

1 Notation & Classification

$$f(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_1 x_1}, \dots) \quad u_{x_k} = \frac{\partial u}{\partial x_k}$$

1.1 Well-Posedness

A problem is called well-posedness, if it satisfies all of the following criteria:

1. **Existence:** The problem has a solution
2. **Uniqueness:** There is no more than one solution
3. **Stability:** A small change in the equation or in the side conditions gives rise to a small change in the solution

1.2 Classification

- **Order:** The order of a PDE is the order of the highest derivative in the equation.
- **Linear:** An PDE is called linear if the unknown function u and it's derivatives occur only in a linear relationship.
- **Semilinear:** A PDE is called semilinear if only the unknown function u occurs in a non-linear relationship, but all the partial derivatives of u occur linear.
- **Quasilinear:** A PDE is called quasilinear if the highest order derivative occurs linear in F , but lower order derivatives of u and u itself occur non-linear.

1.3 Strong vs. Weak Solutions

The Set $C^k(D)$ contains all functions that are k -times differential in D . A Function in the Set $C^k(D)$, that satisfies a PDE of order k is called a **strong** (or **classical**) **solution**. If the solution is not k times differential, it is called a **weak solution**.

1.4 Differential Operators

The operation of an operator L on a function u is denoted by $L[u]$ A differential Operator is defined by partial derivatives of functions. Example:

$$\text{Laplace Operator: } \Delta = \left[\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right]$$

A **linear operator** satisfies the following equation:

$$L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2]$$

1.5 Initial Conditions

A problem is called an **initial value problem** if a condition at $t = t_0$ is given. Example of the heat equation:

$$u_t - \Delta u = 0 \quad u(t = 0, x, y, z) = u_0$$

1.6 Boundary Conditions

Boundary conditions are conditions on the behavior of the solution (or it's derivatives) at the boundary $\partial\Omega$ of the domain Ω .

- **Dirichlet condition:** In this condition, the values at the boundary $\partial\Omega$ are given (e.g. by measurements)

$$u(x, y, z, t) = f(x, y, z, t) \quad (x, y, z) \in \partial\Omega, t > 0$$

- **Neumann condition:** Here, the normal derivative $\partial_n u$ of the unknown function u . $\partial_n u$ denotes the outward normal derivative at $\partial\Omega$

$$\partial_n u(x, y, z, t) = f(x, y, z, t) \quad (x, y, z) \in \partial\Omega, t > 0$$

- **A condition of a third kind** is a combination of the two mentioned above (sometimes also called a Robin condition).

2 First-Order PDE

$$F(x_1, x_2, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_n}) = 0$$

2.1 Method of characteristics

Solve the following initial value problem (of order 2):

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

$$u_x + u_y = 2u + 1 \quad u(x, 0) = 0 \quad a = 1, \quad b = 1, \quad c = 2$$

1. write the characteristic equation and parametric initial condition (for the initial condition, substitute s for x or y , depending on the initial condition.)

$$\frac{\partial x}{\partial t}(t, s) = a(x, y, u) \quad \frac{\partial y}{\partial t}(t, s) = b(x, y, u) \quad \frac{\partial u}{\partial t}(t, s) = c(x, y, u)$$

$$x_t = 1 \quad y_t = 1 \quad u_t = 2u + 1 \quad x(0, s) = s \quad y(0, s) = 0 \quad u(0, s) = 0$$

2. solve the characteristic equation (by simple integrating or by solving the ODE)

$$x(s, t) = t + f_1(s), \quad y(s, t) = t + f_2(s), \quad u(s, t) = f_3(s)e^{2t} + t + f_4(s)$$

Some example of simple ODEs and how to solve them (use superposition):

$$\frac{\partial x}{\partial t} = \alpha \Rightarrow x(t, s) = \alpha t + C(s)$$

$$\frac{\partial x}{\partial t} = \alpha x \Rightarrow x(t, s) = C(s)e^{\alpha t}$$

$$\frac{\partial x}{\partial t} = \alpha x + \beta e^{\alpha t} \Rightarrow x(t, s) = C(s)e^{\alpha t} + \beta e^{\alpha t} \cdot t$$

$$\frac{\partial x}{\partial t} = \alpha x + \beta e^{\gamma t} \Rightarrow x(t, s) = C(s)e^{\alpha t} - \frac{\beta e^{\gamma t}}{\alpha - \gamma} \quad \alpha \neq \gamma$$

$$\frac{\partial^2 x}{\partial t^2} = -\alpha x \Rightarrow x(t, s) = C_1(s) \cos \alpha t + C_2(s) \sin \alpha t$$

