Equations of MHD 2

Which MHD model? Equation-mania

Review of Elementary Kinetic Theory

Boltzmann Equation

$$\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha(\mathbf{r},\mathbf{v},t) + \mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r},\mathbf{v},t) = \left[\frac{\delta f_\alpha(\mathbf{r},\mathbf{v},t)}{\delta t}\right]_{coll}$$

Collisional Boltzmann Equation

Moments of a distribution function $$<\chi(\mathbf{r},\mathbf{v},t)>_\alpha = \frac{1}{n_\alpha(\mathbf{r},\mathbf{v},t)}\int_v \chi(\mathbf{r},\mathbf{v},t)\, f_\alpha(\mathbf{r},\mathbf{v},t)\, d^3v$$

$$<1>_\alpha = 1 \qquad <\mathbf{v}>_\alpha = \mathbf{u}_\alpha(\mathbf{r},t)$$

The **peculiar** velocity or random velocity $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha$

Maxwellian Distribution $$f_\alpha(\mathbf{v}) = n_\alpha\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2}\exp\left(-\frac{m_\alpha(\mathbf{v}-\mathbf{u}_\alpha)^2}{2kT_\alpha}\right)$$

General transport equation

$$\frac{\partial}{\partial t}(n_\alpha <\chi>_\alpha) + \nabla\cdot(n_\alpha <\chi\mathbf{v}>_\alpha) - n_\alpha <\mathbf{a}\cdot\nabla_v\chi>_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha <\chi>_\alpha)\right]_{coll}$$

Review of Elementary Kinetic Theory

Moment integrals

Density $n_\alpha < 1 >_\alpha = n_\alpha(\mathbf{r}, t)$

Bulk Velocity $< \mathbf{v} >_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$

Temperature $\frac{1}{2} m_\alpha < c_\alpha^2 >_\alpha = \frac{3}{2} k T_\alpha$

Pressure tensor $n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mathbf{P}_\alpha$ **Total stress tensor** $n_\alpha m_\alpha < \mathbf{u}_\alpha \mathbf{u}_\alpha >_\alpha = \mathbf{\Pi}_\alpha$

Heat flow $\frac{1}{2} n_\alpha m_\alpha < c_\alpha^2 \mathbf{c}_\alpha >_\alpha = \mathbf{q}_\alpha$

High-order pressure tensor $\frac{1}{2} n_\alpha m_\alpha < c_\alpha^2 \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mu_\alpha$

Heat flow tensor $n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mathbf{Q}_\alpha$

Review of Elementary Kinetic Theory

The pressure tensor

$$\mathbf{P}_\alpha = n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \frac{1}{n_\alpha(\mathbf{r}, \mathbf{v}, t)} \int_v \begin{bmatrix} c_x c_x & c_x c_y & c_x c_z \\ c_y c_x & c_y c_y & c_y c_z \\ c_z c_x & c_z c_y & c_z c_z \end{bmatrix} f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3 v$$

After evaluating the velocity integrals, we get a tensor

$$\mathbf{P}_\alpha = \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$$

Based on the definition, it is straightforward to show that $P_{xy} = P_{yx}, P_{yz} = P_{zy}, P_{zx} = P_{xz}$

So the pressure tensor has only 6 independent variables

Review of Elementary Kinetic Theory

Moment transport equations

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll}$$

# of unknowns	Fluid Moment		
1	**Density**	$< 1 >_\alpha = n_\alpha$	$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \frac{\delta n_s}{\delta t}.$
3	**Bulk Velocity**	$< \mathbf{v} >_\alpha = \mathbf{u}_\alpha$	$n_s m_s \frac{D_s \mathbf{u}_s}{Dt} + \nabla \cdot \mathbf{P}_s - n_s m_s \mathbf{G} - n_s e_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) = \frac{\delta \mathbf{M}_s}{\delta t},$
1	**Temperature**	$\frac{1}{2} m_\alpha < c_\alpha^2 >_\alpha = \frac{3}{2} k T_\alpha$	$\frac{D_s}{Dt}\left(\frac{3}{2} p_s\right) + \frac{3}{2} p_s (\nabla \cdot \mathbf{u}_s) + \nabla \cdot \mathbf{q}_s + \mathbf{P}_s : \nabla \mathbf{u}_s = \frac{\delta E_s}{\delta t},$
6	**Pressure tensor**	$n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mathbf{P}_\alpha$	$\frac{D_s \mathbf{P}_s}{Dt} + \nabla \cdot \mathbf{Q}_s + \mathbf{P}_s (\nabla \cdot \mathbf{u}_s) + \frac{e_s}{m_s} (\mathbf{B} \times \mathbf{P}_s - \mathbf{P}_s \times \mathbf{B}) + \mathbf{P}_s \cdot \nabla \mathbf{u}_s + (\mathbf{P}_s \cdot \nabla \mathbf{u}_s)^T = \frac{\delta \mathbf{P}_s}{\delta t}.$
3	**Heat flow**	$\frac{1}{2} n_\alpha m_\alpha < c_\alpha^2 \mathbf{c}_\alpha >_\alpha = \mathbf{q}_\alpha$	$\frac{D_s \mathbf{q}_s}{Dt} + \mathbf{q}_s \cdot \nabla \mathbf{u}_s + \mathbf{q}_s (\nabla \cdot \mathbf{u}_s) + \mathbf{Q}_s : \nabla \mathbf{u}_s + \nabla \cdot \boldsymbol{\mu}_s + \left[\frac{D_s \mathbf{u}_s}{Dt} - \mathbf{G} - \frac{e_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B})\right] \cdot \left(\boldsymbol{\tau}_s + \frac{5}{2} p_s \mathbf{I}\right) - \frac{e_s}{m_s} \mathbf{q}_s \times \mathbf{B} = \frac{\delta \mathbf{q}_s}{\delta t},$

Heat flow tensor $n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mathbf{Q}_\alpha$

