1 Notation & Classification

$$f(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_1 x_1}, \dots) \qquad u_{x_k} = \frac{\partial u}{\partial x_k}$$

1.1 Well-Posedness

A problem is called well-posedness, if it satisfies all of the following criteria:

1. **Existence**: The problem has a solution
2. **Uniqueness**: There is no more than one solution
3. **Stability**: A small change in the equation or in the side conditions gives rise to a small change in the solution

1.2 Classification

- **Order**: The order of a PDE is the order of the highest derivative in the equation.
- **Linear**: An PDE is called linear if the unknown function u and it's derivatives occur only in a linear relationship.
- **Semilinear**: A PDE is called semilinear if only the unknown function u occurs in a non-linear relationship, but all the partial derivatives of u occur linear.
- **Quasilinear**: A PDE is called quasilinear if the highest order derivative occurs linear in F, but lower order derivatives of u and u itself occur non-linear.

1.3 Strong vs. Weak Solutions

The Set $C^k(D)$ contains all functions that are k-times differential in D. A Function in the Set $C^k(D)$, that satisfies a PDE of order k is called a **strong** (or **classical**) **solution**. If the solution is not k times differential, it is called a **weak solution**.

1.4 Differential Operators

The operation of an operator L on a function u is denoted by $L[u]$ A differential Operator is defined by partial derivatives of functions. Example:

$$\text{Laplace Operator:} \quad \Delta = \left[\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right]$$

A **linear operator** satisfies the following equation:

$$L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2]$$

1.5 Initial Conditions

A problem is called an **initial value problem** if a condition at $t = t_0$ is given. Example of the heat equation:

$$u_t - \Delta u = 0 \qquad u(t = 0, x, y, z) = u_0$$

1.6 Boundary Conditions

Boundary conditions are conditions on the behavior of the solution (or it's derivatives) at the boundary $\partial\Omega$ of the domain Ω.

- **Dirichlet condition**: In this condition, the values at the boundary $\partial\Omega$ are given (e.g. by measurements)

$$u(x, y, z, t) = f(x, y, z, t) \qquad (x, y, z) \in \partial\Omega,\ t > 0$$

- **Neumann condition**: Here, the normal derivative $\partial_n u$ of the unknown function u. $\partial_n u$ denotes the outward normal derivative at $\partial\Omega$

$$\partial_n u(x, y, z, t) = f(x, y, z, t) \qquad (x, y, z) \in \partial\Omega,\ t > 0$$

- **A condition of a third kind** is a combination of the two mentioned above (sometimes also called a Robin condition).

2 First-Order PDE

$$F(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots u_{x_n}) = 0$$

2.1 Method of characteristics

Solve the following initial value problem (of order 2):

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$
$$u_x + u_y = 2u+1 \quad u(x,0)=0 \qquad a=1, \quad b=1, \quad c=2$$

1. write the characteristic equation and parametric initial condition (for the initial condition, substitute s for x or y, depending on the initial condition.)

$$\frac{\partial x}{\partial t}(t,s) = a(x,y,u) \quad \frac{\partial y}{\partial t}(t,s) = a(x,y,u) \quad \frac{\partial u}{\partial t}(t,s) = c(x,y,u)$$
$$x_t = 1 \quad y_t = 1 \quad u_t = 2u+1 \qquad x(0,s)=s \qquad y(0,s)=0 \qquad u(0,s)=0$$

2. solve the characteristic equation (by simple integrating or by solving the ODE)

$$x(s,t) = t + f_1(s), \quad y(s,t) = t + f_2(s), \quad u(s,t) = f_3(s)e^{2t} + t + f_4(s)$$

Some example of simple ODEs and how to solve them (use superposition):

$$\frac{\partial x}{\partial t} = \alpha \Rightarrow x(t,s) = \alpha t + C(s)$$
$$\frac{\partial x}{\partial t} = \alpha x \Rightarrow x(t,s) = C(s)e^{\alpha t}$$
$$\frac{\partial x}{\partial t} = \alpha x + \beta e^{\alpha t} \Rightarrow x(t,s) = C(s)e^{\alpha t} + \beta e^{\alpha t}\cdot t$$
$$\frac{\partial x}{\partial t} = \alpha x + \beta e^{\gamma t} \Rightarrow x(t,s) = C(s)e^{\alpha t} - \frac{\beta e^{\gamma t}}{\alpha - \gamma} \quad \alpha \neq \gamma$$
$$\frac{\partial^2 x}{\partial t^2} = -\alpha x \Rightarrow x(t,s) = C_1(s)\cos\alpha t + C_2(s)\sin\alpha t$$

3. Substitute these solutions into the initial condition to get $f_i(s)$. If some $f_i(s)$ are undefined, normalize the equation for simplicity.