3. Substitute these solutions into the initial condition to get $f_i(s)$. If some $f_i(s)$ are undefined, normalize the equation for simplicity.

$$x(t, s) = t + s \quad y(t, s) = t \quad u(t, s) = f(s)e^{2t} + t - f(s), \quad f(s) = \frac{1}{2}$$

4. Solve for (t, s) as a function of (x, y) and write the solution u :

$$t = y, \quad s = x - y \quad u(x, y) = \frac{1}{2}e^{2y} + y - \frac{1}{2}$$

2.2 Invertibility $(s, t) \rightarrow (x, y)$ (transversality)

The relation $(s, t) \rightarrow (x, y)$ is **locally** invertible, if and only if:

$$\det \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} \bigg|_{x_0, y_0, u_0} = \det \begin{pmatrix} a(x_0, y_0, u_0) & b(x_0, y_0, u_0) \\ \frac{\partial x_0(s)}{\partial s} & \frac{\partial y_0(s)}{\partial s} \end{pmatrix} \neq 0$$

2.3 Existence of a solution

Sometimes, a calculated solution to the PDE is not valid. For this, let $\gamma(s) = (x_0(s), y_0(s), 0)$ be the projection of the initial condition to the (x, y) -plane. if $(x(s, t_0), y(s, t_0), 0) = \gamma(\tilde{s})$ for $t_0 \neq 0$, we must check if the value of the curve equals the value of the initial condition. If $u_0(\tilde{s}) \neq u(s, t_0)$, the solution is not valid.

Example: consider the following PDE:

$$-yu_x + xu_y = u \quad u(x, 0) = x^2, \quad x_0(s) = s, \quad y_0(s) = 0, \quad u_0(s) = s^2$$

$$x(s, t) = s \cos t, \quad y(s, t) = s \sin t, \quad u(s, t) = s^2 e^t$$

Let $t_0 = \pi$ and $\tilde{s} = -s$. Then:

$$x(s, t_0)u = x_0(-s), \quad y(s, t_0) = y_0(-s), \quad u(s, t_0) \neq u_0(-s) \Rightarrow \text{no sol.}$$

2.4 Conservation laws and shock waves

Let's consider the following equation, where y is the time.

$$u_y + uu_x = 0, \quad u(x, 0) = h(x)$$

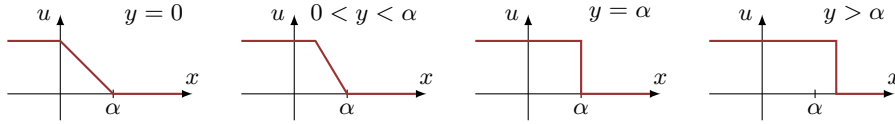
$$x(t, s) = s + t \cdot h(s), \quad y(t, s) = t, \quad u(t, s) = h(s) \Rightarrow u = h(x - uy)$$

The solution is not well defined at points where characteristic curves intersect. From an algebraic perspective:

$$u_x = \frac{h'}{1 + yh'} \quad y_c = \inf \left\{ -\frac{1}{h'(s)} \right\}$$

The classical solution is not defined for $y > y_c$. To extend the solution beyond y_c . The solution u has discontinuities at $u(\gamma(y), y)$. u^- and u^+ is the value of u when we approach the curve γ from the left and from the right, respectively. Then, we get:

$$\gamma(y) = \frac{1}{2}(u^- + u^+)$$



If $h(s)$ (the initial condition) is a piecewise function with a negative jump at x_0 , we can find the solution the following way. First, rewrite the equation:

$$u_y + uu_x = 0, \quad u(x, 0) = h(x) = \begin{cases} u^- & \text{if } x < x_0 \\ u^+ & \text{if } x > x_0 \end{cases}$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}\left(\frac{1}{2}u^2\right) = 0 \Rightarrow \frac{\partial}{\partial x}P(u) + \frac{\partial}{\partial y}Q(u) = 0$$

$$\dot{\gamma} = \frac{Q(u^+) - Q(u^-)}{P(u^+) - P(u^-)} \Rightarrow \gamma : x = x_0 + \dot{\gamma}y \Rightarrow u(x, y) = \begin{cases} u^- & \text{if } x < x_0 + \dot{\gamma}y \\ u^+ & \text{if } x > x_0 + \dot{\gamma}y \end{cases}$$

