

# 1 The Photon

$c \left[ \frac{\text{m}}{\text{s}} \right]$	speed of light
$h \left[ \frac{\text{m}^2 \text{kg}}{\text{s}} \right]$	planc's constant
$e \text{ [C]}$	electorn charge
$m_e \text{ [kg]}$	electron mass
$k_B \left[ \frac{\text{m}^2 \text{kg}}{\text{s}^2 \text{K}} \right]$	bolzmann constant
$\epsilon_0 \left[ \frac{\text{F}}{\text{m}} \right]$	vacuum permittivity

$$c = 2.998 \cdot 10^8 \left[ \frac{\text{m}}{\text{s}} \right]$$

$$h = 6.626 \cdot 10^{-34} \left[ \frac{\text{m}^2 \text{kg}}{\text{s}} \right]$$

$$\hbar = \frac{h}{2\pi}$$

$$e = 1.602 \cdot 10^{-19} \text{ [C]}$$

$$m_e = 9.109 \cdot 10^{-31} \text{ [kg]}$$

$$k_B = 1.381 \cdot 10^{-23} \left[ \frac{\text{m}^2 \text{kg}}{\text{s}^2 \text{K}} \right]$$

$$\epsilon_0 = 8.854 \cdot 10^{-12} \left[ \frac{\text{F}}{\text{m}} \right]$$

$$1 \text{ [eV]} = 1.602 \cdot 10^{-19} \left[ \frac{\text{kg m}^2}{\text{s}^2} \right] \text{ [J]}$$

## 1.1 Photon & Electron

$\lambda \text{ [m]}, \nu \left[ \frac{1}{\text{s}} \right]$	Wavelength, Freq.
$k$	Wavenumber
$E \text{ [J]}$	Energy
$\vec{F}_c \text{ [N]}$	Coulomb Force

$$\lambda = \frac{c}{\nu} \quad \nu = \frac{c}{\lambda} \quad \omega = 2\pi\nu$$

$$k = \frac{2\pi\nu}{c}$$

$$E = h \cdot \nu = \hbar \cdot \omega$$

$$\left| \vec{F}_c \right| = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 r^2}$$

## 1.2 Photoelectric effect

$V \text{ [V]}$	Voltage
$\phi_0 \text{ [eV]}$	Work function
$I \text{ [A]}$	Photo-current
$n \left[ \text{m}^{-3} \right]$	Volume density of electrons
$A \left[ \text{m}^2 \right]$	Area
$v \left[ \frac{\text{m}}{\text{s}} \right]$	velocity of electrons

$$h\nu - \phi_0 = \frac{1}{2}mv^2 = eV$$

$$V(\nu) = \frac{h}{e}\nu - \frac{\phi_0}{e}$$

$$I = nAve$$

## 1.3 Blackbody Radiation

$L \text{ [m]}$	length of blackbody cube	$k_i$	wave constants
$E_x$	Electric field in x-direction	$\langle E \rangle$	Average Energy
$N$	Number of states	$D$	Density of states
$u$	Blackbody radiation	$I$	Power radiated

$$E_x(x, y, z) = E_{0x} \cos(k_x x) \sin(k_y y) \sin(k_z z)$$

$$k_x = n \frac{\pi}{L} \quad k_y = m \frac{\pi}{L} \quad k_z = l \frac{\pi}{L} \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$N(k) = \frac{1}{3\pi^2} k^3 L^3 \quad D(k) = \frac{k^2}{\pi^2}$$

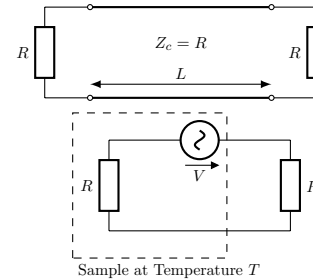
$$u(\omega) = \frac{\omega^2}{\pi^2 c^3} \cdot \frac{\hbar \omega}{\exp\left(\frac{-\hbar \omega}{kT}\right) - 1} d\omega \quad u(\nu) = \frac{8\pi h \nu^3}{c^3 \left( \exp\left(\frac{h\nu}{kT}\right) - 1 \right)} d\nu$$

$$I(\omega) = c \cdot u(\omega)$$

**Equipartition-Theorem:** Each degree of Freedom has an energy of  $kT$

## 1.4 Johnson-Noise

This is the noise created in a one-dimensional circuit (like a coax-cable).



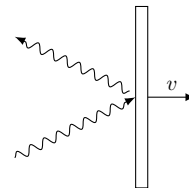
$$\langle V^2 \rangle \quad \text{Noise Voltage}$$

$$\Delta\nu \quad \text{Bandwidth}$$

$$E = E_0 \cdot \sin(k_x \cdot x)$$

$$\langle V^2 \rangle = 4R \cdot k_B T \cdot \Delta\nu$$

## 1.5 Momentum of a photon



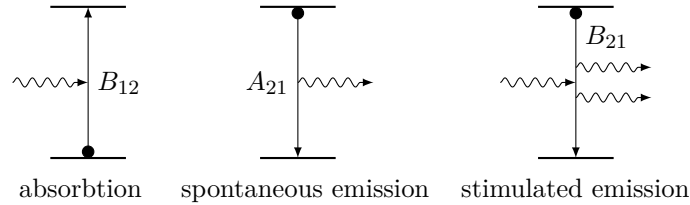
$$p \left[ \frac{\text{kg m}}{\text{s}} \right] \quad \text{momentum}$$

$$p_{\text{absorbing}} = \frac{h\nu}{c} = m \cdot v$$

$$p_{\text{reflecting}} = 2 \cdot \frac{h\nu}{c}$$

$$p = \sqrt{2m_e e \Delta V}$$

## 1.6 Absorption, spontaneous and stimulated emission



$n_1$  Number of electrons in the lower energy state

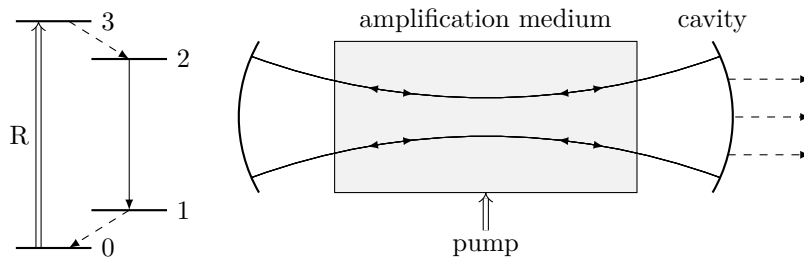
$n_2$  Number of electrons in the higher energy state

$$\frac{dn_2}{dt} = \underbrace{n_1 \cdot u(\nu) \cdot B_{12}}_{\text{absorption}} - \underbrace{n_2 \cdot u(\nu) \cdot B_{21}}_{\text{stimulated emission}} - \underbrace{n_2 \cdot A_{21}}_{\text{spontaneous emission}}$$

$$\frac{n_2}{n_1} = e^{-\frac{h\nu}{k_B T}} = \frac{u(\nu) B_{12}}{u(\nu) B_{21} + A_{21}}$$

$$B_{21} = B_{12} = B \quad A_{21} = \frac{8\pi h \nu^3}{c^3}$$

## 1.7 Laser-optical amplification



Electrons are excited from the ground state “0” to the level “3” by pumping through incoherent radiation. The electrons then fall onto a long-lived state  $n_2$  (State “2”) from level “3”. The pumping can be done either optically by shining a strong incoherent light or by passing a current. It is also assumed that the lower state is quickly emptied by a fast process with lifetime  $\tau_1$ . As a result, the population in state “2” is:

$$n_2 = \frac{R}{A_{21}} \quad \text{whereas} \quad n_1 \approx 0 \quad \text{because} \quad A_{21} < \frac{1}{\tau_1}$$

