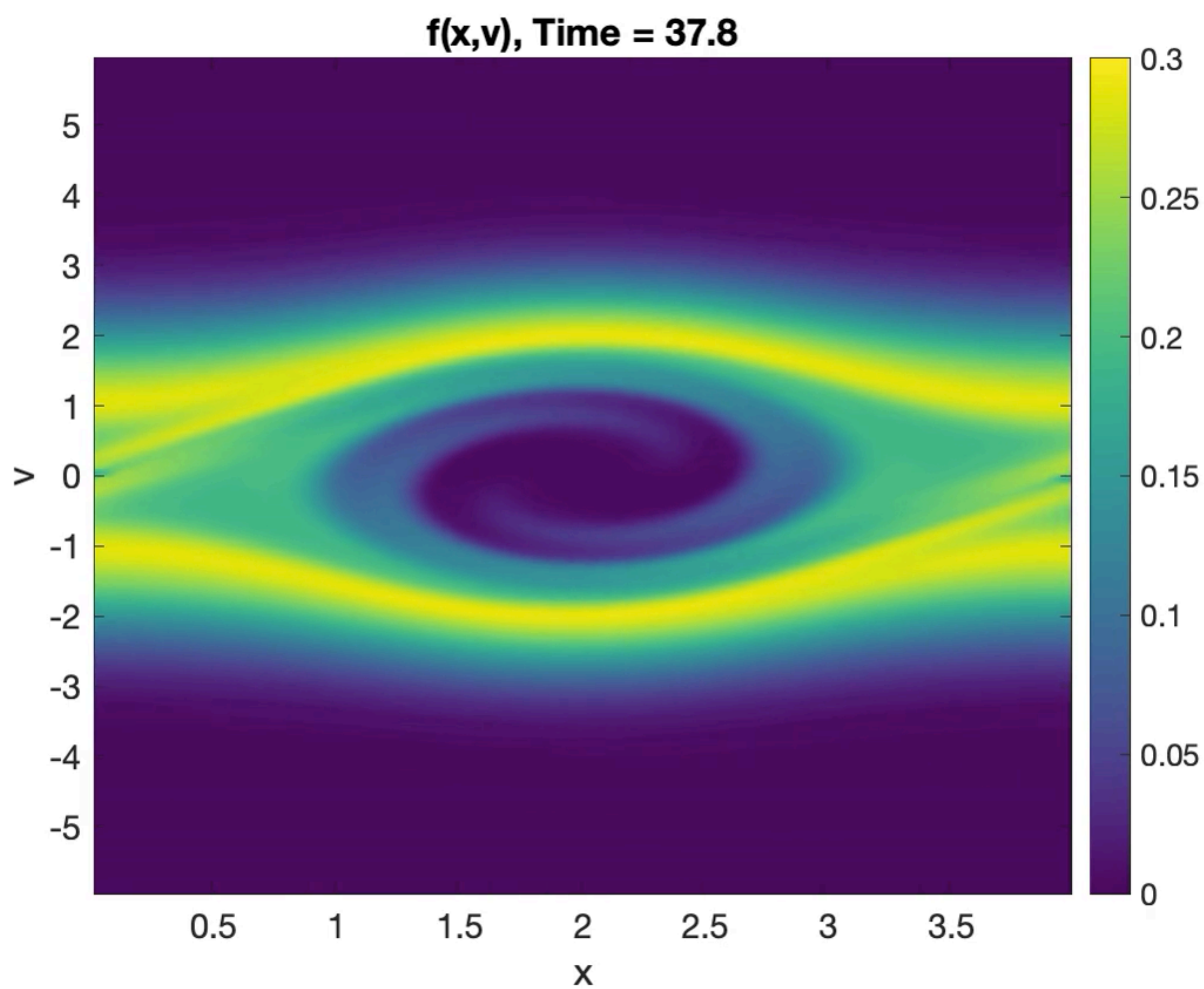
Topic 4

Numerical Solutions to the Electrostatic Vlasov Equations



Outline

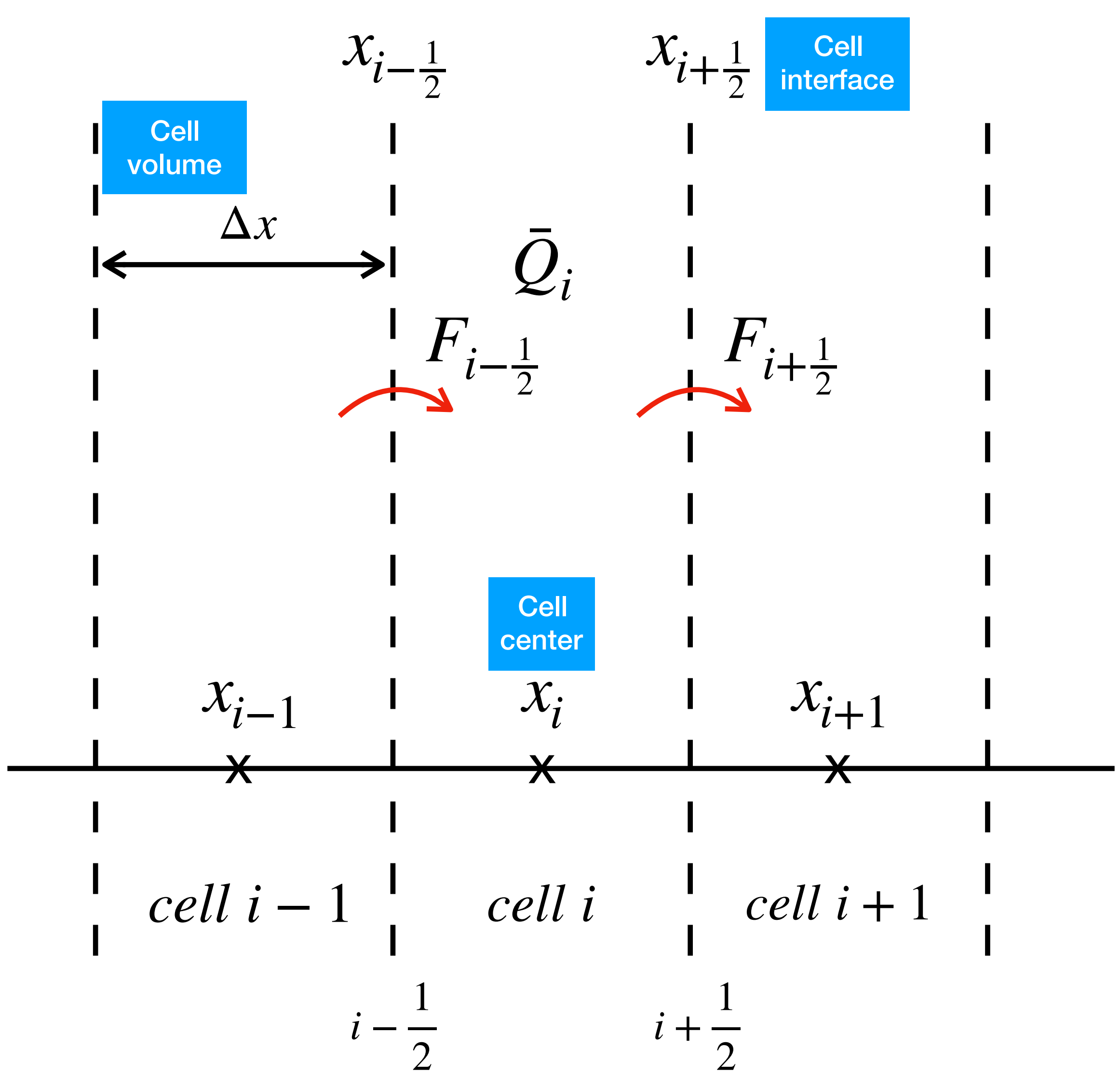
- Review of Finite Volume method
- 2-D Linear Advection
- Vlasov Equations
- Poisson Solving
- Example - Landau Damping

Finite Volume Methods

Basic form

$$\frac{\partial Q}{\partial t} + u_0 \frac{\partial Q}{\partial x} = 0 \xrightarrow{\text{re-write}} \frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0 \qquad F(Q) = u_0 Q \quad \text{for linear advection}$$

Let's discretize the solution domain:



Integrate the PDE in cell i

$$\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} dx \left(\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} \right) = 0$$

$$\longrightarrow \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial Q}{\partial t} dx = - \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial F(Q)}{\partial x} dx$$

$$\longrightarrow \frac{\partial}{\partial t} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} Q dx = - F(Q) \Big|_{i-\frac{1}{2}}^{i+\frac{1}{2}}$$

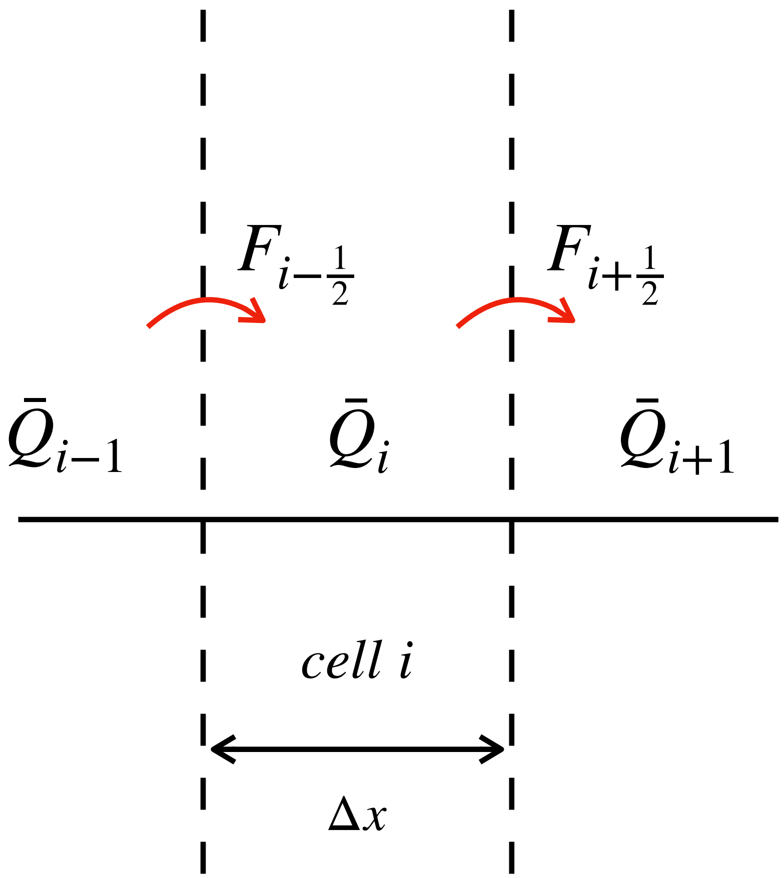
$$\longrightarrow \frac{\partial}{\partial t} \bar{Q} \Delta x = - F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}$$

Rate of mass change

Flux in & out of cell i

The 1st-order upwind scheme is horrible

Flux of the 1st-order upwind method



FV form:

$$\frac{\partial}{\partial t}Q_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

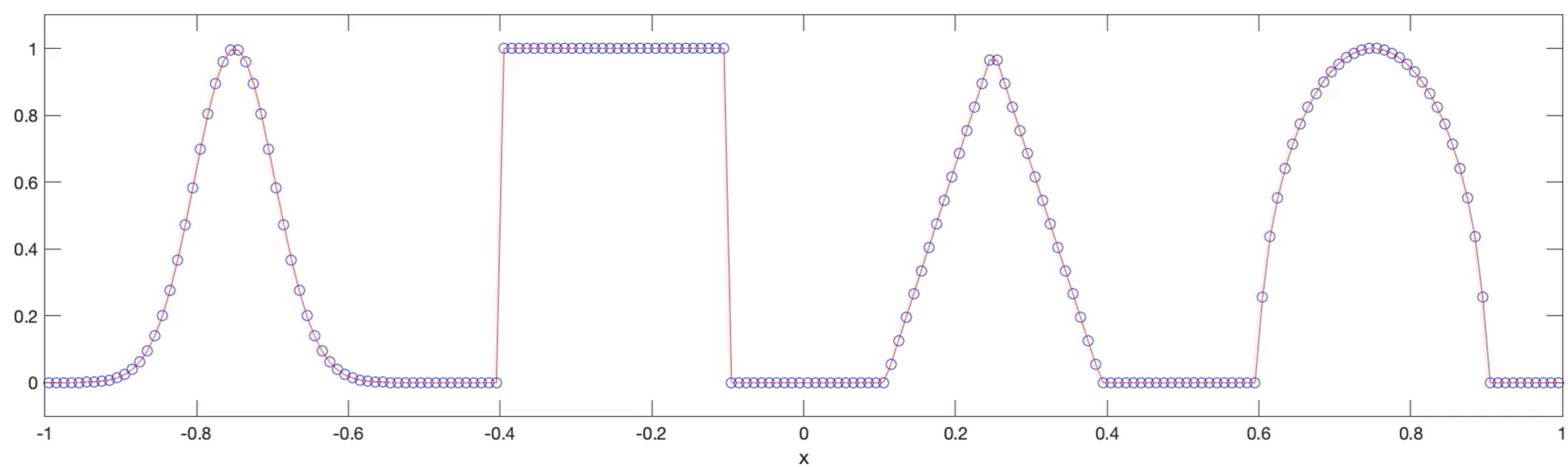
Interface Flux:

$$F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \frac{1}{2}(u_0 Q_i + u_0 Q_{i+1}) - \frac{1}{2}|u_0|(Q_{i+1} - Q_i)$$

Alternatively:

$F(Q) = u_0 Q$

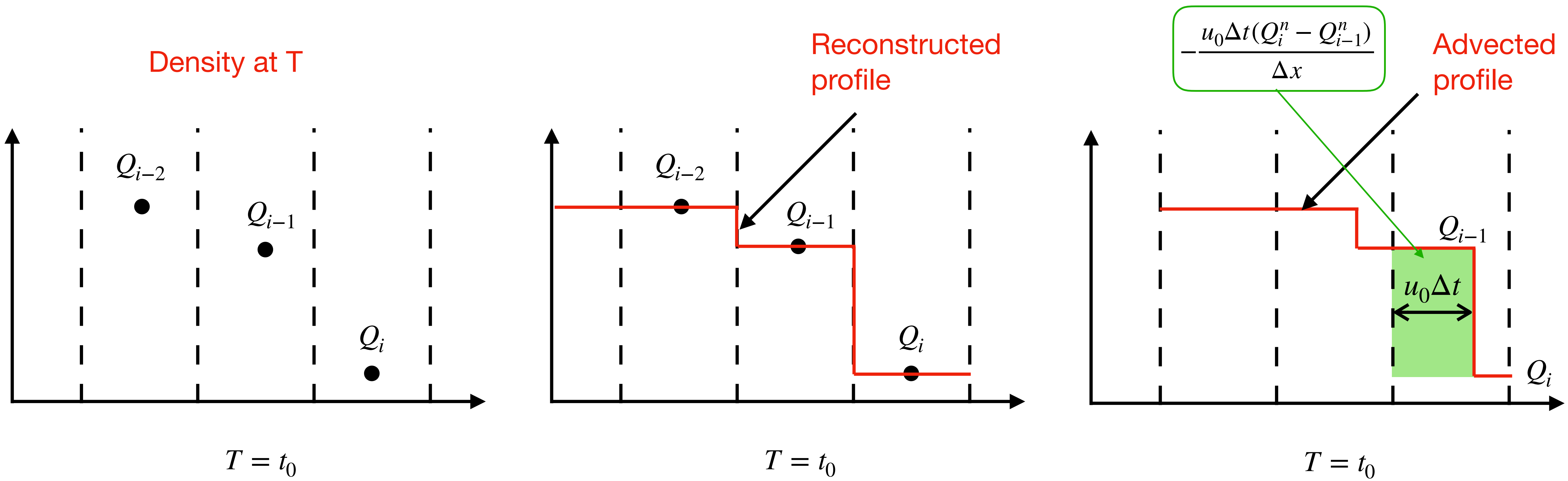
$$F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \frac{1}{2}(F(Q_i) + F(Q_{i+1})) - \frac{1}{2}|u_0|(Q_{i+1} - Q_i)$$



The Advection Nature of the equation

The REA framework

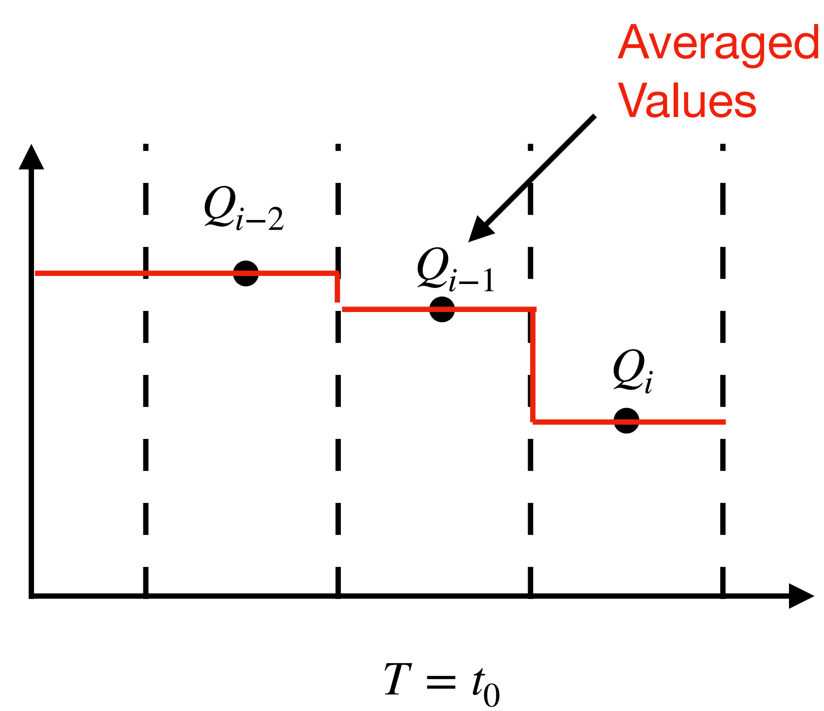
So the upwind scheme is basically an advection



Here's what happened in the upwind method:

1. From Q_i, do a piecewise-constant reconstruction;
2. Move the reconstructed profile by u*delta t
3. Average the shifted profile in each cell to get new Q_i

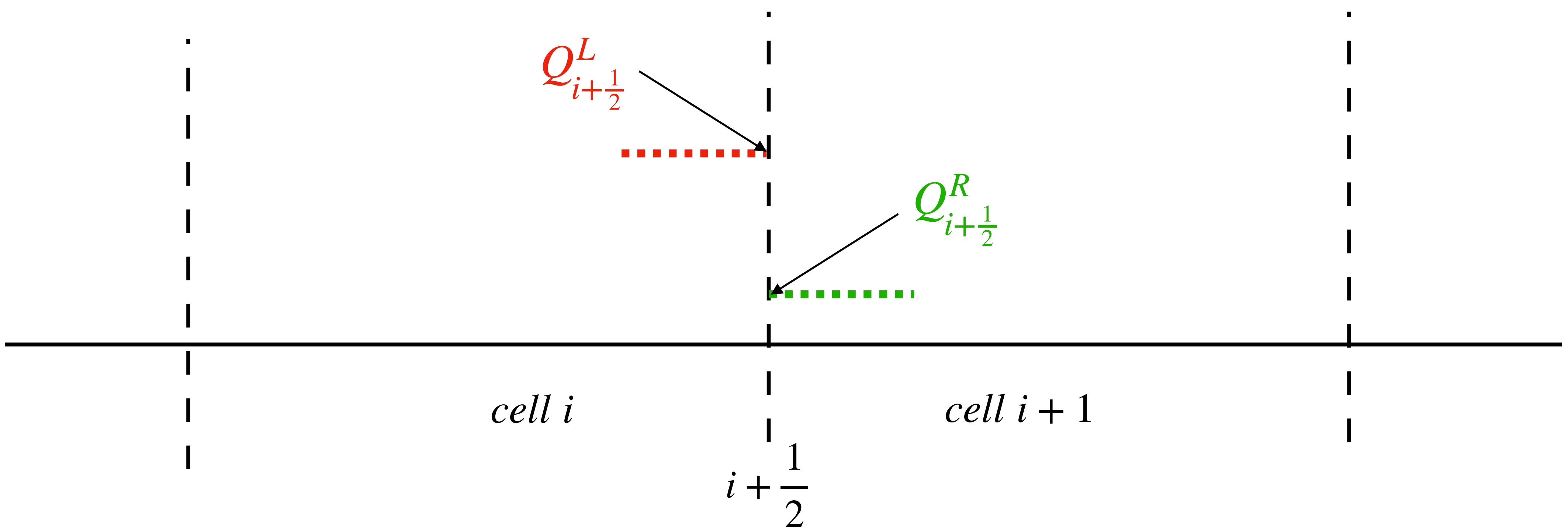
Reconstruct - ***Evolve*** - ***Average*** (REA framework)



Central Schemes

Interface states and wave speed

The use of upwind schemes require the knowledge of the wave speed and direction of the propagation - not always available in non-linear problems. So central schemes are convenient which does not require the information of the wave propagation:



Lax-Friedrichs $$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{\Delta x}{2\Delta t}(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

Rusanov $$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

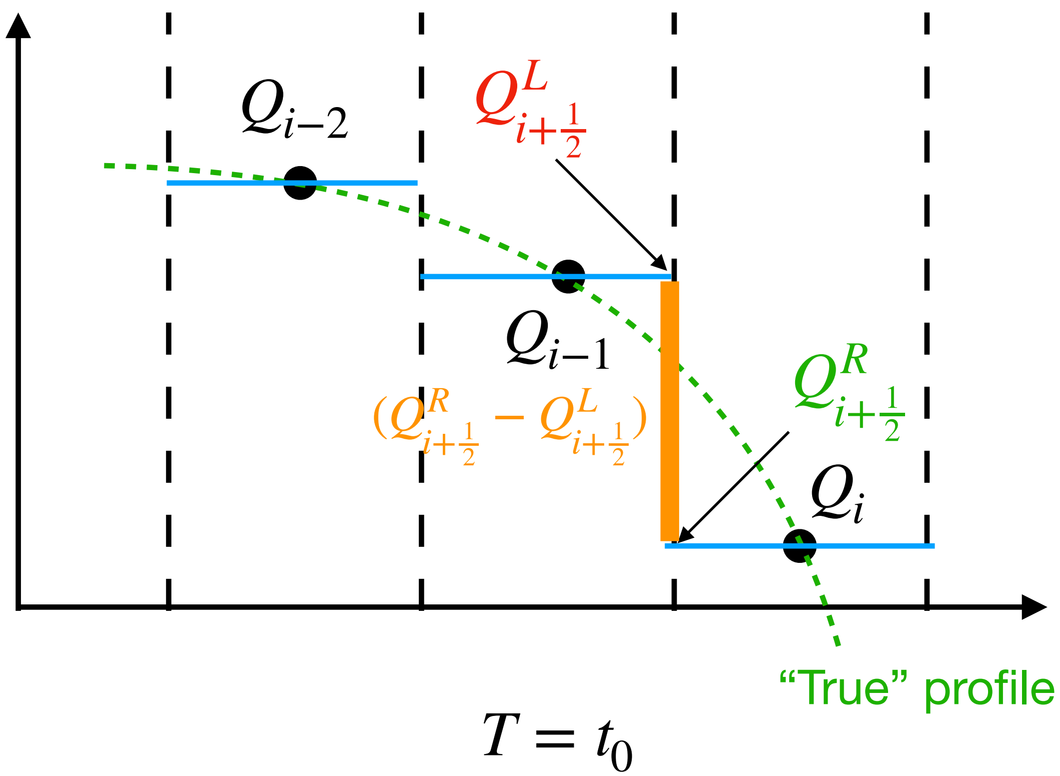
So the numerical diffusion is basically from Q_R - Q_L, how to reduce that?

Finite Volume Methods

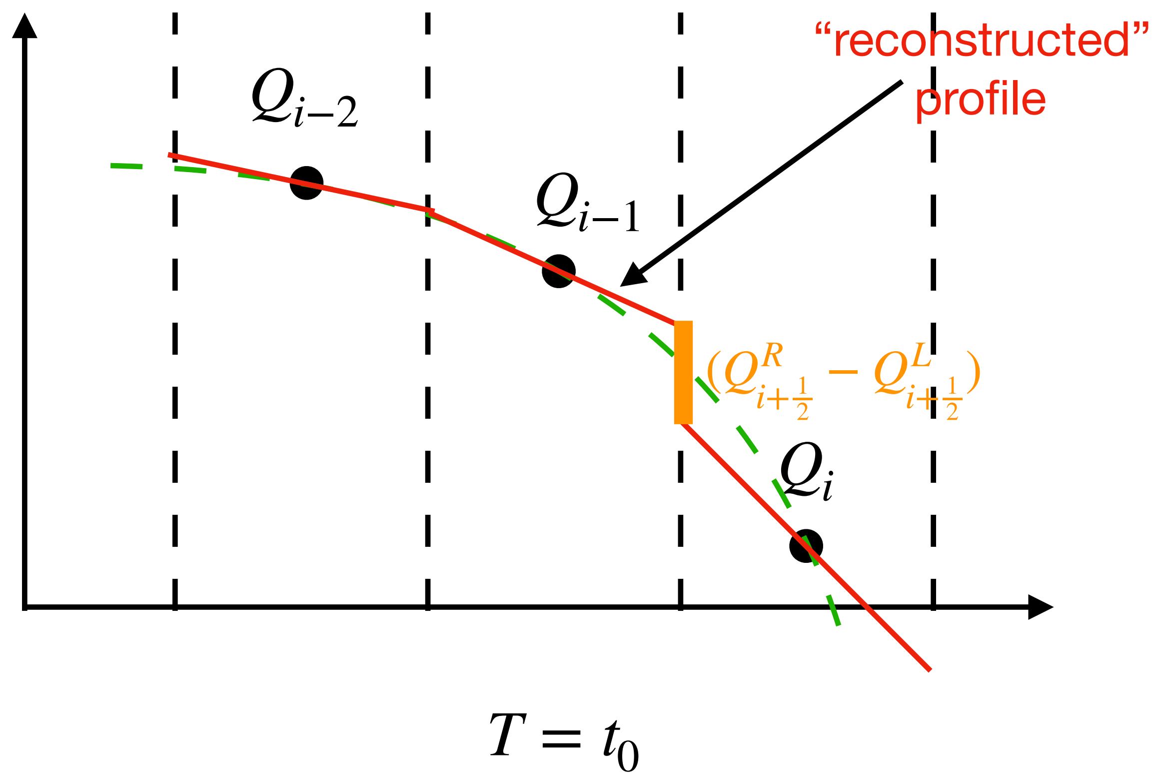
REA with Second-order methods

Now to improve the solution, using a more accurate reconstruction for the profile is needed:

Zeroth-order Reconstruction



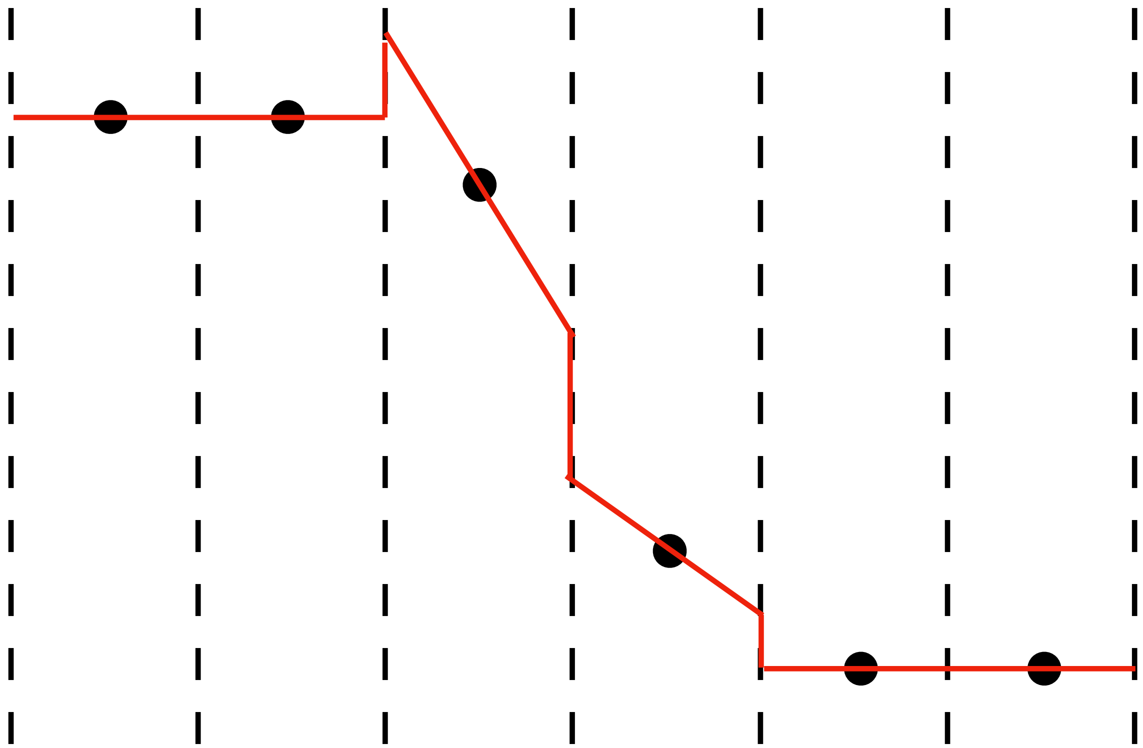
1st-order Reconstruction



Flux is basically: $F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \underbrace{\frac{1}{2}|u_0|\boxed{(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})}}_{\text{Diffusion term}}$

