Homework 3

Farkas Tiberia Giulia

1) a) $\sum_{n\geq 2} \ln\left(\frac{n^2}{1}-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln \frac{n^2-1}{n^2} = \sum_{n\geq 2} \frac{\ln(n+1)}{\ln n^2} + \frac{\ln(n-1)}{\ln n^2}$

$= \sum_{n\geq 2} \ln(n+1) + \ln(n-1) - \ln(n^2)$

$= \sum_{n\geq 2} \ln(n+1) + \ln(n-1) - 2\ln(n)$

$= \sum_{n\geq 2} \ln(n+1) + \sum_{n\geq 2} \ln(n-1) - 2\sum_{n\geq 2} \ln(n)$

$= \lim_{n\to\infty} \Big(\underbrace{\cancel{\ln 3} + \cancel{\ln 4} + \cancel{\ln 5} + \dots}_{\ln(n+1)} + \underbrace{\ln 1 + \cancel{\ln 2} + \cancel{\ln 3} + \dots}_{\ln(n-1)} -$

$- \underbrace{\cancel{2}\ln 2 - 2\cancel{\ln 3} - 2\cancel{\ln 4} - \dots - \cancel{2}\ln n}_{\ln n} \Big)$

$= \lim_{n\to\infty} (\ln 1 - \ln 2)$

$= -\ln 2$

b) $\sum_{n\geq 1} \frac{n+1}{3^n} = S \in (0, +\infty)$

Ratio test: $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{3^{n+1}} \cdot \frac{3^n}{n+1} = \lim_{n\to\infty} \frac{n+2}{3n+3} = \frac{1}{3}$

$< 1 \Rightarrow$ Converges

$\sum_{n\geq 1} \frac{n+1}{3^n} = \sum_{n\geq 1} \frac{n}{3^n} + \sum_{n\geq 1} \frac{1}{3^n} = \sum_{n\geq 1} \frac{n}{3^n} + \frac{1}{2} = \frac{1}{3} \sum_{n\geq 1} \frac{n}{3^{n-1}} + \frac{1}{2}$

$\sum_{n\geq 1} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

$\Rightarrow \sum_{n\geq 1} \frac{n+1}{3^n} = \frac{1}{3} \underbrace{\sum_{n\geq 1} \frac{n}{3^{n-1}}}_{\sum \frac{n+1}{3^n}} + \frac{1}{2} \Rightarrow S = \frac{1}{3} S + \frac{1}{2} \Rightarrow S = \frac{3}{4}$

c) $\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{n}{(n^2+1)^2-n^2} = \sum_{n\geq 1} \frac{n}{(n^2+1-n)(n^2+1+n)}$

$= \frac{1}{2}\sum_{n\geq 1} \frac{2n}{(n^2+1-n)(n^2+1+n)} = \frac{1}{2}\sum_{n\geq 1} \frac{(n^2+1+n)-(n^2+1-n)}{(n^2+1+n)(n^2+1-n)}$

$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n^2+1-n} - \frac{1}{n^2+1+n}\right) = \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)$

$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n(n-1)+1} - \frac{1}{n(n+1)+1}\right)$

$= \frac{1}{2}\lim_{n\to\infty}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \dots + \cancel{\frac{1}{n^2-n+1}} - \underbrace{\frac{1}{n^2+n+1}}_{0}\right)$

$= \frac{1}{2}$

2) The convergence of the series

a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$

Ratio test: $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)^p}\cdot\frac{n^p}{x^n} = \lim_{n\to\infty}\frac{x\cdot n^p}{(n+1)^p} =$

$= \lim_{n\to\infty} x\cdot\left(\frac{n}{n+1}\right)^p = x$

If $x\in(0,1) \Rightarrow \sum_{n\geq 1}\frac{x^n}{n^p}$ is convergent

If $x\in(1,+\infty) \Rightarrow \sum_{n\geq 1}\frac{x^n}{n^p}$ is divergent

If $x=1 \Rightarrow \sum_{n\geq 1}\frac{1}{n^p}$ $\begin{cases}\text{diverges for } p\leq 1\\ \text{converges for } p>1\end{cases}$

b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$

For $n > e^{e^2} \Rightarrow (\ln n)^{\ln n} > n^2 \Rightarrow$

$\Rightarrow \frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$

$\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}} < \sum_{n\geq 2} \frac{1}{n^2}$

