Counting 1-Factors in Regular Bipartite Graphs

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Dec. 5th, 2017

Outline

Basic concepts

- Matching.

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  A set of non loop edges with no shared endpoints.

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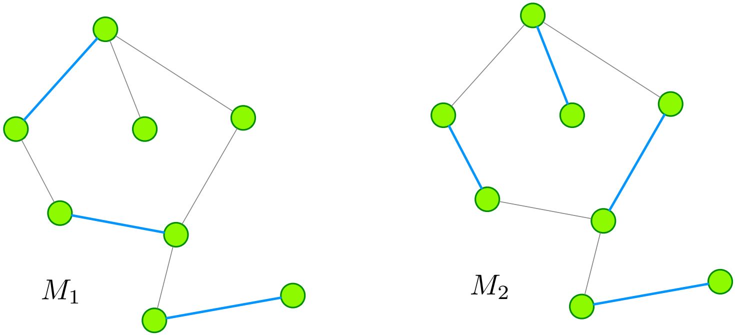

Figure 1: A matching

Basic concepts

- 1-factor (perfect matching)

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  A matching in which every vertex is incident to exactly one edge of the matching. ($n/2$ edges)

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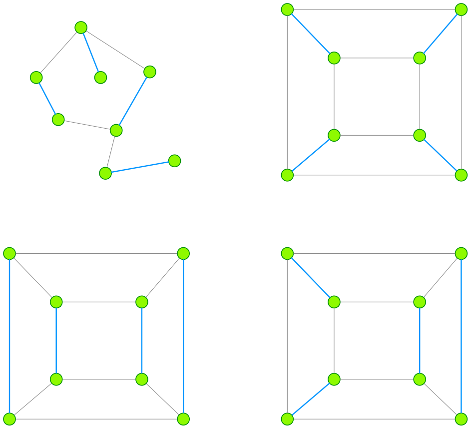

Figure 2: A perfect matching

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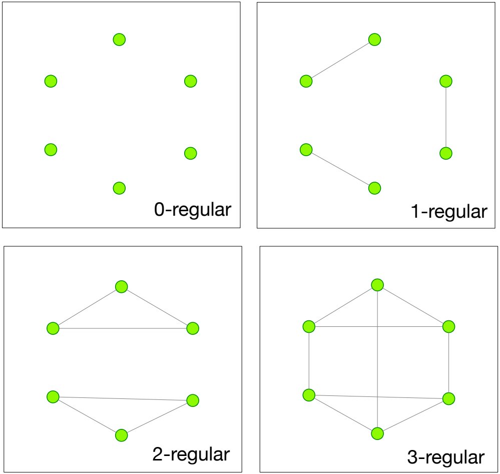

Figure 3: Regular graphs

Basic concepts

- 1-factor in 2-regular bipartite graphs

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- 1-factor in 2-regular bipartite graphs

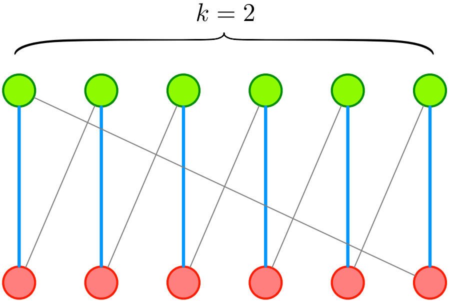

Figure 4: 1- factor in a 2-regular bipartite graph ($2n = 14$)

Basic concepts

- 1-factor in 2-regular bipartite graphs

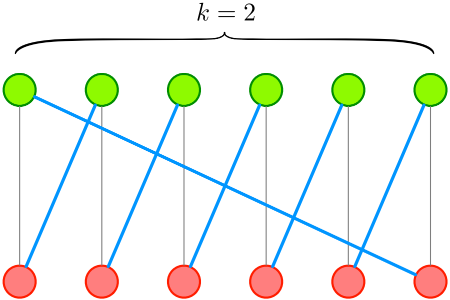

Figure 5: 1- factor in a 2-regular bipartite graph $(2n = 14)$

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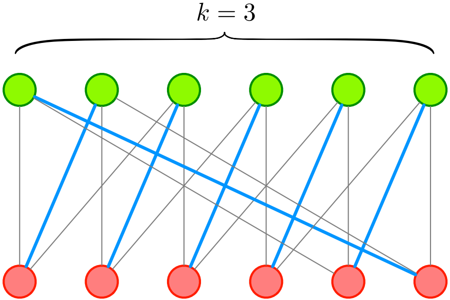