Slope limiters for TVD solutions

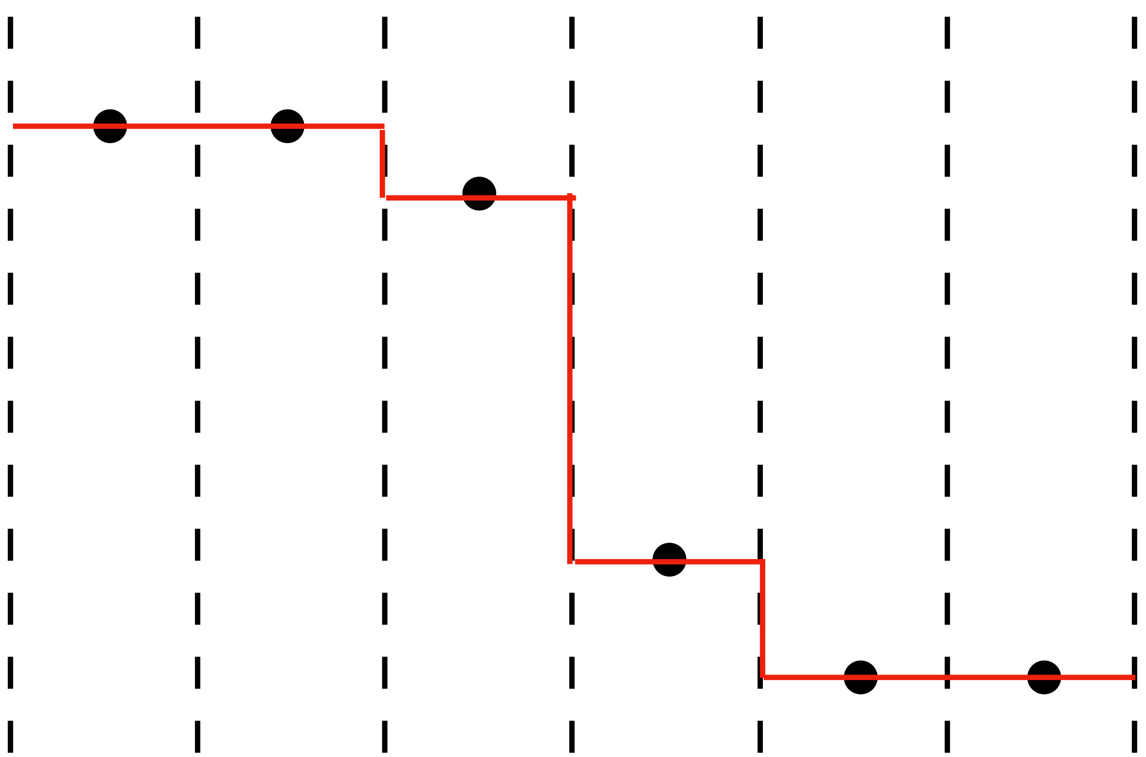
non-TVD reconstruction



$$\sigma_i^n = \frac{Q_{i+1}^n - Q_i^n}{\Delta x}$$ (Lax Wendroff method)

It's second-order but apparently non-TVD!

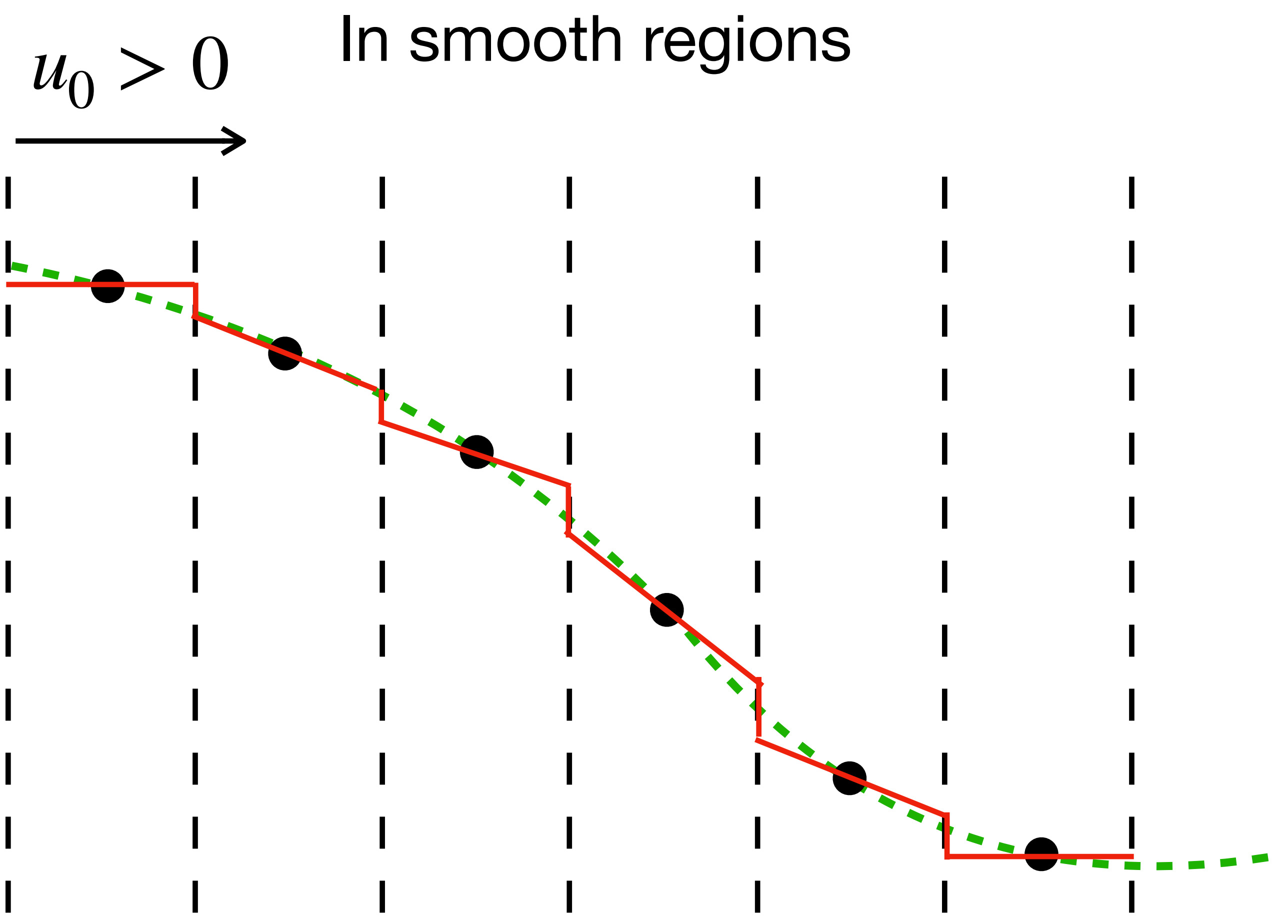
TVD reconstruction (PCM)

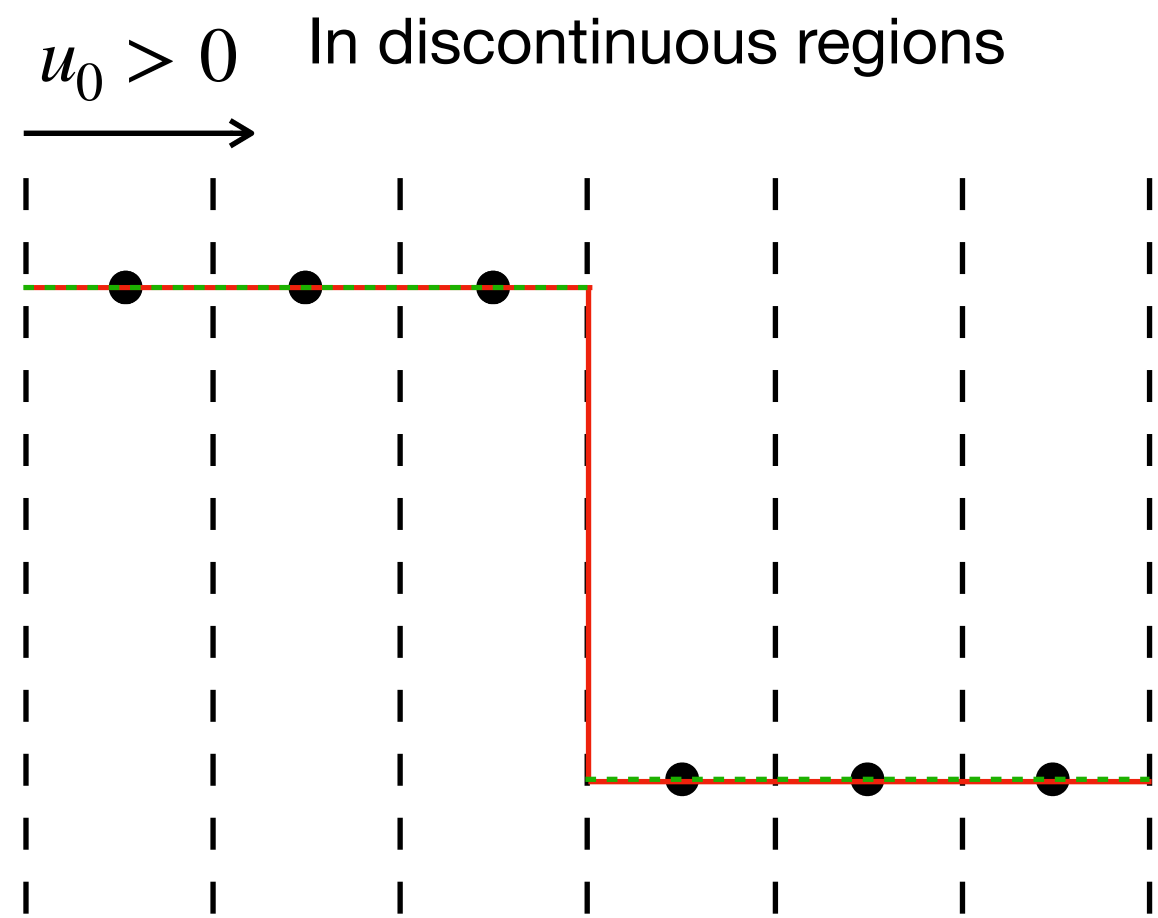


$\sigma_i^n = 0$ (first-order upwind)

It's TVD but apparently first-order

What does slope limiters do?





$$\sigma_i^n = \text{minmod}\left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}, \frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right)$$

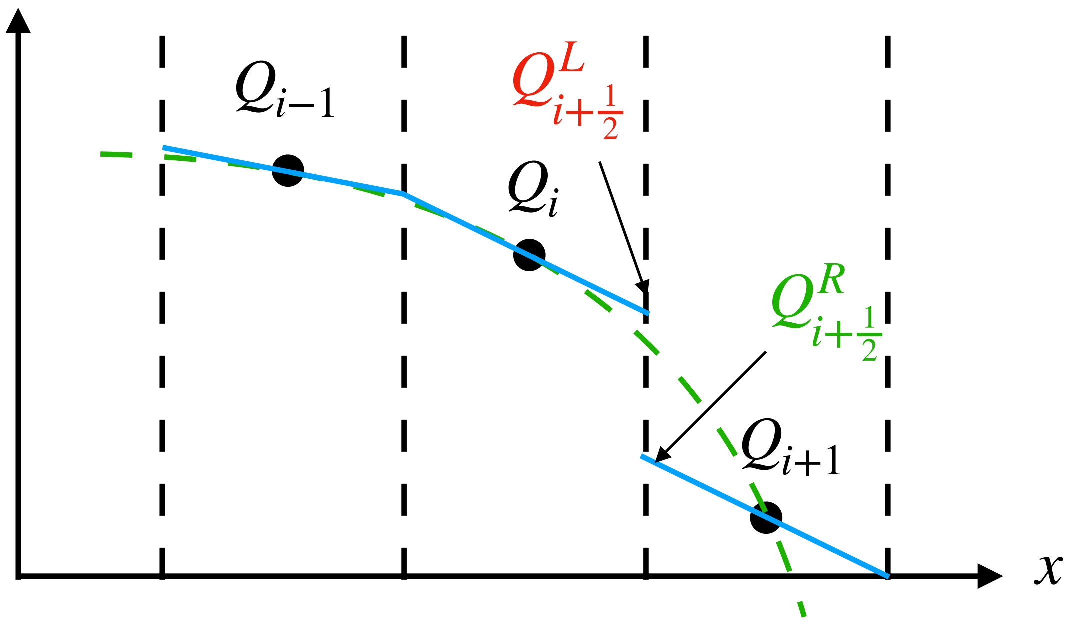
$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0, \\ b & \text{if } |b| < |a| \text{ and } ab > 0, \\ 0 & \text{if } ab \le 0. \end{cases}$$

- In smooth regions, the minmod slope limiter gives a profile that approximates the true profile with piecewise linear functions (2nd-order accuracy)
- In discontinuous regions, the minmod slop limiter chooses the smaller slope which is degenerated to the 1st-order upwind method (guaranteed TVD)

