Homework 7 - Math Analysis

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Statement

The integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

represents the area under the bell curve $y = e^{-x^2}$, and it is related to the normal (Gaussian) probability distribution. It is essential in probability theory and has a wide range of applications.

Considering intervals of the form $[-a, a]$, for increasing values of $a > 0$, show numerically (e.g., trapezium rule) that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Solution

The following Python code numerically approximates the integral using the trapezoidal rule for intervals $[-a, a]$, with increasing values of $a > 0$.

```python
import numpy as np

"""
Considering intervals of the form [ a , a], for
increasing values of a > 0, show numerically (e.g. trapezium rule) that
integral (-inf, inf) exp(-x^2) dx = sqrt(pi)
"""

# Define the function
def f(x):
    return np.exp(-x ** 2)

# Define the trapezoidal rule
def trapez_rule(func, a, b, n):
    h = (b - a) / n
    s = func(a) + func(b)

    for i in range(1, n):
        s += 2 * func(a + i * h)

    return (h / 2) * s

for a in range(0, 20):
    result = trapez_rule(f, -a, a, 1000)
    print(f"For the interval [-{a}, {a}], integral value is ~ {result}")
```

Homework 7

1)

b) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\cdots\sin\frac{(n-1)\pi}{2n}} = L$

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\left(\ln\sin\frac{\pi}{2n} + \ln\sin\frac{2\pi}{2n} + \dots + \ln\sin\frac{(n-1)\pi}{2n}\right) =$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\ln\sin\frac{k\pi}{2n} = \lim_{n\to\infty}\frac{1}{n}\Big(\sum_{k=1}^{n}\ln\sin\frac{k\pi}{2n} - \underbrace{\ln\sin\frac{n\pi}{2n}}_{\ln 1 = 0}\Big)$$

$$= \lim_{n\to\infty}\frac{1}{n}\Big(\underbrace{\sum_{k=1}^{n}\ln\sin\frac{k\pi}{2n}}_{f(x)=\ln\sin \underbrace{x}_{\frac{k\pi}{2n}}}\Big) = \int_0^{\frac{\pi}{2}}\ln\sin x\,dx$$

$$J = \int_0^{\frac{\pi}{2}}\ln(\sin x)dx = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\ln\sin x\,dx + \int_0^{\frac{\pi}{2}}\ln\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx\right)$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\ln\sin x\,dx + \int_0^{\frac{\pi}{2}}\ln\cos x\,dx\right)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\sin x\cos x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\frac{1}{2}\cdot 2\sin x\cos x\,dx$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\ln\sin 2x\,dx + \int_0^{\frac{\pi}{2}}\ln\frac{1}{2}\right)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\sin 2x\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}}\ln 2$$

let $2x = u$

$$= \frac{1}{2\cdot 2}\int_0^{\pi}\ln(\sin u)\,du - \frac{\pi}{4}\ln 2$$

$$J = \frac{J}{2} - \frac{\pi}{4}\ln 2 \Rightarrow J = -\frac{\pi}{2}\ln 2$$

a) $\lim_{n\to\infty} \frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\ldots+n\sqrt[n]{e^n}}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\sqrt[n]{e^k}}{n^2} =$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k\sqrt[n]{e^k}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \underbrace{\frac{k\cdot e^{\frac{k}{n}}}{n}}_{} = \int_0^1 xe^x\,dx$

$f\left(\frac{k}{n}\right) \Rightarrow f(x) = x e^x$

$= xe^x\Big|_0^1 - \int_0^1 e^x\,dx$

$= e - e^x\Big|_0^1$

$= e - e + 1 = 1$

2. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx \quad , \alpha > 0$

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

and that

$\Gamma(n) = (n-1)! \quad , n \in \mathbb{N}^*$

$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}\,dx = -x^\alpha e^{-x}\Big|_0^\infty + \alpha\int_0^\infty x^{\alpha-1} e^{-x}\,dx$

$f(x) = x^\alpha \Rightarrow f'(x) = \alpha x^{\alpha-1}$

$g'(x) = e^{-x} \Rightarrow g(x) = -e^{-x}$

$= \lim_{x\to\infty} -x^\alpha e^{-x} + \alpha\Gamma(\alpha) = \lim_{x\to\infty} \frac{-x^\alpha}{e^x} + \alpha\Gamma(\alpha) \overset{l'H}{=}$

$= \lim_{x\to\infty} \frac{\alpha!}{e^x} + \alpha\Gamma(\alpha) = \alpha\Gamma(\alpha)$

$\Rightarrow \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\Gamma(n) = (n-1)!$

$\Gamma(n) = (n-1)\Gamma(n-1)$ (as we saw)

$= (n-1)(n-2)\,\Gamma(n-2)$

$= \ldots$

$= (n-1)(n-2)\ldots 1 \cdot \int_0^\infty x^0 e^{-x}\,dx$

$= (n-1)(n-2)\ldots \int_0^\infty e^{-x}\,dx$

$= (n-1)(n-2)\ldots 2 \cdot 1 \cdot \left(-\frac{1}{e^x}\right)\Big|_0^\infty$

$= (n-1)(n-2)\ldots 2 \cdot 1$

$\Gamma(n) = (n-1)!$