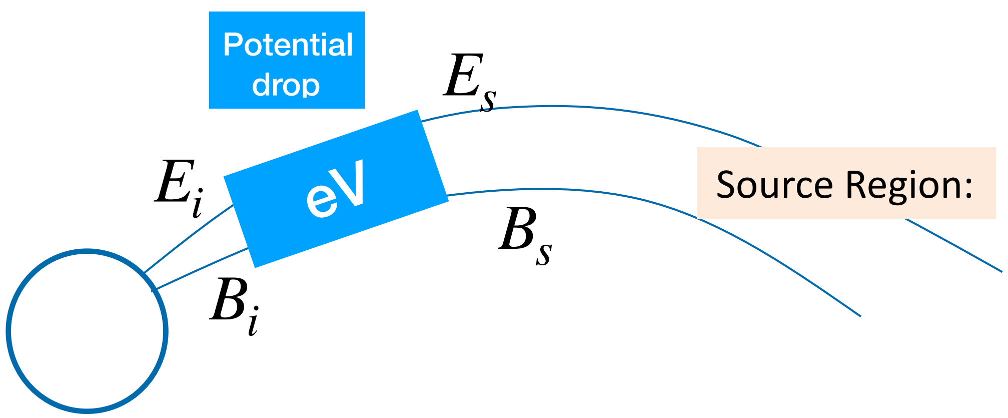
High-order pressure tensor $\frac{1}{2} n_\alpha m_\alpha < c_\alpha^2 \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha = \mu_\alpha$

Eqns from Schunk and Nagy, Chapter 3

Review of Elementary Kinetic Theory

The Knight relation and auroral precipitation

Number flux $n_\alpha < \mathbf{v} >_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$



bi-Maxwellian distribution

$$f(\mathbf{v}) = N_e \left(\frac{m_e}{2\pi}\right)^{3/2} \frac{1}{E_{o,\parallel}^{1/2} E_{o,\perp}} \exp\left(-\frac{\frac{1}{2}mv_\perp^2}{E_{o,\perp}} - \frac{\frac{1}{2}mv_\parallel^2}{E_{o,\parallel}}\right)$$

Conservation of first adiabatic invariant:

$$\frac{\mu}{B} = \frac{\frac{1}{2}mv_\perp^2}{B} = \text{const}$$

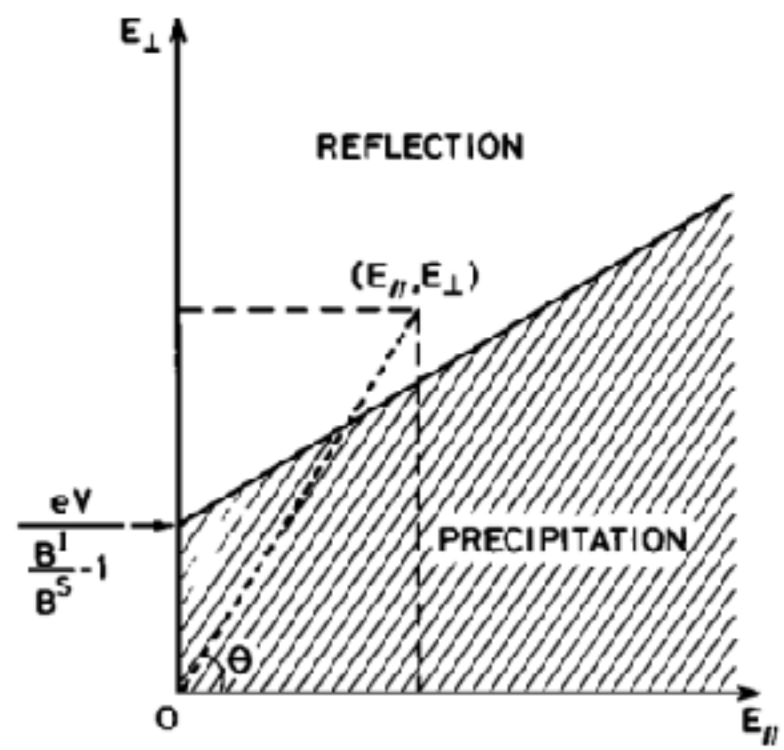
$$\longrightarrow \quad \frac{E_{\perp,S}}{B_S} = \frac{E_{\perp,I}}{B_I}$$

Conservation of energy

$$E_{\perp,I} + E_{\parallel,I} = E_{\perp,S} + E_{\parallel,S} + eV$$

combine:

$$E_{\parallel,I} = E_{\perp,S}\left(1 - \frac{B_I}{B_S}\right) + E_{\parallel,S} + eV >= 0$$



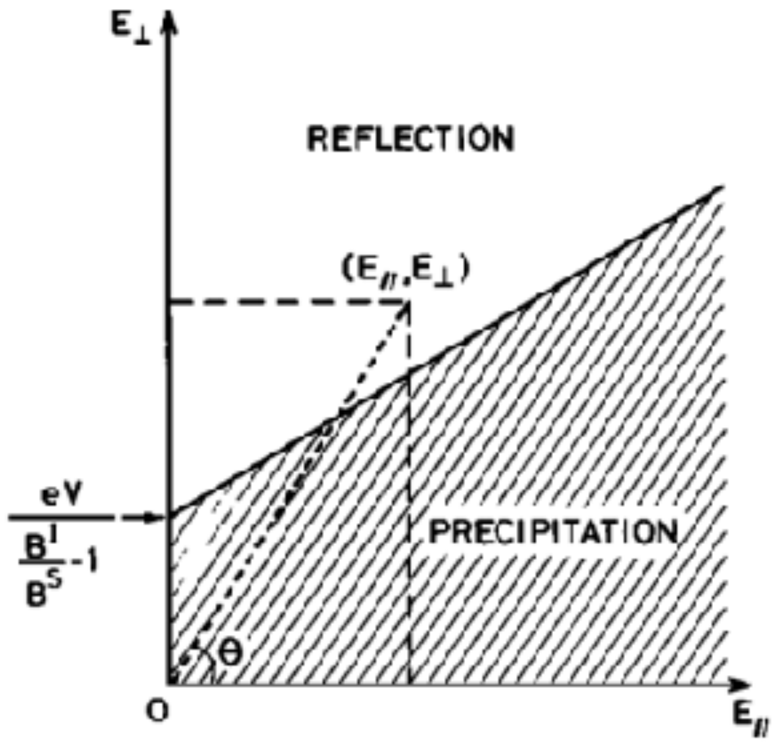
Fridman and Lemaire, [1980]