Summary of the Finite Volume framework

Step 1: Interface Reconstruction

purpose: get interface values



INPUT:	Q_i	
OUTPUT:	$Q^L_{i+\frac{1}{2}}$	$Q^R_{i+\frac{1}{2}}$

Step 2: Flux calculation

$$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

INPUT:	$Q^L_{i+\frac{1}{2}}$	$Q^R_{i+\frac{1}{2}}$
OUTPUT:	$F_{i+\frac{1}{2}}$	

Step 3: Finite Volume Update

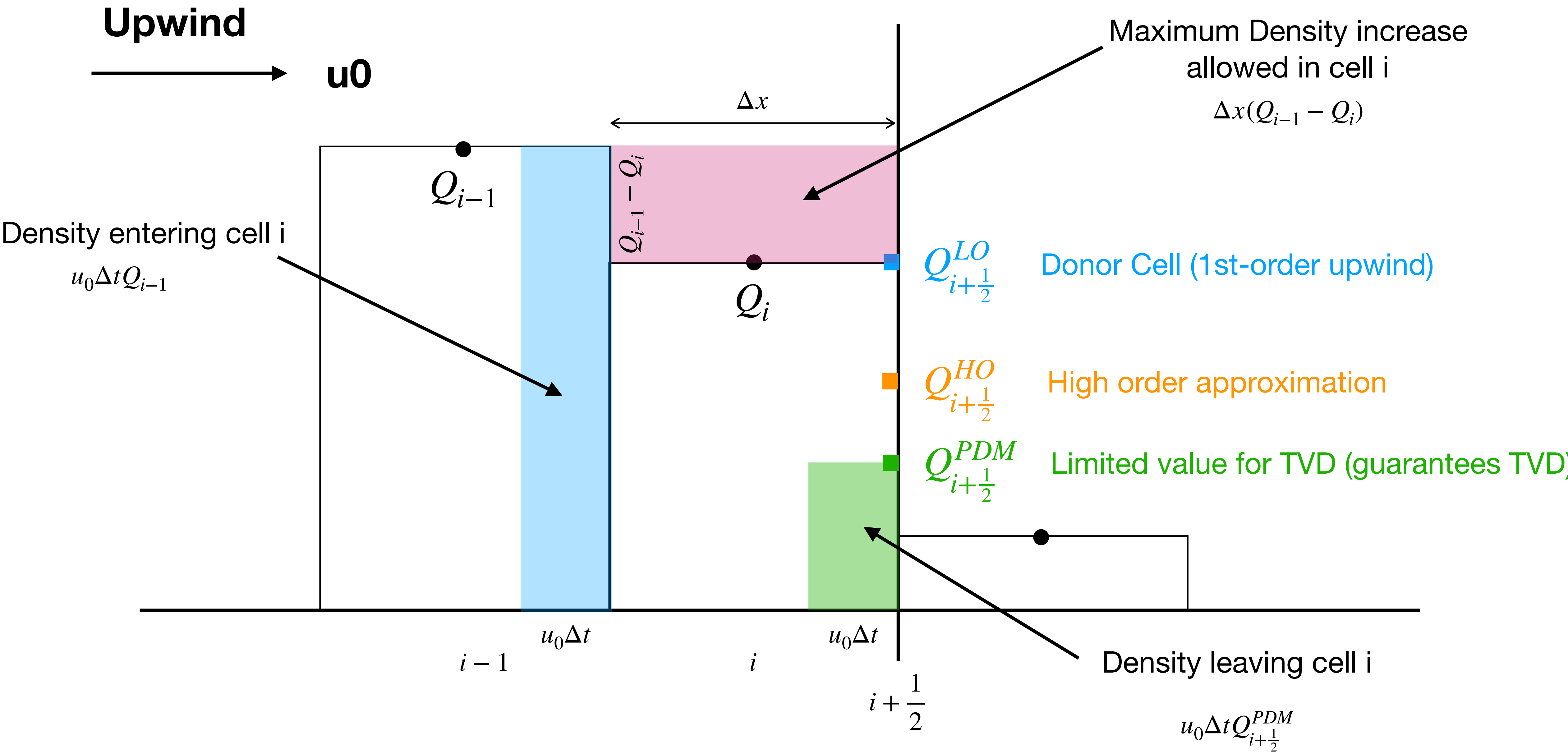
$$\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

INPUT:	Q^n_i	$F_{i+\frac{1}{2}}$
OUTPUT:	Q^{n+1}_i	

$$Q^n_i \xrightarrow{\text{Reconstruct}} Q^L_{i+\frac{1}{2}},\ Q^R_{i+\frac{1}{2}} \xrightarrow{\text{Flux calculation}} F_{i+\frac{1}{2}} \xrightarrow{\text{F-V update}} Q^{n+1}_i$$

The Partial Donor Cell Method

How to "correct" the left interface state $Q^{L}_{i+\frac{1}{2}}$



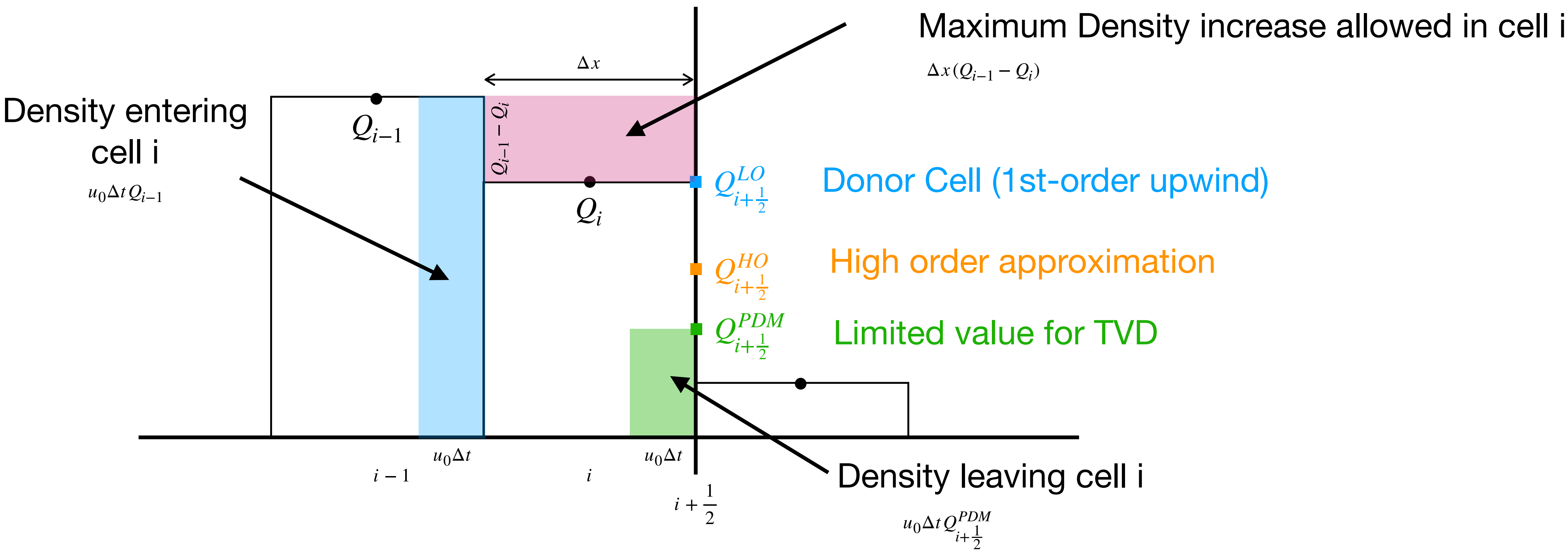
Now we can find the $Q^{PDM}_{i+\frac{1}{2}}$, which is the "limited" value for TVD:

$$u_0 \Delta t Q_{i-1} - u_0 \Delta t Q^{PDM}_{i+\frac{1}{2}} = \Delta x (Q_{i-1} - Q_i) \longrightarrow Q^{PDM}_{i+\frac{1}{2}} = \frac{1}{\epsilon} Q_i + (1 - \frac{1}{\epsilon}) Q_{i-1}$$

If we use any interface value < $Q^{PDM}_{i+\frac{1}{2}}$, cell i goes overshoot (or undershoot)

The Partial Donor Cell Method

How to “correct” the interface flux



Now we have three candidates for interface values at i+1/2: $Q^{PDM}_{i+\frac{1}{2}}$ $Q^{HO}_{i+\frac{1}{2}}$ $Q^{LO}_{i+\frac{1}{2}}$

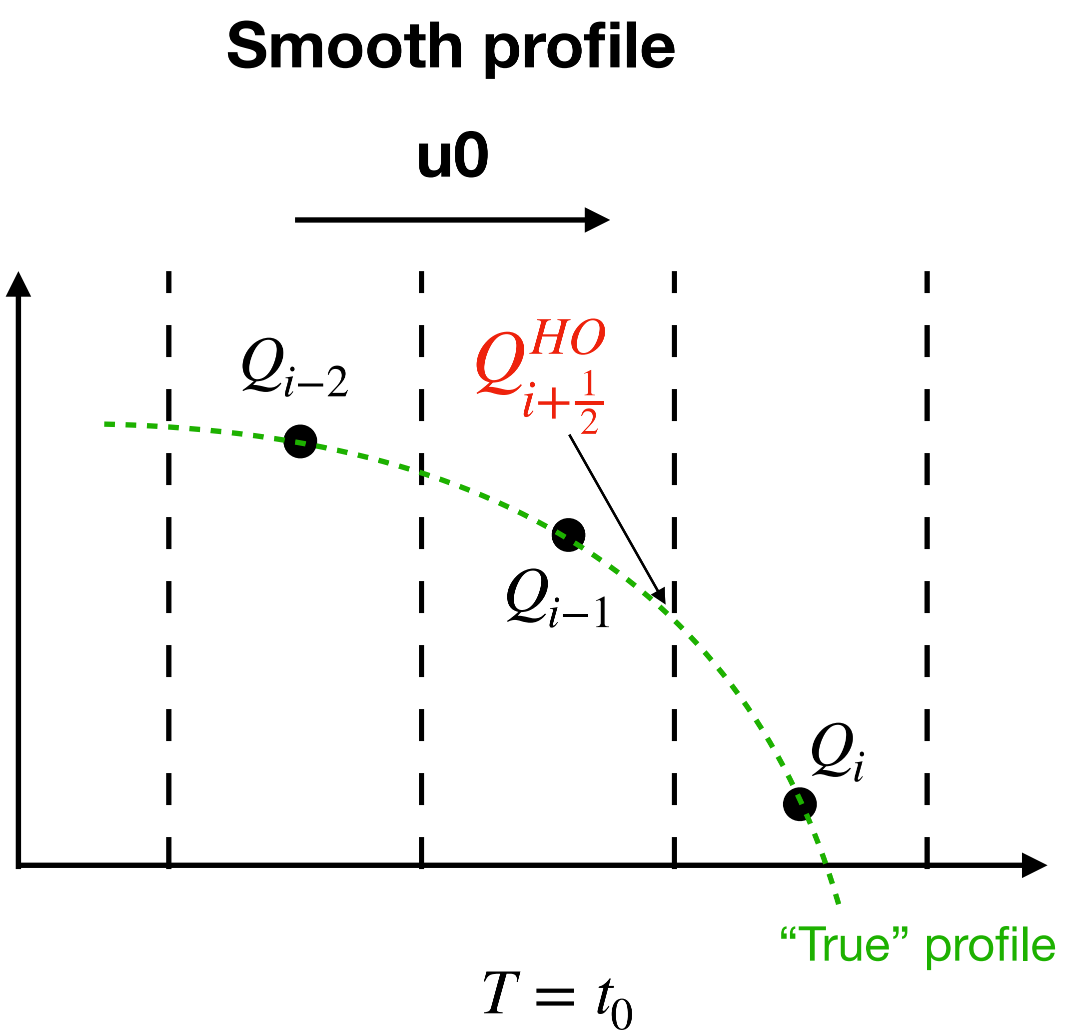
Which one to use?

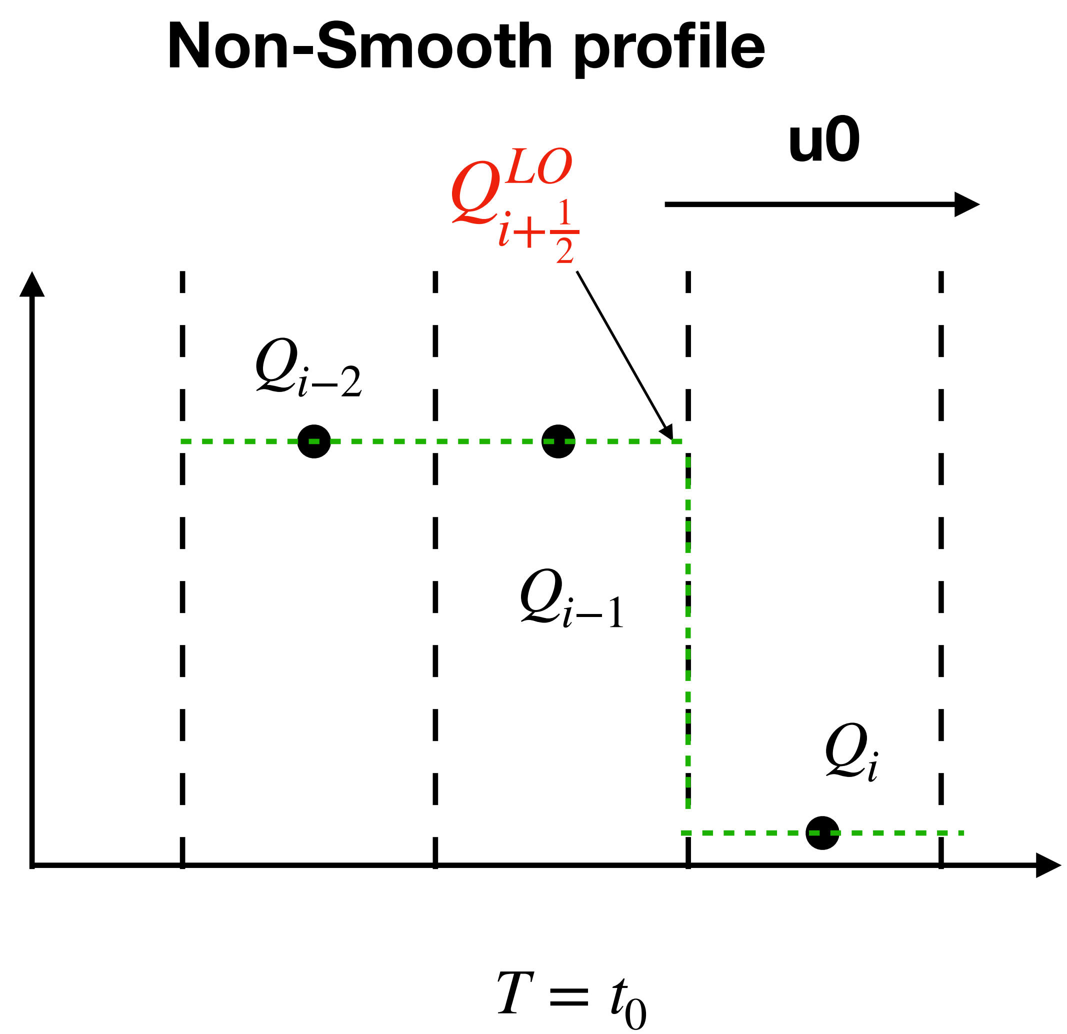
The most intuitive choice is to always use the value in the middle:

$$Q^{L}_{i+\frac{1}{2}} = \mathrm{median}(Q^{LO}_{i+\frac{1}{2}}, Q^{HO}_{i+\frac{1}{2}}, Q^{PDM}_{i+\frac{1}{2}})$$

