5 Singular Value Decomposition

Exercise 5.1
Calculate by hand the SVD of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix},$$

and, next (homework),

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise 5.2
Let A be a matrix of size $n \times m$ with the SVD : $A = U\Sigma V^T$. Assume that A is of rank r and $n > m$.

1. Show that A can be rewritten as

$$A = \sum_{k=1}^{r} \sigma_k u_k v_k^T \tag{1}$$

   where u_k is the k-th column of U and v_k is the k-th column of V.
2. Calculate the rank of each matrix $u_k v_k^T$.
3. Show that

$$A = U_r \Sigma_r V_r^T$$

   where $U_r = [u_1 \cdots u_r]$, $V_r = [v_1 \cdots v_r]$ and $\Sigma_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$.
4. Homework : show the equality (1) when $n \leq m$.

Exercise 5.3
Let A be a matrix of size $n \times m$ and rank r with the SVD decomposition $A = U\Sigma V^T$. We assume the equality and the notations in (1) whatever the values of n and m.

1. Show that $Av_k = 0$ for all $k > r$.
2. Show that $Av_k \neq 0$ for all $k \leq r$.
3. Let K be the linear space such that $K = \{x | Ax = 0\}$. Show that the set of right singular vectors $\{v_1, \ldots, v_r\}$ form a basis of K.
4. Let $x \in \mathbb{R}^m$ and $y = Ax$. Show that there exists some real $c_1, \ldots, c_r$ such that

$$Ax = \sum_{k=1}^{r} c_k u_k.$$

5. Let F be the linear space such that $F = \{y \in \mathbb{R}^n | y = Ax,\ x \in \mathbb{R}^m\}$. Show that the vectors $\{u_1, \ldots, u_r\}$ form a basis of F.
6. Knowing the SVD decomposition of A, how can we calculate easily the rank of A ?

Exercise 5.4

The goal is to apply the SVD to an image with Python Jupyter. The image "lena512.png" is on the Jalon website.

1. Run and explain the following code :

```
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
from PIL import Image # import Python Imaging Library (PIL)
imggray = Image.open('lena512.pgm')
plt.imshow(imggray, cmap='gray')
imgmat = np.matrix(imggray)
print(imgmat)
```

2. Compute the SVD of the image by using "np.linalg.svd".
3. Code the best image approximation of rank one. Plot this approximation.
   Hints : you can use the command "np.diag" to create a diagonal matrix.
4. Code and plot the best image approximations of rank $r \in \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$. What do you observe ? Is the SVD useful to compress the image in order to decrease its storage space ?
5. Give an estimate of the compression rate with respect to r. The compression rate is defined as the compressed size divided by the uncompressed size. Plot the compression rate as a function of r.