## 1.8 Fermi Energy of a metal

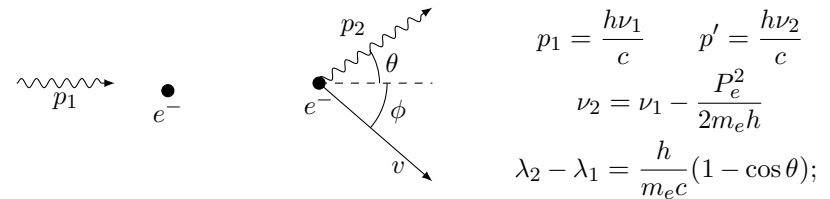
$$E_F = \frac{\hbar^2}{2m_e} \left( 3n\pi^2 \right)^{2/3} \quad n = \frac{\rho}{m} = \frac{\rho \cdot N_A}{m_{mol}}$$

Where  $m_e$  [kg] is the mass of the electron,  $m$  [kg] is the mass of an single atom of the metal,  $m_{mol}$   $\left[ \frac{\text{kg}}{\text{mol}} \right]$  is the atomic weight,  $n$   $\left[ \frac{1}{\text{m}^3} \right]$  is the number of atoms per unit of volume and  $\rho$   $\left[ \frac{\text{kg}}{\text{m}^3} \right]$  is the density of the metal.

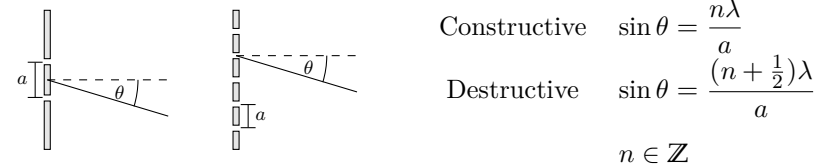
## 2 Wave mechanics

	frequency	wavelength	momentum	energy
Particle		$\lambda_b = \frac{h}{p}$	$p = mv$	$E = \frac{1}{2}mv^2$
Wave	$\omega$	$\lambda = \frac{2\pi c}{\omega}$	$p = \frac{\hbar\omega}{c}$	$E = \hbar\omega$

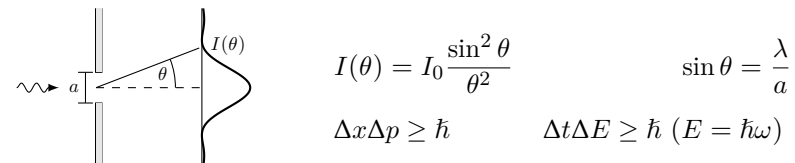
### 2.1 Compton Scattering



### 2.2 Double Slit and Bragg Diffraction



### 2.3 Single slit and uncertainty relation



## 2.4 Bohr-Sommerfeld quantisation

Every single particle must satisfy the following equation. The quantized energy levels below relate to the hydrogen atom

$$\int_{length} p \cdot ds = n \cdot h \quad n \in \mathbb{N}$$

$$E_n = -\frac{Z^2}{n^2} \cdot \frac{m_e e^4}{8\epsilon_0^2 h^2} = -\frac{Z^2}{n^2} \cdot E_{ry}$$

$$r_n = \frac{n^2}{Z} \cdot \frac{2\epsilon_0 h}{m_e e^2} = \frac{n^2}{Z} \cdot a_0$$

$$E_{ry} = 13.6 \text{ [eV]}$$

$$a_0 = 5.292 \cdot 10^{-11} \text{ [m]}$$

$p$  Momentum of particle  
 $E_n$  Energy of the nth state  
 $E_{ry}$  Rydberg Energy  
 $a_0$  Bohr-radius  
 $Z$  Number of protons

## 3 Quantum Mechanics

### 3.1 Wave function

$$\psi(\mathbf{x}, t) : \mathbb{R}^4 \rightarrow \mathbb{C} \quad \iiint |\psi(\mathbf{x}, t)|^2 d^3r = 1$$

$$\psi(\mathbf{x}, t) = a\psi_1(\mathbf{x}, t) + b\psi_2(\mathbf{x}, t), \quad |a|^2 + |b|^2 = 1$$

$$P(x)dx = |\psi(x)|^2 dx \quad P_{ab} = \int_a^b |\psi(x)|^2 dx \quad \langle x \rangle = \int_{-\infty}^{\infty} x |\psi(x)|^2 dx$$

### 3.2 The Schrödinger equation

$$i\hbar \cdot \frac{\partial \Psi}{\partial t}(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \cdot \nabla^2 \Psi(\mathbf{x}, t) + V(\mathbf{x}, t) \Psi(\mathbf{x}, t)$$

$$\Psi = A \cdot e^{i(\mathbf{k}\mathbf{x} - \omega t)} \quad \mathbf{k} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$E = \omega\hbar = \frac{\hbar^2 k^2}{2m}, \quad k^2 = |\mathbf{k}|^2$$

The wave function and its derivative must be continuous where the potential  $V(x, t)$  is finite.

#### 3.2.1 Phase and Group Velocity

phase velocity  $v_\varphi$  (phase movement), group velocity  $v_g$  (movement of wave packet)

$$v_\varphi = \frac{\omega}{k} \quad v_g = \frac{\partial \omega}{\partial k} \quad \text{Particle wave: } v_\varphi \cdot 2 = v_g$$

#### 3.2.2 Stationary (Time independent) States

In a stationary state, the wave function is a product of a function  $\varphi(\mathbf{x})$  independent of time and a function  $\chi(t)$  independent of space.