$$x(t,s) = t+s \qquad y(t,s) = t \qquad u(t,s) = f(s)e^{2t} + t - f(s),\ f(s) = \frac{1}{2}$$

4. Solve for (t,s) as a function of (x,y) and write the solution u:

$$t = y, \qquad s = x-y \qquad u(x,y) = \frac{1}{2}e^{2y} + y - \frac{1}{2}$$

2.2 Invertibility $(s,t) \to (x,y)$ (transversality)

The relation $(s,t) \to (x,y)$ is **locally** invertable, if and only if:

$$\det\begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix}\Bigg|_{x_0,y_0,u_0} = \det\begin{pmatrix} a(x_0,y_0,u_0) & b(x_0,y_0,u_0) \\ \frac{\partial x_0(s)}{\partial s} & \frac{\partial y_0(s)}{\partial s} \end{pmatrix} \neq 0$$

2.3 Existence of a solution

Sometimes, a calculated solution to the PDE is not valid. For this, let $\gamma(s) = (x_0(s), y_0(s), 0)$ be the projection of the initial condition to the (x,y)-plane. if $(x(s,t_0), y(s,t_0), 0) = \gamma(\tilde{s})$ for $t_0 \neq 0$, we must check if the value of the curve equals the value of the initial condition. If $u_0(\tilde{s}) \neq u(s,t_0)$, the solution is not valid.
Example: consider the following PDE:

$$-yu_x + xu_y = u \quad u(x,0) = x^2, \qquad x_0(s) = s,\ y_0(s) = 0,\ u_0(s) = s^2$$
$$x(s,t) = s\cos t, \quad y(s,t) = s\sin(t), \quad u(s,t) = s^2 e^t$$

Let $t_0 = \pi$ and $\tilde{s} = -s$. Then:

$$x(s,t_0)u = x_0(-s),\ y(s,t_0) = y_0(-s),\ u(s,t_0) \neq u_0(-s) \Rightarrow \text{no sol.}$$

2.4 Conservation laws and shock waves

Let's consider the following equation, where y is the time.

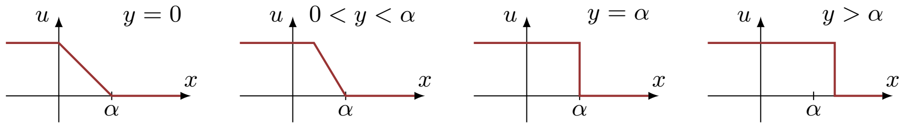
$$u_y + uu_x = 0, \quad u(x,0) = h(x)$$
$$x(t,s) = s + t\cdot h(s), \quad y(t,s) = t, \quad u(t,s) = h(s) \Rightarrow u = h(x-uy)$$

The solution is not well defined at points where characteristic curves intersect. From an algebraic perspective:

$$u_x = \frac{h'}{1+yh'} \qquad y_c = -\frac{1}{\inf\{h'(s)\}}$$

The classical solution is not defined for $y > y_c$. To extend the solution beyond y_c. The solution u has discontinuities at $u(\gamma(y), y)$. u^- and u^+ is the value of u when we approach the curve γ from the left and from the right, respectively. Then, we get:

$$\gamma(y) = \frac{1}{2}(u^- + u^+)$$



If $h(s)$ (the initial condition) is a piecewise function with a negative jump at x_0, we can find the solution the following way. First, rewrite the equation:

$$u_y + uu_x = 0, \quad u(x,0) = h(x) = \begin{cases} u^- & \text{if } x < x_0 \\ u^+ & \text{if } x > x_0 \end{cases}$$

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial y} u = 0 \;\Rightarrow\; \frac{\partial}{\partial x} P(u) + \frac{\partial}{\partial y} Q(u) = 0$$

$$\dot{\gamma} = \frac{P(u^+) - P(u^-)}{Q(u^+) - Q(u^-)} \;\Rightarrow\; \gamma : x = x_0 + \dot{\gamma} y \;\Rightarrow\; u(x,y) = \begin{cases} u^- & \text{if } x < x_0 + \dot{\gamma} y \\ u^+ & \text{if } x > x_0 + \dot{\gamma} y \end{cases}$$

3 Second-Order PDE

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \quad a,b,c,d,e,f,g : \mathbb{R}^2 \mapsto \mathbb{R}$$

$$D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \delta(L) = -\det(D), \quad \begin{cases} \det(D) < 0 & L[u] \text{ is hyperbolic} \\ \det(D) = 0 & L[u] \text{ is parabolic} \\ \det(D) > 0 & L[u] \text{ is elliptic} \end{cases}$$