Figure 6: 1-factor in a 3-regular (cubic) bipartite graph ($2n = 14$)

Basic concepts

- 1-factor in 3-regular bipartite graphs

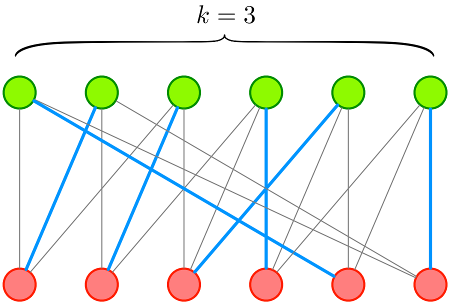

Figure 7: 1-factor in a 3-regular (cubic) bipartite graph ($2n = 14$)

Basic concepts

- 1-factor in 4-regular bipartite graphs

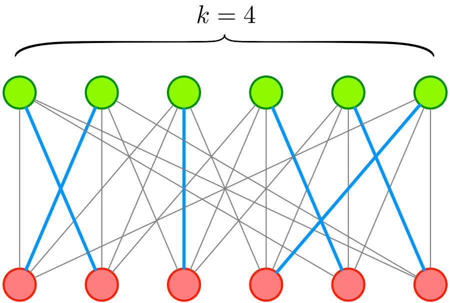

Figure 8: 1-factor in a 4-regular bipartite graph ($2n = 14$)

How many 1-factors in a k-regular bipatite as $n \to \infty$?

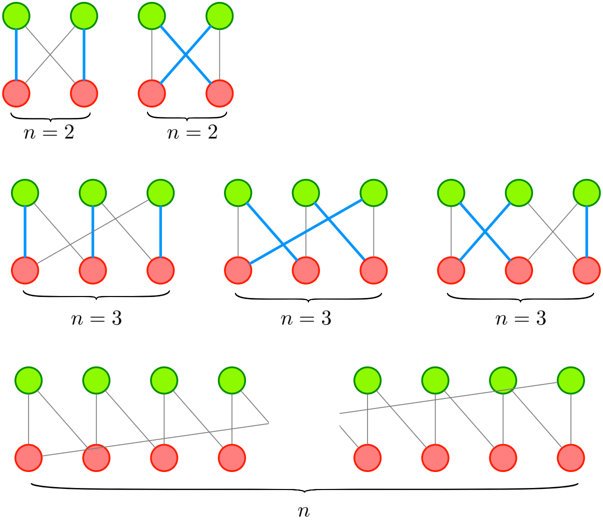

Figure 9: k-regular bipartite ($k = 2$)

The problem

What is the number of 1-factors for an abitrary k-regular bipartite graph on $2n$ vertices?

Results

- Any k-regular bipartite graph on $2n$ vertices has at least

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n \tag{1}$$

1-factors[1].

[1] Schrijver A. Counting 1-factors in regular bipartite graphs[J]. Journal of Combinatorial Theory, Series B, 1998, 72(1): 122-135.

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- We will prove a simple cubic case (i.e., $k = 3$, $\Phi(G) \geq (\frac{4}{3})^n$).

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- We will prove a simple cubic case (i.e., $k = 3$, $\Phi(G) \geq (\frac{4}{3})^n$).
- $\Phi(G)$ denotes the number of 1-factors in G.