3 Second-Order PDE

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \quad a, b, c, d, e, f, g : \mathbb{R}^2 \mapsto \mathbb{R}$$

$$D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \delta(L) = -\det(D), \quad \begin{cases} \det(D) < 0 & L[u] \text{ is hyperbolic} \\ \det(D) = 0 & L[u] \text{ is parabolic} \\ \det(D) > 0 & L[u] \text{ is elliptic} \end{cases}$$

1. Apply a transformation $(\xi, \eta) \Leftrightarrow (x, y)$ (canonical transformation), such that:

$$\xi(x, y), \quad \eta(x, y), \quad \det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} \neq 0$$

$$\tilde{D} = \begin{pmatrix} A(\xi, \eta) & B(\xi, \eta) \\ B(\xi, \eta) & C(\xi, \eta) \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

$$\tilde{D}_{\text{hyp}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{D}_{\text{par}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{D}_{\text{ell}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hyperbolic: $A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$
 $C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$

Parabolic : $C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \frac{1}{a}(a\eta_x + b\eta_y)^2 = 0$
 $a\eta_x + b\eta_y = 0$ (Solve this first order PDE for $s(x, y)$)

Elliptic : $A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 1$
 $C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 1$
 $B(\xi, \eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0$

Hint: $\xi_x^2 + 2\xi_x\xi_y + \xi_y^2 = 0 \Rightarrow \xi_x = 1, \xi_y = -1 \Rightarrow t = x - y$
Requirement: The determinant of the jacobian matrix cannot vanish.

2. Apply this transformation to write down the canonical form (chain rule):

$$w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \quad u(x, y) = w(t(x, y), s(x, y))$$

$$u_x = w_t\xi_x + w_s\eta_x \quad u_y = w_t\xi_y + w_s\eta_y$$

$$u_{xx} = w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_{tt}t_{xx} + w_{ts}s_{xx}$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_{tt}t_{yy} + w_{ts}s_{yy}$$

$$u_{xy} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_{tt}t_{xy} + w_{ts}s_{xy}$$

3. Now, the equation looks much easier. The following solutions can be found to the three types of equations:

Hyperbolic: $w_{\xi\eta} = 0 \Rightarrow w(\xi, \eta) = F(t) + G(s)$

Parabolic: $w_{\xi\xi} = 0$

Elliptic: $w_{\xi\xi} + w_{\eta\eta} = 0$

3.1 Example: Wave Equation

The general solution to the wave equation is: $(F(\xi)$ and $G(\eta)$ must be two times continuously differentiable!)

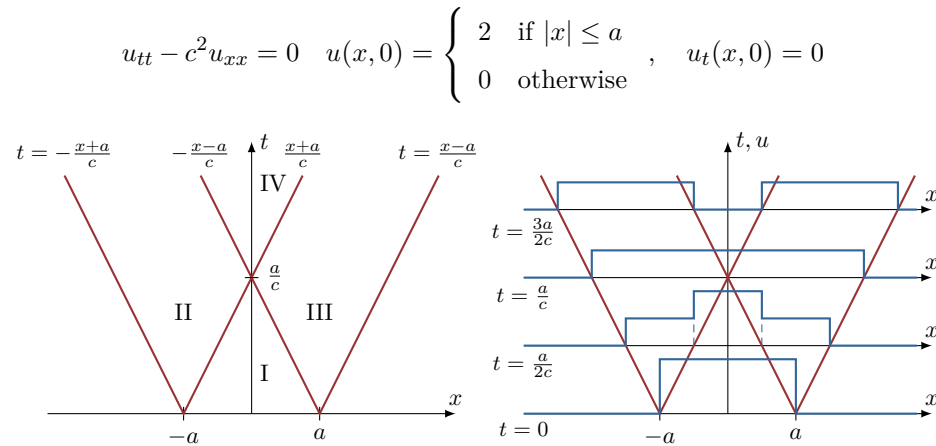
$$u(x, t) : \quad u_{tt} - c^2 u_{xx} = 0 \quad \xi = x + ct, \quad \eta = x - ct \Rightarrow -4c^2 u_{\xi\eta} = 0$$

$$w(\xi, \eta) = F(\xi) + G(\eta) \quad u(x, t) = F(x + ct) + G(x - ct)$$

With the initial condition $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, we get the following equation (D'Alembert wave equation):

$$u(t, x) = \frac{1}{2} (f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

If the initial conditions are not continuous, we can use the graphical method to derive the solution.



