Compact Sets.

Ticao Zhang, Oct 11, 2017

Theorem 5.4. If M is compact then M is closed.

Proof. For any $x_o \in M^c$ we consider the cover

$$U_{\epsilon_{x_0}} = (-\infty, x_0 - \epsilon_{x_0}) \bigcup (x_0 + \epsilon_{x_0}, +\infty), \tag{1}$$

where ϵ_{x_0} ranges all positive real numbers, therefore we can always find a collection of sets so that

$$M \subset \bigcup_{\epsilon_{x_0} \in \Lambda} U_{\epsilon_{x_0}}, \tag{2}$$

We note that M is compact, thus

$$M \subset U_{\epsilon_{x_0},1} \bigcup U_{\epsilon_{x_0},1} \bigcup ... \bigcup U_{\epsilon_{x_0},n} = U_\epsilon, \tag{3}$$

where $\epsilon = \min\{\epsilon_{x_0,1}\epsilon_{x_0,2}, ..., \epsilon_{x_0,n}\}$.

We get

$$(x_0 - \epsilon, x_0 + \epsilon) \bigcap M = \emptyset. \tag{4}$$

or

$$(x_0 - \epsilon, x_0 + \epsilon) \subset M^c. \tag{5}$$

In other words, for any x in M^c, there is a segment containing x contained in M^c. This implies that M^c is open. According to Theorem 1.6, M is closed. □

Here is another way of proof

Proof. Take any $x_0 \in M^c$, for every $y \in M$ there is an open segment U_y that contains y and an open segment V_y that contains x_0, s.t. $U_y \bigcap V_y = \emptyset$. The sets $\{U_y\}$ is an open cover of M and since M is compact, there are points $y_1, y_2, ..., y_m \in M$ s.t. $M \subseteq \bigcup_{i=1}^m U_{y_i}$. Let $U = \bigcup_{i=1}^m U_{y_i}$ and $V = \bigcap_{i=1}^m V_{y_i}$. Then U and V are open and $U \bigcap V = \emptyset$. Therefore, $V \subseteq M^c$. Since x_0 is arbitrary, M^c is open and M is closed. □