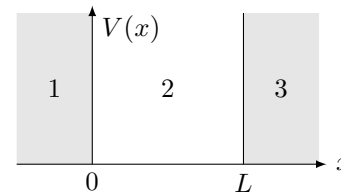
$$\Psi_n(\mathbf{x}, t) = \psi_n(\mathbf{x}) \cdot \chi_n(t) = \psi_n(\mathbf{x}) \cdot e^{-i\frac{E_n}{\hbar}t}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi_n(\mathbf{x}) + V(\mathbf{x})\psi_n(\mathbf{x}) = \psi_n(\mathbf{x}) \cdot E_n$$

$$\iiint |\Psi|^2 d^3\mathbf{x} = \iiint |\psi|^2 d^3\mathbf{x} = 1$$

$$\Psi(\mathbf{x}, t) = \sum a_n \psi_n(\mathbf{x}) \cdot e^{-i\frac{E_n}{\hbar}t} \quad \sum |a_n|^2 = 1$$

#### 3.2.3 Example: 1D infinite potential well



$$\Psi_1 = \Psi_3 = 0$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_2(x, t) = E \psi_2(x, t)$$

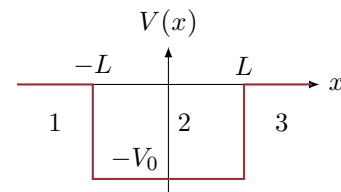
$$\psi_2 = A \sin(kx) + B \cos(kx)$$

$$\text{Boundary cond.: } \psi_2(0) = \psi_2(L) = 0$$

$$\psi_{2n} = A \cdot \sin(k_n x) \quad \Psi_{2n} = A \cdot \sin(k_n x) \cdot e^{-i\frac{E_n}{\hbar}t}, \quad \text{Normalize: } A = \sqrt{\frac{2}{L}}$$

$$E_n = n^2 \cdot \frac{\hbar^2 \pi^2}{2mL} = n^2 \cdot E_0, \quad k_n = \frac{n\pi}{L} \left[ \frac{1}{\text{m}} \right]$$

#### 3.2.4 Example: 1D finite potential well



The Energy  $E$  can be either bigger or smaller than 0. If  $E > 0$ , the wave function will decay exponentially in region 1 and 3. If  $E < 0$ , the wave will propagate away from the potential well.

**Inside the well:** The general solution to the rearranged Schrödinger's is:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_2(x) = (E - V_0) \psi_2(x)$$

$$\psi_2(x) = A_2 e^{ikx} + A'_2 e^{-ikx} \quad E = \frac{k^2 \hbar^2}{2m} \quad k = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

**Outside the well:** There are two cases, which can apply:

1.  $E > 0$ : **Unbound state**

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_1(x) = E \psi_1(x) \quad \psi_1 = A_1 e^{ikx} + A'_1 e^{-ikx} \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

The unbound state does not make sense to be investigated, because the particle is free to be anywhere. In the following, only the unbound state is considered.

2.  $E < 0$ : **Bound state**

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_1(x) = E \psi_1(x) \quad \psi_1 = B_1 e^{\delta x} + B'_1 e^{-\delta x} \quad \delta = \sqrt{-\frac{2mE}{\hbar^2}}$$

We see that as  $x \rightarrow -\infty$ , the Term  $B'_1$ , as well as  $B_3$  approaches  $\infty$ . Since the wave function cannot approach  $\infty$ ,  $B'_1 = B_3 = 0$  is a condition.

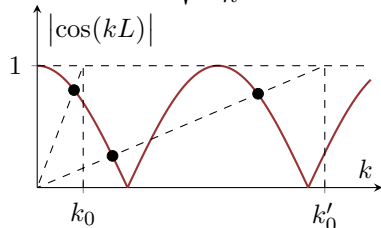
**Boundary conditions:** We require, that the wave function is continuous, as well as it's spacial derivative. Therefore, we have:

$$\begin{aligned} \psi_1(-L) &= \psi_2(-L) & \psi_2(L) &= \psi_3(L) \\ \frac{\partial}{\partial x} \psi_1(-L) &= \frac{\partial}{\partial x} \psi_2(-L) & \frac{\partial}{\partial x} \psi_2(L) &= \frac{\partial}{\partial x} \psi_3(L) \end{aligned}$$

**Even solutions:** only even (cosine) components

$$|\cos(kL)| = \frac{k}{k_0}, \quad \tan(kL) > 0$$

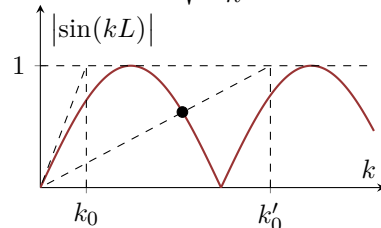
$$k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$



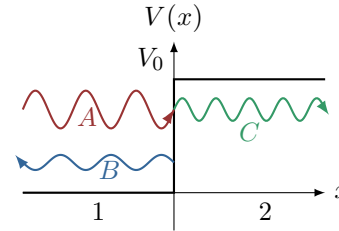
**Odd solutions:** only odd (sine) components

$$|\sin(kL)| = \frac{k}{k_0}, \quad \tan(kL) > 0$$

$$k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$



### 3.3 Example: 1D potential step function



An incoming plane wave from the left hits a potential step at  $x = 0$ . In region 1, two waves are added together, one is traveling to the right and one to the left. If  $E > V_0$ , the wave is transmitted to region 2. if  $E < V_0$ , the wave decays exponentially in region 2.

In **Region 1**, the general solution to the Schrödinger equation is:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_1(x) = E \psi_1(x), \quad \psi_1(x) = A e^{ik_1 x} + B e^{-ik_1 x}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

In **Region 2**, there are two cases, which can apply:

1.  $E > V_0$ : **Transmission**

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_2 = (E - V_0) \psi_2(x) \quad \psi_2 = C e^{ik_2 x}, \quad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

2.  $E < V_0$ : **Complete reflection**

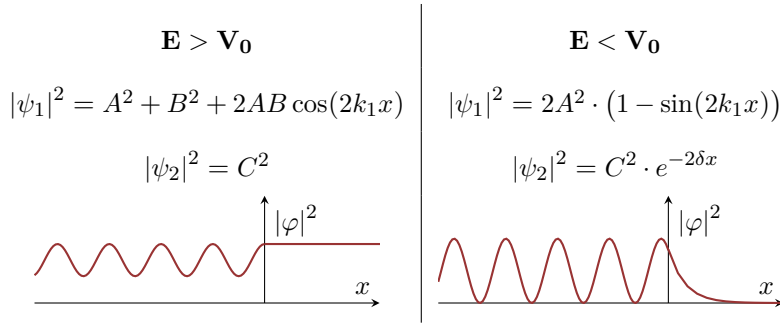
$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_2 = (E - V_0) \psi_2(x) \quad \psi_2 = C e^{\delta_2 x}, \quad \delta_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Applying the **initial conditions**, which require the wave function and it's derivative to be continuous at  $x = 0$ , we get the following expression for  $A$ ,  $B$ ,  $C$ :

$$\psi_1(x=0) = \psi_2(x=0) \quad \frac{\partial}{\partial x} \psi_1(x=0) = \frac{\partial}{\partial x} \psi_2(x=0)$$

$E > V_0$	$E < V_0$
$A + B = C$	$A + B = C$
$k_1(A - B) = k_2 C$	$A = B$

The **probability density function**  $|\psi(x, t)|^2 = |\varphi(x)|^2 = \varphi \cdot \varphi^*$  can then be computed and sketched:

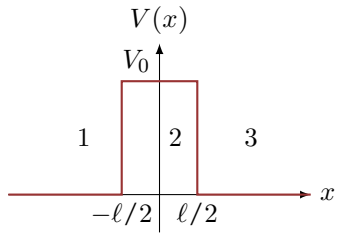


To find the **transmission coefficient**  $T$  and the **reflection coefficient**  $R$ , we normalize  $A = 1$ . Then, we can define  $B = \sqrt{R}$  and  $C = \sqrt{T}$ . Then, we can solve for  $R$  and  $T$ :

$$T = \frac{4k_1k_2}{(k_1 + k_2)^2} \quad R = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2$$

If  $E < V_0$ , nothing is transmitted and therefore  $T = 0$  and  $R = 1$ .