$$F_N = \int_{precip} v_\parallel f(\mathbf{v}) d\mathbf{v}$$

$$F_E = \int_{precip} \frac{1}{2} m v^2 v_\parallel f(\mathbf{v}) d\mathbf{v}$$

Review of Elementary Kinetic Theory

The Knight relation and auroral precipitation



Integrate the precipitation region in velocity space

$$F_N = \int_{precip} v_\parallel f(\mathbf{v}) d\mathbf{v}$$

$$f(\mathbf{v}) = N_e \left(\frac{m_e}{2\pi}\right)^{3/2} \frac{1}{E_{o,\parallel}^{1/2} E_{o,\perp}} \exp\left(-\frac{\frac{1}{2}mv_\perp^2}{E_{o,\perp}} - \frac{\frac{1}{2}mv_\parallel^2}{E_{o,\parallel}}\right)$$

$$\longrightarrow \quad F_N = \frac{B_I}{B_S} N_e \left(\frac{E_{o,\parallel}}{2\pi m_e}\right)^{1/2} \left[1 - \frac{\exp(-xeV/E_{0,\parallel})}{1+x}\right]$$

$$x = \frac{E_{0,\parallel}}{E_{0,\perp}} \frac{1}{B_I/B_S - 1}$$

So, **IF** the precipitating number flux F_N is carried by field-aligned currents $J_\parallel$

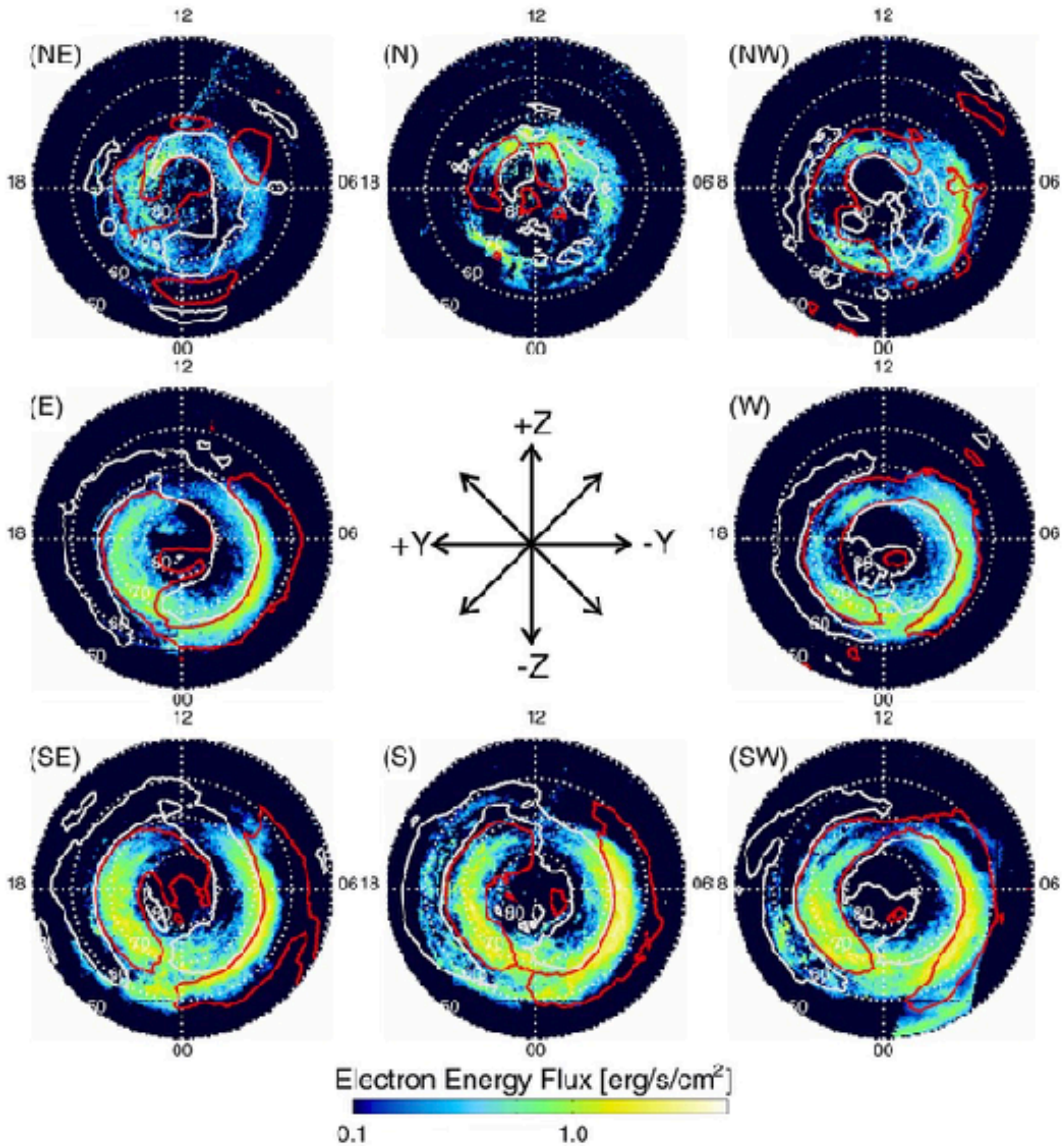
$$\frac{J_\parallel}{e} = \frac{B_I}{B_S} N_e \left(\frac{E_{o,\parallel}}{2\pi m_e}\right)^{1/2} \left[1 - \frac{\exp(-xeV/E_{0,\parallel})}{1+x}\right]$$