1. Apply a transformation $(\xi, \eta) \leftrightarrows (x, y)$ (canonical transformation), such that:

$$\xi(x,y), \quad \eta(x,y), \qquad \det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \neq 0$$

$$\tilde{D} = \begin{pmatrix} A(\xi,\eta) & B(\xi,\eta) \\ B(\xi,\eta) & C(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

$$\tilde{D}_{\text{hyp}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{D}_{\text{par}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{D}_{\text{ell}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hyberbolic:	$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$
	$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$
Parabolic :	$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \frac{1}{a}(a\eta_x + b\eta_y)^2 = 0$
	$a\eta_x + b\eta_y = 0$ (Solve this first order PDE for $s(x,y)$)
Elliptic :	$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 1$
	$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 1$
	$B(\xi,\eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0$
Hint:	$\xi_x^2 + 2\xi_x\xi_y + \xi_y^2 = 0 \Rightarrow \xi_x = 1,\ \xi_y = -1 \Rightarrow \xi = x - y$
Requirement:	The determinant of the jakobian matrix cannot vanish.

2. Apply this transformation to write down the canonical form (chain rule):

$$w(\xi,\eta) = u\left(x(\xi,\eta), y(\xi,\eta)\right) \qquad u(x,y) = w\left(\xi(x,y), \eta(x,y)\right)$$
$$u_x = w_t\xi_x + w_s\eta_x \qquad u_y = w_t\xi_y + w_s\eta_y$$
$$u_{xx} = w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_t t_{xx} + w_s s_{xx}$$
$$u_{yy} = w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_t t_{yy} + w_s s_{yy}$$
$$u_{xy} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_t t_{xy} + w_s s_{xy}$$

3. Now, the equation looks much easier. The following solutions can be found to the three types of equations:

Hyperbolic:	$w_{\xi\eta} = 0 \;\Rightarrow\; w(\xi,\eta) = F(\xi) + G(\eta)$
Parabolic:	$w_{\xi\xi} = 0$
Elliptic:	$w_{\xi\xi} + w_{\eta\eta} = 0$

3.1 Example: Wave Equation

The general solution to the wave equation is: ($F(\xi)$ and $G(\eta)$ must be two times continuously differentiable!)

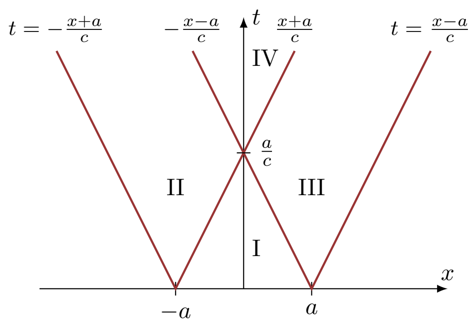
$$u(x,t): \quad u_{tt} - c^2 u_{xx} = 0 \quad \xi = x + ct,\ \eta = x - ct \ \Rightarrow\ -4c^2 w_{\xi\eta} = 0$$
$$w(\xi,\eta) = F(\xi) + G(\eta) \quad u(x,t) = F(x+ct) + G(x-ct)$$

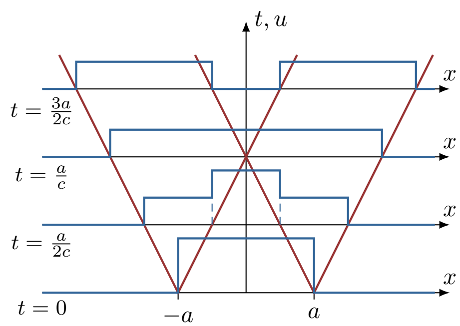
With the initial condition $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we get the following equation (D'Alembert wave equation):

$$u(t,x) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

If the initial conditions are not continuous, we can use the graphical method to derive the solution.

$$u_{tt} - c^2 u_{xx} = 0 \quad u(x,0) = \begin{cases} 2 & \text{if } |x| \le a \\ 0 & \text{otherwise} \end{cases}, \quad u_t(x,0) = 0$$





3.2 Non-Homogeneous Wave equation

For the non-homogeneous case, we must integrate the Force $f(x,t)$ over the triangle corresponding to the domain of dependence (Region I above).

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$
$$u(t,x) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-\tilde{t})}^{x+c(t-\tilde{t})} F(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t}$$

3.2.1 Properties of the Wave Equation

if $f(x)$, $g(x)$ and $F(x,t)$ are odd, even or periodic, the solution $u(x,t)$ is also odd, even or periodic respectively.