$\downarrow$

converges $\Rightarrow$

$\Rightarrow \sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$ converges

c) $\sum_{n\geq 1} (\sqrt[n]{n} - 1) = \sum_{n\geq 1} n^{\frac{1}{n}} - 1$

$n^{\frac{1}{n}} - 1 > \frac{1}{n}$

$n^{\frac{1}{n}} > \frac{1}{n} + 1$

$n > \left(\frac{1}{n} + 1\right)^n \Rightarrow$

$\lim_{n\to\infty} \left(\frac{1}{n} + 1\right)^n = e \Rightarrow \left(\frac{1}{n} + 1\right)$ is increasing

$\Rightarrow \sum_{n\to\infty} \left(\frac{1}{n} + 1\right)^n = \infty$

$\sum_{n\to\infty} \sqrt[n]{n} - 1 > \sum_{n\to\infty} \frac{1}{n} \to$ divergent $\Rightarrow$

$\Rightarrow \sum_{n\to\infty} \sqrt[n]{n} - 1$ is divergent

d) $\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\cdot\frac{1}{2n+1}$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{\cancel{1\cdot 3}\cdot\ldots\cdot(2n+1)}{\cancel{2\cdot 4}\cdot\ldots\cdot 2n+2}\cdot\frac{1}{2n+3}\cdot(2n+1)\cdot\frac{\cancel{2\cdot 4}\cdot\ldots\cdot\cancel{2n}}{\cancel{1\cdot 3}\cdot\ldots\cdot\cancel{(2n-1)}}$$

$$=\lim_{n\to\infty}\frac{(2n+1)^2}{(2n+2)(2n+3)}=\lim_{n\to\infty}\frac{4n^2+4n+1}{4n^2+10n+6}=1$$

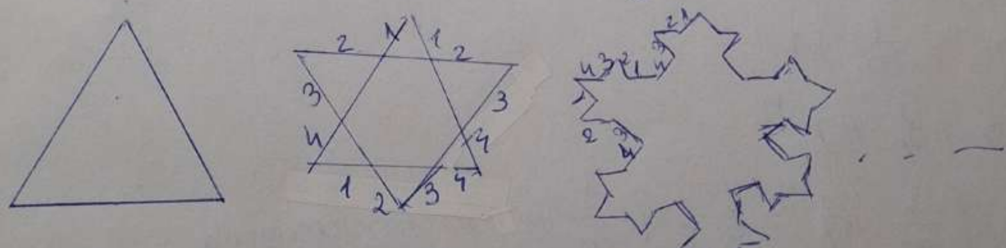
Raabe - Duhamel

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-\frac{4n^2+4n+1}{}\right)=$$

$$=\lim_{n\to\infty} n\cdot\frac{6n+5}{4n^2+4n+1}=\lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1}=\frac{6}{4}>1 \Rightarrow$$

the series is convergent

3. ~ Known as the Koch Snowflake ~



First step: figure with three sides

Second step: one side of the previous figure becomes four sides of the actual figure $\Rightarrow$ figure with $3\cdot 4=12$ sides

Third step: one side of the previous figure becomes four sides of the actual figure $\Rightarrow$ figure with $4\cdot 12=4\cdot 4\cdot 3=4^2\cdot 3=48$ sides

⋮

We suppose that $p(n)=3\cdot 4^n$ represents the number of the sides of the initial figure after n iteration and $p(n)$ is true, and demonstrate that $p(n+1)=3\cdot 4^{n+1}$ is also true.

$$p(n+1) = p(n) \cdot 4$$
$$p(n+1) = 3 \cdot 4^n \cdot 4$$
$$p(n+1) = 3 \cdot 4^{n+1} \quad \text{True}$$

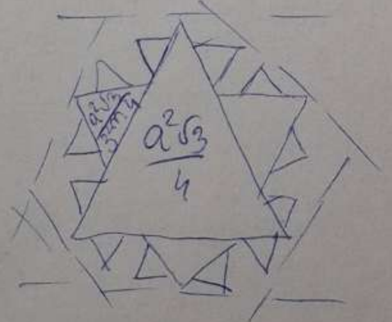
$\Rightarrow$ At iteration n, there would be $3 \cdot 4^n$ sides.