### 3.3.1 Example: 1D finite potential barrier



An incoming plane wave from the left hits a potential barrier with length  $l$ . The Transmission coefficient tells, how much of the wave can continue at the other side of the barrier (quantum tunneling).

In **Region 1 and 3**, the general expression for the wave equation is the following:

$$\psi_j(x) = A_j e^{ik_j x} + A'_j e^{-ik_j x}, \quad k_j = \sqrt{\frac{2mE}{\hbar^2}}, \quad j \in \{1, 3\}$$

In **Region 2**, the expression is depending on  $V_0$ . There are two cases:

1.  **$E < V_0$** :  $\varphi_2 = B_2 e^{\delta_2 x} + B'_2 e^{-\delta_2 x}$ ,  $\delta_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$
2.  **$E > V_0$** :  $\varphi_2 = A_2 e^{ik_2 x} + A'_2 e^{-ik_2 x}$ ,  $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$

Apply **boundary conditions** at  $x = -\ell/2$  and  $x = \ell/2$  in order to determine all constants. If the wave is only traveling from left to right, then  $A'_3 = 0$ .

$$\psi_1(-\ell/2) = \psi_2(-\ell/2), \quad \psi_2(\ell/2) = \psi_3(\ell/2)$$

$$\frac{\partial}{\partial x} \psi_1(-\ell/2) = \frac{\partial}{\partial x} \psi_2(-\ell/2), \quad \frac{\partial}{\partial x} \psi_2(\ell/2) = \frac{\partial}{\partial x} \psi_3(\ell/2)$$

Then, the **transmission coefficient**  $T$  and the **reflection coefficient**  $R$  can be calculated as following:

$$R = \left( \frac{A_1}{A'_1} \right)^2, \quad T = \left( \frac{A_3}{A_1} \right)^2$$

$$T = \frac{E < V_0}{4E(V_0 - E) + V_0^2 \sinh^2(\delta_2 \ell)} \quad T = \frac{E > V_0}{4E(V_0 - E) + V_0^2 \sin^2(k_2 \ell)}$$

If  **$E > V_0$** , the transmission coefficient has a maximum. If  $k_2 \ell = n\pi \Rightarrow T = 1$  (**resonance**). The minimum of  $T$  is at:  $k_2 \ell = \pi/2 + n\pi$ .

If  $\delta_2 \ell \gg 1$ , the transmission coefficient is proportional to:  $T \propto e^{-2\delta_2 \ell}$

## 4 Wave Function Space (Hilbert Space)

### 4.1 Inner Product

The inner product  $\langle \psi_1 | \psi_2 \rangle$  is defined like the scalar product for vectors. If the inner product  $\langle \psi_1 | \psi_2 \rangle = 0$ ,  $\psi_1, \psi_2$  are **orthogonal**.

$$\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^*(\mathbf{x}, t) \psi_2(\mathbf{x}, t) d^3 \mathbf{x}$$

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(\mathbf{x}, t) \psi(\mathbf{x}, t) d^3 \mathbf{x} = \int_{-\infty}^{\infty} |\psi(\mathbf{x}, t)|^2 d^3 \mathbf{x} = 1$$

### 4.2 Fourier Transform

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}} \varphi(p) dp, \quad \varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}} \psi(x) dx$$

$$\psi(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} e^{\frac{i\vec{p}\vec{x}}{\hbar}} \varphi(\vec{p}) d\vec{p}, \quad \varphi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} e^{\frac{i\vec{p}\vec{x}}{\hbar}} \psi(\vec{x}) d\vec{x}$$

$$\int_{-\infty}^{\infty} \psi_1^*(x) \cdot \psi_2(x) \cdot dx = \int_{-\infty}^{\infty} \varphi_1^*(p) \cdot \varphi_2(p) \cdot dp$$

## 5 Observable Measurements, Time-dependence

Doing a measurement in quantum mechanics (observable) can be interpreted as applying an operator  $\hat{A}$  on the wave function  $\psi(\mathbf{x}, t)$ . For example, to compute the expected position  $\langle \mathbf{x} \rangle_\psi$ , we apply the operator  $\hat{\mathbf{x}} = \mathbf{x}$  to average the wave function:

$$\langle \mathbf{x} \rangle_\Psi = \iiint \Psi^*(\mathbf{x}, t) \cdot \mathbf{x} \cdot \Psi(\mathbf{x}, t) d^3\mathbf{x} = \iiint \mathbf{x} \cdot |\Psi(\mathbf{x}, t)|^2 d^3\mathbf{x}$$

Name	Operator
Position	$\hat{\mathbf{x}} = [\mathbf{x}]$
Momentum	$\hat{\mathbf{p}} = [-i\hbar \nabla]$
Hamiltonian	$\hat{H} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right]$

$$\nabla = \left[ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \right]^T$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

### 5.1 Canonical commutation relation

The commutators is a way of describing the effect of the order, in which multiple operators are applied.

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \quad [\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}], \quad [\hat{A}, \hat{A}] = 0$$

$$[\hat{A}, (\hat{B} + \hat{C})] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$$

$$[\hat{p}_x, \hat{p}_y] = 0, \quad [\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{z}, \hat{p}_x] = [\hat{z}, \hat{p}_y] = 0$$

### 5.2 Eigenstates and Eigenvalues

An Observable has an Operator  $\hat{A}$ . a state  $u_n(x)$  is called an eigenstate the operator applied on the wave function acts like a scalar multiplication to it. Then, the measurement of the general state  $\psi(x)$  is a superposition of all the eigenstates.

$$\hat{A}u_n(x) = a_n u_n(x), \quad \int_{-\infty}^{\infty} u_n^*(x) \hat{A}u_n(x) dx = a_n \quad \hat{A}\psi(x) = \sum_n c_n u_n(x)$$

### 5.3 Harmonic Oscillator

A Quantum mechanical harmonic oscillator can be interpreted as the solution to the Schrödinger equation:

$$\left[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x) = E\psi(x), \quad V(x) = \frac{1}{2}kx^2 = \frac{m\omega^2}{2}x^2$$

To simplify the equation, we define a new length scale and energy:

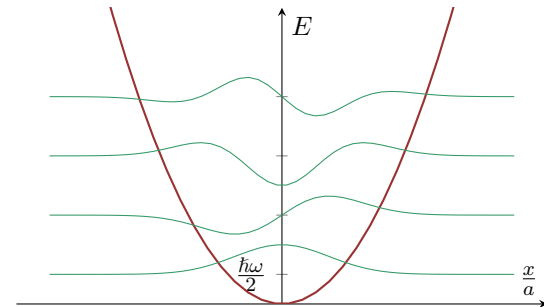
$$a = \sqrt{\frac{\hbar}{m\omega}}, \quad \tilde{x} = \frac{x}{a}, \quad \tilde{E} = \frac{E}{\hbar\omega} \Rightarrow \frac{1}{2} \left[ -\frac{\partial^2}{\partial \tilde{x}^2} + \tilde{x}^2 \right] \varphi(\tilde{x}) = \tilde{E} \varphi(\tilde{x})$$

Then, the solutions to the equation is:

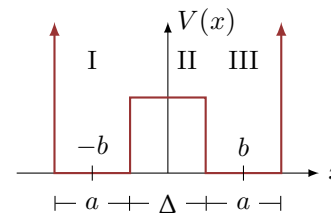
$$E_n = \left( n + \frac{1}{2} \right) \hbar\omega, \quad \psi(\tilde{x}) = c_n H_n(\tilde{x}) e^{-\tilde{x}^2/2}, \quad H_n(\tilde{x}) = (-1)^n e^{\tilde{x}^2} \cdot \frac{\partial^n}{\partial \tilde{x}^n} e^{-\tilde{x}^2}$$

$$H_0(\tilde{x}) = 1, \quad H_1(\tilde{x}) = 2\tilde{x}, \quad H_2(\tilde{x}) = 4\tilde{x}^2 - 2, \quad H_3(\tilde{x}) = 8\tilde{x}^3 - 12\tilde{x}$$

$$\Psi_n(x) = \frac{1}{\sqrt[4]{\pi} \sqrt{2^n n!} a} \cdot H_n\left(\frac{x}{a}\right) e^{-\frac{x^2}{2a^2}}$$



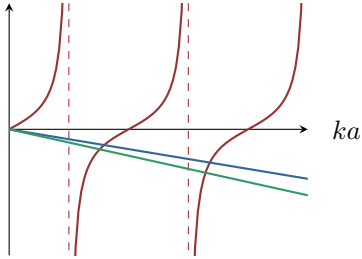
### 5.4 The coupled quantum well



This is the simplified potential of an ammonia molecule  $\text{NH}_3$ . The wave function outside the well ( $|x| > b + \frac{a}{2}$ ) is zero. There exists a symmetric, as well as an antisymmetric solution. We consider the case:  $E < V_0$

$$\psi_{II} = \begin{cases} \mu \cosh(\delta x) & \text{symmetric} \\ \mu \sinh(\delta x) & \text{antisymmetric} \end{cases} \quad k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \delta = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$\tan(ka)$



$$\begin{aligned} \text{symmetric: } \varepsilon_s &= \frac{1 + e^{-\delta\Delta}}{\delta a} \\ \text{antisymmetric: } \varepsilon_a &= \frac{1 - e^{-\delta\Delta}}{\delta a} \\ \tan(ka) &= -ka\varepsilon = -ka \frac{1 \pm e^{-\delta\Delta}}{\delta a} \end{aligned}$$

Now, we can create a superposition of both the symmetric and the antisymmetric case:

$$\psi_{sI} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right), \quad \psi_{sIII} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right)$$

$$\psi_{aI} = -\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right), \quad \psi_{aIII} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right)$$

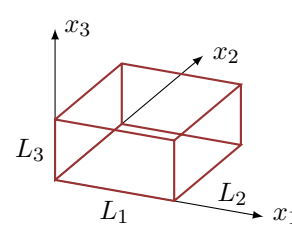
$$\Psi_L = \frac{1}{\sqrt{2}}(\Psi_s - \Psi_a), \quad \Psi_R = \frac{1}{\sqrt{2}}(\Psi_s + \Psi_a)$$

$$\Psi_L(x, t) = \frac{1}{\sqrt{2}} e^{-i\omega_s t} \left( \psi_s(x) - e^{-i(\omega_a - \omega_s)t} \psi_a(x) \right)$$

$$\omega_a = \frac{E_a}{\hbar}, \quad \omega_s = \frac{E_s}{\hbar}, \quad E_a - E_s = \frac{\hbar^2 \pi^2}{2m\delta a^2} \cdot 8e^{-\delta\Delta}$$

From the formula describing the wave equation, we can see that at  $t_0$ , the particle can only be found in region I, and after some time  $t_{1/2}$ , the particle can only be found in region III. The particle has tunneled from one side to the other. Now, we can define a period  $T = \frac{2\pi\hbar}{E_a - E_s}$

## 6 Schrödinger Equation in 3D



$$V(x_i) = \begin{cases} 0 & \text{if } 0 < x_i < L_i \\ \infty & \text{otherwise} \end{cases}$$

$$V(x, y, z) = V(x_1) + V(x_2) + V(x_3)$$

$$\psi(x_1, x_2, x_3) = \psi_1(x_1) \cdot \psi_2(x_2) \cdot \psi_3(x_3)$$

$$-\frac{\hbar^2}{2m} \left[ \frac{\psi_1''(x_1)}{\psi_1(x_1)} + \frac{\psi_2''(x_2)}{\psi_2(x_2)} + \frac{\psi_3''(x_3)}{\psi_3(x_3)} \right] + V(x_1) + V(x_2) + V(x_3) = E$$

This equation can be separated into three smaller equations for every spacial dimension  $x_i$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} \psi_i(x_i) + V(x_i) \psi_i(x_i) = E_i \psi_i(x_i)$$

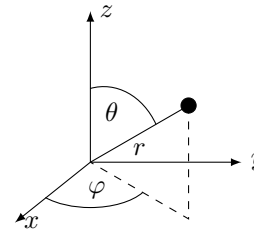
$$E_i^{(n_i)} = n_i^2 \frac{\hbar^2 \pi^2}{2mL_i^2}, \quad \psi_i^{(n_i)} = A \cdot \sin\left(\frac{\pi n_i x_i}{L_i}\right)$$

After normalizing, the wave function can be written as:

$$\psi(x_1, x_2, x_3) = \sqrt{\frac{8}{L_1 L_2 L_3}} \sin\left(\frac{\pi n_1 x_1}{L_1}\right) \sin\left(\frac{\pi n_2 x_2}{L_2}\right) \sin\left(\frac{\pi n_3 x_3}{L_3}\right)$$

When  $L_1 = L_2 = L_3$ , there sometimes exists multiple states (**degeneracies**) for the same energy  $E = E_1 + E_2 + E_3$ . Now, we can generate new solutions to the wave function via superposition of those states. In general, degeneracies arise from symmetries (obvious or hidden).