$f(x) = f(-x)$	$g(x) = g(-x)$	$F(x,t) = F(-x,t)$	$\Rightarrow$	$u(x,t) = u(-x,t)$
$f(x) = -f(-x)$	$g(x) = -g(-x)$	$F(x,t) = -F(-x,t)$	$\Rightarrow$	$u(x,t) = -u(-x,t)$
$f(x) = f(x+nP)$	$g(x) = g(x+nP)$	$F(x,t) = F(x+nP,t)$	$\Rightarrow$	$u(x,t) = u(x+nP,t)$

3.3 Method of Separation of Variables

3.3.1 Heat Equation

A good example to show the method of separation is the heat equation. Here, we look at a box (in x-direction). Outside of the box, the value u is 0.

$$\frac{1}{\kappa}u_t = u_{xx},\ 0 \le x \le L,\ t > 0, \quad u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0$$

The initial condition $f(x)$ must satisfy: $f(0) = f(L) = 0$, To find a solution, we try the following ansatz:

$$u(x,t) = X(x)\cdot T(t) \ \Rightarrow\ \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

The left side of the equation is only depending on the time t, and the right side only on x. Therefore, it must be equal to a constant $-\lambda$. Now, we have the following equations:

$$X''(x) = -\lambda X(x) \qquad X(0) = X(L) = 0 \qquad T'(t) = -\lambda\kappa T(t)$$

The only case, we get non-trivial solutions is to have $\lambda > 0$. Then, we can write a solution for $X(x)$, where we must also impose the boundary conditions to get the following expression:

$$X_n(x) = C\sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Such a solution $X_n(x)$ is called an **eigenfunction**, and λ an **eigenvalue** Solving the ODE for $T(t)$ gives:

$$T_n'(t) = -\lambda_n\kappa T_n(t) \ \Rightarrow\ T_n(t) = B_n e^{-\left(\frac{n\pi}{L}\right)^2\kappa t}$$

Now, we can write the solution to the PDE. Because of linearity, we can create a superposition of all eigenfunctions. Then, setting $t = 0$, we can see, that $f(x)$ is expressed as a Fourier series.

$$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}$$

$$u(x,t) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}, \quad f(x) = \sum_{n=1}^{N} B_n \sin\left(\frac{n\pi}{L}x\right)$$

3.3.2 Wave Equation

An other example is the wave equation. x is restricted to a certain range.

$$u_{tt} - c^2 u_{xx} = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x), \quad 0 < x < L,\ t > 0$$

$$u(x,t) = T(t) \cdot X(x) \ \Rightarrow\ \frac{X_{xx}}{X} = \frac{T_{tt}}{c^2 T} \ \Rightarrow\ X_{xx} = -\lambda X, \quad T_{tt} = -c^2 \lambda T$$

- **Dirichlet boundary condition:** $u(0,t) = u(L,t) = 0$.

  We first solve the equation for $X(x)$. Implying the boundary condition and requiring a non trivial solution, we get $\lambda > 0$ and $\alpha = 0$.

$$X(x) = \begin{cases} \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ \alpha + \beta x & \text{if } \lambda = 0 \\ \alpha \cos\sqrt{\lambda}x + \beta \sin\sqrt{\lambda}x & \text{if } \lambda > 0 \end{cases}, \ X(0) = X(L) = 0$$

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

  Using this λ_n, we can solve the time-part of the equation:

$$T(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

  Now,we can write the general solution to the equation:

$$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x),\ A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi}{L}x\right) = g(x),\ B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

- **Neumann boundary condition:** $u_x(0,t) = u_x(L,t) = 0$.

  We first solve the equation for $X(x)$. Implying the boundary condition and requiring a non trivial solution, we get $\lambda >= 0$ and $\beta = 0$:

$$X(x) = \begin{cases} \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ \alpha + \beta x & \text{if } \lambda = 0 \\ \alpha \cos\sqrt{\lambda}x + \beta \sin\sqrt{\lambda}x & \text{if } \lambda > 0 \end{cases}, \ X_x(0) = X_x(L) = 0$$

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right), \quad X_0(x) = 1, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

  Using this λ_n, we can solve the time-part of the equation:

$$T(t) = A_0 + B_0 t + A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

  Now,we can write the general solution to the equation:

$$u(x,t) = A_0 + B_0 t + \sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad A_0 = \frac{1}{L}\int_0^L f(x) dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad B_0 = \frac{1}{L}\int_0^L g(x) dx$$

3.3.3 Inhomogeneous Equations

Inhomogeneous Equations with homogeneous boundary conditions:

$$u_t t - c^2 u_x x = F(x),\ u(0,t) = u(L,t) = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x)$$

1. Choose ansatz as if equation was homogeneous (see Dirichlet or Neumann boundary condition):

$$X''(x) = -n^2 X(x) \ \Rightarrow\ u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \quad X(x) = \sin\frac{n\pi x}{L}$$

2. Substitute the genera solution into the equation:

$$\sum_{n=1}^{\infty}\left(T_n'' + n^2c^2T_n\right)\sin\frac{n\pi x}{L} = F(x) = \sin\frac{m\pi x}{L}\sin\omega t$$