Let take the initial length of a side $= a$. In the next iteration the length of the side will be $\frac{a}{3}$ and so on.

Using mathematical induction (just like I did above), we get to the conclusion that the length of a side of the figure after n iterations is: $a \cdot \frac{1}{3^n}$.

$\Rightarrow$ Perimeter of the figure = n sides $\cdot$ length of the sides

$$P = 3 \cdot 4^n \cdot a \cdot \frac{1}{3^n}$$
$$P = 3a \cdot \left(\frac{4}{3}\right)^n$$
$$\lim_{n \to \infty} p = \lim_{n \to \infty} 3a \cdot \underbrace{\left(\frac{4}{3}\right)}_{>1}^n = +\infty$$



Area of equilateral $\Delta = \frac{l^2\sqrt{3}}{4}$

Area after 0 iteration: $\frac{a^2\sqrt{3}}{4}$

Area after 1 iteration: $\frac{a^2\sqrt{3}}{4} + 3 \cdot \frac{a^2\sqrt{3}}{3^2 \cdot 4}$.

Area after 2 iteration: $\frac{a^2\sqrt{3}}{4} + 3 \cdot \frac{a^2\sqrt{3}}{3^2 \cdot 4} + 12 \, \frac{a^2\sqrt{3}}{9^2 \cdot 4}$

With each step we will have 4 times more triangles than before.

$$A = \frac{a^2\sqrt{3}}{4}\left(1 + 3 \cdot \left(\frac{1}{3}\right)^2 + 3 \cdot 4 \cdot \left(\frac{1}{3^2}\right)^2 + 3 \cdot 4^2 \cdot \left(\frac{1}{3^3}\right)^2 + \dots\right)$$

$$A = \frac{1}{4} \cdot \frac{a^2\sqrt{3}}{4}\left(4 + 3\cdot 4\left(\frac{1}{3}\right)^2 + 3\cdot 4^2\left(\frac{1}{3^2}\right)^2 + 3\cdot 4^3\left(\frac{1}{3^3}\right)^2 + \dots\right)$$

$$A = \frac{\sqrt{3}a^2}{16}\left(4 + 3\cdot 4\left(\frac{1}{9}\right)^1 + 3\cdot 4^2\cdot\left(\frac{1}{9}\right)^2 + 3\cdot 4^3\left(\frac{1}{9}\right)^3 + \dots\right)$$

$$A = \frac{\sqrt{3}a^2}{16}\left(4 + 3\cdot\frac{4}{9} + 3\left(\frac{4}{9}\right)^2 + 3\cdot\left(\frac{4}{9}\right)^3 + \dots\right)$$

$$A = \frac{\sqrt{3}a^2}{16}\left(4 + 3\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \dots\right)\right)$$

$$A = \frac{\sqrt{3}a^2}{16}\left(4 + 3\sum_{n\geq 1}\left(\frac{4}{9}\right)^n\right)$$

$$\sum_{n\geq 1}\left(\frac{4}{9}\right)^n = \frac{1}{1-\frac{4}{9}} - 1 = \frac{9}{5} - 1 = \frac{4}{5}$$

$$A = \frac{\sqrt{3}a^2}{16}\left(4 + 3\cdot\frac{4}{5}\right)$$

$$A = \frac{\sqrt{3}a^2}{\cancel{16}_1}\cdot\frac{\cancel{32}^{2}}{5}$$

$$A = \frac{2a^2\sqrt{3}}{5}$$