Notes on BBH numerical code

Binary black hole collisions

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1 BSSN formulation: χ-version

For $G_{\mu\nu} = 0 \;\Rightarrow \rho = j^k = S_{ij} = 0$ and geometric units ($c = G = 1$):

$$
\begin{aligned}
\partial_t \chi &= \pounds_{\vec{\beta}}\, \chi + \tfrac{2}{3}\chi \left(\alpha K - \partial_i \beta^i\right) \,, \quad \textit{[conformal factor]} && (1)\\
\partial_t \alpha &= \pounds_{\vec{\beta}}\, \alpha - 2\alpha K \,, \quad \textit{[lapse: } 1 + \textit{log gauge]} && (2)\\
\partial_t \tilde{\gamma}_{ij} &= \pounds_{\vec{\beta}}\, \tilde{\gamma}_{ij} - 2\alpha \tilde{A}_{ij} \,, \quad \textit{[spatial metric]} && (3)\\
\partial_t K &= \pounds_{\vec{\beta}}\, K - D^2\alpha + \alpha \left(\tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{1}{3}K^2\right) \,, \quad \textit{[mean curvature]} && (4)\\
\partial_t \tilde{\Gamma}^i &= \pounds_{\vec{\beta}}\, \tilde{\Gamma}^i + \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \tfrac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k - 2\tilde{A}^{ij}\partial_j\alpha && (5)\\
&+ 2\alpha \left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \tfrac{3}{2}\chi^{-1}\tilde{A}^{ij}\partial_j\chi - \tfrac{2}{3}\tilde{\gamma}^{ij}\partial_j K\right) \,, \quad \textit{[constraints]} &&\\
\partial_t B^i &= \beta^j\partial_j B^i - \eta B^i + (\partial_t - \beta^j\partial_j\,)\tilde{\Gamma}^i && (6)\\
\partial_t \beta^i &= \beta^j\partial_j\beta^i + \tfrac{3}{4}B^i \,, \quad \textit{[shift: gamma} - \textit{driver gauge]} && (7)\\
\partial_t \tilde{A}_{ij} &= \pounds_{\vec{\beta}}\, \tilde{A}_{ij} + \alpha \left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\,\tilde{A}^k{}_j\right) &&\\
&+ \chi \left(-D_iD_j\alpha + \alpha^{(3)}R_{ij}\right)^{TF} \,, \quad \textit{[extrinsic curvature]} && (8)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(3)}R_{ij} &= \tilde{R}_{ij} + R^{\chi}_{ij} && (9)\\
\tilde{R}_{ij} &= -\frac{1}{2}\tilde{\gamma}^{kl}\partial_k\partial_l\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\,\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\,\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{kl}\left(2\tilde{\Gamma}^m_{k(i}\,\tilde{\Gamma}_{j)ml} + \tilde{\Gamma}^m_{il}\,\tilde{\Gamma}_{(ij)k}\right) && (10)\\
R^{\chi}_{ij} &= \frac{1}{2\chi}\tilde{D}_i\tilde{D}_j\chi + \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{D}^2\chi - \frac{1}{4\chi^2}\tilde{D}_i\chi\,\tilde{D}_j\chi - \frac{3}{4\chi^2}\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k\chi && (11)\\
R^{TF}_{ij} &= R_{ij} - \frac{1}{3}R\gamma_{ij} && (12)\\
R &= \tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2 && (13)
\end{aligned}
$$

Reference: [1](on pag. 425) and [2].

2 Initial conditions

Head-on collision:

$$\alpha = 1 \tag{14}$$

$$\beta^i = 0\,, \quad \textit{[geodesic slicing]} \tag{15}$$

$$K = 0\,, \quad \textit{[maximal slicing]} \tag{16}$$

$$\tilde{\gamma}_{ij} = \eta_{ij}\,, \quad \textit{[conformal flatness]} \tag{17}$$

$$\psi = 1 + \frac{r_s}{4}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,, \quad \textit{[Brill – Lindquist]} \tag{18}$$

$$\tilde{A}_{ij} = 0 \tag{19}$$

Spiral collision:

$$\alpha = \chi^{-2} \tag{20}$$

$$\beta^i = 0\,, \quad \textit{[geodesic slicing]} \tag{21}$$

$$K = 0\,, \quad \textit{[maximal slicing]} \tag{22}$$

$$\tilde{\gamma}_{ij} = \eta_{ij}\,, \quad \textit{[conformal flatness]} \tag{23}$$

$$\psi_{BL} = 1 + \frac{r_s}{4}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\,, \quad \textit{[Brill – Lindquist]} \tag{24}$$

