Seminar 10 - Linear maps and matrices

V, V' - K v.s.

$f : V \to V'$ - linear map

$\hookrightarrow \forall v_1, v_2 \in V$: $f(v_1+v_2) = f(v_1)+f(v_2)$

$f(kv) = k\, f(v)$

$B = (v_1, \dots, v_n)$ basis of V

$B' = (v_1', \dots, v_n')$ basis of V'

$$
! \quad [f]_{BB'} = \begin{array}{c} \text{column} \\ \left([f(v_1)]_{B'} \,\middle|\, \cdots \,\middle|\, [f(v_n)]_{B'} \right) \end{array}
$$

10.2) $f \in Hom(\mathbb{R}^3, \mathbb{R}^2)$

$f(x,y,z) = (y, -x)$

$B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$

$B' = (v_1', v_2') = ((1,1), (1,-2))$

$E' = (e_1', e_2') = ((1,0), (0,1))$

Determine $[f]_{BE'}$ and $[f]_{BB'}$

$$[f]_{BE'} = \left([f(v_1)]_{E'} \mid [f(v_2)]_{E'} \mid [f(v_3)]_{E'} \right)$$

$$[f(v_1)]_{E'} = [(1,-1)] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$[f(v_2)]_{E'} = [(1,0)] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[f(v_3)]_{E'} = [(0,-1)] = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$[f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$[f]_{BB'} = \begin{pmatrix} \ldots \end{pmatrix}$$

$$[f(v_1)]_{B'} = [(1,-1)]_{B'} = \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \end{bmatrix}$$

$$\begin{cases} 1 = a + b \\ -1 = a - 2b \end{cases}$$

$$2 = 3b \Rightarrow b = \frac{2}{3}$$

$$a = 1 - \frac{2}{3} = \frac{1}{3}$$

$$[f(v_2)]_{B'} = [(1,0)]_{B'} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \text{ where}$$

$$(1,0) = \alpha \cdot (1,1) + \beta \cdot (1,-2)$$

$$= (\alpha + \beta \,,\, \alpha - 2\beta)$$

$$\Rightarrow \begin{cases} 1 = \alpha + \beta \Leftrightarrow \alpha = 1 - \beta \\ 0 = \alpha - 2\beta \Leftrightarrow 1 - 3\beta = 0 \end{cases}$$

$$\Rightarrow \beta = \frac{1}{3}$$

$$\alpha = \frac{2}{3}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\left[(0,-1)\right] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{cases} 0 = \alpha + \beta \Rightarrow \alpha = -\beta \Rightarrow \alpha = -\frac{1}{3} \\ -1 = \alpha - 2\beta \Rightarrow \beta = \frac{1}{3} \end{cases}$$

$$[f]_{B,B'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

10.5.) $\mathbb{R}_2[x] = \{ f \in \mathbb{R}[x] \mid \deg f \leq 2 \}$

$E = (1, x, x^2)$

$B = (1, x-1, x^2+1)$

bases of $\mathbb{R}_2[x]$

Consider $\varphi \in \mathrm{End}(\mathbb{R}_2[x])$

Determine $[\varphi]_E$, $[\varphi]_B$

$$\varphi(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1) + (a_1 + a_2)x + (a_0 + a_2)x^2$$

$$[\varphi]_E = \left([\varphi(1)]_E \,\middle|\, [\varphi(x)]_E \,\middle|\, [\varphi(x^2)]_E \right)$$

$$\varphi(1) = 1 + x^2 \Rightarrow [\varphi(1)]_E = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\varphi(x) = 1 + x \rightarrow [\varphi(x)] = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\varphi(x^2) = x + x^2 = [\varphi(x^2)]_E = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$[\varphi]_B = \left([\varphi(1)]_B \,\middle|\, [\varphi(x-1)]_B \,\middle|\, [\varphi(x^2+1)]_B \right)$$

$$\varphi(1) = 1 + x^2 \Rightarrow [\varphi(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\varphi(x-1) = x - x^2 \Rightarrow [\varphi(x-1)] = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\varphi(x^2+1) = 1 \overset{x}{+} 2x^2 \to [\varphi(x^2+1)] = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\to \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

V, V' – K.v.s.,

B, B' – bases

$\forall\, v \in V$:

$$\boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$$!