Improve the Donor Cell Method

The partial donor cell method is much less diffusive because it tries to use the high-order approximation whenever possible:





Idea: in smooth structure region, use a high-order approximation for accuracy

$Q^{HO}_{i+\frac{1}{2}}$ (arbitrary high order)

in non-smooth structure region, use a low-order upwind value for stability (TVD)

Extending to Multi-Dimensional Problems

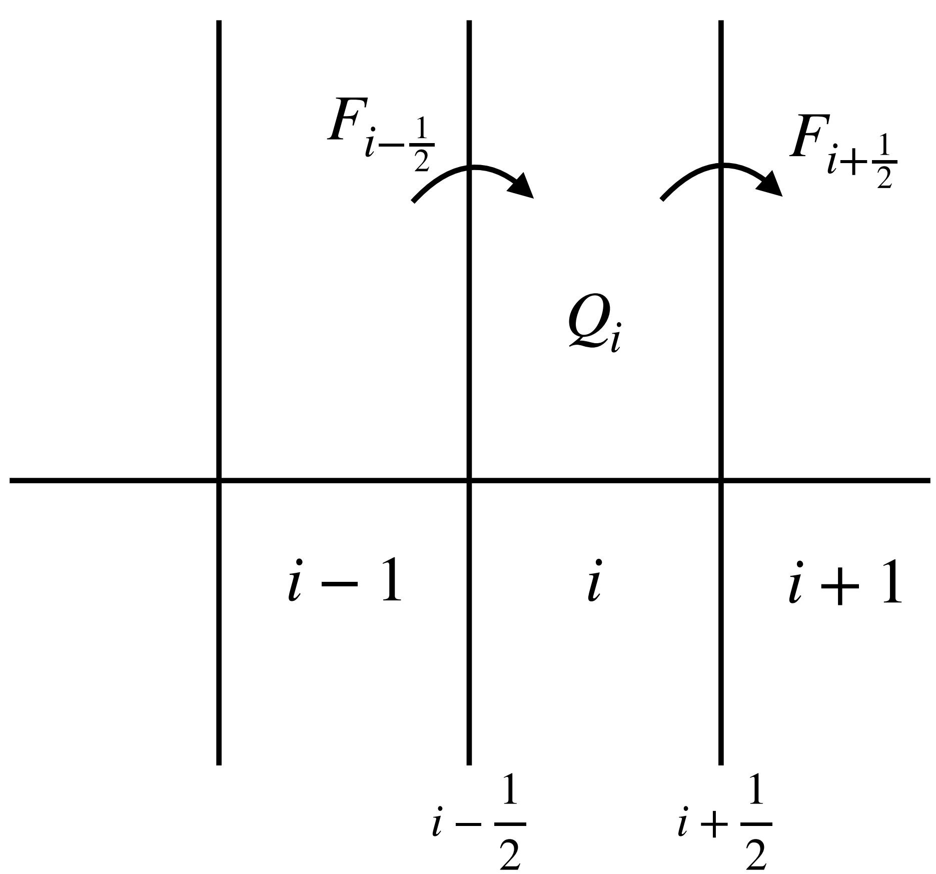
2-D Linear Advection $\frac{\partial}{\partial t}\rho + \mathbf{u}_0 \cdot \nabla \rho = 0$

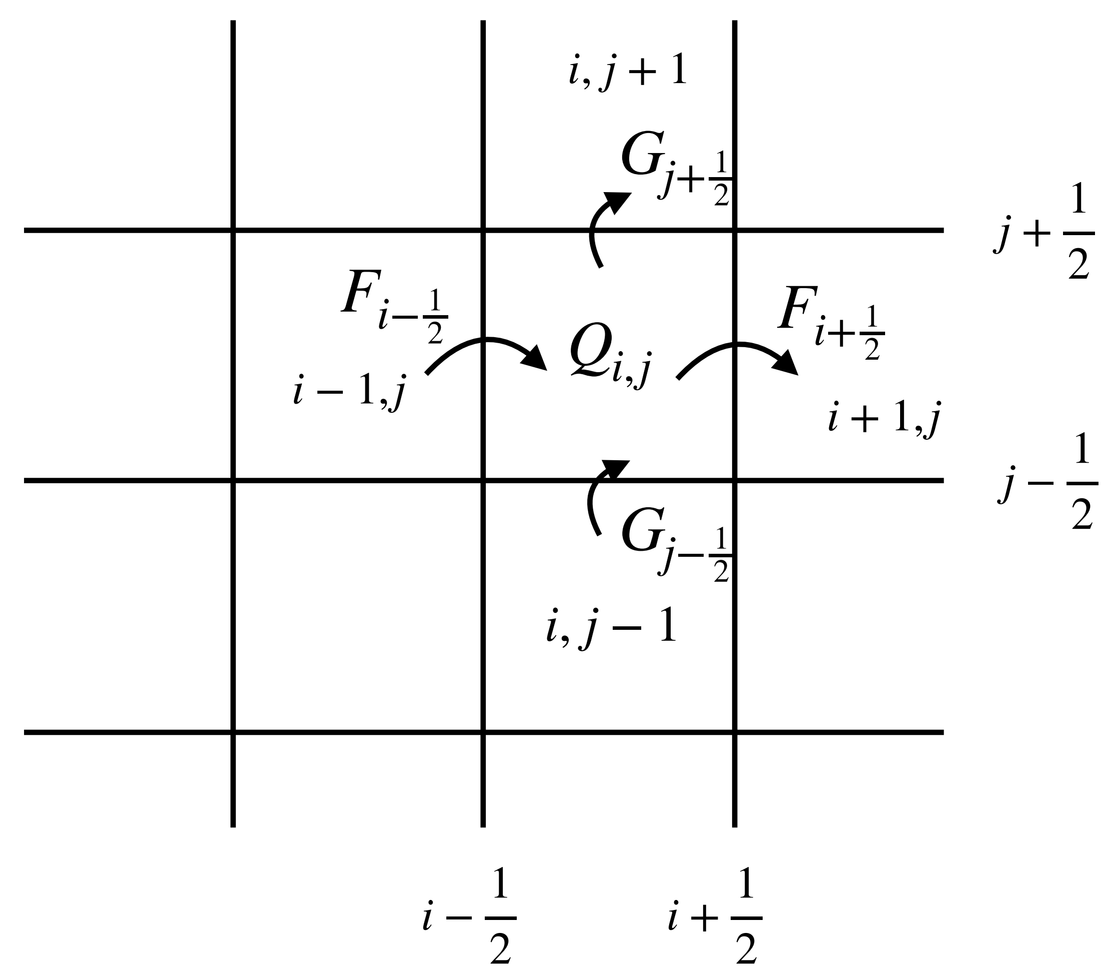
Now both rho and u are functions of x and y: $\rho(x, y), \mathbf{u}_0 = (u_x, u_y)$

The equation is still linear and is a simple extension of the 1-D equation:

1-D $\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} = 0$

2-D $\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} + \frac{\partial}{\partial y}\underbrace{(u_y\rho)}_{G(\rho)} = 0$

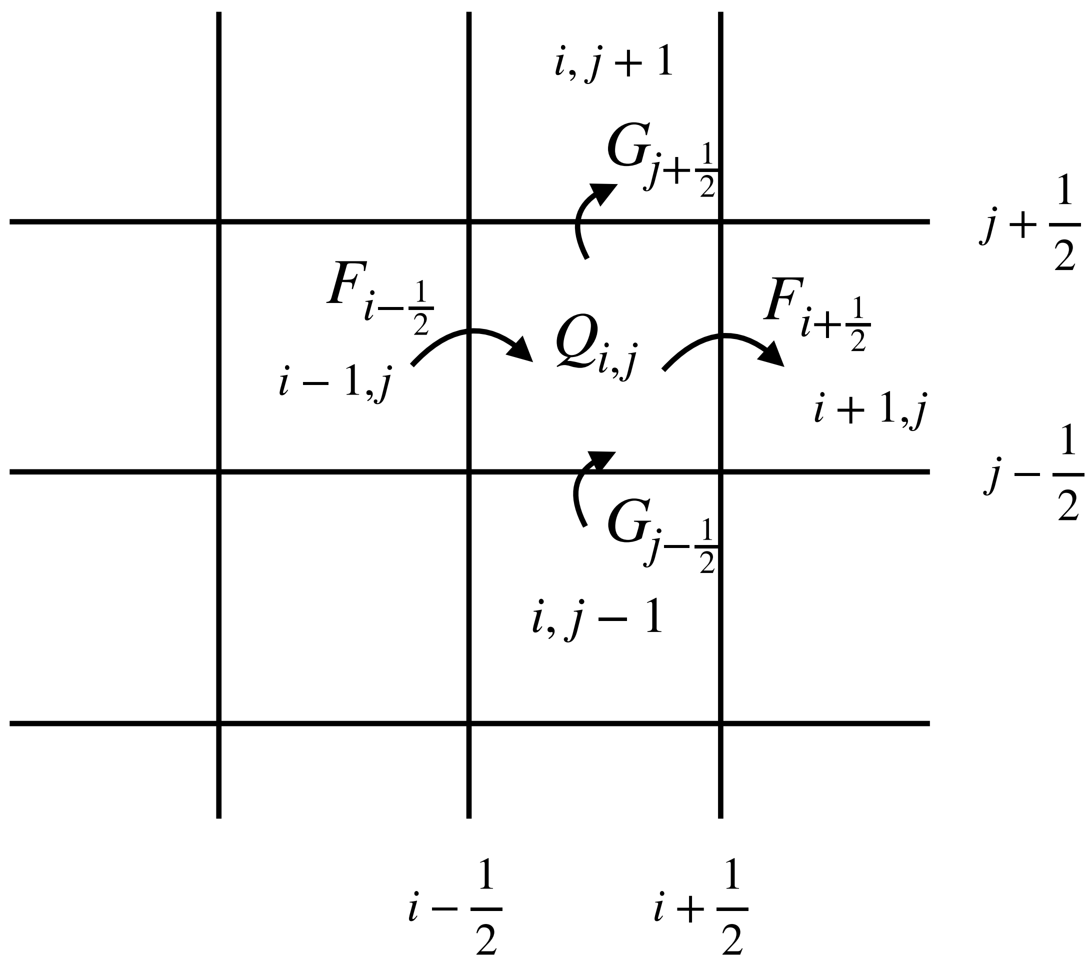




Extending to Multi-Dimensional Problems

Two-Dimensional Algorithm for Advection

2-D $$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} + \frac{\partial}{\partial y}\underbrace{(u_y\rho)}_{G(\rho)} = 0$$



Finite-Volume form

$$\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) - \frac{1}{\Delta y}(G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}})$$

Initialization

Step 1: Interface Reconstruction in x-direction

Step 2: Interface flux in x-direction

Step 3: Interface Reconstruction in y-direction

Step 4: Interface flux in y-direction

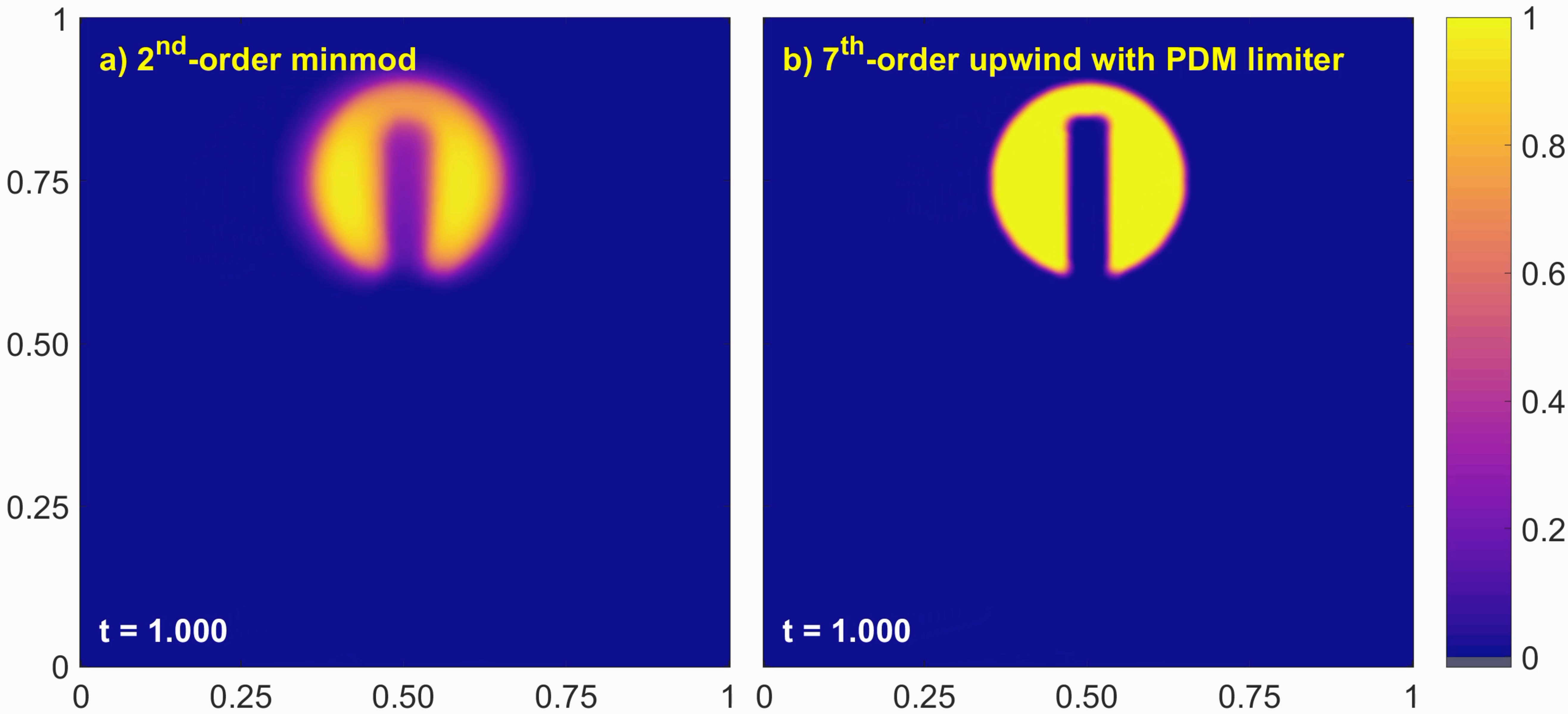
Step 5: finite-volume update

Boundary Conditions

Extending to Multi-Dimensional Problems

2-D Advection

$$\frac{\partial Q}{\partial t} + \mathbf{u}_0 \cdot \nabla Q = 0$$



Vlasov Equations

Boltzmann Equation:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = \cancelto{0}{\left(\frac{\delta f_s}{\partial t}\right)_c}$$