$$\psi = \psi_{BL} + u\,, \quad \textit{[Brandt – Brugmann]} \tag{25}$$

$$\nabla^2 u = -\frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij}\,, \quad \textit{[Hamiltonian constraint]} \tag{26}$$

$$\tilde{A}_{ij} = \frac{3}{2}\sum_{q=1}^{2}\frac{1}{r_q^2}\left\{2\,p^q_{(i}\,l^q_{j)} - (\eta_{ij} - l^q_i\,l^q_j)\,p^q_k\,l^k_q + \frac{4}{r}\,l^q_{(i}\,\epsilon_{j)km}\,s^k_q\,l^m_q\right\}\,, \quad \textit{[Bowen – York]} \tag{27}$$

3 Conformal transformation

$$\tilde{\gamma}_{ij} = \chi\,\gamma_{ij} \tag{28}$$
$$\tilde{A}_{ij} = \chi\left(K_{ij} - \tfrac{1}{3}K\gamma_{ij}\right) \tag{29}$$

$$\chi \doteq e^{-4\phi} = \psi^{-4} \;\rightarrow\; \phi = -\frac{1}{4}\ln\chi \tag{30}$$
$$\phi = \frac{1}{12}\ln\gamma \;\rightarrow\; \gamma^{1/2} = \chi^{-3/2} \tag{31}$$

3.1 Derivative transformation:

Rescaling ϕ to χ:

$$\tilde{D}_i\phi = -\frac{1}{4\chi}\tilde{D}_i\chi \tag{32}$$
$$\tilde{D}_i\tilde{D}_j\phi = \frac{1}{4\chi^2}\tilde{D}_i\chi\tilde{D}_j\chi - \frac{1}{4\chi}\tilde{D}_i\tilde{D}_j\chi \tag{33}$$

Conformal derivative:

$$\tilde{D}_i = D_i \tag{34}$$
$$\tilde{D}^i = \chi^{-1}D^i \tag{35}$$

Gradient operator:

$$\tilde{D}_i\phi = D_i\phi = \partial_i\phi \tag{36}$$

Divergent operator:

$$D_i\,v^i = \gamma^{-1/2}\,\partial_i\left(\gamma^{1/2}\,v^i\right) = \chi^{3/2}\,\partial_i\left(\chi^{-3/2}\,v^i\right) \tag{37}$$

$$\tilde{D}_j\,v^i = D_j\,v^i - C^i_{\ jk}v^k \tag{38}$$

$$\tilde{D}_i\,v^i = D_i\,v^i - C^i_{\ ij}v^j = D_i\,v^i - \frac{2}{3}\chi^{-1}\partial_i\chi\,v^i = \partial_i\,v^i \tag{39}$$

Reference: [3](on pag. 152).

Laplacian operator:

$$D^2\alpha = D_i D^i \alpha = D_i(\partial^i \alpha) = \chi^{3/2}\,\partial_i\left(\chi^{-3/2}\,\partial^i\alpha\right) \tag{40}$$

$$= \partial^2\alpha - \frac{3}{2}\chi^{-1}\partial_i\chi\,\gamma^{ij}\,\partial_j\alpha \tag{41}$$

$$\tilde{D}^2\alpha = \tilde{D}_i(\tilde{D}^i\alpha) = \partial_i(\tilde{D}^i\alpha) = \partial_i(\chi^{-1}\,\partial^i\alpha) \tag{42}$$

$$= \chi^{-1}\partial^2\alpha - \chi^{-2}\partial_i\chi\,\gamma^{ij}\,\partial_j\alpha \tag{43}$$

Gradient-gradient operator:

$$D_iD_j\alpha = D_i(\partial_j\alpha) = \partial_i\partial_j\alpha - {}^{(3)}\Gamma^k_{ij}\,\partial_k\alpha \tag{44}$$

$$= \partial_i\partial_j\alpha - (\tilde{\Gamma}^k_{ij} + C^k_{ij})\,\partial_k\alpha \tag{45}$$

$$(D_iD_j\alpha)^{TF} = D_iD_j\alpha - \frac{1}{3}\,tr(D_iD_j\alpha)\,\gamma_{ij} \tag{46}$$

$$\tilde{D}_i\tilde{D}_j\alpha = \tilde{D}_i(\partial_j\alpha) = D_i(\partial_j\alpha) + C^k_{ij}\,(\partial_k\alpha) \tag{47}$$

$$= \partial_i\partial_j\alpha - (\tilde{\Gamma}^k_{ij} + C^k_{ij})\,\partial_k\alpha + C^k_{ij}\,(\partial_k\alpha) \tag{48}$$

$$= \partial_i\partial_j\alpha - \tilde{\Gamma}^k_{ij}\,\partial_k\alpha \tag{49}$$

Reference: [3](on pag. 100 and 161).