3.2 Non-Homogeneous Wave equation

For the non-homogeneous case, we must integrate the Force $f(x, t)$ over the triangle corresponding to the domain of dependence (Region I above).

$$u_{tt} - c^2 u_{xx} = F(x, t), \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

$$u(t, x) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

$$+ \frac{1}{2c} \int_0^t \int_{x-c(t-\tilde{t})}^{x+c(t-\tilde{t})} F(\tilde{x}, \tilde{t}) d\tilde{x} d\tilde{t}$$

3.2.1 Properties of the Wave Equation

if $f(x)$, $g(x)$ and $F(x, t)$ are odd, even or periodic, the solution $u(x, t)$ is also odd, even or periodic respectively.

$$\begin{array}{llll} f(x) = f(-x) & g(x) = g(-x) & F(x, t) = F(-x, t) & \Rightarrow u(x, t) = u(-x, t) \\ f(x) = -f(-x) & g(x) = -g(-x) & F(x, t) = -F(-x, t) & \Rightarrow u(x, t) = -u(-x, t) \\ f(x) = f(x + nP) & g(x) = g(x + nP) & F(x, t) = F(x + nP, t) & \Rightarrow u(x, t) = u(x + nP, t) \end{array}$$

3.3 Method of Separation of Variables

3.3.1 Heat Equation

A good example to show the method of separation is the heat equation. Here, we look at a box (in x -direction). Outside of the box, the value u is 0.

$$\frac{1}{\kappa} u_t = u_{xx}, \quad 0 \leq x \leq L, \quad t > 0, \quad u(x, 0) = f(x), \quad u(0, t) = u(L, t) = 0$$

The initial condition $f(x)$ must satisfy: $f(0) = f(L) = 0$. To find a solution, we try the following ansatz:

$$u(x, t) = X(x) \cdot T(t) \Rightarrow \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

The left side of the equation is only depending on the time t , and the right side only on x . Therefore, it must be equal to a constant $-\lambda$. Now, we have the following equations:

$$X''(x) = -\lambda X(x) \quad X(0) = X(L) = 0 \quad T'(t) = -\lambda \kappa T(t)$$

The only case, we get non-trivial solutions is to have $\lambda > 0$. Then, we can write a solution for $X(x)$, where we must also impose the boundary conditions to get the following expression:

$$X_n(x) = C \sin\left(\frac{n\pi}{L} x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Such a solution $X_n(x)$ is called an **eigenfunction**, and λ an **eigenvalue**. Solving the ODE for $T(t)$ gives:

$$T'_n(t) = -\lambda_n \kappa T_n(t) \Rightarrow T_n(t) = B_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}$$

Now, we can write the solution to the PDE. Because of linearity, we can create a superposition of all eigenfunctions. Then, setting $t = 0$, we can see, that $f(x)$ is expressed as a Fourier series.

$$u_n(x, t) = B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}$$

$$u(x, t) = \sum_{n=1}^N B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t}, \quad f(x) = \sum_{n=1}^N B_n \sin\left(\frac{n\pi}{L}x\right)$$

3.3.2 Wave Equation

An other example is the wave equation. x is restricted to a certain range.

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L, \quad t > 0$$

$$u(x, t) = T(t) \cdot X(x) \Rightarrow \frac{X_{xx}}{X} = \frac{T_{tt}}{c^2 T} \Rightarrow X_{xx} = -\lambda X, \quad T_{tt} = -c^2 \lambda T$$

- **Dirichlet boundary condition:** $u(0, t) = u(L, t) = 0$.

We first solve the equation for $X(x)$. Impling the boundary condition and requiring a non trivial solution, we get $\lambda > 0$ and $\alpha = 0$.

$$X(x) = \begin{cases} \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ \alpha + \beta x & \text{if } \lambda = 0 \\ \alpha \cos \sqrt{\lambda}x + \beta \sin \sqrt{\lambda}x & \text{if } \lambda > 0 \end{cases}, \quad X(0) = X(L) = 0$$

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

Using this λ_n , we can solve the time-part of the equation:

$$T(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

Now, we can write the general solution to the equation:

$$u(x, t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right) \right]$$

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x), \quad A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi}{L}x\right) = g(x), \quad B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

- **Neumann boundary condition:** $u_x(0, t) = u_x(L, t) = 0$.