3. Write the equation for $T_n(t)$ down for the two possibilities and solve them separately:

$$n \neq m:\; T_n'' + n^2c^2T_n = 0 \xrightarrow{\text{IC}} T_n(t)$$
$$n = m:\; T_m'' + m^2c^2T_m = \sin\omega t$$
$$\Rightarrow\; T_m(t) = \frac{1}{\omega^2 - m^2c^2}\left(\frac{\omega}{mc}\sin\frac{cm\pi t}{L} - \sin\omega t\right)$$

Inhomogeneous Equations with inhomogeneous boundary conditions:

$$r(x)m(t)u_t - \left[(p(x)u_x)_x + q(x)u\right] = F(x,t),\; 0 < x < L$$

$$\alpha u(0,t) + \beta u_x(0,t) = a(t),\; \gamma u(L,t) + \delta u_x(L,t) = b(t),\; u(x,0) = f(x)$$

1. Determine all the parameters mentioned above.

2. Choose an ansatz for $w(x,t)$ from the following table:

boundary condition		$w(x,t)$
Dirichlet	$\beta = \delta = 0$	$w(x,t) = a(t) + \frac{x}{L}\left(b(t) - a(t)\right)$
Neumann	$\alpha = \gamma = 0$	$w(x,t) = xa(t) + \frac{x^2}{2L}\left(b(t) - a(t)\right)$
Mixed	$\beta = \gamma = 0$	$w(x,t) = a(t) + xb(t)$
Mixed	$\alpha = \delta = 0$	$w(x,t) = (x-L)a(t) + b(t)$

3. Write down the transformed equations: $v(x,t) = u(x,t) - w(x,t)$

$$r(x)m(t)v_t - \left[(p(x)v_x)_x + q(x)v\right] = \tilde{F}(x),\; 0 < x < L$$

$$\tilde{F}(x,t) = F(x,t) - r(x)m(t)w_t + \left[(p(x)w_x)_x + q(x)w\right]$$

$$\alpha v(0,t) + \beta v_x(0,t) = 0,\; \gamma v(L,t) + \delta v_x(L,t) = 0,\; v(x,0) = f(x) - w(x,0)$$

4. Solve the inhomogeneous equation with homogeneous boundary conditions.

3.4 Laplace Equation

The Laplace equation is the homogeneous equation $\Delta u = 0$. Solutions to this equation inside the region D are called **harmonic functions** in D. Here, the variables of $u(x,y)$ are spacial (and not time). ν is the unit vector perpendicular to the boundary ∂D, pointing outwards of the region D.
The Poisson equation is $\Delta u = F$, to which we can have the following boundary conditions:

- **Dirichlet problem**: $u(x,y) = g(x,y),\; (x,y) \in \partial D$
- **Neumann problem**: $\frac{\partial u}{\partial \nu}(x,y) = g(x,y),\; (x,y) \in \partial D$. Necessary condition for the existence of a solution to the Neumann problem is: $\int_D F = \int_{\partial D} g$
- **Robin problem**: $u(x,y) + \alpha\frac{\partial u}{\partial \nu}(x,y),\; (x,y) \in \partial D$

3.4.1 Harmonic Functions

D is a bounded region with boundary ∂D. u is a harmonic function solving the equation $\Delta u = 0$.
Weak maximum principle: the maximal and minimal values of u inside D occur on the boundary ∂D:

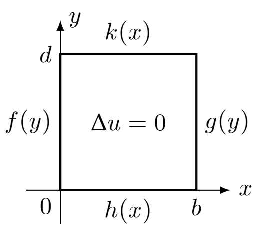
$$\min_{\partial D} u \leq u(x,y) \leq \max_{\partial D} u$$

Strong maximum principle: if u attains it's maximum (or minimum) value inside D, then u is constant.
Mean Value principle: Let $B_R(x_0,y_0) \subseteq D$ be a circular region with radius R and centered at (x_0,y_0) completely contained in D. The function u is harmonic if and only if:

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)d\theta$$

Uniqueness of solutions: Let u solve the equation $\Delta u = F$ inside D with the boundary condition $u(x,y) = g(x,y),\; (x,y) \in \partial D$, then u is unique

3.4.2 Solving with Separation of Variables



To solve the problem, we divide it into two simpler problems, where u_1 is the solution if $h(x) = 0$ and $k(x) = 0$, and u_2 if $f(y) = 0$ and $g(y) = 0$.

For this to work, the boundary condition at the corner must be zero, If not, see next section.

