Seminar 9

Applications of Gaussian elimination

1. Solving linear systems
2. finding the rank of a matrix
3. inverting a matrix
4. extracting a basis from a system of generators

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2. $\forall A, B \in \mathcal{M}_{m,n}(K)$

if $A \sim B$, then $\operatorname{rank} A = \operatorname{rank} B$

So $\forall A \in \mathcal{M}_{m,n}(K) \Rightarrow$

$\Rightarrow$ rank A = rank (echelon (A)) =

= # non-zero rows in the echelon form

3.1. Compute by applying elementary transformations the rank of the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_3, L_1}{\Longleftrightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\underset{\substack{L_2 -= 2L_1 \\ L_4 -= 2L_1}}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_3 -= L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underset{\substack{L_3 -= L_4 \\ L_2 -= \frac{1}{2} L_4 \\ L_1 -= \frac{1}{2} L_4}}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_1 -= \frac{1}{2} L_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rank} A = 3$$

3) $$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} , \alpha, \beta \in \mathbb{R}$$

$$\underset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{\substack{L_2 - \beta L_1 \\ L_3 - 2L_1}}{\sim}$$

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \underset{L_2 \leftrightarrow L_3}{\sim}$$

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$$

if $\alpha = 0 \Rightarrow$

$$B \sim \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix}$$

$$\underset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

若 $\alpha \neq 0$：

$$B \sim \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -2/\alpha & 1/\alpha \\ 0 & 0 & \gamma & d \end{pmatrix}$$

$$\gamma = \left(3 - 3\beta - 1 - \beta\alpha\right)\frac{2}{\alpha}$$

$d = 4-3\beta$ ……

$\mathrm{rank}(A) = 2$ iff $\gamma = \delta = 0$

$\Leftrightarrow \cdots \cdots \Rightarrow$

$$\begin{cases} 2\alpha - 3\alpha\beta + 2 - 2\alpha\beta = 0 \\ 4\alpha - 3\alpha\beta + \alpha\beta - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3\alpha - 5\alpha\beta = -2 \\ 4\alpha - 2\alpha\beta = 1 \end{cases}$$

$\cdots$

③ Inverting a matrix

$A \in \mathcal{M}_n(\mathbb{K})$

$$\left(A \mid I_n \right) \overset{\text{Gaussian-Jordan}}{\sim} \dots \sim \left(I \mid A^{-1} \right)$$

5. Compute the inverse by identifying transformation

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{L_2 -= 2L_1 \\ L_3 -= 3L_1}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$\underset{L_2 \cdot -\frac{1}{5}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$L_3 += 12L_2 \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right)$$

$L_3 \cdot 5$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$L_2 \mathrel{-}= \frac{3}{5} L_3$

$L_1 \mathrel{-}= 2 L_3$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -\frac{38}{5} & -10 & \\ 0 & 1 & 0 & ; & -3 & \\ 0 & 0 & 1 & 9 & (-12) & 5 \end{array}\right)$$

$L_2 \mathrel{-}= 4 L_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

(4) $V = K^n$

$v_1, \dots, v_m \in V$

then if echelon $\left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}\right) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

then $B = (w_1, \dots, w_r)$ is a basis for $\langle v_1, \dots, v_m \rangle$

3. Determine the dimension of the subspaces $S, T, S+T, S \cap T$ and find basis for all

$S = \langle (1,0,4), (2,1,0), (1,1,-4) \rangle$

$T = \langle (-3,-2,4), (5,2,4), (-2,0,-8) \rangle$

Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$

$B_S = \{ (1,0,4), (0,1,-8) \}$

$$B_T = \{\ldots\} = \begin{pmatrix} 3 & -2 & 4 \\ 0 & -1 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim T = 2 \Rightarrow B_T = \{(3,-2,4),(0,-1,8)\}$$

$$B_{S+T} = \{(1,0,4),(0,1,-8),(-3,-2,4),(0,-1,8)\}$$

$$= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ -3 & -2 & 4 \\ 0 & -1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & -2 & 16 \\ 0 & -1 & 8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{S+T} = \{(1,0,4),(0,1,-8)\}$$

$$\left.\begin{array}{l} \dim S+T = 2 \\ \dim S = 2 \\ \dim T = 2 \end{array}\right| \Rightarrow S = T$$