Evolution of the plasma distribution function

$$\mathbf{F}_s = q_s / m_s (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lorentz Force

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} + \nabla \times \mathbf{E} = 0$$

Faraday's Law

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_s q_s \int_{-\infty}^{+\infty} v f_s d^3 v$$

Ampere-Maxwell's Law

Electrostatic $\nabla \times \mathbf{E} = 0 \longrightarrow \frac{\partial \mathbf{B}}{\partial \mathbf{t}} = 0 \xrightarrow{\text{Use}} \nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon}$ Gauss' Law

Electrostatic Vlasov equations

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon} \sum_s q_s \int_{-\infty}^{+\infty} f_s d^3 v$$

Vlasov-Poisson Equations

Electrostatic Vlasov equations

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon} \sum_s q_s \int_{-\infty}^{+\infty} f_s d^3 v$$

If we use a potential form for the E field: $\mathbf{E} = -\nabla \phi$

The Electrostatic Vlasov system becomes:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = 0 \qquad \text{Advection}$$

$$\nabla^2 \phi = -\frac{1}{\epsilon} \underbrace{\sum_s q_s \int_{-\infty}^{+\infty} f_s d^3 v}_{\text{Charge density}} \qquad \text{Poisson}$$

This is the so-called **Vlasov-Poisson** system

Vlasov-Poisson Equations in 1D

1D1V Vlasov-Poisson

Configuration space: x

Velocity space: v_x

$\longrightarrow$ $f_s = f(x, v)$

Electric field: E $\longrightarrow$ $E = E(x)$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial E}{\partial x} = \frac{1}{\epsilon_0}\rho_e$$

$$\rho_e = e\int_{-\infty}^{+\infty} f d^3 v$$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{1}{\epsilon_0}\rho_e$$

$$\rho_e = e\int_{-\infty}^{+\infty} f d^3 v$$

Vlasov-Poisson Equations in 1D

Normalize the Equations

A natural choice for the normalization of the Vlasov-Poisson system of equations would redefine all the relevant quantities as follows,

$$t = \omega_{pe}^{-1}\tilde{t}$$

$$x = d_e\tilde{x}$$

$$v = c\tilde{v}$$

$$q = e$$

$$m = m_e\tilde{m}$$

$$n = n_0\tilde{n}$$

$$E = \frac{en_0d_e}{\epsilon_0}\tilde{E}$$

$$\longrightarrow$$

$$\frac{\partial f}{\partial \tilde{t}} + v\frac{\partial f}{\partial \tilde{x}} + \tilde{E}\frac{\partial f}{\partial \tilde{v}} = 0$$

$$\frac{\partial \tilde{E}}{\partial \tilde{x}} = \int_{-\infty}^{+\infty} f d\tilde{v}$$

The Vlasov Equation: 2D linear advection

Recall the linear advection equation: 2-D

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} + \frac{\partial}{\partial y}\underbrace{(u_y\rho)}_{G(\rho)} = 0$$

Vlasov equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial v} = 0$$

Now here v and x are independent variables - v is not a function of x, so in the x-direction it's simply an linear advection equation (in configuration space)

The less intuitive part is the second term - eE/m is the acceleration in the velocity space, since eE/m is not a function of v, so the second term is also a linear advection (in velocity space)

$$\frac{\partial f}{\partial t} + \overset{u_x}{v}\frac{\partial f}{\partial x} + \overset{u_y}{E}\frac{\partial f}{\partial v} = 0 \quad \longrightarrow \quad \frac{\partial f}{\partial t} + u_x\frac{\partial f}{\partial x} + u_y\frac{\partial f}{\partial y} = 0$$

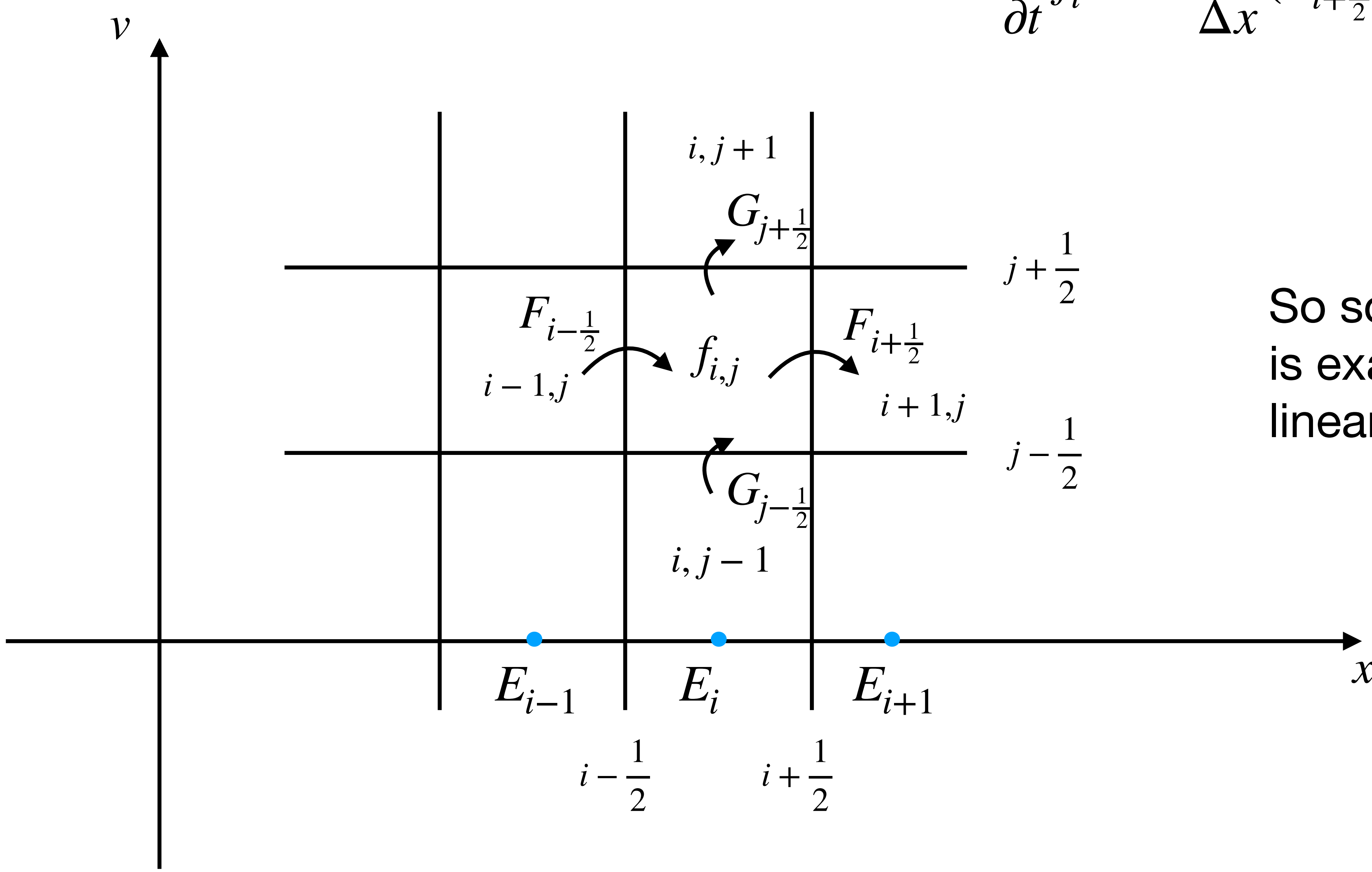
y

The Vlasov Equation: 2D linear advection

Finite Volume Vlasov Solver

$$\frac{\partial f}{\partial t} + \overset{u_x}{v}\frac{\partial f}{\partial x} + \overset{u_y}{E}\frac{\partial f}{\partial v} = 0 \longrightarrow \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\underbrace{(vf)}_{F(f)} + \frac{\partial}{\partial v}\underbrace{(Ef)}_{G(f)} = 0$$

$$\xrightarrow{\text{FV}} \frac{\partial}{\partial t} f_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) - \frac{1}{\Delta y}(G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}})$$



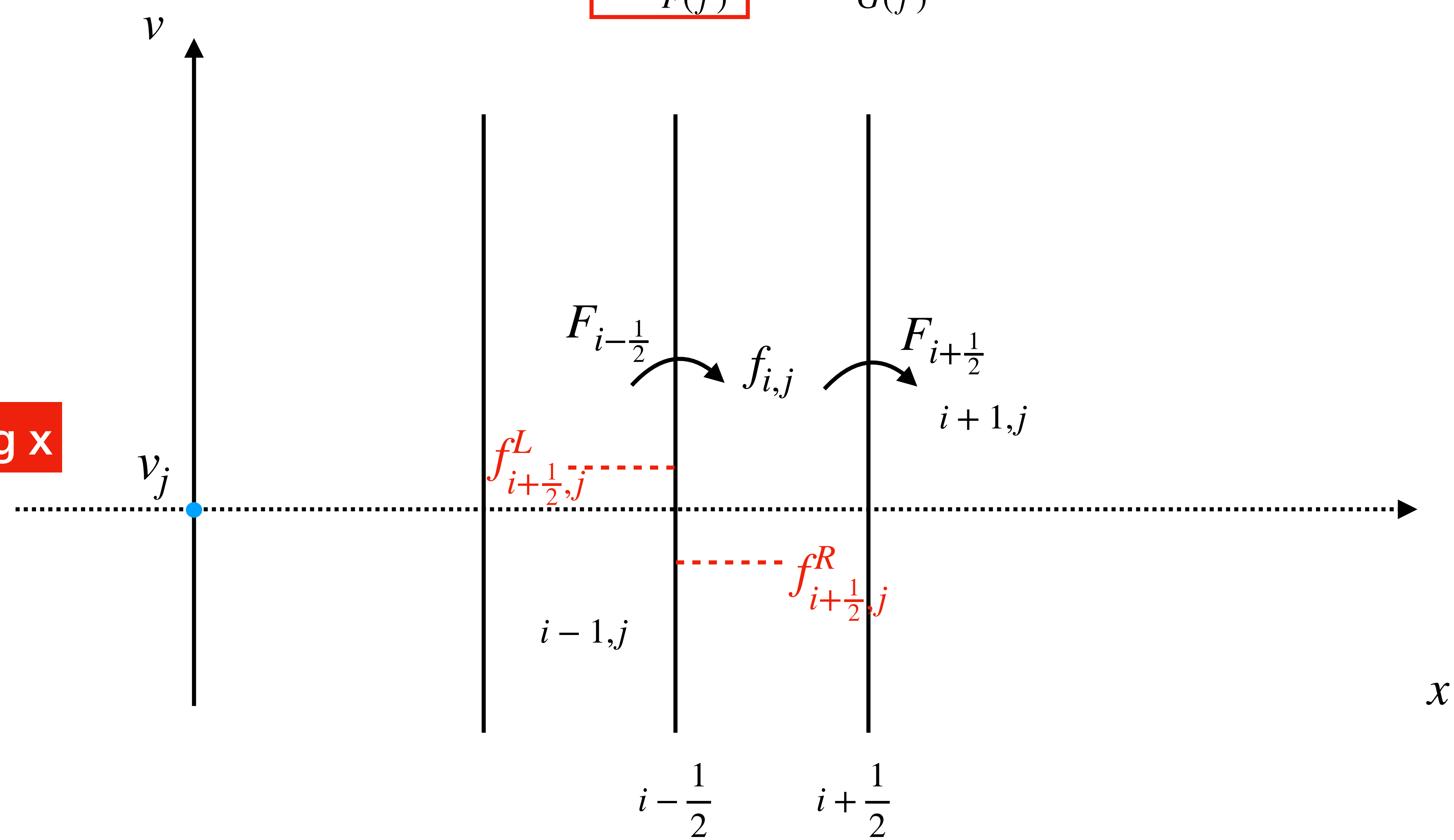
So solving the Vlasov equation is exactly the same as a 2-D linear advection problem!