Let's solve for Dirichlet and Neumann BC with $k = h = 0$ (in the other case, just flip x and y):

$$\Delta u = 0 \qquad u(x,y) = X(x) \cdot Y(y) \;\Rightarrow\; \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Dirichlet: $u(0,y) = f,\ u(b,y) = g$	**Neumann**: $u_x(0,y) = f,\ u_x(b,y) = g$
$Y_n(y) = \sin\left(\frac{n\pi}{d} y\right),\ n \in \mathbb{N}$	$Y_n(y) = \cos\left(\frac{n\pi}{d} y\right),\ n \in \mathbb{N}_0$

$$u(x,y) = \sum_n X_n(x) = \sum_n A_n \sinh\left(\frac{n\pi}{d} x\right) + B_n \sinh\left(\frac{n\pi}{d}(x-b)\right)$$

Using this basis (sinh), we have achieved that the term with A_n is zero at $x = 0$ and the term B_n is zero at $x = b$. To calculate the constants:

Dirichlet:

$$f(x) = u(0,y) = \sum_n B_n \sinh\left(-\frac{n\pi}{d} b\right) \sin\left(\frac{n\pi}{d} y\right) = \sum_n \beta_n \sin\left(\frac{n\pi}{d} y\right)$$

$$g(x) = u(b,y) = \sum_n A_n \sinh\left(+\frac{n\pi}{d} b\right) \sin\left(\frac{n\pi}{d} y\right) = \sum_n \alpha_n \sin\left(\frac{n\pi}{d} y\right)$$

$$\beta_n = \frac{2}{d} \int_0^d f(y) \sin\left(\frac{n\pi}{d} y\right) dy, \quad \alpha_n = \frac{2}{d} \int_0^b g(y) \sin\left(\frac{n\pi}{d} y\right) dy$$

Neumann:

$$f(x) = u_x(0,y) = \sum_n \frac{n\pi}{d} B_n \sinh\left(-\frac{n\pi}{d} b\right) \cos\left(\frac{n\pi}{d} y\right) = \sum_n \beta_n \cos\left(\frac{n\pi}{d} y\right)$$

$$g(x) = u_x(b,y) = \sum_n \frac{n\pi}{d} A_n \sinh\left(+\frac{n\pi}{d} b\right) \cos\left(\frac{n\pi}{d} y\right) = \sum_n \alpha_n \cos\left(\frac{n\pi}{d} y\right)$$

$$\beta_n = \frac{2}{d} \int_0^d f(y) \cos\left(\frac{n\pi}{d} y\right) dy, \quad \alpha_n = \frac{2}{d} \int_0^b g(y) \cos\left(\frac{n\pi}{d} y\right) dy$$

3.4.3 Discontinuity on the boundary

If $G : \partial D \to \mathbb{R}$ is the boundary condition and G is continuous and if we need to separate the problem into two smaller problems, we can get discontinuities. To avoid this, we define a new problem with boundary conditions: $H := G - P$

$$H(x,y) = G(x,y) - P(x,y),\ P(x,y) = \gamma_0 + \gamma_1 x + \gamma_2 y + \gamma_3 xy, \quad \Delta P = 0$$

Now, we choose $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ such that $H(x,y) = 0$ at the vertices of rectangle:

$$\gamma_0 = G(0,0),\ \gamma_1 = G(b,0) - G(0,0),\ \gamma_2 = G(0,d) - G(0,0),$$

$$\gamma_3 = G(b,d) - G(0,d) - G(b,0) + G(0,0)$$

3.4.4 Circular Boundary

If we have a circular boundary with radius $\widehat{r}$, we have $u(r,\phi)$ where $x = r\cos\phi$ and $y = r\sin\phi$. We can write the Laplacian as:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = 0, \quad u(r,\phi) = R(r) \cdot \Phi(\phi),\ \Phi(-\pi) = \Phi(\pi)$$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda \;\Rightarrow\; \Phi'' = -\lambda\Phi, \quad R'' + \frac{R'}{r} - \frac{\lambda R}{r^2} = 0$$

$$\Phi(\phi) = A\cos(n\phi) + B\sin(n\phi),\ R(r) = r^n,\ n \in \mathbb{N}_0,\ \lambda = n^2$$

$$u(r,\phi) = \frac{A_0}{2} + \sum_{n\in\mathbb{N}} r^n \left(A_n \cos(n\phi) + b_n \sin(n\phi)\right)$$

$$u(\widehat{r},\phi) = h(\phi) = \frac{A_0}{2} + \sum_{n\in\mathbb{N}} \widehat{r}^n \left(A_n \cos(n\phi) + b_n \sin(n\phi)\right)$$

$$A_n = \frac{1}{\widehat{r}^n \pi} \int_{-\pi}^{\pi} h(\phi)\cos(n\phi) d\phi, \quad B_n = \frac{1}{\widehat{r}^n \pi} \int_{-\pi}^{\pi} h(\phi)\sin(n\phi) d\phi$$

