Seminar 8 - Linear systems

$$(S)\ \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\underbrace{A}_{\substack{m\times n \\ \text{matrix}}} \cdot \overbrace{X}^{n\times 1 \text{ vector}} = \underbrace{b}_{\substack{m\times 1 \\ \text{mat}}}$$

$$x_1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Kronecker-Capelli-Rouché Theorem

System S compatible $\Leftrightarrow$

$\Leftrightarrow$ rank A = rank $\bar{A}$

$$\bar{A} = \left(A \mid b \right)$$

$\Leftrightarrow$ all of the characteristic minors are 0

8.2. (1) $\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix}$$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right)$$

Determine rank A

<u>Minor of rank 3</u>

$$d_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = 1 - 6 - 4 - (2 + 2 + 6) = 1 - 10 - 10 = -19 \neq 0$$

$\Rightarrow$ rank $A = 3$

$$\left.\begin{array}{l} \left.\begin{array}{l} \text{rank } \bar{A} \leq 3 \\ \text{rank } \bar{A} \geq \text{rank } A \end{array}\right\} \Rightarrow \text{rank } \bar{A} = 3 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ system is compatible

Let $x_4 = t$

$$\begin{cases} x_1 + x_2 + x_3 = 5 + 2t & (1) \\ 2x_1 + x_2 - 2x_3 = 1 - t & (2) \\ 2x_1 - 3x_2 + x_3 = 3 - 2t & (3) \end{cases}$$

$$x_1 = \frac{\Delta x_1}{\Delta}$$

$$\Delta x_1 = \begin{vmatrix} 5+2t & 1 & 1 \\ 1-t & 1 & -2 \\ 3-2t & -3 & 1 \end{vmatrix} =$$

$$= 5+2t - 3 + 3t - 6 + 4t$$

$$- (3 - 2t + 30 + 12t + 1 - t)$$

$$= -4 \cancel{+ 5t} - 34 \cancel{- 5t}$$

$= -30$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -15$$

$$x_1 = \frac{-30}{-15} = 2$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta}$$

$$x_3 = \frac{\Delta_{x_3}}{\Delta}$$

Gaussian elimination (echelon transformation)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -2 & 1 & 2 & 3 \end{array}\right) \sim$$

$L_2 = L_2 - 2L_1$ $\sim$ $L_3 = L_3 - 2L_1$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & -5 & -1 & 6 & -7 \end{array}\right) \sim$$

$L_3 = L_3 - 5L_2$ $\sim$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & 0 & 19 & -19 & 38 \end{array}\right)$$

row echelon form

Gauss method + revert to the system and solve it manually

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ -x_2 - x_3 + 5x_4 = -5 \\ 13x_3 - 15x_4 = 30 \end{cases}$$

Gauss - Jordan method

Continue by making 0's above the pivots

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -2 & 4 & -5 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 3 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$$

$x_4 = t$

$$\begin{cases} x_1 = 2 \\ x_2 = 1 + t \\ x_3 = 2 + t \\ x_4 = t \end{cases}$$

5. 1) $\begin{cases} 2x + 2y + 2z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

$$\overline{A} = \left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

Applying the gauss. method

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \sim$$

L_1 and L_2 can be swapped (to have 1 as pivot)

$L_2 = L_2 - \frac{1}{2} L_1$

$L_3 = L_3 + \frac{1}{2} L_1$

$$\sim \left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 0 & -2 & -1 & -\frac{1}{2} \\ 0 & 3 & 2 & \frac{7}{2} \end{array}\right)$$

$L_3 = L_3 - \left(-\frac{3}{2}\right) L_2$

$$\sim \left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 0 & -2 & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{11}{4} \end{array}\right)$$

Applying it further

$$\underset{\sim}{L_2} = L_2 + 2L_3$$

$$L_1 = L_1 - 4L_3$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & -8 \\ 0 & -2 & 0 & 5 \\ 0 & 0 & \frac{1}{2} & \frac{11}{4} \end{array}\right) \sim$$

$$L_1 = L_1 + L_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 0 & -3 \\ 0 & -2 & 0 & 5 \\ 0 & 0 & \frac{1}{2} & \frac{11}{4} \end{array}\right)$$

$$\begin{cases} 2x_1 = -3 \\ -2x_2 = 5 \\ \frac{1}{2}x_3 = \frac{11}{4} \end{cases} \iff \begin{cases} x_1 = -\frac{3}{2} \\ x_2 = -\frac{5}{2} \\ x_3 = \frac{11}{2} \end{cases}$$