The Vlasov Equation: 2D linear advection

in x-direction (configuration space)

$$\frac{\partial f}{\partial t} + \overset{u_x}{v}\frac{\partial f}{\partial x} + \overset{u_y}{E}\frac{\partial f}{\partial v} = 0 \longrightarrow \frac{\partial f}{\partial t} + \boxed{\frac{\partial}{\partial x}\underbrace{(vf)}_{F(f)}} + \frac{\partial}{\partial v}\underbrace{(Ef)}_{G(f)} = 0$$

v_j = const along x



Reconstruction in x-dir:

$$f^L_{i+\frac{1}{2},j} \quad f^R_{i+\frac{1}{2},j}$$

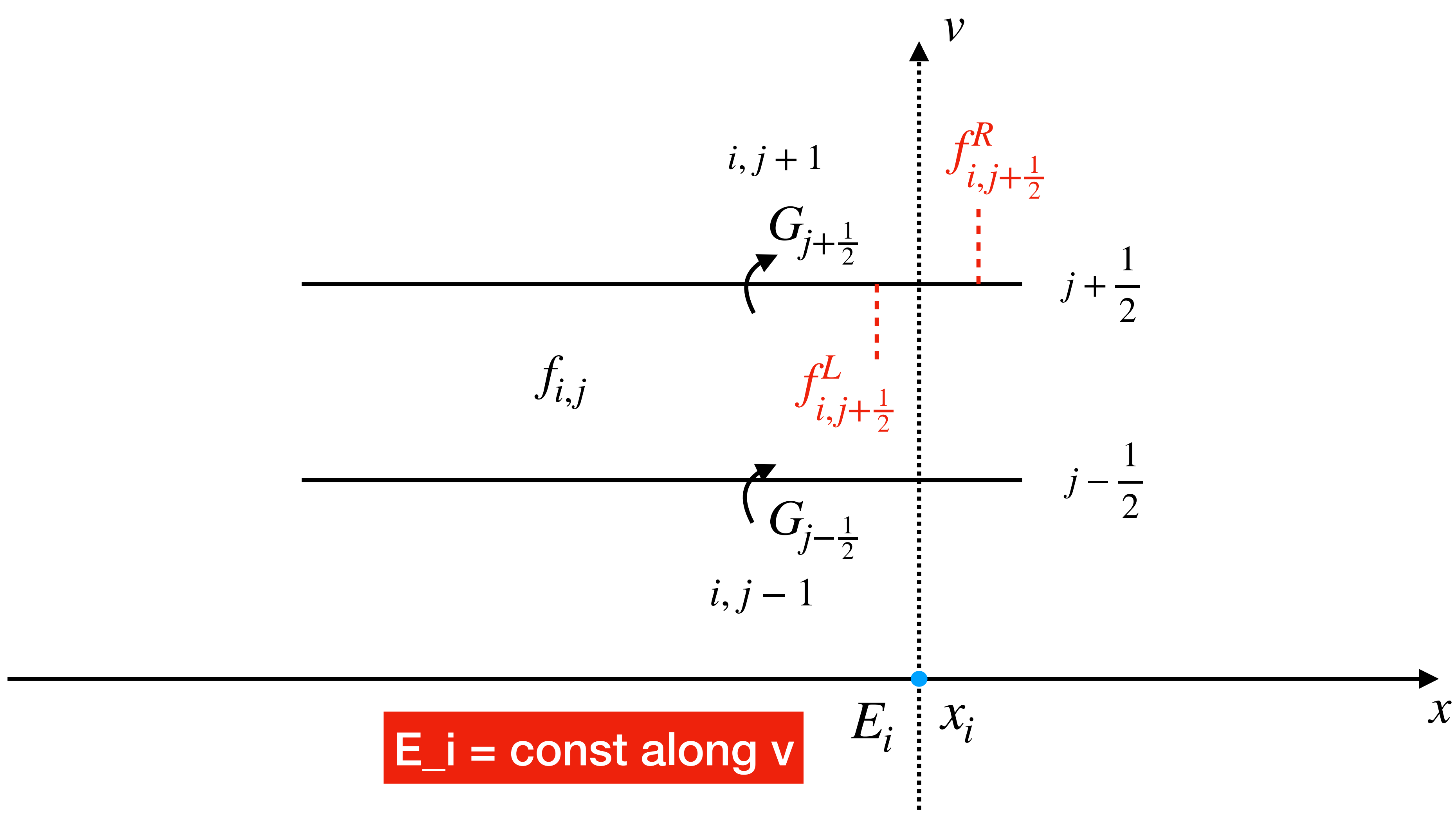
Interface Flux:

$$\longrightarrow \quad F_{i+\frac{1}{2}} = \frac{1}{2}(v_j f^L_{i+\frac{1}{2},j} + v_j f^R_{i+\frac{1}{2},j}) - \frac{1}{2}|v_j|(f^L_{i+\frac{1}{2},j} - f^R_{i+\frac{1}{2},j})$$

The Vlasov Equation: 2D linear advection

in v-direction (velocity space)

$$\frac{\partial f}{\partial t} + \overset{u_x}{v}\frac{\partial f}{\partial x} + \overset{u_y}{E}\frac{\partial f}{\partial v} = 0 \longrightarrow \frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\underbrace{(vf)}_{F(f)} + \boxed{\frac{\partial}{\partial v}\underbrace{(Ef)}_{G(f)}} = 0$$



Reconstruction in x-dir:

Interface Flux:

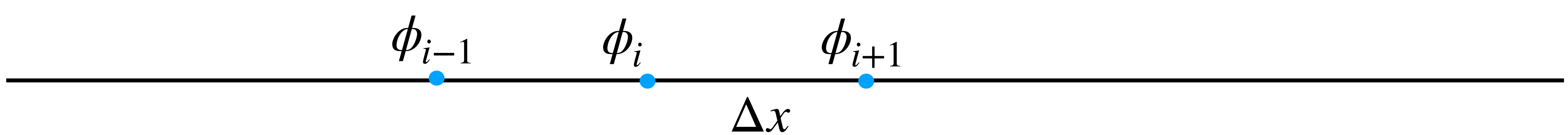
$$f^L_{i,j+\frac{1}{2}} \quad f^R_{i,j+\frac{1}{2}} \longrightarrow G_{i,j+\frac{1}{2}} = \frac{1}{2}(E_i f^L_{i,j+\frac{1}{2}} + E_i f^R_{i,j+\frac{1}{2}}) - \frac{1}{2}|E_i|(f^L_{i,j+\frac{1}{2}} - f^R_{i,j+\frac{1}{2}})$$

The Poisson Equation: 1D Diffusion

The normalized Poisson equation for electric potential: $$\frac{\partial^2 \phi}{\partial x^2} = -\rho_e$$

Use central difference approximation for the LHS:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} = \rho_{e,i} \longrightarrow \phi_{i+1} - 2\phi_i + \phi_{i-1} = \Delta x^2 \rho_{e,i}$$



So we can write a series of algebra equations:

x

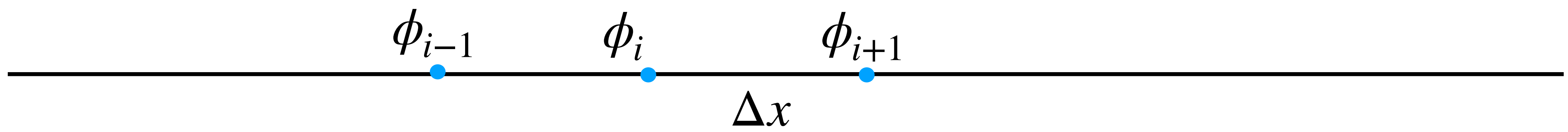
$$\phi_3 - 2\phi_2 + \phi_1 = -\Delta x^2 \rho_{e,2}$$

$$\phi_4 - 2\phi_3 + \phi_2 = -\Delta x^2 \rho_{e,3}$$

...

$$\phi_N - 2\phi_{N-1} + \phi_{N-2} = -\Delta x^2 \rho_{e,N-1}$$

The Poisson Equation: 1D Diffusion



So we can write a series of algebra equations:

$$\phi_3 - 2\phi_2 + \phi_1 = -\Delta x^2 \rho_{e,2}$$

$$\phi_4 - 2\phi_3 + \phi_2 = -\Delta x^2 \rho_{e,3}$$

...

$$\phi_N - 2\phi_{N-1} + \phi_{N-2} = -\Delta x^2 \rho_{e,N-1}$$

$$\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \end{bmatrix} \quad T = \begin{bmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} \Delta x^2 \rho_{e,1} \\ \Delta x^2 \rho_{e,2} \\ \Delta x^2 \rho_{e,3} \\ \vdots \\ \Delta x^2 \rho_{e,N} \end{bmatrix} \longrightarrow T \cdot \Phi = b$$

Boundary condition

Can also solve Gauss' law directly

What's used in the code - works for periodic boundary

Note that $\frac{\partial E}{\partial x} = -\rho_e$ is a linear equation. The E field at any point is the sum of the E caused by charge at each grid point.

For periodic boundary, the average electric field due to chase at any grid point must be zero

To get E, we integrate the ODE: $\frac{\partial E}{\partial x} = \rho_e$

$$E(x) = \int_0^x \rho_e(x')dx' + C \longrightarrow C = \langle E(x) \rangle$$

This is simply because

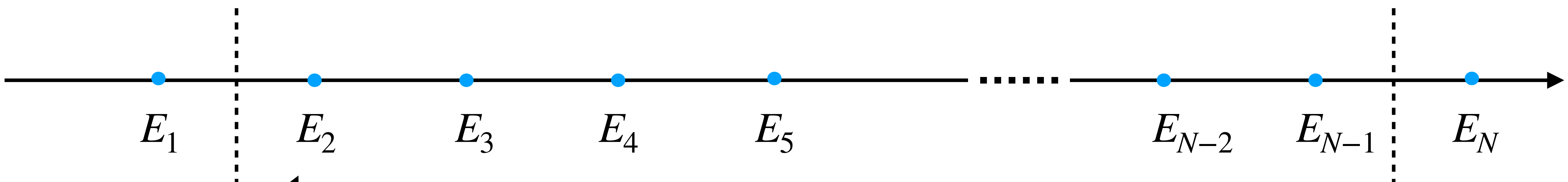
$$\text{Average } E = \frac{\int E dx}{\int dx} = \frac{-\int_0^1 \frac{d\phi}{dx} dx}{1} = -(\phi(x=1) - \phi(x=0)) \equiv 0$$

Can also solve Gauss' law directly

What's used in the code - works for periodic boundary

To get E, we integrate the ODE:

$$E(x) = \int_0^x \rho_e(x')dx' + C \longrightarrow C = \langle E(x) \rangle$$



Assuming 0
For the moment

$$E_2 = 0 \xrightarrow{\text{Iterate}} E_3 = E_2 + \left(\frac{\rho_2 + \rho_3}{2}\right)\Delta x$$

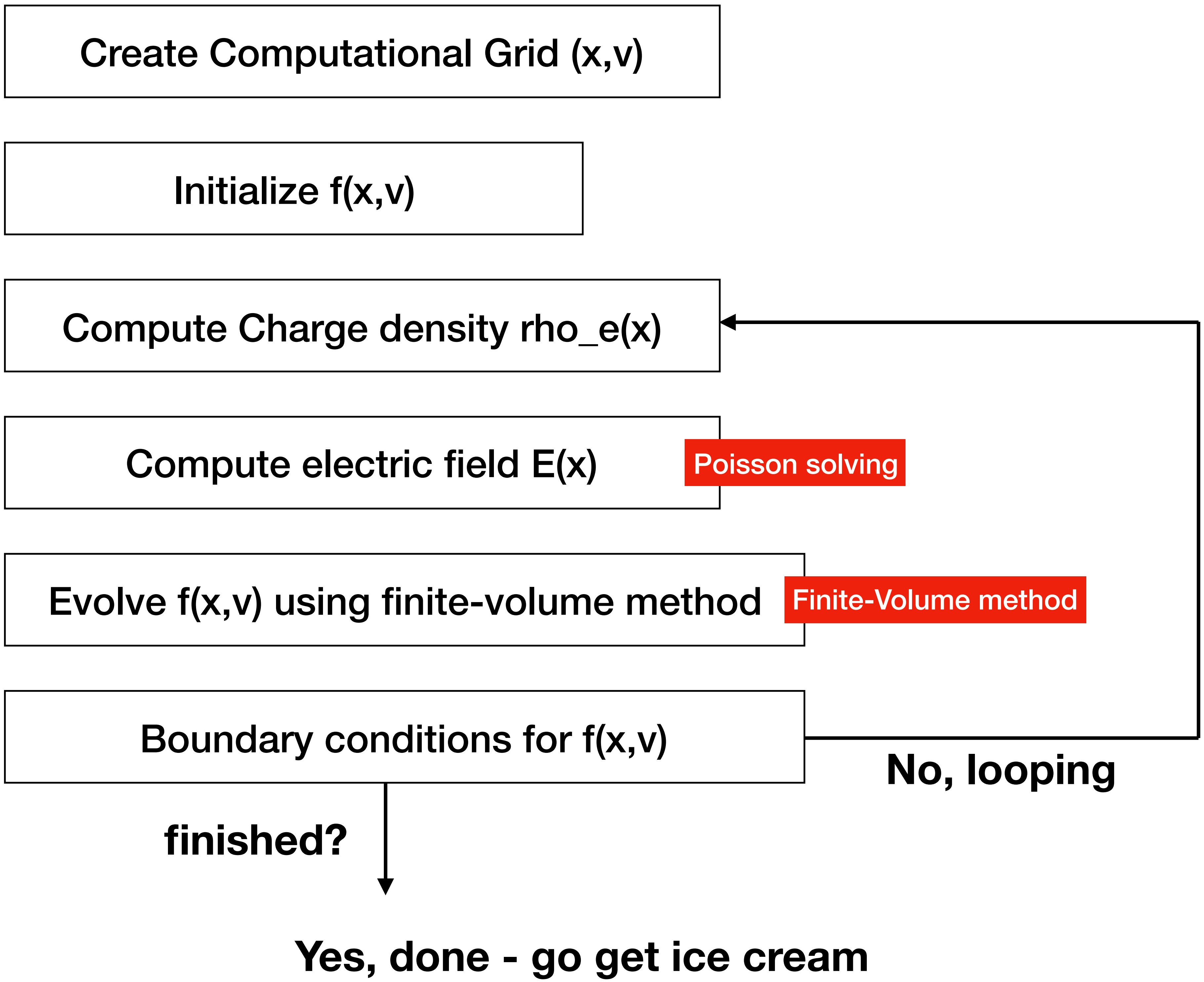
$$E_4 = E_3 + \left(\frac{\rho_3 + \rho_4}{2}\right)\Delta x$$

......

$$E_{N-1} = E_{N-2} + \left(\frac{\rho_{N-2} + \rho_{N-1}}{2}\right)\Delta x$$

Then calculate average E: $\langle E \rangle = \frac{1}{N}\sum_{i=2}^{N-1} E_i \xrightarrow{\text{Final E}} E_i = E_i - \langle E \rangle$

Put things together



Physics

Landau Damping

Dispersion Relation

Starting from the normalized Vlasov-Poisson equations:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial E}{\partial x} = -\rho_e$$

Perturb the distribution function as $f = f_0 + f_1 \qquad E = \underbrace{\cancel{E_0}}_{0} + E_1$

$$\frac{\partial(\overbrace{\cancel{f_0}}^{0} + f_1)}{\partial t} + v\frac{\partial(\overbrace{\cancel{f_0}}^{0} + f_1)}{\partial x} + E_1\frac{\partial(f_0 + \overbrace{\cancel{f_1}}^{\text{Small}})}{\partial v} = 0$$

$$\longrightarrow \quad \frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + E_1\frac{\partial f_0}{\partial v} = 0$$

$$\xrightarrow{f_1 \sim e^{i(kx-\omega t)}} \quad -i\omega f_1 + ikvf_1 + E_1\frac{\partial f_0}{\partial v} = 0 \quad \longrightarrow \quad \boxed{f_1 = iE_1\frac{\partial f_0/\partial v}{\omega - kv}}$$

Physics

Landau Damping

Dispersion Relation

Poisson's equation is also very straightforward

$$\frac{\partial E}{\partial x} = -\rho_e \longrightarrow \frac{\partial E}{\partial x} = \int_{-\infty}^{+\infty} f dv \xrightarrow{E_1 \sim e^{i(kx-\omega t)}} \boxed{ikE_1 = \int_{-\infty}^{+\infty} f_1 dv}$$