We first solve the equation for $X(x)$. Impling the boundary condition and requiring a non trivial solution, we get $\lambda \geq 0$ and $\beta = 0$:

$$X(x) = \begin{cases} \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x} & \text{if } \lambda < 0 \\ \alpha + \beta x & \text{if } \lambda = 0 \\ \alpha \cos \sqrt{\lambda}x + \beta \sin \sqrt{\lambda}x & \text{if } \lambda > 0 \end{cases}, \quad X_x(0) = X_x(L) = 0$$

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right), \quad X_0(x) = 1, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}$$

Using this λ_n , we can solve the time-part of the equation:

$$T(t) = A_0 + B_0 t + A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

Now, we can write the general solution to the equation:

$$u(x, t) = A_0 + B_0 t + \sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right) \right]$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad A_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad B_0 = \frac{1}{L} \int_0^L g(x) dx$$

3.3.3 Inhomogeneous Equations

Inhomogeneous Equations with homogeneous boundary conditions:

$$u_{tt} - c^2 u_{xx} = F(x), \quad u(0, t) = u(L, t) = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

1. Choose ansatz as if equation was homogeneous (see Dirichlet or Neumann boundary condition):

$$X''(x) = -n^2 X(x) \Rightarrow u(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \quad X(x) = \sin \frac{n\pi x}{L}$$

2. Substitute the general solution into the equation:

$$\sum_{n=1}^{\infty} \left(T_n'' + n^2 c^2 T_n \right) \sin \frac{n\pi x}{L} = F(x) = \sin \frac{m\pi x}{L} \sin \omega t$$

3. Write the equation for $T_n(t)$ down for the two possibilities and solve them separately:

$$\begin{aligned} n \neq m : T_n'' + n^2 c^2 T_n &= 0 \xrightarrow{\text{IC}} T_n(t) \\ n = m : T_m'' + m^2 c^2 T_m &= \sin \omega t \\ \Rightarrow T_m(t) &= \frac{1}{\omega^2 - m^2 c^2} \left(\frac{\omega}{mc} \sin \frac{cm\pi t}{L} - \sin \omega t \right) \end{aligned}$$

Inhomogeneous Equations with inhomogeneous boundary conditions:

$$r(x)m(t)u_t - [(p(x)u_x)_x + q(x)u] = F(x, t), \quad 0 < x < L$$

$$\alpha u(0, t) + \beta u_x(0, t) = a(t), \quad \gamma u(L, t) + \delta u_x(L, t) = b(t), \quad u(x, 0) = f(x)$$

1. Determine all the parameters mentioned above.
2. Choose an ansatz for $w(x, t)$ from the following table:

boundary condition		$w(x, t)$
Dirichlet	$\beta = \delta = 0$	$w(x, t) = a(t) + \frac{x}{L} (b(t) - a(t))$
Neumann	$\alpha = \gamma = 0$	$w(x, t) = xa(t) + \frac{x^2}{2L} (b(t) - a(t))$
Mixed	$\beta = \gamma = 0$	$w(x, t) = a(t) + xb(t)$
Mixed	$\alpha = \delta = 0$	$w(x, t) = (x - L)a(t) + b(t)$

3. Write down the transformed equations: $v(x, t) = u(x, t) - w(x, t)$

$$r(x)m(t)v_t - [(p(x)v_x)_x + q(x)v] = \tilde{F}(x), \quad 0 < x < L$$

$$\tilde{F}(x, t) = F(x, t) - r(x)m(t)w_t + [(p(x)w_x)_x + q(x)w]$$

$$\alpha v(0, t) + \beta v_x(0, t) = 0, \quad \gamma v(L, t) + \delta v_x(L, t) = 0, \quad v(x, 0) = f(x) - w(x, 0)$$

4. Solve the inhomogeneous equation with homogeneous boundary conditions.

3.4 Laplace Equation

The Laplace equation is the homogeneous equation $\Delta u = 0$. Solutions to this equation inside the region D are called **harmonic functions** in D . Here, the variables of $u(x, y)$ are spacial (and not time). ν is the unit vector perpendicular to the boundary ∂D , pointing outwards of the region D .

