HomeWork 2

FARKAS
TIBERIA-GIULIA

1. $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

$(\forall)\ \varepsilon > 0, (\exists)\ N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon, (\forall) n \geq N_\varepsilon$

Let $\varepsilon > 0$ Take $N_\varepsilon = \left[\frac{1-\varepsilon}{4\varepsilon}\right] + 1$

$\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon$

$\left|\frac{2n+2-2n-3}{4n+6}\right| < \varepsilon$

$\left|\frac{-1}{4n+6}\right| < \varepsilon$

$\frac{1}{4n+6} < \varepsilon$

$4n+6 > \frac{1}{\varepsilon}$

$4n > \frac{1}{\varepsilon} - 6$

$4n > \frac{1-\varepsilon}{\varepsilon}$

$n > \frac{1-\varepsilon}{4\varepsilon}, (\forall) n \geq N_\varepsilon > \frac{1-\varepsilon}{4\varepsilon}$

2.

a) $(1+2+\dots+n)^{\frac{1}{n}}$

$\lim_{n\to\infty} (1+2+\dots+n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}} =$

$= \lim_{n\to\infty} \left(\frac{n^2+n}{2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\frac{1}{n}} \cdot \lim_{n\to\infty} (n^2+n)^{\frac{1}{n}}$

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$= \lim_{n\to\infty} e^{\ln(n^2+n)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(n^2+n)}$

$= \lim_{n\to\infty} e^{\frac{2}{n}\ln\left(n+\frac{1}{n}\right)} = \lim_{n\to\infty} e^{\frac{2\ln n}{n}} = 1$

$\frac{1}{n} \to 0$, $\frac{2\ln n}{n} \to 0$

b) $\left(\frac{\ln(n+1)}{\ln n}\right)^n$

$$\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln n}\right)^n = \lim_{n\to\infty}\left(1+\frac{\ln(n+1)}{\ln n}-1\right)^n =$$

$$= \lim_{n\to\infty}\left(1+\frac{\ln\frac{n+1}{n}}{\ln n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{\ln\frac{n+1}{n}}{\ln n}\right)^{\frac{\ln n}{\ln\frac{n+1}{n}}}\right]^{\frac{n\ln\frac{n+1}{n}}{\ln n}}$$

$$= e^{\lim_{n\to\infty}\frac{n\ln\frac{n+1}{n}}{\ln n}} = e^0 = \underline{1}$$

$$\lim_{n\to\infty}\frac{\left(\ln\frac{n+1}{n}\right)^n}{\ln n} = \frac{\lim_{n\to\infty}\left(\ln\frac{n+1}{n}\right)^n}{\lim_{n\to\infty}\ln n} = \frac{1}{\infty} = 0$$

$$\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)^n = \ln\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \ln\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =$$

$$= \ln e = 1$$

c) $\frac{n^n}{1+2^2+3^3+\dots+n^n}$

$$\lim_{n\to\infty}\frac{n^n}{1+2^2+3^3+\dots+n^n}$$

Let $a_n = n^n$ and $b_n = 1+2^2+3^3+\dots+n^n$

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} =$$

$$= \lim_{n\to\infty} 1-\frac{n^n}{(n+1)^n\cdot(n+1)} = \lim_{n\to\infty} 1-\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}$$

$$= \lim_{n\to\infty} 1-\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\lim_{n\to\infty}\frac{1}{n+1} = 1-\frac{1}{e}\cdot 0 = \underline{1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}} = e^{-1} = \frac{1}{e}$$

3. $x_n = \frac{\sin(n)}{n}$

x_n - bounded, monotone, convergent?

$\lim_{n \to \infty} x_n = ?$

$-1 \le \sin n \le 1 \mid \cdot \frac{1}{n}$

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\Rightarrow \frac{\sin n}{n} \in (0,1] \Rightarrow x_n$ bounded

$$\lim_{n \to \infty} -\frac{1}{n} \le \lim_{n \to \infty} \frac{\sin n}{n} \le \lim_{n \to \infty} \frac{1}{n}$$

$0 \quad 0 \quad 0$

$\Rightarrow \lim_{n \to \infty} x_n = 0 \Rightarrow x_n$ convergent

Let $f : \mathbb{R}^* \to (0,1]$, $f(n) = \frac{\sin n}{n}$

$$f'(n) = \frac{(\sin n)' n - \sin n \cdot n'}{n^2} = \frac{n \cos n - \sin n}{n^2}$$

$\Rightarrow f$ is not monotone because it increases and decreases depending on cos and sin

$\Rightarrow x_n$ is not monotone

4. $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$

x_n - decreasing, bounded $\Rightarrow$ convergent

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln (n+1) + \ln n$$
$$= \frac{1}{n+1} - \ln \frac{n+1}{n}$$

Let compare $\frac{1}{n+1}$ and $\ln \frac{n+1}{n}$

We assume that $\frac{1}{n+1} - \ln \frac{n+1}{n} < 0$

$$\frac{1}{n+1} < \ln \frac{n+1}{n} \quad | \hat{\ } e$$

$$e^{\frac{1}{n+1}} < e^{\ln \frac{n+1}{n}}$$

$$\sqrt[n]{e} < \frac{n+1}{n}$$

$$\sqrt[n+1]{e} < 1 + \frac{1}{n}$$ True for $(\forall)\ n \geq 1 \Rightarrow$

$\Rightarrow x_{n+1} - x_n = \frac{1}{n+1} - \ln \frac{n+1}{n} < 0 \Rightarrow$

x_n decreasing (1)

$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \leq x_1 = 1$

It is known that $x_n \to \gamma$, known as the Euler-Mascheroni constant, where $\gamma > 0$

$\Rightarrow x_n > 0 \Rightarrow x_n \in (0, 1] \Rightarrow x_n$ is bounded (2)

From (1), (2) $\Rightarrow x_n$ converges