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We will prove

For any 3-regular (cubic) bipartite graph G on $2n$ vertices

$$\Phi(G) \geq (\frac{4}{3})^n. \tag{2}$$

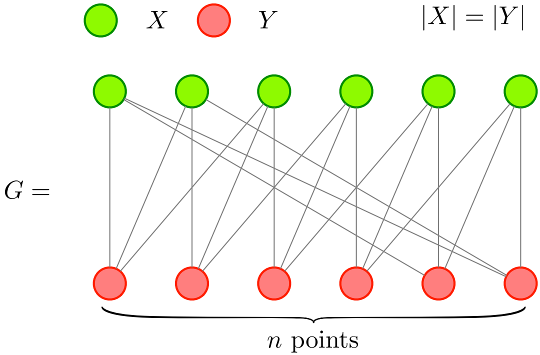

Figure 10: A cubic bipartite graph on $2n$ vertices.

A sketch of proof

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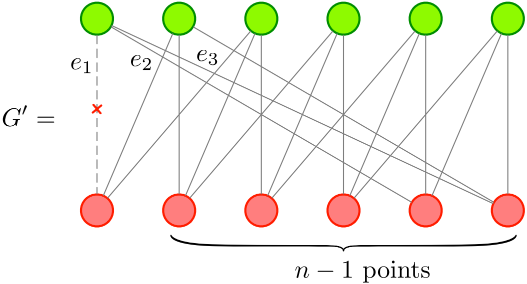

Figure 11: $G' = G - e_i, (i = 1, 2, 3)$.

A sketch of proof

2 If the assertion formulated previous is true, since every perfect matching of G is counted in exactly two graphs $G - e_i$, this implies that G contains at least $3(\frac{4}{3})^{n-1} > (\frac{4}{3})^n$ 1-factors.

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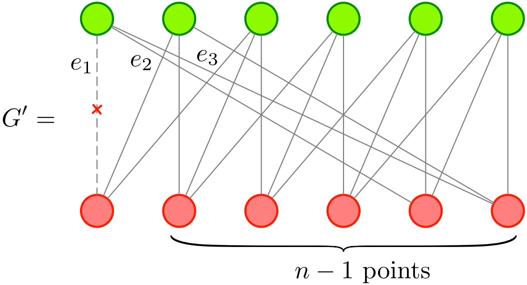

Figure 12: $G' = G - e_i, (i = 1, 2, 3)$.

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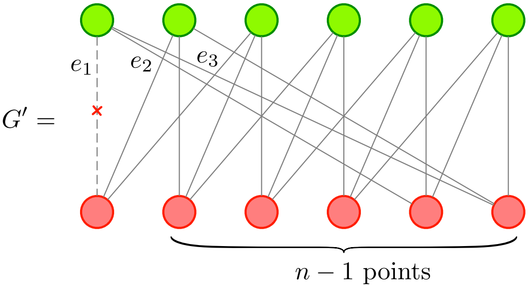

Figure 12: $G' = G - e_i, (i = 1, 2, 3)$.

That is, $\Phi(G') \geq 2(\frac{4}{3})^{n-1} \Longrightarrow \Phi(G) > (\frac{4}{3})^n$.

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3 Let u be a point of G' of degree 2, and let uv_1 and uv_2 be the two lines incident with u.

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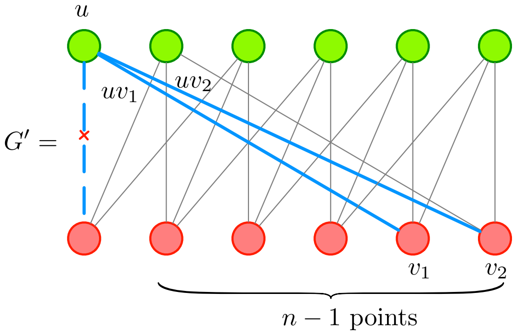

Figure 13: G': $n-1$ point of degree 3 and 1 point of degree 2 in each side.