### 6.1 Schrödinger Equation in spherical coordinates



$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

$$\psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) \cdot Y_{\ell}^m(\theta, \varphi) = R_{n\ell}(r) \cdot P_{\ell}^m(\cos \theta) e^{im\varphi}$$

The angular part  $Y_{\ell}^m(\theta, \varphi)$  can be written as:

$$P_{\ell}^m(x) = (i - x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} P_{\ell}(x) \quad P_{\ell}(x) = \frac{1}{2^{\ell} \cdot \ell!} \frac{\partial^{\ell}}{dx^{\ell}} (x^2 - 1)^{\ell}$$

The solution to  $Y$  will be a **spherical harmonic**. Finally, we must apply the normalization

$$\int_0^{\infty} |R(r)|^2 r^2 dr = 1, \quad \int_{\theta=0}^{\pi} \int_{\varphi=-\pi}^{\pi} |Y_{\ell}^m(\theta, \varphi)|^2 \sin \theta d\varphi d\theta = 1$$

These solutions are the same as **spherical harmonics**. They form an **orthogonal basis**, meaning that every well-behaved function  $f(\theta, \varphi)$  can be expressed as a superposition of those harmonics.

### 6.1.1 Hydrogen Atom

The radial part  $R_{n\ell}$  of the hydrogen atom with potential  $V(r) = \frac{-e^2}{4\pi\epsilon_0 r}$  can be written as:

$$R_{n\ell}(r) = \frac{1}{r} \rho^{\ell+1} e^{-\rho} v(\rho), \quad \rho = \frac{r}{na_0}, \quad a_0 = \frac{4\pi\epsilon_0 \hbar^2}{me^2} \approx 5.29 \cdot 10^{-11} \text{ [m]}$$

$$\psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) Y_{\ell}^m(\theta, \varphi) \quad j_{max} = (n - \ell - 1) \geq 0 \quad |m| \leq \ell$$

$$E = -\frac{E_{Ry}}{n^2} \approx -\frac{13.6}{n^2} \text{ [eV]}$$

$v(\rho)$  is a polynomial of degree  $j_{max}$  with coefficients:  $C_{g+1} = \frac{2(g+l+1-n)}{(g+1)(g+2l+2)} C_g$ . For state  $n$ , there are  $d(n) = n^2$  different solutions (**degeneracies**). The **effective radius** is  $na_0$ . The **probability** of finding an electron between  $r$  and  $r + dr$  is:

$$p(r)dr = r^2 |R_{n\ell}(r)|^2 dr$$

### 6.1.2 Quantum Numbers

$n$  is the main quantum number,  $\ell$  is the orbital quantum number and  $m$  is the magnetic quantum number (projection of angular momentum). Chemists give the different  $\ell$ 's different names.

- $\ell = 0$ : the orbital is called an s-state ( $\max p(r)dr$  is at  $r = 0$ ).
- $\ell = 1$ : the orbital is called an p-state ( $p(r=0)dr = 0$ ).
- $\ell = 2$ : the orbital is called an d-state.

## 7 Angular Momentum and Spin

$$\begin{aligned} \hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y & \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z & \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x & \hat{\mathbf{L}} &= \det \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix} \\ [\hat{L}_x, \hat{L}_y] &= i\hbar \hat{L}_z & [\hat{L}_y, \hat{L}_z] &= i\hbar \hat{L}_x & [\hat{L}_z, \hat{L}_x] &= i\hbar \hat{L}_y \end{aligned}$$

$$\hat{L}_x = i\hbar \left( \sin \varphi \frac{\partial}{\partial \theta} + \frac{\cos \theta \cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi} \right), \quad \hat{L}_y = -i\hbar \left( \cos \varphi \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi} \right)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]$$

Angular momentum operators do not commute. In order to get commutable operators, we introduce  $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$$[\hat{L}^2, \hat{L}_x] = 0, \quad [\hat{L}^2, \hat{L}_y] = 0, \quad [\hat{L}^2, \hat{L}_z] = 0$$

$$\hat{L}^2 Y_{\ell}^m(\theta, \varphi) = \hbar^2 \ell(\ell+1) Y_{\ell}^m(\theta, \varphi), \quad \hat{L}_z Y_{\ell}^m(\theta, \varphi) = \hbar m Y_{\ell}^m(\theta, \varphi)$$

### 7.1 Ladder Operator

If a ladder operators  $\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$  are used in the following way: Suppose, we have a wave function  $\psi$ , which is simultaneously an eigenfunction of  $\hat{L}^2$  and  $\hat{L}_z$ . Then,  $\hat{L}_{\pm}\psi$  is also an eigenfunction of  $\hat{L}^2$  and  $\hat{L}_z$  with the following eigenvalues:

$$\begin{aligned} \hat{L}^2 \psi &= \lambda \psi, \quad \hat{L}_z \psi = \mu \psi & \hat{L}^2 (\hat{L}_{\pm} \psi) &= \lambda (\hat{L}_{\pm} \psi), \quad \hat{L}_z (\hat{L}_{\pm} \psi) = (\mu \pm \hbar) (\hat{L}_{\pm} \psi) \\ \hat{L}_+ Y_{\ell}^m &= \hbar \sqrt{\ell(\ell+1) - m(m+1)} Y_{\ell}^{m+1} & \hat{L}_- Y_{\ell}^m &= \hbar \sqrt{\ell(\ell+1) - m(m-1)} Y_{\ell}^{m-1} \end{aligned}$$

### 7.2 Spin

Idea:  $\ell = \frac{1}{2} \rightarrow m = \pm \frac{1}{2}$ . Instead of using  $\ell$ , we use  $s$  to describe the spin. The operators  $\hat{L}_i$  are now called  $\hat{S}_i$ . We define the spin as  $|s, m_s\rangle$ :

$$\chi_+ = \left| \frac{1}{2}, \frac{1}{2} \right\rangle = |\uparrow\rangle, \quad \chi_- = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = |\downarrow\rangle, \quad \chi = a\chi_+ + \beta\chi_- = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1$$

Since we have now only two eigenstates  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , we can write:

$$\hat{S}^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



If we separate the spin in all three directions, we can write:

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle + |\downarrow_z\rangle), \quad |\downarrow_x\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle - |\downarrow_z\rangle)$$

Now, we define the ladder operators in the same way as for the angular momentum:

$$\hat{S}_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}, \quad \hat{S}_+ |\uparrow\rangle = \hat{S}_- |\downarrow\rangle = 0, \quad \hat{S}_+ |\downarrow\rangle = \hbar |\uparrow\rangle, \quad \hat{S}_- |\uparrow\rangle = \hbar |\downarrow\rangle$$

We can write states in dirac notation as:  $|\ell, m\rangle$ . For spins, we get:

$$|1, 1\rangle = |\uparrow\uparrow\rangle, \quad |1, -1\rangle = |\downarrow\downarrow\rangle, \quad |1, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |0, 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

### 7.3 Identical Particles

When we have two particles with wave function  $\varphi_a(\vec{x}_1)$  and  $\varphi_b(\vec{x}_2)$ , we can combine them in two different ways:

- **Fermion:**  $\psi_- = \frac{1}{\sqrt{2}} (\varphi_a(\vec{x}_1)\varphi_b(\vec{x}_2) - \varphi_b(\vec{x}_1)\varphi_a(\vec{x}_2))$ : symmetric (electron)
- **Boson:**  $\psi_+ = \frac{1}{\sqrt{2}} (\varphi_a(\vec{x}_1)\varphi_b(\vec{x}_2) + \varphi_b(\vec{x}_1)\varphi_a(\vec{x}_2))$ : antisymmetric (photon)

Pauli exclusion principle: **Two fermions cannot occupy the same identical state**

#### 7.3.1 Exchange Interactions

We have two particles, which have normalized and orthogonal wave functions. We have three different interactions:

1. The particles are **distinguishable**:  $\psi = \psi_a(x_1)\psi_b(x_2)$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} - 2 \langle x \rangle_{\psi_a} \langle x \rangle_{\psi_b}$$

2. **Symmetric** wave function:  $\psi_+ = \frac{1}{\sqrt{2}} (|\psi_a\rangle |\psi_b\rangle + |\psi_b\rangle |\psi_a\rangle)$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} - 2 \langle x \rangle_{\psi_a} \langle x \rangle_{\psi_b} - 2 \left| \langle \psi_a | x | \psi_b \rangle \right|^2$$

3. **Antisymmetric** wave function:  $\psi_- = \frac{1}{\sqrt{2}} (|\psi_a\rangle |\psi_b\rangle - |\psi_b\rangle |\psi_a\rangle)$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} - 2 \langle x \rangle_{\psi_a} \langle x \rangle_{\psi_b} + 2 \left| \langle \psi_a | x | \psi_b \rangle \right|^2$$

### 7.4 Many Electrons: Atomic Shells

To write the wave function of an electron, we use the notation indexed by:  $|n, \ell, m\rangle$ .

shell $n$	subshell $\ell$	max $e^-$ in subshell	max $e^-$ in shell
K	1s	2	2
L	2s	2	2 + 6 = 8
	2p	6	
M	3s	2	2 + 6 + 10 = 18
	3p	6	
	3d	10	

Here, the number of degeneracies per shell is displayed. Remember, we have  $n \geq 1$ ,  $0 \leq \ell < n$ ,  $-\ell \leq m \leq \ell$ , and for every different state, the electron can have either spin up or spin down. So, the number of electrons in a sub shell is the number of degeneracies.

### 7.5 Term Symbol

$$^{2S+1}L_J$$

$S$  is the total spin quantum number,  $2S + 1$  is the spin multiplicity,  $J$  ( $m$ ) is the total angular momentum quantum number and  $L$  is the orbital quantum number in spectroscopic notation:  $L = 0 \rightarrow S$ ,  $L = 1 \rightarrow P$ ,  $L = 2 \rightarrow D \dots$

## 8 Quantum Statistics and Solid States

### 8.1 Chemical Potential

For the chemical potential  $\mu$ , the flux of particles (instead of energy) is important.

$$\mu = \frac{\partial F}{\partial N}, \quad F = U - TS$$

where  $F$  is the free energy,  $U$  is the total energy,  $T$  is the temperature,  $S$  is the entropy and  $N$  is the number of particles. In other words: **Chemical Potential is the energy exchanged when a particle is added or removed.**

### 8.2 Fermi-Dirac & Einstein Statistics

What is the probability of a given state to be occupied? And if we have multiple particles, how many electrons are in a state?

$$\text{Fermions: } f_F(E, \mu, T) = \frac{1}{\exp\left(\frac{E - \mu}{kT}\right) + 1}, \quad \text{Bosons: } f_B(E, \mu, T) = \frac{1}{\exp\left(\frac{E - \mu}{kT}\right) - 1}$$

As the energy  $E \gg \mu$ , both distributions are the same: the classical Boltzmann distribution. If the electrons are far away, we do not need to consider the interaction between single particles.

### 8.3 Fermi-Dirac statistics of a free electron gas

The number of particles  $N$  given an energy  $E$  with temperature  $T = 0$  is:

$$N(E) = \frac{L^3}{3\pi^2} \frac{(2mE)^{\frac{3}{2}}}{\hbar^2} = \frac{4}{3}\pi k^3 \cdot \frac{1}{8} \cdot \left(\frac{L}{\pi}\right)^3 \cdot 2$$

This equation is derived by multiplying the Volume of a sphere (in the positive  $k$  sector) with radius  $k$  with the volume of a single state and the number of possible spins (2). The Density of states  $D(E)$  is given by:

$$D(E) = \frac{\partial N(E)}{\partial E} \cdot \frac{1}{V} = \frac{1}{2\pi^2} \left(\frac{2m}{\hbar}\right)^{\frac{3}{2}} \sqrt{E}$$

### 8.4 Bloch's Theorem

In a periodic potential, the wave equation can be written as a periodic function  $u_{nk}(x)$  multiplied by a complex exponential:

$$\psi_{nk}(x) = e^{ikx} u_{nk}(x)$$

We can apply this theorem the band formation to conclude the following. As soon as we have a crystal, the electrons can be in a continuous band of energies:

$$E = E_0 - 2A \cos(ka) = E(k)$$

where  $A$  is the inter-atomic coupling. We can see that the band width is directly proportional to  $A$ .

## 9 Useful formulas

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \quad \int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a} \quad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$

$$\int x^n e^{cx} dx = e^{cx} \sum_{i=0}^n (-1)^{n-i} \frac{n!}{i! c^{n-i+1}} x^i \quad \int_0^{\infty} x^n e^{-cx} dx = \frac{n!}{c^{n+1}}$$

$$\text{Gaussian: } G = A \cdot e^{\frac{-x^2}{2\sigma^2}}$$

### 9.1 Trigonometry

$$\begin{aligned} \sin(2\alpha) &= 2 \sin \alpha \cos \alpha & \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ \sin(\alpha \pm \beta) &= \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) & \cos(\alpha \pm \beta) &= \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \\ \sin(\alpha) \pm \sin(\beta) &= 2 \sin \frac{\alpha \pm \beta}{2} \cos \frac{\alpha \mp \beta}{2} & \cos(\alpha) + \cos(\beta) &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \cos(\alpha) - \cos(\beta) &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} & \sin(\alpha) \sin(\beta) &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos(\alpha) \cos(\beta) &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) & \sin(\alpha) \cos(\beta) &= \frac{1}{2} (\sin(\alpha - \beta) + \sin(\alpha + \beta)) \\ \sin^2 \alpha &= \frac{1}{2} (1 - \cos 2\alpha) & \cos^2 \alpha &= \frac{1}{2} (1 + \cos 2\alpha) \\ \sin^3 \alpha &= \frac{1}{4} (3 \sin \alpha - \sin 3\alpha) & \cos^3 \alpha &= \frac{1}{4} (3 \cos \alpha + \cos 3\alpha) \\ \frac{\sin 2\alpha}{\sin \alpha} &= 2 \cos \alpha & \sin \alpha \cos \alpha &= \frac{1}{2} \sin 2\alpha \\ c^2 &= a^2 + b^2 - 2ab \cos \gamma & \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} &= 2r = \frac{u}{\pi} \end{aligned}$$