The Poisson equation is $\Delta u = F$, to which we can have the following boundary conditions:

- **Dirichlet problem:** $u(x, y) = g(x, y), \quad (x, y) \in \partial D$
- **Neumann problem:** $\frac{\partial u}{\partial \nu}(x, y) = g(x, y), \quad (x, y) \in \partial D$. Necessary condition for the existence of a solution to the Neumann problem is: $\int_D F = \int_{\partial D} g$
- **Robin problem:** $u(x, y) + \alpha \frac{\partial u}{\partial \nu}(x, y), \quad (x, y) \in \partial D$

3.4.1 Harmonic Functions

D is a bounded region with boundary ∂D . u is a harmonic function solving the equation $\Delta u = 0$.

Weak maximum principle: the maximal and minimal values of u inside D occur on the boundary ∂D :

$$\min_{\partial D} u \leq u(x, y) \leq \max_{\partial D} u$$

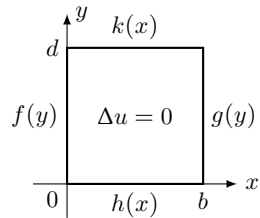
Strong maximum principle: if u attains its maximum (or minimum) value inside D , then u is constant.

Mean Value principle: Let $B_R(x_0, y_0) \subseteq D$ be a circular region with radius R and centered at (x_0, y_0) completely contained in D . The function u is harmonic if and only if:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R \cos \theta, y_0 + R \sin \theta) d\theta$$

Uniqueness of solutions: Let u solve the equation $\Delta u = F$ inside D with the boundary condition $u(x, y) = g(x, y), \quad (x, y) \in \partial D$, then u is unique

3.4.2 Solving with Separation of Variables



To solve the problem, we divide it into two simpler problems, where u_1 is the solution if $h(x) = 0$ and $k(x) = 0$, and u_2 if $f(y) = 0$ and $g(y) = 0$.

For this to work, the boundary condition at the corner must be zero, If not, see next section.

Let's solve for Dirichlet and Neumann BC with $k = h = 0$ (in the other case, just flip x and y):

$$\Delta u = 0 \quad u(x, y) = X(x) \cdot Y(y) \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Dirichlet: $u(0, y) = f, \quad u(b, y) = g$ $Y_n(y) = \sin\left(\frac{n\pi}{d}y\right), \quad n \in \mathbb{N}$ $u(x, y) = \sum_n X_n(x) = \sum_n A_n \sinh\left(\frac{n\pi}{d}x\right) + B_n \sinh\left(\frac{n\pi}{d}(x-b)\right)$	Neumann: $u_x(0, y) = f, \quad u_x(b, y) = g$ $Y_n(y) = \cos\left(\frac{n\pi}{d}y\right), \quad n \in \mathbb{N}_0$
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Using this basis (sinh), we have achieved that the term with A_n is zero at $x = 0$ and the term B_n is zero at $x = b$. To calculate the constants:

Dirichlet:

$$\begin{aligned} f(x) = u(0, y) &= \sum_n B_n \sinh\left(-\frac{n\pi}{d}b\right) \sin\left(\frac{n\pi}{d}y\right) = \sum_n \beta_n \sin\left(\frac{n\pi}{d}y\right) \\ g(x) = u(b, y) &= \sum_n A_n \sinh\left(+\frac{n\pi}{d}b\right) \sin\left(\frac{n\pi}{d}y\right) = \sum_n \alpha_n \sin\left(\frac{n\pi}{d}y\right) \\ \beta_n &= \frac{2}{d} \int_0^d f(y) \sin\left(\frac{n\pi}{d}y\right) dy, \quad \alpha_n = \frac{2}{d} \int_0^b g(y) \sin\left(\frac{n\pi}{d}y\right) dy \end{aligned}$$

Neumann:

$$\begin{aligned} f(x) = u_x(0, y) &= \sum_n \frac{n\pi}{d} B_n \sinh\left(-\frac{n\pi}{d}b\right) \cos\left(\frac{n\pi}{d}y\right) = \sum_n \beta_n \cos\left(\frac{n\pi}{d}y\right) \\ g(x) = u_x(b, y) &= \sum_n \frac{n\pi}{d} A_n \sinh\left(+\frac{n\pi}{d}b\right) \cos\left(\frac{n\pi}{d}y\right) = \sum_n \alpha_n \cos\left(\frac{n\pi}{d}y\right) \\ \beta_n &= \frac{2}{d} \int_0^d f(y) \cos\left(\frac{n\pi}{d}y\right) dy, \quad \alpha_n = \frac{2}{d} \int_0^b g(y) \cos\left(\frac{n\pi}{d}y\right) dy \end{aligned}$$

