HW 3

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Problem 1

(a) We generate parameters for c and h in table 1 and table 2, respectively.

Table 1: value of c

c_1	c_2	c_3	c_4
31	28	40	44

Table 2: value of h

h_1	h_2	h_3
427	369	224

We formulate the LP as

$$\begin{aligned} \max \quad & 31x_1 + 28x_2 + 40x_3 + 44x_4 \\ \text{s. t.} \quad & 2x_1 + x_2 + 10x_3 + x_4 \leq 427 \\ & 10x_1 + x_2 + x_3 \leq 369 \\ & 2x_2 + x_3 + 3x_4 \leq 224 \\ & x_1 + x_2 \geq 10 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned} \tag{1}$$

We solve this problem using *LINDO* software. The solution is as follows

```
Global optimal solution found.
Objective value:                              5083.394
Infeasibilities:                              0.000000
Total solver iterations:                             3
Elapsed runtime seconds:                          0.03
```

```
Model Class:                                    LP

Total variables:                  4
Nonlinear variables:              0
Integer variables:                0

Total constraints:                5
Nonlinear constraints:            0

Total nonzeros:                  16
Nonlinear nonzeros:               0

                    Variable           Value        Reduced Cost
                          X1        33.95775            0.000000
                          X2        0.000000            4.746479
                          X3        29.42254            0.000000
                          X4        64.85915            0.000000

                         Row    Slack or Surplus     Dual Price
                           1        5083.394            1.000000
                           2        0.000000            2.348592
                           3        0.000000            2.630282
                           4        0.000000            13.88380
                           5        23.95775            0.000000
```

Therefore, the total profit is 5083.39 (in thousands of dollars), and the optimal value for each product is

Table 3: Production for each product

A	B	C	D
34.0	0	29.4	64.9

(b) The dual problem can be formulated as

$$\begin{aligned} \min \quad & 427y_1 + 369y_2 + 224y_3 + 10y_4 \\ \text{s. t.} \quad & 2y_1 + 10y_2 + y_4 \geq 31 \\ & y_1 + y_2 + 2y_3 + y_4 \geq 28 \\ & 10y_1 + y_2 + y_3 \geq 40 \\ & y_1 + 3y_3 \geq 44 \\ & y_1, y_2, y_3 \geq 0, y_4 \leq 0 \end{aligned} \tag{2}$$

or in *LINDO*

```
MIN= 427 * _2 + 369 * _3 + 224 * _4 + 10 * _5;
```

```
[X1] 2 * _2 + 10 * _3 + _5 >= 31;
[X2] _2 + _3 + 2 * _4 + _5 >= 28;
[X3] 10 * _2 + _3 + _4 >= 40;
[X4] _2 + 3 * _4 >= 44;
@BND( -1e+30, _5 , 0);
END
```

The optimal values for dual variables can be seen in the results printed in red rectangle in (a).

(c) Using the previous results,

i. Machine III, because the shadow price for constraints III (Machine III) has the largest value, 13.88380, which means an additional 1 hour for h_3 would brings an extra increase in the total profit for about 13.88380. Therefore, I would be willing to pay for less than 13.88380 (in thousands of dollars) for an extra 1 hour for Machine III.

ii. Yes. As can be seen, the slack for this contract (constraints 4 in our LP) is 23.95775, which means we can increase the minimum combined production of A and B by a number of 23.95775 while not decrasing the current total profit (or change the basis variables). Therefore, the current terms are profitable for the company and they should not renegotiate it.

iii. As can be seen, we are now not producing product B. The reduced cost for product B is 4.746479, which means forcing the company to produce 1 ton of product B would make the company lose 4.746479 (in thousands of dollars). Therefore, if we are enforce a production of 0.5 ton of product B, we would lose

$$4.746479 \times 0.5 = 2.3732395 \tag{3}$$

in thousand of dollars.

Problem 2

Solution

(a) We formulate the dual problem as

$$
\begin{aligned}
\max \quad & 4y_1 + 20y_2 + 6y_3 \\
\text{s. t.} \quad & -y_1 + y_2 + y_3 \le 0 \\
& y_1 + y_3 \le 1 \\
& 2y_1 + 2y_3 \le -1 \\
& 2y_2 - 2y_3 \le 2 \\
& y_2 \le 0.
\end{aligned} \tag{4}
$$

or in *LINDO*

```
MODEL:
  MAX= 4 * _2 + 20 * _3 + 6 * _4;
 [X2] _2 + _4 <= 1;
 [X3] 2 * _2 + 2 * _4 <= - 1;
 [X4] - 2 * _3 - 2 * _4 <= 2;
 [X1] - _2 + _3 + _4 <= 0;
 @FREE( _2); @BND( -1e+30, _3, 0); @FREE( _4);
 END
```

(b) To check the complementary slackness, we introduce slack variables $w_i \ge 0 (i = 1, 2, 3)$ and $z_j \ge 0 (j = 1, 2, 3, 4)$. The primal problem now becomes

$$
\begin{aligned}
\min \quad & x_2 - x_3 + 2x_4 \\
\text{s. t.} \quad & -x_1 + x_2 + 2x_3 = 4 + w_1 \\
& x_1 + 2x_4 + w_2 = 20 \\
& x_1 + x_2 + 2x_3 - 2x_4 = 6 + w_3 \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{5}
$$

and its dual

$$
\begin{aligned}
\max \quad & 4y_1 + 20y_2 + 6y_3 \\
\text{s. t.} \quad & -y_1 + y_2 + y_3 + z_1 = 0 \\
& y_1 + y_3 + z_2 = 1 \\
& 2y_1 + 2y_3 + z_3 = -1 \\
& 2y_2 - 2y_3 + z_4 = 2 \\
& y_2 \le 0.
\end{aligned} \tag{6}
$$

Put $x = [2, 6, 0, 1]$ into (5) and (6) and use the complementary slackness, we get $y_3 = -1$ and $y_1 = -1$, which contradicts $y_1 + y_3 = 1$. Therefore, $x = [2, 6, 0, 1]$ is not optimal

(c)

Put $x = [1, 0, 2.5, 0]$ into (5) and (6), we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2.5 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 19 \\ 0 \end{bmatrix},$$

Since the complementary slackness holds, $[1, 0, 2.5, 0]$ is optimal.

We run a python code to test it. The solution $[1, 0, 2.5, 0]$ is indeed optimal.

```
from scipy.optimize import linprog
c=[0,1,-1,2]
A_ub=[[1,0,0,2]]
A_eq=[[-1,1,2,0],[1,1,2,-2]]
b_uq=[20]
b_eq=[4,6]
x0_bnds = [0,None]
x1_bnds = [0,None]
x2_bnds = [0,None]
x3_bnds = [0,None]

b_ub=[20]

res = linprog(c,A_ub,b_ub,A_eq,b_eq,bounds=(x0_bnds,x1_bnds,x2_bnds,x3_bnds,))
print(res)
```