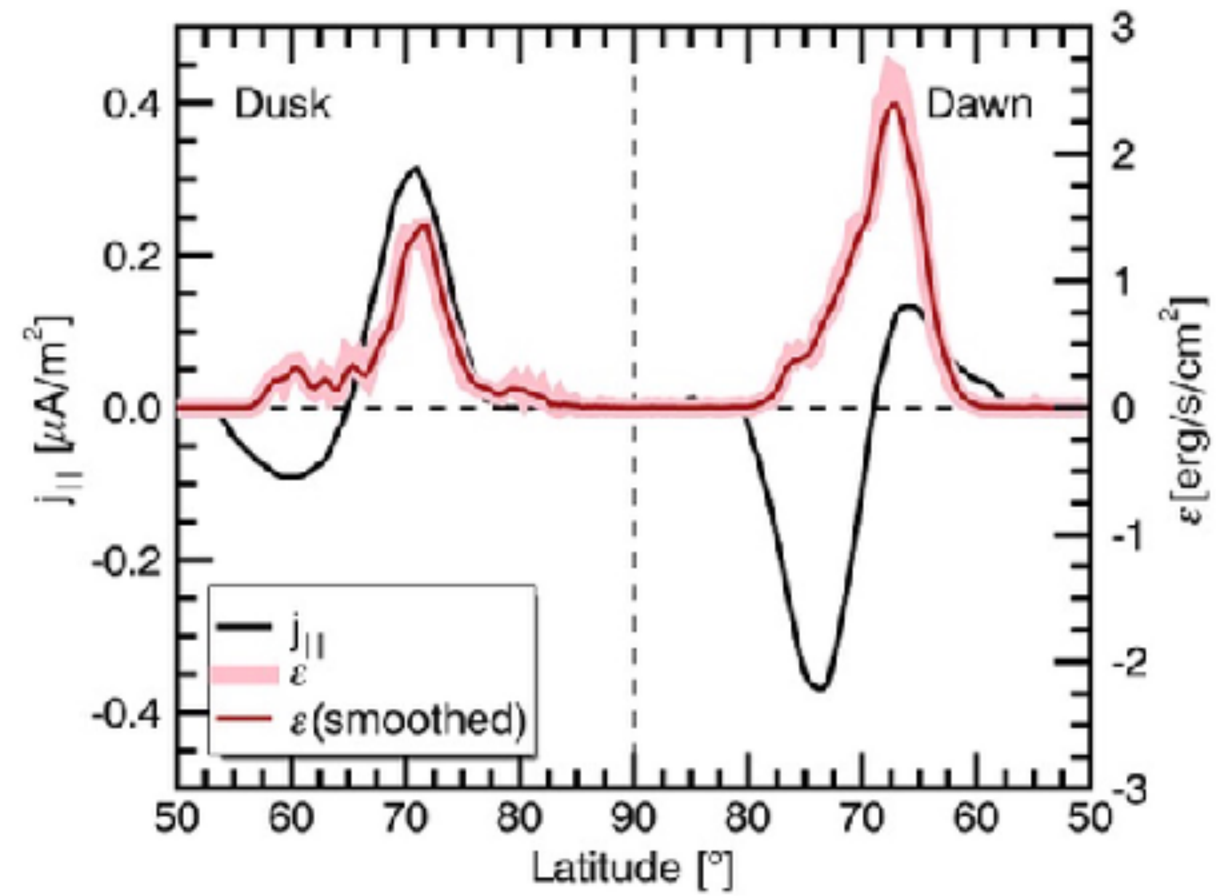
$$\xrightarrow{e^x \sim 1+x} \quad J_\parallel \approx \frac{B_I/B_S}{B_I/B_S - 1} N_e \left(\frac{E_{o,\parallel}}{2\pi m_e}\right)^{1/2} \frac{eV}{E_{0,\perp}}$$

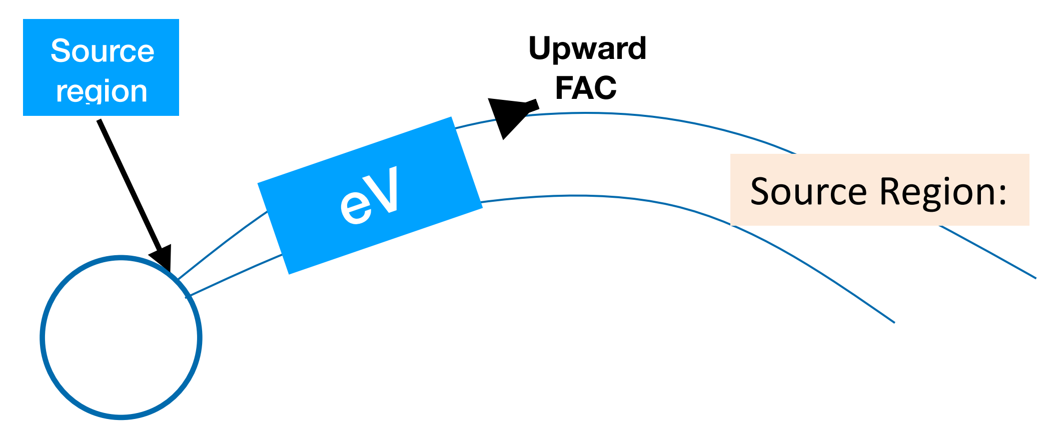
Knight-relation or Current-voltage relation

Note: Field-aligned current $\neq$ auroral precipitation

Are FACs aurora?







Korth et al., 2017

Macroscopic equations

Moment of the general transport equation

General transport equation

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll}$$

The 5-moment equations $(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha = \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix},\ \mathbf{q}_\alpha = 3)$

$$\frac{1}{2} n_\alpha m_\alpha < c_\alpha^2 \mathbf{c}_\alpha >_\alpha$$

Assumption: isotropic, equilibrium, Maxwellian $\mathbf{P}_\alpha = \begin{bmatrix} p_\alpha & 0 & 0 \\ 0 & p_\alpha & 0 \\ 0 & 0 & p_\alpha \end{bmatrix},\ \mathbf{q}_\alpha \equiv 0$

$$n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha$$

Mass equation $$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Momentum equation $$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\nabla p_\alpha + n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Pressure equation $$\frac{D}{Dt}\left(\frac{3}{2} p_\alpha\right) + \frac{5}{2} \nabla \cdot \mathbf{u}_\alpha = M_\alpha + \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha$$

Macroscopic equations

Primitive form versus conservative form

Primitive form

Mass equation $$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Velocity equation $$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\nabla p_\alpha + n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Pressure equation $$\frac{D}{Dt}\left(\frac{3}{2}p_\alpha\right) + \frac{5}{2}\nabla \cdot \mathbf{u}_\alpha = M_\alpha + \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \frac{1}{2}u_\alpha^2 S_\alpha$$

The 5-moment equations
$(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ p_\alpha = 1)$

conservative form

Mass equation $$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Momentum equation $$\frac{\partial n_\alpha m_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{I} p_\alpha\right) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Energy equation $$\frac{\partial \epsilon_\alpha}{\partial t} + \nabla \cdot \left[\mathbf{u}_\alpha\left(\epsilon_\alpha + p_\alpha\right)\right] = q_\alpha n_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha + n_\alpha m_\alpha \mathbf{g} \cdot \mathbf{u}_\alpha + M_\alpha + \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \frac{1}{2}u_\alpha^2 S_\alpha$$

$$\epsilon_\alpha = \frac{1}{2}n_\alpha m_\alpha u_\alpha^2 + \frac{p_\alpha}{\gamma - 1}$$ Plasma energy

Macroscopic equations

The cold-plasma model (4-moment?)