A sketch of proof

(1) $v_1 = v_2,\ G'' = G' - u - v_1$

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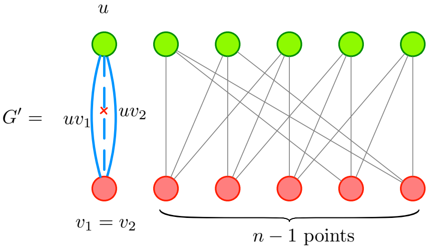

Figure 14: (a)

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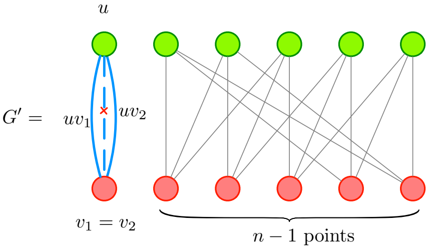

Figure 14: (a)

b. G'' has only 1 point of degree 2 in each color class.

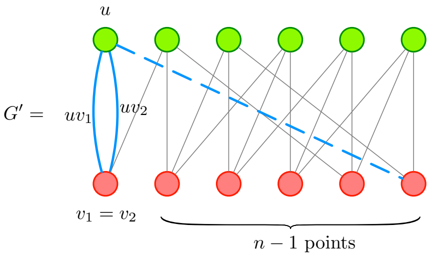

Figure 15: (b).

A sketch of proof

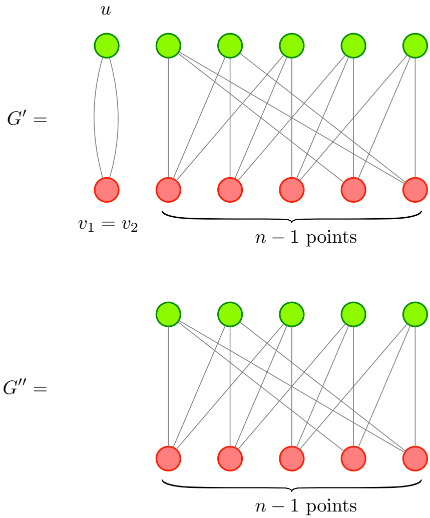

Figure 16: $G'' = G' - u - v_1$.

- By the induction hypothesis, $\Phi(G'') \geq 2(\frac{4}{3})^{n-2}$.

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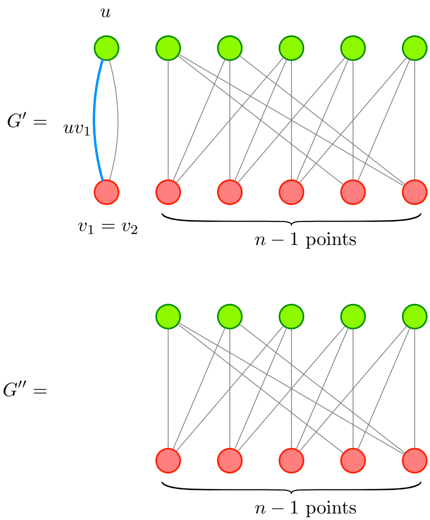

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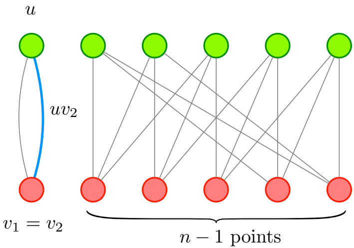

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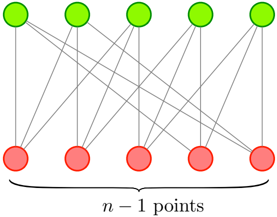

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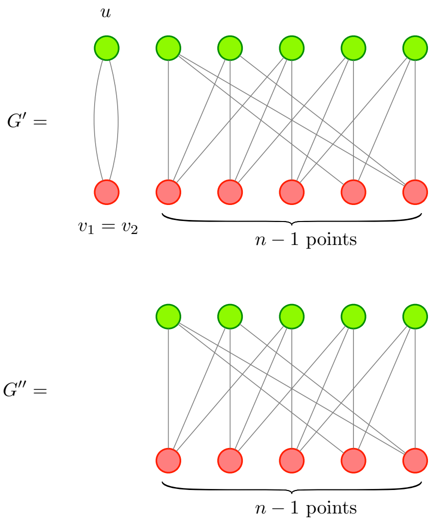

