \wedge \neg r)}}_{\text{absorption}} \vee (p \wedge \neg r)$$

$$\equiv \neg q \vee (p \wedge \neg r)$$

Cube $\neg q$: $T \Rightarrow F$

models of $\neg U_7$

$\Rightarrow$ antimodels of U_7

	p	q	r
i_1	T	F	T
i_2	T	F	F
i_3	F	F	T
i_4	F	F	F

Cube : $p \wedge \neg r = T \Rightarrow$

$\Rightarrow$ $p = T$

$\neg r = T \Rightarrow r = F$

	p	q	r
i_5	T	T	F
i_6	T	F	F

$\equiv$