# 10 Periodic Table of the Elements

2.20 <b>H</b> Hydrogen 1.00784– 1.00811																		2 <b>He</b> Helium 4.002602(2)					
3 0.98 2s <b>Li</b> Lithium 6.938–6.997	4 1.57 2s <b>Be</b> Beryllium 9.0121831(5)	<div><div>Zenegss</div><div>Sy</div><div>Name</div><div>saw</div></div> <div>Z = atomic number; eneg = electronegativity; ss = subshell; Sy = Symbol, Name = element name, saw = standard atomic weight</div>																5 2.04 2p <b>B</b> Boron 10.806–10.821	6 2.55 2p <b>C</b> Carbon 12.0096– 12.0116	7 3.04 2p <b>N</b> Nitrogen 14.00643– 14.00728	8 3.44 2p <b>O</b> Oxygen 15.99903– 15.99977	9 3.98 2p <b>F</b> Fluorine 18.998403163(6)	10 2p <b>Ne</b> Neon 20.1797(6)
11 0.93 3s <b>Na</b> Sodium 22.98976928(2)	12 1.31 3s <b>Mg</b> Magnesium 24.304–24.307																	13 1.61 3p <b>Al</b> Aluminium 26.9815385(7)	14 1.90 3p <b>Si</b> Silicon 28.084–28.086	15 2.19 3p <b>P</b> Phosphorus 30.973761998(5)	16 2.58 3p <b>S</b> Sulphur 32.059–32.076	17 3.16 3p <b>Cl</b> Chlorine 35.446–35.457	18 3p <b>Ar</b> Argon 39.948(1)
19 0.82 4s <b>K</b> Potassium 39.0983(1)	20 1.00 4s <b>Ca</b> Calcium 40.078(4)	21 1.36 3d <b>Sc</b> Scandium 44.955908(5)	22 1.54 3d <b>Ti</b> Titanium 47.867(1)	23 1.63 3d <b>V</b> Vanadium 50.9415(1)	24 1.66 3d* <b>Cr</b> Chromium 51.9961(6)	25 1.55 3d <b>Mn</b> Manganese 54.938044(3)	26 1.83 3d <b>Fe</b> Iron 55.845(2)	27 1.88 3d <b>Co</b> Cobalt 58.933194(4)	28 1.91 3d <b>Ni</b> Nickel 58.6934(4)	29 1.90 3d* <b>Cu</b> Copper 63.546(3)	30 1.65 3d <b>Zn</b> Zinc 65.38(2)	31 1.81 4p <b>Ga</b> Gallium 69.723(1)	32 2.01 4p <b>Ge</b> Germanium 72.630(8)	33 2.18 4p <b>As</b> Arsenic 74.921595(6)	34 2.55 4p <b>Se</b> Selenium 78.971(8)	35 2.96 4p <b>Br</b> Bromine 79.901–79.907	36 3.00 4p <b>Kr</b> Krypton 83.798(2)						
37 0.82 5s <b>Rb</b> Rubidium 85.4678(3)	38 0.95 5s <b>Sr</b> Strontium 87.62(1)	39 1.22 4d <b>Y</b> Yttrium 88.90584(2)	40 1.33 4d <b>Zr</b> Zirconium 91.224(2)	41 1.6 4d* <b>Nb</b> Niobium 92.90637(2)	42 2.16 4d* <b>Mo</b> Molybdenum 95.95(1)	43 1.9 4d <b>Tc</b> Technetium (98)	44 2.2 4d* <b>Ru</b> Ruthenium 101.07(2)	45 2.28 4d* <b>Rh</b> Rhodium 102.90550(2)	46 2.20 4d* <b>Pd</b> Palladium 106.42(1)	47 1.93 4d* <b>Ag</b> Silver 107.8682(2)	48 1.69 4d <b>Cd</b> Cadmium 112.414(4)	49 1.78 5p <b>In</b> Indium 114.818(1)	50 1.96 5p <b>Sn</b> Tin 118.710(7)	51 2.05 5p <b>Sb</b> Antimony 121.760(1)	52 2.1 5p <b>Te</b> Tellurium 127.60(3)	53 2.66 5p <b>I</b> Iodine 126.90447(3)	54 2.60 5p <b>Xe</b> Xenon 131.293(6)						
55 0.79 6s <b>Cs</b> Cesium 132.90545196(6)	56 0.89 6s <b>Ba</b> Barium 137.327(7)	57-71 * Lanthanides	72 1.3 5d <b>Hf</b> Hafnium 178.49(2)	73 1.5 5d <b>Ta</b> Tantalum 180.94788(2)	74 2.36 5d <b>W</b> Tungsten 183.84(1)	75 1.9 5d <b>Re</b> Rhenium 186.207(1)	76 2.2 5d <b>Os</b> Osmium 190.23(3)	77 2.20 5d <b>Ir</b> Iridium 192.217(3)	78 2.28 5d* <b>Pt</b> Platinum 195.084(9)	79 2.54 5d* <b>Au</b> Gold 196.966569(5)	80 2.00 5d <b>Hg</b> Mercury 200.592(3)	81 1.62 6p <b>Tl</b> Thallium 204.382– 204.385	82 1.87 6p <b>Pb</b> Lead 207.2(1)	83 2.02 6p <b>Bi</b> Bismuth 208.98040(1)	84 2.0 6p <b>Po</b> Polonium (209)	85 2.2 6p <b>At</b> Astatine (210)	86 2.2 6p <b>Rn</b> Radon (222)						
87 0.7 7s <b>Fr</b> Francium (223)	88 0.9 7s <b>Ra</b> Radium (226)	89-103 ** Actinides	104 6d <b>Rf</b> Rutherfordium (261)	105 6d <b>Db</b> Dubnium (268)	106 6d <b>Sg</b> Seaborgium (269)	107 6d <b>Bh</b> Bohrium (270)	108 6d <b>Hs</b> Hassium (269)	109 6d <b>Mt</b> Meitnerium (278)	110 6d <b>Ds</b> Darmstadtium (281)	111 6d <b>Rg</b> Roentgenium (282)	112 6d <b>Cn</b> Copernicium (285)	113 7p <b>Nh</b> Nihonium (286)	114 7p <b>Fl</b> Flerovium (289)	115 7p <b>Mc</b> Moscovium (289)	116 7p <b>Lv</b> Livermorium (293)	117 7p <b>Ts</b> Tennessine (294)	118 7p <b>Og</b> Oganesson (294)						
*  																							

Standard atomic weights taken from the Commission on Isotopic Abundances and Atomic Weights ([ciaaw.org/atomic-weights.htm](http://ciaaw.org/atomic-weights.htm)). Adapted from Ivan Griffin's L<sup>A</sup>T<sub>E</sub>X Periodic Table. © 2017 Paul Danese

An asterisk (\*) next to a subshell indicates an anomalous (Aufbau rule-breaking) ground state electron configuration.