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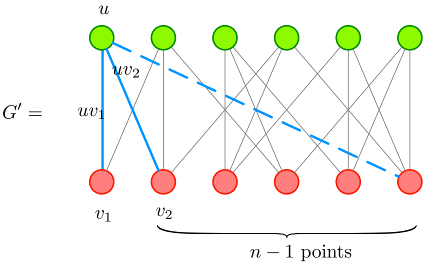

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b. v_1 and v_2 have degree 3 in G'

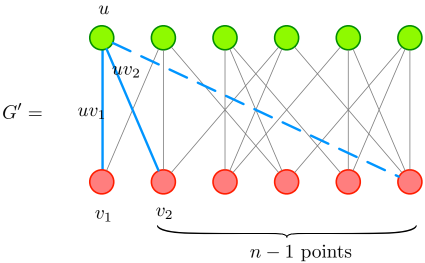

Figure 17: (a)

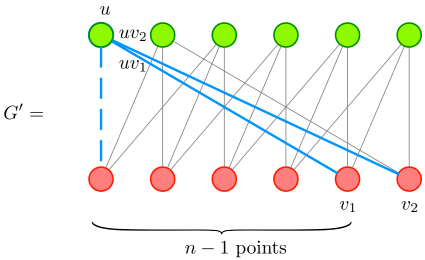

Figure 18: (b).

A sketch of proof

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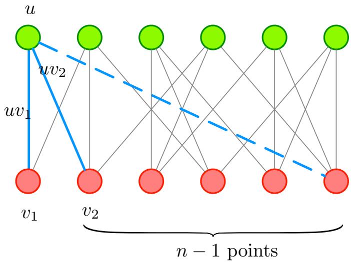
$G' =$ 

A sketch of proof

a. one of v_1 and v_2 has degree 2 in G'

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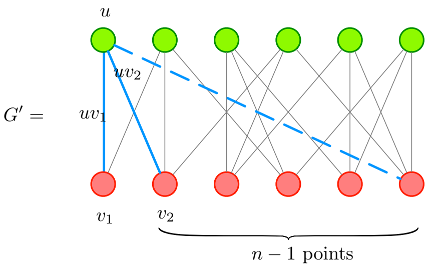

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a. one of v_1 and v_2 has degree 2 in G'

- Delete u,
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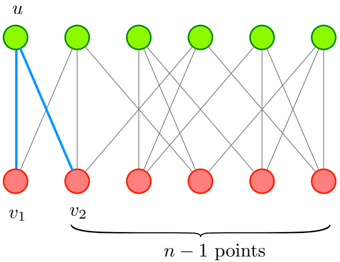
$G' =$ 

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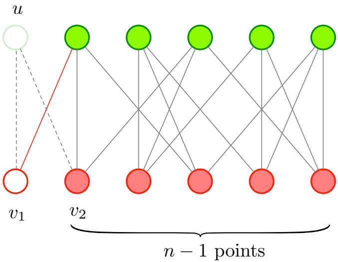
$G' =$ 

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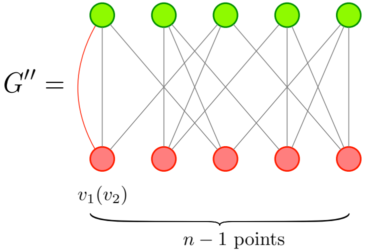


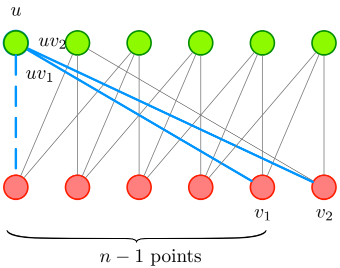
$$\Phi(G'') \geq 3 \cdot (\frac{4}{3})^{n-2} > 2 \cdot (\frac{4}{3})^{n-1}.$$

Note that $\Phi(G) = \Phi(G'')$ since they correspond each other in a natural way.