10.4. $f \in \mathrm{End}(\mathbb{R}^4)$

with the following matrix in the canonical basis:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 7 \end{pmatrix}$$

i) Show that $v = (1, 4, 1, -1) \in \operatorname{Ker} f$ and
$v' = (2, -2, 4, 2) \in \operatorname{Im} f$

ii) Determine a basis and dimension for $\operatorname{Ker} f$
and $\operatorname{Im} f$

iii) Determine f

i) $v = (1, 4, 1, -1) \in \operatorname{Ker} f \Rightarrow$

$v \in \operatorname{Ker} f \Leftrightarrow f(v) = 0 \Rightarrow$

$$\Rightarrow [f(v)]_E = 0 \Leftrightarrow$$

$$\Leftrightarrow [f(v)]_E = [f]_E \cdot [v]_E = 0$$

$$[f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v \in \operatorname{Ker} f$$

$$v' = (2, -2, 4, 2) \in \operatorname{Im} f \iff \exists u \in \mathbb{R}^4 : f(u) = v'$$

$$\iff \exists u \in \mathbb{R}^4 : [f(u)]_E = [v']_E \iff$$

$$\iff \exists u \in \mathbb{R}^4 : [f]_E \cdot [u]_E = [v']_E$$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\iff \begin{pmatrix} a+b-3c+2d \\ -a+b+c+4d \\ 2a+b-5c+d \\ a+2b-4c+5d \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\iff \begin{cases} a+b-3c+2d = 2 \\ -a+b+c+4d = -2 \\ 2a+b-5c+d = 4 \\ a+2b-4c+5d = 2 \end{cases}$$

let $A = \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 2 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right)$

$$\underset{\substack{L_2 += L_1 \\ L_3 -= 2L_1 \\ L_4 -= L_1}}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$\sim$
$L_2 \leftrightarrow L_4$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right)$$

$\sim$
$L_3 += L_2$
$L_4 -= 2L_2$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row echelon form $\Rightarrow$ the system is compatible

$\Rightarrow \exists\, u : f(u) = v' \Rightarrow v' \in \operatorname{Im} f$

ii) $\operatorname{Ker} f = (v_1, \dots, v_m)$, $f(v_k) = 0 \Rightarrow$

$\Rightarrow v = (x_1, x_2, x_3, x_4)$; $[f]_E \cdot [v]_E = 0$

$$[f]_E = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -7 & 3 & 0 \end{array}\right)$$

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$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \end{array}\right)$$

$$\begin{cases} x_1 + x_2 - 3\alpha + 2\beta = 0 \Rightarrow x_1 = \alpha - 3\beta + \cancel{3\alpha} + 2\beta \quad = 2\alpha - \beta \\ -x_2 = \alpha - 3\beta \end{cases}$$

$$\operatorname{Ker} f = \{ (2\alpha - \beta, \alpha - 3\beta, \alpha, \beta) \}$$

$$= \langle (2, 1, 1, 0), (-1, -3, 0, 1) \rangle$$

- check linear independence $\Rightarrow$

$$\Rightarrow \dim(\operatorname{Ker} f) = 2$$

$$\operatorname{Im} f = \left\{ w = (a,b,c,d) \,\middle|\, \exists\, v = (x,y,z,t) : f(v) = w \right\}$$

$$= \left\{ w = (a,b,c,d) \,\middle|\, \exists\, v = (x,y,z,t) : [f]_E \cdot [v]_E = [w]_E \right\}$$

$$= \left\{ w = (a,b,c,d) \,\middle|\, \exists\, v : \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\}$$

$$= \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array} \right)$$

$$\underset{L_4 \leftrightarrow L_2}{\sim}\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right)$$

$$\sim\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & -3a+c+d \\ 0 & 0 & 0 & 0 & 3a+b-d \end{array}\right)$$

System is comp. iff:

$$\begin{cases} -3a+c+d = 0 \\ 3a+b-d = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a = \alpha \\ b = -3\alpha+\beta \\ c = 3\alpha-\beta \\ d = \beta \end{cases}$$

$$\operatorname{Im} f = \left\{ (\alpha, -3\alpha+\beta, 3\alpha-\beta, \beta) \mid \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\langle \left(1, -3, 3 \mid 0\right), \left(0, 1, -1, 1\right) \right\rangle$$

lin indep $\Rightarrow$ basis $\Rightarrow$ dim Im f $= 2$