Conformal tensor:

$$C^i{}_{jk} = -\frac{1}{2}\chi^{-1}\left(\delta^i_j\tilde{D}_k\chi + \delta^i_k\tilde{D}_j\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{il}\tilde{D}_l\chi\right) \tag{50}$$

$$= -\frac{1}{2}\chi^{-1}\left(\delta^i_j\partial_k\chi + \delta^i_k\partial_j\chi - \gamma_{jk}\gamma^{il}\partial_l\chi\right) \qquad [\gamma_{jk}\gamma^{il} = \tilde{\gamma}_{jk}\tilde{\gamma}^{il}] \tag{51}$$

4 Equation adjustments

Conformal metric: [not implemented]

$$\tilde{\gamma}_{ij} \to \tilde{\gamma}^{-1/3}\tilde{\gamma}_{ij} \tag{52}$$

$$\tilde{\gamma}_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}, \Rightarrow \ \tilde{\gamma} = 058 + 156 + 273 - 246 - 570 - 831$$

Conformal extrinsic curvature: [not implemented]

$$\tilde{A}_{ij} \to \tilde{A}_{ij} - \frac{1}{3}\, tr(\tilde{A}_{ij})\, \tilde{\gamma}_{ij} \tag{53}$$

Gamma-functions: [implemented] For non-derivative terms:

$$\tilde{\Gamma}^i = -\partial_j \tilde{\gamma}^{ij} \tag{54}$$

At the punctures: [implemented]

$$\chi^p_{ab} \doteqdot \delta(\chi_{a+1,b} + \chi_{a-1,b} + \chi_{a,b+1} + \chi_{a,b-1}) \qquad [a, b \text{ are spacial index}] \tag{55}$$

From equation

$$\dot{x}^i_p = -\beta^i(x^i_p)\,, \tag{56}$$

we replace the value of χ at the punctures, $\chi(x^i) = \chi_p$.

5 Differential stuffs

1^{st} derivative:

$$\partial_x u_i^n \simeq \frac{1}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) . \tag{57}$$

2^{nd} derivative:

$$\partial_x^2 u_i^n \simeq \frac{1}{\Delta x^2}\left(u_{i+1}^n - u_i^n + u_{i-1}^n\right) , \tag{58}$$
$$\partial_x \partial_y u_{ij}^n \simeq \frac{1}{4\Delta x^2}\left(u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n\right) . \tag{59}$$

Advective terms: $(\beta^i \partial_i)$

$$\partial_x u_i^n \simeq \frac{1}{12\Delta x}\left(u_{i+3}^n - 6u_{i+2}^n + 18u_{i+1}^n - 10u_i^n - 3u_{i-1}^n\right) , \quad \beta^x > 0 , \tag{60}$$
$$\partial_x u_i^n \simeq \frac{1}{12\Delta x}\left(3u_{i+1}^n + 10u_i^n - 18u_{i-1}^n + 6u_{i-2}^n - u_{i-3}^n\right) , \quad \beta^x < 0 . \tag{61}$$

Reference: complementar [4](on pag. 200).

Interative Crank-Nicholson method: (ICN)

$$v_i^1 = u_i^n + \Delta t\, \mathcal{S}(u_i^n) , \tag{62}$$
$$v_i^a = u_i^n + \frac{\Delta t}{2}\left[\mathcal{S}(u_i^n) + \mathcal{S}(v_i^{a-1})\right] , \tag{63}$$
$$u_i^{n+1} = v_i^{(\mathcal{Q})} , \tag{64}$$

where $a = 2, ..., \mathcal{Q}$ is the interaction index. We used $\mathcal{Q} = 3$.

Reference: [5, 6].

Kreiss-Oliger dissipation:

$$\partial_0 u = \mathcal{S}(u) - \frac{\epsilon}{\Delta x}(-1)^{\mathcal{Q}} \Delta_k^{2\mathcal{Q}} u , \tag{65}$$

where $0 < \epsilon \Delta t / \Delta x < 1/2^{2\mathcal{Q}-1}$ and $\Delta_k^{2\mathcal{Q}} \doteqdot (\Delta_k^+ \Delta_k^-)^{\mathcal{Q}}$. For $\mathcal{Q} = 2$,

$$\Delta_i^4 u = u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n \tag{66}$$

Reference: [1](on pag. 358).