The simplest closed system of macroscopic transport equations that can be formed is known as the ***cold*** plasma model. This simple model encompasses only the equations of conservation of mass and of momentum.

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\cancelto{0}{\nabla p_\alpha} + n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\cancel{\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha + \frac{1}{2}n_\alpha m_\alpha u_\alpha^2\right) + \nabla \cdot \left[\frac{1}{2}n_\alpha m_\alpha < v^2 \mathbf{v} >_\alpha\right] - n_\alpha < \mathbf{F} \cdot \mathbf{v} >_\alpha = M_\alpha}$$

No thermal effect

This is known as the cold plasma model

Production/loss

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Momentum transfer

$$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = + n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Q: is this a good model for ionospheric plasma?

Macroscopic equations

The warm-plasma model

In the warm plasma model, the simplifying approximation is introduced in the equation of conservation of energy, in which we neglect the term involving the **heat flux** vector. Thus, the approximation consists in taking $\nabla \cdot q_\alpha = 0$, which means that the processes occurring in the plasma are such that there is no thermal energy flux. This approximation is also called the **adiabatic approximation**.

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\nabla p_\alpha + n_\alpha q_\alpha(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\frac{D}{Dt}\left(\frac{3}{2}p_\alpha\right) + \frac{5}{2}\nabla \cdot \mathbf{u}_\alpha = M_\alpha + \mathbf{u}_\alpha \cdot \mathbf{A}_\alpha + \frac{1}{2}u_\alpha^2 S_\alpha$$

When collision is negligible, the energy equation is reduced to the **adiabatic equation**

$$p_\alpha \rho_\alpha^{-\gamma} = \text{const}$$

Generally, the warm plasma model gives a more precise description of the behavior of plasma phenomena as compared to the cold plasma model.

Q: is this a good model for ionospheric/magnetospheric plasma?

Think: What is the “hot plasma model?”

Macroscopic equations

The 6-moment equations $(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ p_{||} = 1,\ p_\perp = 1)$

Assumption: anisotropic, equilibrium, bi-Maxwellian

$$(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha \sim \begin{bmatrix} p_{\perp\alpha} & 0 & 0 \\ 0 & p_{\perp\alpha} & 0 \\ 0 & 0 & p_{||\alpha} \end{bmatrix} = 2,\ \mathbf{q}_\alpha \equiv 0)$$

This is very common in collisionless plasma - Perpendicular and parallel motions are separated by the magnetic field

Now the MHD equations become

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$\frac{\partial n_\alpha m_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{I} p_{\perp\alpha} + \underbrace{\left(p_{||\alpha} - p_{\perp\alpha} \right)}_{\text{Mirror force}} \mathbf{bb} \right) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\frac{\partial p_{||\alpha}}{\partial t} + \nabla \cdot (p_{||\alpha} \mathbf{u}_\alpha) = -2 p_{||\alpha} \mathbf{b} \cdot (\mathbf{B} \cdot \nabla) \mathbf{u}_s$$

Paralle pressure eqn

$$\frac{\partial p_{\perp\alpha}}{\partial t} + \nabla \cdot (p_{\perp\alpha} \mathbf{u}_\alpha) = -p_{\perp\alpha} \nabla \cdot \mathbf{u}_\alpha + p_{\perp\alpha} \mathbf{b} \cdot (\mathbf{B} \cdot \nabla) \mathbf{u}_s$$

perp pressure eqn

Mirror force and transverse acceleration

Pressure tensor:

$$\mathbf{P}_\alpha = \begin{bmatrix} p_{\perp\alpha} & 0 & 0 \\ 0 & p_{\perp\alpha} & 0 \\ 0 & 0 & p_{\|\alpha} \end{bmatrix}$$

$$= p_{\perp\alpha}\mathbf{I} + (p_{\|\alpha} - p_{\perp\alpha})\mathbf{bb}$$

Assumption is valid when the anisotropy is generated by a strong magnetic field

Force is the divergence of a stress tensor:

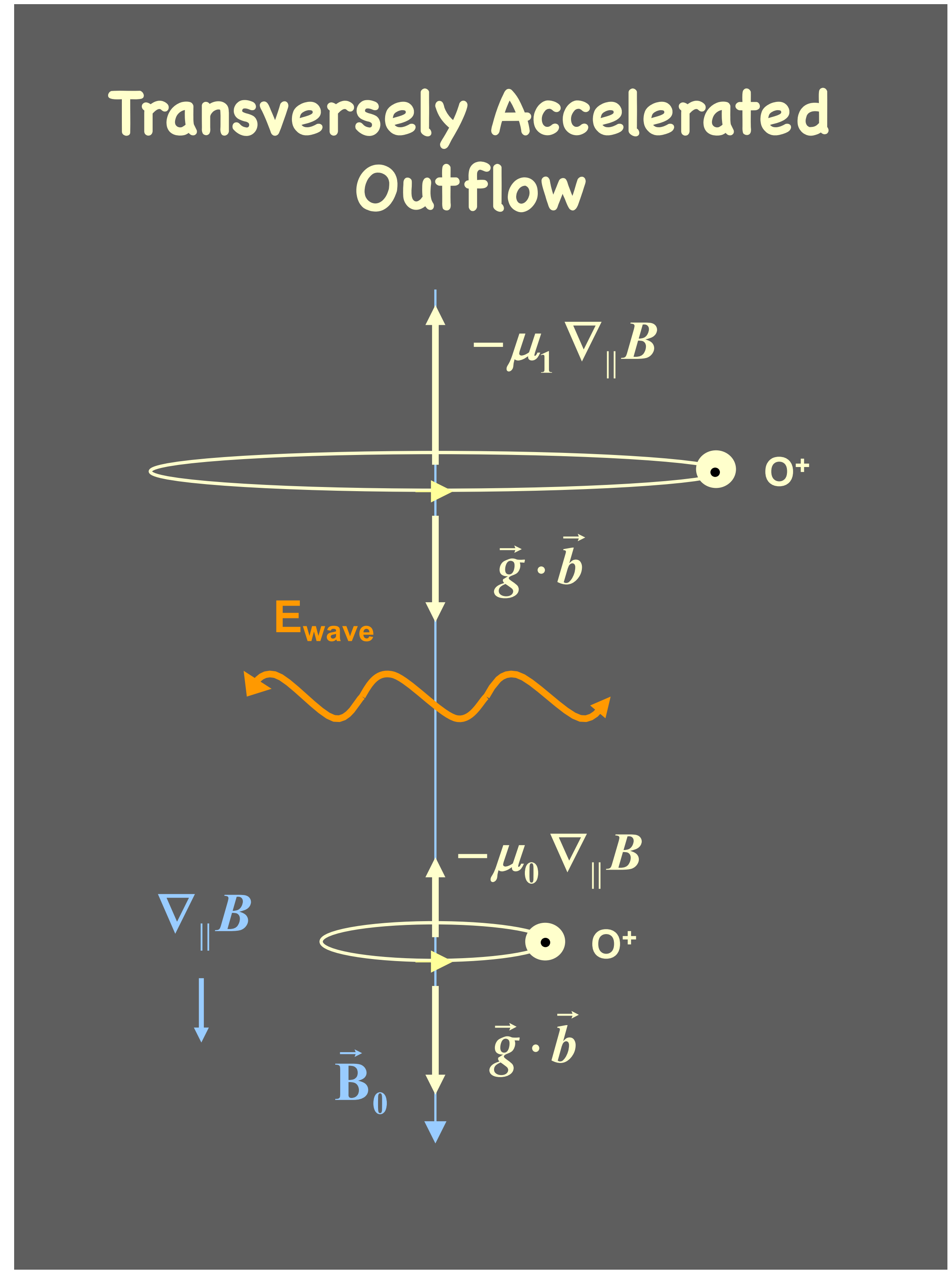
$$\nabla \cdot \mathbf{P}_\alpha = \nabla \cdot [p_{\perp\alpha}\mathbf{I} + (p_{\|\alpha} - p_{\perp\alpha})\mathbf{bb}]$$

$$= \nabla p_{\perp\alpha} + (\mathbf{B}\cdot\nabla)\left[\frac{p_{\|\alpha} - p_{\perp\alpha}}{B}\mathbf{b}\right]$$

Along B

Mirror force

The physical meaning of mirror force is from the fact that p_perp and p_parallel changes in the opposite direction along diverging/converging magnetic field lines



Macroscopic equations

The 8-moment equations $(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha = 2,\ \mathbf{q}_\alpha = 2)$

Assumption: anisotropic, equilibrium, non-Maxwellian

$$(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \ \mathbf{P}_\alpha \sim \begin{bmatrix} p_{\perp\alpha} & 0 & 0 \\ 0 & p_{\perp\alpha} & 0 \\ 0 & 0 & p_{\parallel\alpha} \end{bmatrix} = 2,\ \ \mathbf{q}_\alpha \sim (\mathbf{q}_\perp + \mathbf{q}_\parallel) = 2$$

e.g., ion-conic distribution

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$\frac{\partial n_\alpha m_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{I} p_{\perp\alpha} + \left(p_{\parallel\alpha} - p_{\perp\alpha} \right) \mathbf{bb} \right) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\frac{\partial p_{\parallel\alpha}}{\partial t} + \nabla \cdot (p_{\parallel\alpha} \mathbf{u}_\alpha) = -2 p_{\parallel\alpha} \mathbf{b} \cdot (\mathbf{B} \cdot \nabla) \mathbf{u}_s$$

$$\frac{\partial p_{\perp\alpha}}{\partial t} + \nabla \cdot (p_{\perp\alpha} \mathbf{u}_\alpha) = -p_{\perp\alpha} \nabla \cdot \mathbf{u}_\alpha + p_{\perp\alpha} \mathbf{b} \cdot (\mathbf{B} \cdot \nabla) \mathbf{u}_s$$

$$\frac{D\mathbf{q}_\alpha}{Dt} + \frac{7}{5} \mathbf{q}_\alpha \cdot \nabla \mathbf{u}_\alpha + \frac{7}{5} \mathbf{q}_\alpha \nabla \cdot \mathbf{u}_\alpha + \frac{2}{5} \nabla \mathbf{u}_\alpha \cdot \mathbf{q}_\alpha + \frac{5}{2} \frac{k p_\alpha \nabla T_\alpha}{m_\alpha} + \frac{q_\alpha}{m_\alpha} \mathbf{q}_\alpha \times \mathbf{B} = \frac{\delta \mathbf{q}_\alpha}{\delta t}$$

Macroscopic equations

The 10-moment equations $(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha = 6,\ \mathbf{q}_\alpha = 0)$

Assumption: anisotropic, equilibrium, Maxwellian

$$(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha \sim \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = 6,\ \mathbf{q}_\alpha \equiv 0)$$

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$\frac{\partial n_\alpha m_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha\right) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\frac{D\mathbf{P}_\alpha}{Dt} + \mathbf{P}_\alpha \nabla \cdot \mathbf{u}_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{B} \times \mathbf{P}_\alpha - \mathbf{P}_\alpha \times \mathbf{B}) + \mathbf{P}_\alpha \cdot \nabla \mathbf{u}_\alpha + (\mathbf{P}_\alpha \cdot \nabla \mathbf{u}_\alpha)^T = \frac{\delta \mathbf{P}_\alpha}{\delta t}$$

closure: 10-moment Gaussian closure

Macroscopic equations

The 13-moment equations $(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha = 6,\ \mathbf{q}_\alpha = 3)$

Assumption: anisotropic, equilibrium, truncated-Maxwellian

$$(n_\alpha = 1,\ \mathbf{u}_\alpha = 3,\ \mathbf{P}_\alpha \sim \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = 6,\ \mathbf{q}_\alpha = 3)$$