Now combine with the f_1 linearization: $f_1 = iE_1 \frac{\partial f_0 / \partial v}{\omega - kv}$

$$\longrightarrow \boxed{1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial f_0 / \partial v}{v - \omega/k} df}$$

$\omega_p = 1$ In normalized units

The above integral is not straightforward to evaluate because of the singularity at v = w/k

Vlasov's solution: Plasma Langmuir wave (incomplete)

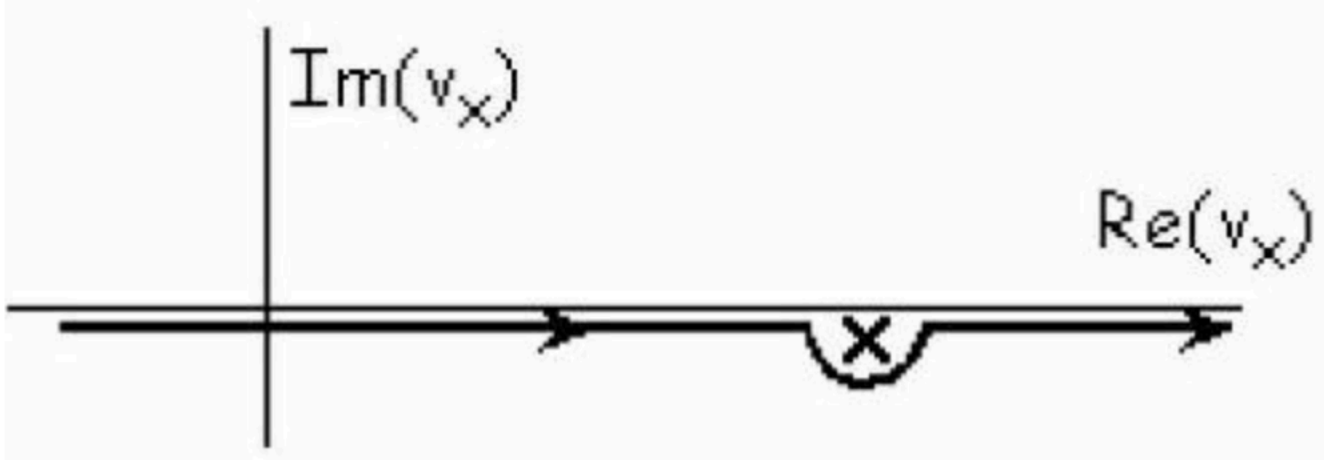
Landau's solution: electron Damping (correct)

Physics

Landau Damping

Landau Contour

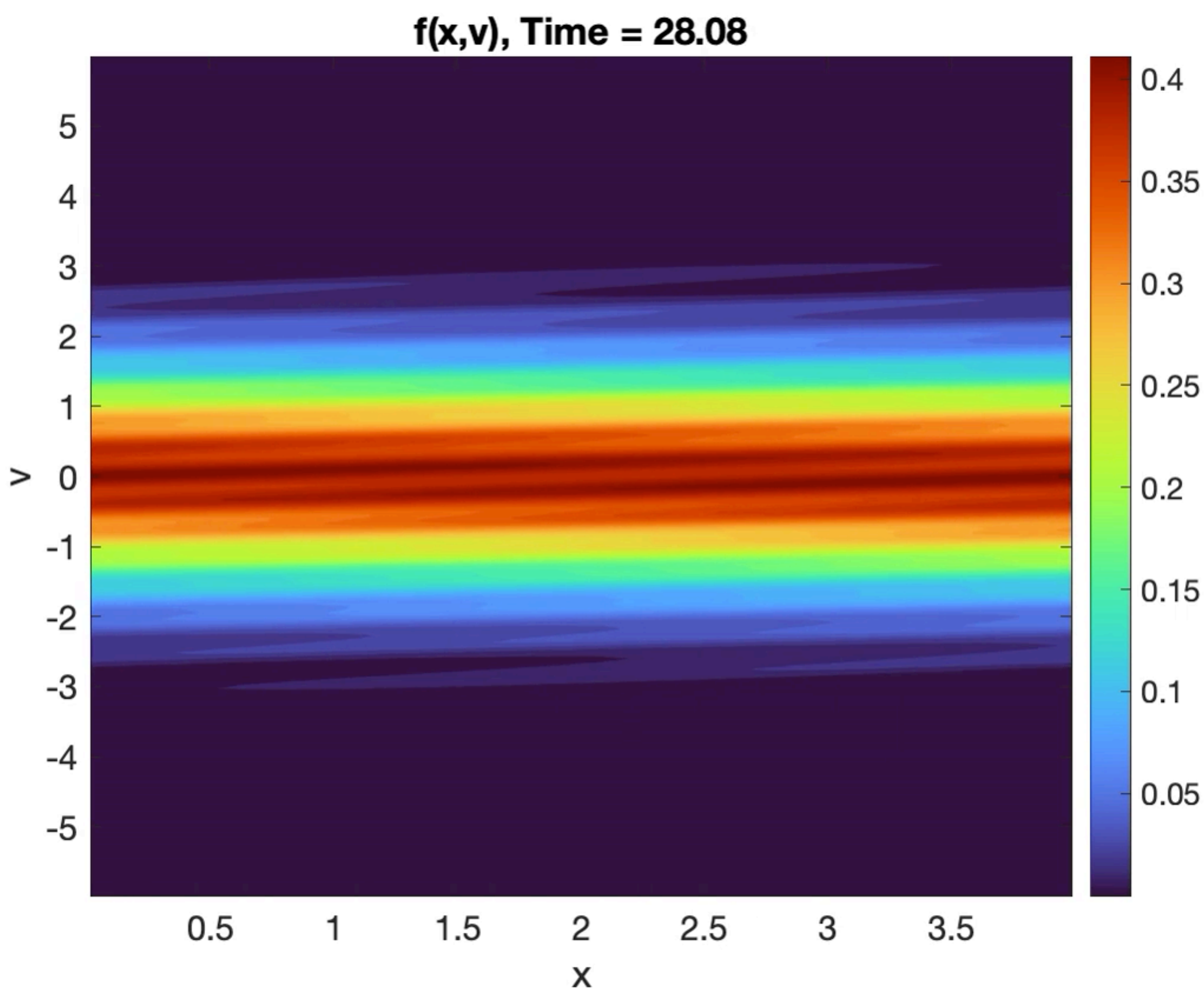
$$1 = \frac{\omega_p^2}{k^2}\left(\mathscr{P}\int_{-\infty}^{+\infty}\frac{\partial f_0/\partial v}{v-\omega/k}df + \pi i\frac{\partial f_0}{\partial v}\bigg|_{v=\omega/k}\right)$$

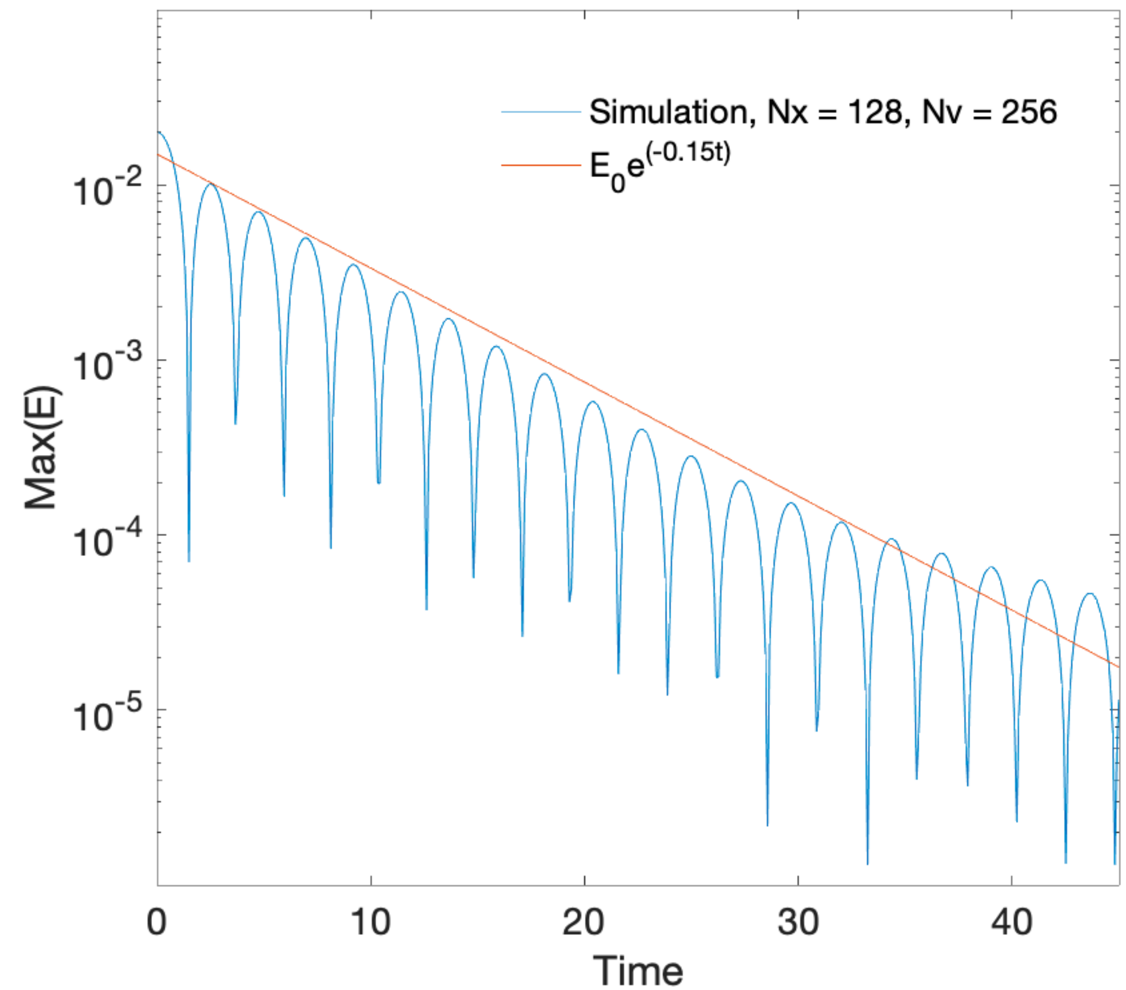


The Landau Contour

Landau Damping

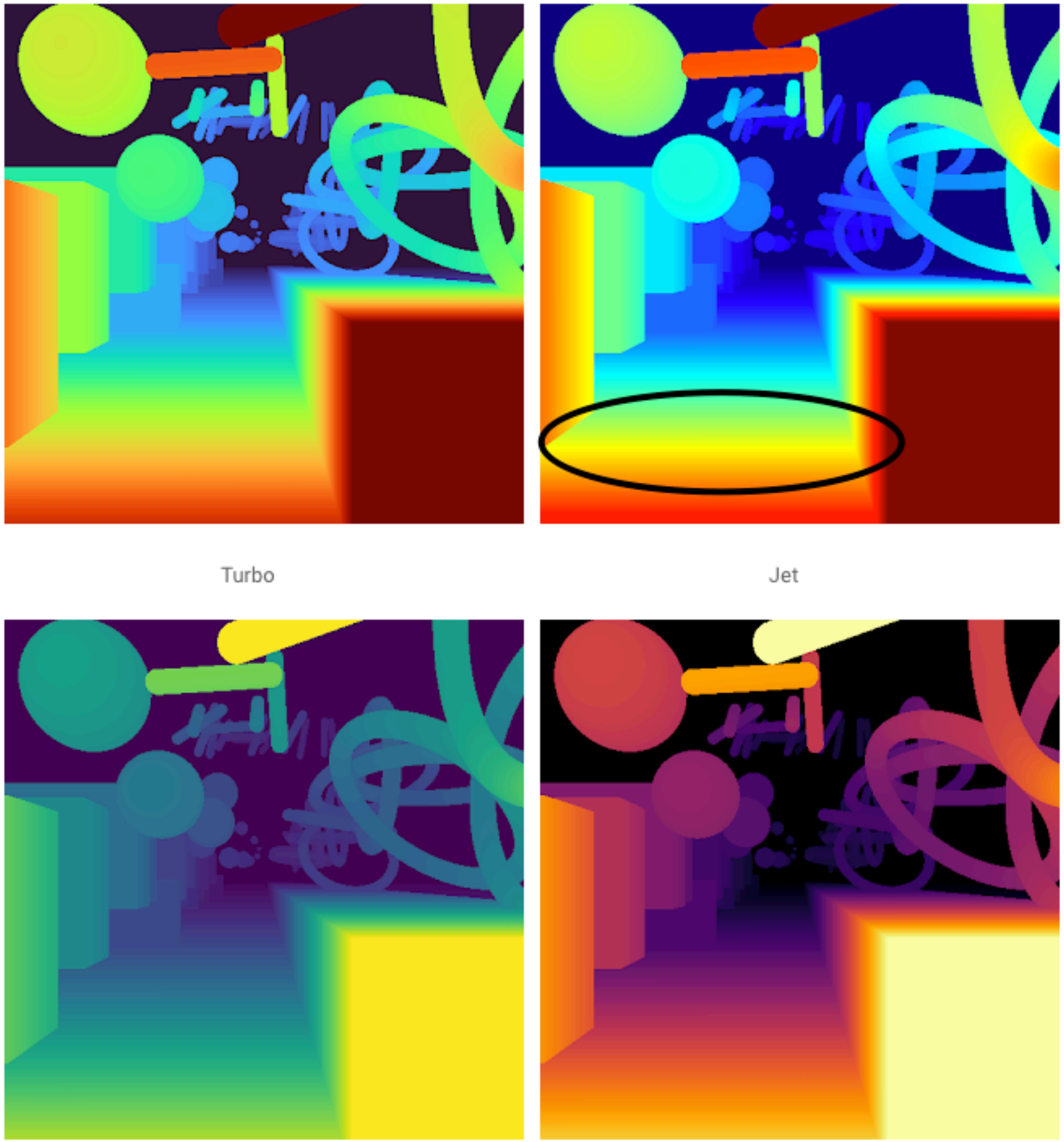
Simulation results





$$f(v,x)\Big|_{t=0} = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[1 + \cos(kx)\right] \qquad A = 0.05,\ \ k = 0.5 \qquad Nx = 128,\ \ Nv = 256$$

Choice of Colormaps



Turbo Jet

Viridis Inferno

rainbow colormap?

Vlasov-Poisson Simulations

Two-stream Instability