A sketch of proof

b. v_1 and v_2 have degree 3 in G'.

- Delete u. merge v_1 and v_2 to obtain G''

$G' =$ 

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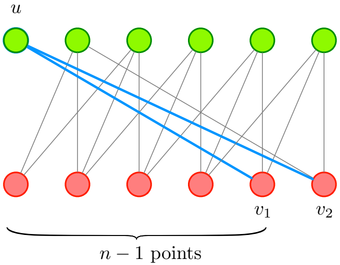

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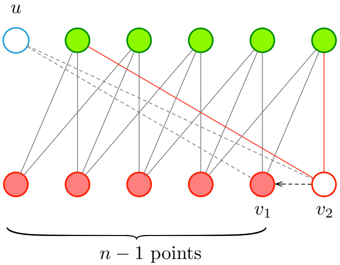

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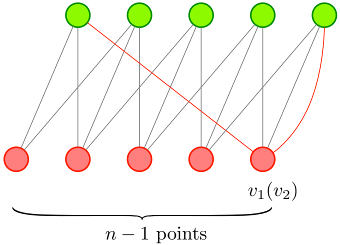
- Also, $\Phi(G) = \Phi(G'')$.
- $v_1 = v_2$ has degree 4 in G''.
- Let $v_1 w_i, (i = 1, 2, 3, 4)$ be the 4 lines of G'' incident with v_1. Then $G'' - v_1 w_i$ has one point of degree 2 in each color class, and all other points are of degree 3.

$G'' =$ 

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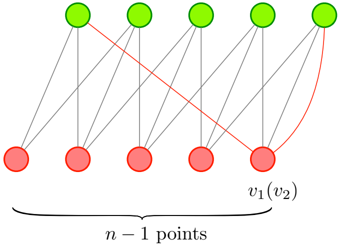
- $v_1 = v_2$ has degree 4 in G''.
- By the induction hypothesis, each graph $G'' - v_1 w_i$ has at least $2 \cdot (\frac{4}{3})^{n-2}$ 1-factors. Adding this for $i = 1, 2, 3, 4$, we count every perfect matching of G' 3 times.

$G'' =$



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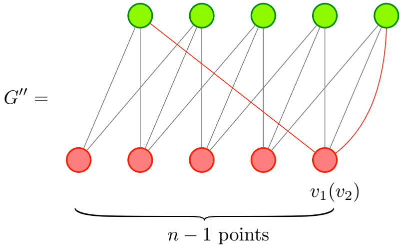
- Delete u. merge v_1 and v_2 to obtain G''
- Also, $\Phi(G) = \Phi(G'')$.
- $v_1 = v_2$ has degree 4 in G''.
- Thus

$$\Phi(G'') \geq 4 \cdot 2 \cdot (\frac{4}{3})^{n-2}/3$$
$$= 2 \cdot (\frac{4}{3})^{n-1}.$$



Conclusion

Result

The number of 1-factors in cubic bipartite graph on $2n$ points grow exponentially with n.

$$\Phi(G) \geq (\frac{4}{3})^n.$$

The relationship with the permenent of a matrix

- The number of 1-factors in a bipartite graph is equal to the *permanent* of its 0-1 adjacency matrix, given by

$$\mathrm{perm} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)} \tag{3}$$

where S_n is the set of permutations of the set $\{1, 2, ..., n\}$.

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- Each non-zero term in the definition of perm($\mathbf{A}$) corresponds to a perfect matching of G and visa versa.
- To evaluate a permanent of a matrix is NP-hard!But there are some nice inequalities involving permanents which are useful.

Reference I

Plummer, Michael D and Lovász, László
Matching theory.
Elsevier, 1986.

A. Schrijver.
Counting 1-factors in regular bipartite graphs[J].
Journal of Combinatorial Theory, Series B, 72(1): 122-135, 1998.

Thansks