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$\frac{\partial n_\alpha m_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla \cdot \left(n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha\right) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$\frac{D\mathbf{P}_\alpha}{Dt} + \mathbf{P}_\alpha \nabla \cdot \mathbf{u}_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{B} \times \mathbf{P}_\alpha - \mathbf{P}_\alpha \times \mathbf{B}) + \mathbf{P}_\alpha \cdot \nabla \mathbf{u}_\alpha + (\mathbf{P}_\alpha \cdot \nabla \mathbf{u}_\alpha)^T = \frac{\delta \mathbf{P}_\alpha}{\delta t}$$

$$\frac{D\mathbf{q}_\alpha}{Dt} + \frac{7}{5}\mathbf{q}_\alpha \cdot \nabla \mathbf{u}_\alpha + \frac{7}{5}\mathbf{q}_\alpha \nabla \cdot \mathbf{u}_\alpha + \frac{2}{5}\nabla \mathbf{u}_\alpha \cdot \mathbf{q}_\alpha + \frac{5}{2}\frac{k p_\alpha \nabla T_\alpha}{m_\alpha} + \frac{q_\alpha}{m_\alpha}\mathbf{q}_\alpha \times \mathbf{B} = \frac{\delta \mathbf{q}_\alpha}{\delta t}$$

closure: truncated Gaussian closure

Two-fluid MHD equations

Electron fluid moment equations:

production/loss

Collisional momentum transfer

$$\frac{\partial}{\partial t}(n_e m_e) + \nabla \cdot (n_e m_e \mathbf{u}_e) = S_e$$

$$\sim n_e m_e \nu_e (\mathbf{u}_e - \mathbf{u}_i) = \frac{n_e e}{\sigma}\mathbf{J}$$

$$n_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_\alpha m_e \mathbf{g} + \mathbf{A}_e - \mathbf{u}_e S_e$$

$$\frac{D}{Dt}\left(\frac{3}{2}p_e\right) + \frac{5}{2}\nabla \cdot \mathbf{u}_e = \mathbf{u}_e \cdot \mathbf{A}_e + \frac{1}{2}u_e^2 S_e + M_e$$

$$M_e \approx \frac{J^2}{\sigma} - M_i$$ Heating

Ion fluid moment equations:

$$\frac{\partial}{\partial t}(n_i m_i) + \nabla \cdot (n_i m_i \mathbf{u}_i) = S_i$$

$$-\mathbf{A}_e = -\frac{n_e e}{\sigma}\mathbf{J}$$

$$n_i m_i \frac{D\mathbf{u}_i}{Dt} = -\nabla p_i + n_i e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + n_i m_i \mathbf{g} + \mathbf{A}_i - \mathbf{u}_i S_i$$

$$\frac{D}{Dt}\left(\frac{3}{2}p_i\right) + \frac{5}{2}\nabla \cdot \mathbf{u}_i = \mathbf{u}_i \cdot \mathbf{A}_i + \frac{1}{2}u_i^2 S_i + M_i$$

$$M_i = \frac{3m_e}{m_i} n_e \nu_e (T_e - T_i)$$

Two-fluid MHD equations

Scales resolved in the momentum equation:

$$n_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_\alpha m_e \mathbf{g} + \mathbf{A}_e - \mathbf{u}_e S_e$$

Ignore gravity collision source

$$\xrightarrow{} \underbrace{\frac{D\mathbf{u}_e}{Dt} = -\frac{\nabla p_e}{n_e m_e}}_{\text{Electron acoustic}} - \frac{e}{m_e}(\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$$

$\mathbf{u}_e \times \mathbf{B} \rightarrow \sim \frac{eB}{m_e} = \omega_e$ Electron gyro freq

$\mathbf{E} \xrightarrow{\text{Mass eqn}} \omega_p$ Plasma freq

Q: plasma freq in F-region?

unmagnetized, electrostatic limit

$$n_e m_e \frac{\cancelto{0}{D\mathbf{u}_e}}{Dt} = -\nabla p_e - n_e e(\cancelto{-\nabla\phi}{\mathbf{E}} + \cancelto{0}{\mathbf{u}_e \times \mathbf{B}}) + \cancelto{0}{n_\alpha m_e \mathbf{g}} + \cancelto{0}{\mathbf{A}_e - \mathbf{u}_e S_e}$$

Ignore gravity collision source

$$\longrightarrow 0 = -ne\nabla\phi - \nabla p_e \xrightarrow[p_e = n_e k T_e]{\text{Isothermal}} \nabla\left[\frac{e\phi}{T} + \ln n_e\right] = 0 \longrightarrow n_e(\mathbf{r}) = n_0 e^{\frac{-e\phi((r))}{kT_e}}$$

Boltzmann eqn

Collision-dominant limit

$$n_e m_e \frac{\cancelto{0}{D\mathbf{u}_e}}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \cancelto{0}{n_\alpha m_e \mathbf{g}} + n_e m_e \nu_e(\mathbf{u}_e - \mathbf{u}_i) - \cancelto{\text{Ignore}}{\mathbf{u}_e S_e}$$

$$\longrightarrow 0 = -\nabla p_e - n_e e\mathbf{E} + n_e m_e \nu_e(\mathbf{u}_e - \mathbf{u}_i) \longrightarrow \mathbf{E} = -\frac{\nabla p_e}{n_e e} + \frac{1}{\sigma}\mathbf{J}$$

Ohm's law

Multi-fluid Hall MHD equations

We get the following fluid equations for each species (including e, i)

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{u_i} = 0$$

Mass conservation

$$\frac{\partial n_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i \mathbf{u_i} \mathbf{u_i} + \nabla p_i - \frac{n_i q_i}{m_i}(\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

Momentum conservation

$$\frac{\partial \mathscr{E}_i}{\partial t} + \nabla \cdot \mathbf{u_i} \left(\mathscr{E}_i + p_i\right) - q_i n_i \mathbf{u}_i \cdot (\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

Energy conservation

$$\mathscr{E}_i = \frac{1}{2} u_i^2 + \frac{p_i}{\gamma - 1}$$

Looking at the electron momentum equation:

$$\boxed{\frac{\partial n_e \mathbf{u_e}}{\partial t} + \nabla \cdot n_e \mathbf{u_e} \mathbf{u_e}} + \frac{1}{m_e} \nabla p_e + \frac{n_e e}{m_e}(\mathbf{E} + \mathbf{u_e} \times \mathbf{B}) = 0$$

Intertial terms:
Langmuir scale!