3.4.3 Discontinuity on the boundary

If $G : \partial D \rightarrow \mathbb{R}$ is the boundary condition and G is continuous and if we need to separate the problem into two smaller problems, we can get discontinuities. To avoid this, we define a new problem with boundary conditions: $H := G - P$

$$H(x, y) = G(x, y) - P(x, y), \quad P(x, y) = \gamma_0 + \gamma_1 x + \gamma_2 y + \gamma_3 xy, \quad \Delta P = 0$$

Now, we choose $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ such that $H(x, y) = 0$ at the vertices of rectangle:

$$\begin{aligned} \gamma_0 &= G(0, 0), \quad \gamma_1 = G(b, 0) - G(0, 0), \quad \gamma_2 = G(0, d) - G(0, 0), \\ \gamma_3 &= G(b, d) - G(0, d) - G(b, 0) + G(0, 0) \end{aligned}$$

3.4.4 Circular Boundary

If we have a circular boundary with radius \hat{r} , we have $u(r, \phi)$ where $x = r \cos \phi$ and $y = r \sin \phi$. We can write the Laplacian as:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} = 0, \quad u(r, \phi) = R(r) \cdot \Phi(\phi), \quad \Phi(-\pi) = \Phi(\pi)$$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{\Phi''}{\Phi} = \lambda \Rightarrow \Phi'' = -\lambda \Phi, \quad R'' + \frac{R'}{r} - \frac{\lambda R}{r^2} = 0$$

$$\Phi(\phi) = A \cos(n\phi) + B \sin(n\phi), \quad R(r) = r^n, \quad n \in \mathbb{N}_0, \quad \lambda = n^2$$

$$u(r, \phi) = \frac{A_0}{2} + \sum_{n \in \mathbb{N}} r^n (A_n \cos(n\phi) + b_n \sin(n\phi))$$

$$u(\hat{r}, \phi) = h(\phi) = \frac{A_0}{2} + \sum_{n \in \mathbb{N}} \hat{r}^n (A_n \cos(n\phi) + b_n \sin(n\phi))$$

$$A_n = \frac{1}{\hat{r}^n \pi} \int_{-\pi}^{\pi} h(\phi) \cos(n\phi) d\phi, \quad B_n = \frac{1}{\hat{r}^n \pi} \int_{-\pi}^{\pi} h(\phi) \sin(n\phi) d\phi$$

Remark: if the circle does exclude the center, $R(r)$ can be either R_1 or R_2 :

$$R_1 = \begin{cases} r^n \\ r^{-n} \end{cases} \quad R_2 = \begin{cases} 1 \\ \ln r \end{cases}$$

$$R_n(r) = C_n r^n + D_n r^{-n}, \quad \forall n \in \mathbb{N}, \quad R_0(r) = R_0 + D_0 \ln r$$

4 General formulas

4.1 Ordinary Differential Equation ODE

4.1.1 linear first order ODE

Solve the ODE: $u'(x) + \lambda u(x) = f(x)$. First, solve for the **homogeneous** solution: $u_h(x) = Ae^{-\lambda x}$. For the **particular** solution, use the following ansatz:

- If $f(x)$ is a polynomial of degree n , then $u_p(x)$ is also a polynomial of degree n .
- If $f(x) = be^{kx}$, then $u_p(x) = Be^{kx}$.
- If $f(x) = b \cos(kx)$ or $f(x) = b \sin(kx)$, then $u_p(x) = B \cos(kx) + C \sin(kx)$

Finally, the solution to the equation is: $u(x) = u_h(x) + u_p(x)$

4.1.2 Nonlinear first order ODE

$$\frac{du}{dx} = f(u) \Rightarrow \frac{1}{f(u)} du = 1 dx \Rightarrow \int \frac{1}{f(u)} du = \int 1 dx$$

4.2 Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \quad a_0 = \frac{1}{L}$$

4.3 Trigonometry

$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$	$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$
$\sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha)$	$\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha)$
$\sin^3 \alpha = \frac{1}{4} (3 \sin \alpha - \sin 3\alpha)$	$\cos^3 \alpha = \frac{1}{4} (3 \cos \alpha + \cos 3\alpha)$
$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$	$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$	$2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$