6 GW extraction

From Newman-Penrose scalar

$$\Psi_4 \doteq -C_{\alpha\mu\beta\nu} n^{\alpha} \bar{m}^{\mu} n^{\beta} \bar{m}^{\nu} = \ddot{h}_+ - i\,\ddot{h}_x \,, \tag{67}$$

using the Gram-Schmidt orthonormalization,

$$\begin{aligned} \vec{u} &= [x, y, z] \,, &(68)\\ \vec{v} &= [-y, x, 0] \,, &(69)\\ \vec{w} &= [xz, yz, -(x^2+y^2)] \,, &(70) \end{aligned}$$

where the tetrad vector are

$$\begin{aligned} n^0 &= \frac{1}{\sqrt{2}\alpha} \,, \quad n^i = -\frac{1}{\sqrt{2}} \left(\beta^i/\alpha + u^i\right) \,, &(71)\\ l^0 &= \frac{1}{\sqrt{2}\alpha} \,, \quad l^i = -\frac{1}{\sqrt{2}} \left(\beta^i/\alpha - u^i\right) \,, &(72)\\ m^0 &= 0 \,, \quad m^i = \frac{1}{\sqrt{2}} \left(v^i + i w^i\right) \,. &(73)\\ & &(74) \end{aligned}$$

Therefore,

$$\begin{aligned} \Psi_4 &= -\frac{1}{2}[n^0 n^0 (R_{jl} - K_{jp} K^p_l + K K_{jl}) + 4 n^0 n^k D_{[k} K_{l]j} \\ &+ n^i n^k (R_{ijkl} + 2 K_{i[k} K_{l]j})](v^j - i w^j)(v^l - i w^l) \,. \end{aligned} \tag{75}$$

where

$$\begin{aligned} R_{ijkl} &= e^{4\phi}(\tilde{R}_{ijkl} + 4\tilde{\gamma}_{l[i}\tilde{D}_{j]}\tilde{D}_k\phi - 4\tilde{\gamma}_{k[i}\tilde{D}_{j]}\tilde{D}_l\phi \\ &- 16\,\partial_{[i}\phi\,\tilde{\gamma}_{j][k}\partial_{l]}\phi - 8\tilde{\gamma}_{k[i}\tilde{\gamma}_{j]l}\tilde{\gamma}^{mn}\partial_m\phi\,\partial_n\phi) \,, &(76)\\ R_{ij} &= \tilde{R}_{ij} - 2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\phi + 4\partial_i\phi\,\partial_j\phi \\ &- 4\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\phi\,\partial_n\phi &(77)\\ K_{ij} &= e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{3}K\tilde{\gamma}_{ij}\right) \\ D_i K_{jk} &= e^{4\phi}(\partial_i\tilde{A}_{jk} - 2\tilde{\Gamma}^l_{i(j}\tilde{A}_{k)l} + \frac{1}{3}\tilde{\gamma}_{jk}\partial_i K) \\ &- 4K_{i(j}\partial_{k)} + 4\tilde{\gamma}_{i(j}K_{k)m}\tilde{\gamma}^{ml}\partial_l\phi \,. &(78) \end{aligned}$$

Reference: [2, 7] and [8] (strain from Ψ_4).

Spherical harmonics:

$$\begin{aligned} \Psi_4(t,\theta,\phi) &= \sum_{l,m} A_{lm}(t) Y^{-2}_{lm}(\theta,\phi) \,, &(79)\\ A_{lm}(t) &= \int \Psi_4(t,\theta,\phi)\bar{Y}^{-2}_{lm}(\theta,\phi) d\Omega \,. &(80) \end{aligned}$$

Reference: [9] and [10](why decompose into spherical harmonics).

$$Y_{2-2}^{-2} = \sqrt{\frac{5}{64\pi}}(1-\cos\theta)^2 e^{-2i\phi}\,, \tag{81}$$

$$Y_{2-1}^{-2} = \sqrt{\frac{5}{16\pi}}\sin\theta(1-\cos\theta)e^{-i\phi}\,, \tag{82}$$

$$Y_{20}^{-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\,, \tag{83}$$

$$Y_{21}^{-2} = \sqrt{\frac{5}{16\pi}}\sin\theta(1+\cos\theta)e^{i\phi}\,, \tag{84}$$

$$Y_{22}^{-2} = \sqrt{\frac{5}{64\pi}}(1+\cos\theta)^2 e^{2i\phi}\,. \tag{85}$$

Reference: [11].

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