Verification

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \quad L_1 \; L_2 \; \longleftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 2 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \quad \begin{array}{l} \underbrace{L_2 = L_2 - 2L_1} \\ L_3 = L_3 + L_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 2 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right) \quad L_2 \; L_3 \; \longleftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 2 & 1 \end{array}\right) \quad L_3 = L_3 - 4L_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ & & & 3 \\ & & & -11 \end{array}\right.$$

... I arrive at the same answers

5) 3)
$$\begin{cases} x+y+z = 3 \\ x-y+z = 1 \\ 2x-y+2z = 3 \end{cases}$$

$y+z = 4$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right)$$

$\sim$

$L_2 = L_2 - L_1$

$L_3 = L_3 - 2L_1$

$L_4 = L_4 - L_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

$L_2 \leftrightarrow L_4$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & -3 & 0 & -3 \\ 0 & -2 & 0 & -2 \end{array}\right)$$

$\sim$

$L_3 = L_3 - 3L_2$

$L_4 = L_4 - 2L_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

We notice we'd have $0 \cdot x + 0 \cdot y + 0 \cdot z = -6$

impossible $\Rightarrow$ incompatible system

(or incompatibility rule)

made up by Tudor Micu

8.6.

$$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \in \mathbb{R} \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right)$$

$$L_1 \ L_2 \atop \longleftrightarrow \quad \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right)$$

$L_2 = L_2 - 2L_1$

$\sim$

$L_3 = L_3 - L_1$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & -3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda - 2 \end{array}\right)$$

... I will get : $\begin{cases} \text{if } \lambda \neq 5 \Rightarrow \text{sys. inc.} \\ \text{if } \lambda = 5 \Rightarrow \\ \Rightarrow \begin{cases} \boxed{x_1 + 2x_2} - x_3 + 4x_4 = 2 \\ \boxed{-3x_2} + 3x_3 - 7x_4 = - \end{cases} \end{cases}$

Let $x_3 = \alpha$, $x_4 = \beta$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 2 + \alpha - 4\beta \Rightarrow x_1 = 2 + \alpha - 4\beta - \dfrac{-6 - 6\alpha + 14\beta}{-3} \\ x_2 = \dfrac{-3 - 3\alpha + 7\beta}{-3} \end{cases}$$

$$-3x_1 = \cancel{-6} - 3\alpha + 12\beta$$
$$\cancel{+6} + 6\alpha - 14\beta$$
$$= 3\alpha - 2\beta$$

$$\Rightarrow x_1 = 2\alpha + \frac{2}{3}\beta$$

8.7) $\left\{\begin{array}{l} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a \end{array}\right.$, $a \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right)$$

$$\xleftrightarrow{L_1 \, L_2} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right)$$

$L_2 = L_2 - aL_1$

$\sim$

$L_3 = L_3 - L_1$

$$\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a \\ 0 & 1-a & a-1 & a^2-a \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & (1-a)(1+a) & 1-a & 1-a \\ 0 & 1-a & -(1-a) & -a(1-a) \end{array}\right)$$

If $\alpha = 1 \Rightarrow 1 - \alpha = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x + y + z = 1 \\ y = \gamma \\ z = \beta \end{cases} \Rightarrow x = 1 - \gamma - \beta$$

For $\alpha \neq 1$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & \alpha & 1 & \alpha \\ 0 & (1-\alpha)(1+\alpha) & 1-\alpha & (1+\alpha)(1-\alpha) \\ 0 & 1-\alpha & -(1-\alpha) & -\alpha(1-\alpha) \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1+a & 1 & 1+a \\ 0 & 1 & -1 & -a \end{array}\right)$$

$$L_2 \Leftrightarrow L_3 \quad \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 1+a & 1 & 1+a \end{array}\right)$$

$$L_3 = L_3 - (1+a) L_2 \quad \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 0 & a+2 & 1+2a+a^2 \end{array}\right)$$

if $a = -2 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right)$ impossible if $a \neq -2$

$$\begin{cases} x_1 + a x_2 + x_3 = a & (1) \\ x_2 - x_3 = -a \iff x_2 = \dfrac{(a+1)^2}{a+2} - a = \\ (a+2) x_3 = (a+1)^2 \Rightarrow x_3 = \dfrac{(a+1)^2}{a+2} \end{cases}$$

$$x_2 = \frac{a^2+2a+1 - a^2 - 2a}{a+2} = \frac{1}{a+2}$$

$$(1) \iff x_1 = a - \frac{a}{a+2} - \frac{(a+1)^2}{a+2}$$

$$= \frac{\cancel{a^2} + \cancel{2a} - a - \cancel{a^2} - \cancel{2a} - 1}{a+2}$$

$$x_1 = -\frac{a+1}{a+2}$$