Remark: if the circle does exclude the center, $R(r)$ can be either R_1 or R_2:

$$R_1 = \begin{cases} r^n \\ r^{-n} \end{cases} \qquad R_2 = \begin{cases} 1 \\ \ln r \end{cases}$$

$$R_n(r) = C_n r^n + D_n r^{-n},\ \forall n \in \mathbb{N}, \qquad R_0(r) = R_0 + D_0 \ln r$$

4 General formulas

4.1 Ordinary Differential Equation ODE

4.1.1 General Solutions

- The solution to the first order linear ODE $y' + p(x)y = q(x)$ is:

$$y(x) = e^{P(x)}\left(\int q(x)e^{P(x)}dx + C\right) \quad \text{width} \quad P(x) = \int_0^x p(\tilde{x})d\tilde{x}$$

- The solution of the second order linear ODE $y'' + \lambda y = 0$ is:

$$y(x) = \begin{cases} \alpha e^{\sqrt{-\lambda}x} + \beta e^{\sqrt{-\lambda}x} = \tilde{\alpha}\cosh\left(\sqrt{-\lambda}x\right) + \tilde{\beta}\sinh\left(\sqrt{-\lambda}x\right) & \lambda < 0 \\ \alpha + \beta x & \lambda = 0 \\ \alpha\cos\left(\sqrt{\lambda}x\right) + \beta\sin\left(\sqrt{\lambda}x\right) & \lambda > 0 \end{cases}$$

- The solution to the following Euler (equidimensional) equation is:

$$Ax'2y'' + Bxy' + Cy = 0 \qquad Ar_{1,2}(r_{1,2} - 1) + Br_{1,2} + C = 0$$

$$y(x) = \begin{cases} \alpha x^{r_1} + \beta x^{r_2} & r_1,\ r_2 \in \mathbb{R},\ r_1 \neq r_2 \\ \alpha x^r + \beta x^r \ln x & r_1,\ r_2 \in \mathbb{R},\ r_1 = r_2 = r \\ \alpha x^\lambda \cos(\mu \ln x) + \beta x^\lambda \sin(\mu \ln x) & r_1 = \lambda + i\mu \in \mathbb{C} \end{cases}$$

4.1.2 Nonlinear first order ODE

$$\frac{du}{dx} = f(u) \ \Rightarrow \ \frac{1}{f(u)}du = 1dx \ \Rightarrow \ \int \frac{1}{f(u)}du = \int 1dx$$

4.1.3 Higher Order: Method of characteristics

$$y^{(n)} + \alpha_{n-1}y^{(n-1)} + \dots + \alpha_1 y' + \alpha_0 y = f(x)$$

First, we try to find the homogeneous solution $y_h(x)$, where $f(x) = 0$

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_1\lambda + \alpha_0 = 0 \ \Rightarrow \ (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

To write down the solution, we need to know the multiplicity of a root. If a root is unique, we can just add a exponential. If there are only real roots which are all different, we can write:

$$y_h(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + \dots + C_n e^{\lambda_n x}$$

If we have multiple roots that are the same, we need to multiply the second exponential factor by x, the third by x^2 and so on.

$$y_h(x) = C_1 e^{-\lambda x} + C_2 x e^{-\lambda x} + C_3 x^2 e^{-\lambda x} \cdots$$

If we have two complex roots, e.g. $\lambda = a + ib$, then we write the following:

$$y_h(x) = C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$$

For the particular solution y_p, we use the following ansatz:

- If $f(x)$ is a polynomial of degree n, then $u_p(x)$ is also a polynomial of degree n.
- If $f(x) = be^{kx}$, then $u_p(x) = Be^{kx}$.
- If $f(x) = b\cos(kx)$ or $f(x) = b\sin(kx)$, then $u_p(x) = B\cos(kx) + C\sin(kx)$

After substituting these into the equation and solving it, we get $y_p(x)$. Finally, the solution to the problem is: $y(x) = y_h(x) + y_p(x)$

4.2 Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} \qquad a_0 = \frac{1}{L}\int_0^L f(x)dx$$

4.3 Trigonometry

$$\begin{aligned} \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ \sin^2\alpha &= \tfrac{1}{2}(1 - \cos 2\alpha) \\ \sin^3\alpha &= \tfrac{1}{4}(3\sin\alpha - \sin 3\alpha) \\ \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ 2\sin\alpha\sin\beta &= \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ \sin\alpha + \sin\beta &= 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2} \end{aligned}$$

$$\begin{aligned} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ \cos^2\alpha &= \tfrac{1}{2}(1 + \cos 2\alpha) \\ \cos^3\alpha &= \tfrac{1}{4}(3\cos\alpha + \cos 3\alpha) \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ 2\cos\alpha\cos\beta &= \cos(\alpha - \beta) + \cos(\alpha + \beta) \\ \cos\alpha + \cos\beta &= 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2} \end{aligned}$$

$$\begin{aligned} \int x^n \sin ax\ dx &= -\tfrac{x^n}{a}\cos ax + \tfrac{n}{a}\int x^{n-1}\cos ax\ dx \\ \int x^n \cos ax\ dx &= \tfrac{x^n}{a}\sin ax - \tfrac{n}{a}\int x^{n-1}\sin ax\ dx \\ \int e^{ax}\sin bx\ dx &= \tfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \\ \int e^{ax}\cos bx\ dx &= \tfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \end{aligned}$$

4.4 Operators

$$\nabla u = \operatorname{grad} u = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} u$$

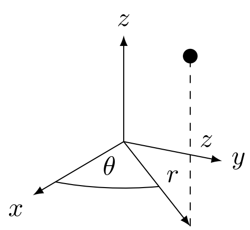
$$\operatorname{div} \vec{u} = \nabla \vec{u} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_x & u_y & u_z \end{pmatrix}^\intercal = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{curl} \vec{u} = \nabla \times \vec{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \vec{e}_z$$

$$\Delta u = \nabla^2 u = \operatorname{div}(\nabla u) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u$$

4.4.1 Differential Operators in Polar Coordinates

We use the notation e_r and e_θ to denote unit vectors in radial and angular direction, respectively, and e_z to denote a unit vector in the z direction. A vector $\vec{u}$ is expressed as $\vec{u} = u_r e_r + u_\theta e_\theta$. We also use $V(r, \theta)$ to denote a scalar function.



$$\nabla V = \frac{\partial V}{\partial r} e_r + \frac{1}{r} \frac{\partial V}{\partial \theta} e_\theta$$

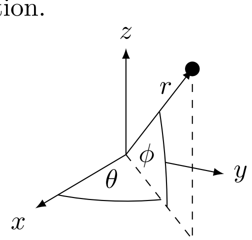
$$\nabla \cdot \vec{u} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(\frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) e_z$$

$$\Delta V = \vec{\nabla} \cdot \vec{\nabla} V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2}$$

4.4.2 Differential Operators in Spherical Coordinates

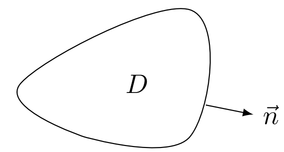
We use the notation e_r, e_θ and e_ϕ to denote unit vectors in the radial, vertical angular direction, and horizontal angular direction, respectively. A vector $\vec{u}$ is expressed as $\vec{u} = u_r e_r + u_\theta e_\theta + u_\phi e_\phi$. We also use $V(r, \theta, \phi)$ to denote a scalar function.



$$\nabla V = \frac{\partial V}{\partial r} e_r + \frac{1}{r} \frac{\partial V}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} e_\phi$$

$$\Delta V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial V}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 V}{\partial \theta^2}$$

4.5 Integration Formulas



$$\iint_D \nabla u \; dx \; dy = \oint_{\partial D} u \cdot \vec{n} \; ds$$

$$\iint_D (Q_x - P_y) \; dx \; dy = \oint_{\partial D} (P \; dx + Q \; dy)$$

$$\iint_D (u \Delta u + \nabla u \cdot \nabla v) dx \; dy = \oint_{\partial D} v \frac{\partial u}{\partial \vec{n}} \; ds$$

$$\iint_D (v \Delta u - u \Delta v) \; dx \; dy = \oint_{\partial D} \left(v \frac{\partial u}{\partial \vec{n}} - u \frac{\partial v}{\partial \vec{n}} \right) ds$$

$$\frac{\partial}{\partial t} \int_{a(t)}^{b(t)} G(\tilde{x}, t) d\tilde{x} = G(b(t), t) b'(t) - G(a(t), t) a'(t) + \int_{a(t)}^{b(t)} \frac{\partial G(\tilde{x}, t)}{\partial t} \; d\tilde{x}$$

4.6 Green's Identities

$$\Delta u = \operatorname{div}(\nabla u) \;\Rightarrow\; \iint_D \Delta u \; dx \; dy = \iint_D \operatorname{div}(\nabla u) \; dx \; dy = \oint_{\partial D} \nabla u \cdot \vec{n} \; ds$$

$$\iint_D (u \Delta v + v \Delta u) \; dx \; dy = \oint_{\partial D} (u \nabla v \cdot \vec{n} + v \nabla u \cdot v) \; ds$$

$$\operatorname{div}(v \nabla u) = \nabla v \nabla u + v \Delta u \;\Rightarrow\; \iint_D \nabla v \nabla u = \oint_{\partial D} v \nabla u \cdot \vec{n} \; ds - \iint_D v \Delta u \; dx \; dy$$