Ambipolar Electricfield

So let's say electron mass = 0

$$\mathbf{E} = -\mathbf{u_e} \times \mathbf{B} + \frac{1}{ne} \nabla p_e$$

The question now is we don't know $\mathbf{u_e}$, but we know the total current J:

$$\mathrm{j} = \sum_i n_i q_i \mathbf{u_i} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha - ne\mathbf{u_e} \longrightarrow \mathbf{E} = -\mathbf{u}_j \times \mathbf{B} - \frac{1}{ne}\left(\mathbf{j} \times \mathbf{B} - \nabla p_e\right)$$

Ideal term Hall term Ambipolar

Multi-fluid Hall MHD equations

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{u_i} = 0$$

$$\frac{\partial n_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i \mathbf{u_i} \mathbf{u_i} + \nabla p_i - \frac{n_i q_i}{m_i}(\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

$$\frac{\partial \mathscr{E}_i}{\partial t} + \nabla \cdot \mathbf{u_i} \left(\mathscr{E}_i + p_i\right) - n_i q_i \mathbf{u}_i \cdot (\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

$$\mathbf{E} = -\mathbf{u}_j \times \mathbf{B} - \frac{1}{ne}\left(\mathbf{j} \times \mathbf{B} - \nabla p_e\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Mean velocity of the ion current

$$\mathbf{u}_j = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha / \sum_\alpha n_\alpha q_\alpha$$

Multi-fluid Hall MHD equations have coupling terms in the Lorentz force go like

$$\frac{\rho_\alpha m_\alpha}{m_\alpha}(\mathbf{u}_\alpha - \mathbf{u_J}) \times \mathbf{B}$$

Proportional to ion gryofrequency

Not necessary for global magnetosphere problems

Multi-fluid BATSRUS

Glocer et al., [2010]

Sometimes people also include an electron pressure equation:

$$\frac{D}{Dt}\left(\frac{3}{2}p_e\right) + \frac{5}{2}\nabla \cdot \mathbf{u}_e = M_e$$

Multi-fluid Winglee MHD code

Winglee et al., [1998]

Or simply setting the electron pressure to be a portion of ion pressure

$$p_e = \beta \cdot \sum_i p_i$$

Q: what is multi-fluid LFM/GAMERA?

Multi-fluid Ideal MHD equations

The trick in GAMERA

The motion of ions follow basic hydrodynamic and plasma processes:

$$\mathbf{u}_i = \mathbf{u}_{i,\parallel} + \mathbf{u}_{E\times B} + \mathbf{u}_{Gyro}$$

Total velocity | Parallel | ExB drift | Gyro motion | **Scale dependent**

τ_{MHD} | $1/\Omega_i$

Zeroth-order drift | **first-order drift**

In global magnetosphere simulations, the time scale of interest is always much greater than ion gryoscales

$$\frac{1}{\Omega_{ci}} \ll \tau_{MHD} \xrightarrow[\varepsilon \sim 1/\tau_{MHD}]{\text{Expansion about}} \mathbf{u}_\alpha = \mathbf{u}_\perp + \mathbf{u}_\parallel + \varepsilon \mathbf{u}_\alpha^d$$

We get the following equation in the primitive form for the zero-th order: $\frac{\partial}{\partial t} \sim \varepsilon$

Parallel | ExB drift | 0

ε^2 term $\rightarrow 0$

$$\rho_\alpha \frac{\partial(\mathbf{u}_\parallel + \mathbf{u}_\perp + \cancel{\varepsilon \mathbf{u}^{\mathbf{d}}_\alpha})}{\partial t} + \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla p_\alpha - \frac{\rho_\alpha q_\alpha}{m_\alpha}\left(\mathbf{u}^{\mathbf{d}}_\alpha \times \mathbf{B} + \frac{1}{ne}\nabla_\parallel p_e\right) = 0$$

Drift force (unknown)

Now if we know $\frac{\partial \mathbf{u}_\perp}{\partial t}$ then the drift force is solved and the equations are complete

A new way to do Multi-fluid MHD

Idea: use the total momentum equation by summing the following species eqn:

$$\sum_\alpha \left[\rho_\alpha \frac{\partial(\mathbf{u}_\perp + \mathbf{u}_{\alpha\|})}{\partial t} + \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla p_\alpha - \frac{\rho_\alpha q_\alpha}{m_\alpha}(\mathbf{u}_\alpha^\mathbf{d} \times \mathbf{B} + \frac{1}{ne}\nabla_\| p_e) = 0 \right]$$

We get

$$\rho \frac{\partial \mathbf{u}_\perp}{\partial t} + \sum_\alpha \rho_\alpha \frac{\partial \mathbf{u}_\alpha \|}{\partial t} + \sum_\alpha \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \sum_i \nabla p_i - \mathbf{j} \times \mathbf{B} = 0$$

Note that the summation for the pressure includes the electron pressure and that the summation over the Lorentz force leads to JxB exactly, now we can calculate the u_perp term as:

$$\left(\frac{\partial \mathbf{u}_\perp}{\partial t}\right)_\perp = -\frac{1}{\rho}\left(\sum_\alpha \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \sum_i \nabla p_i\right) - \sum_\alpha \frac{\rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{B}}{B^2}\left(\frac{\partial \mathbf{B}}{\partial t}\right)_\perp - \mathbf{j} \times \mathbf{B}$$

now, substitute to the ion momentum equations, we get the drift term explicitly:

A new way to do Multi-fluid MHD

$$\rho_\alpha \frac{\partial(\mathbf{u}_\perp + \mathbf{u}_{\alpha\|})}{\partial t} + \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla p_\alpha - \frac{\rho_\alpha q_\alpha}{m_\alpha}(\mathbf{u}_\alpha^{\mathbf{d}} \times \mathbf{B} + \frac{1}{ne}\nabla_{\|} p_e) = 0$$

$$\boxed{\left(\frac{\partial \mathbf{u}_\perp}{\partial t}\right)_\perp = -\frac{1}{\rho}\left(\sum_\alpha \rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \sum_i \nabla p_i\right) - \sigma_\alpha \frac{\rho_\alpha \mathbf{u}_\alpha \cdot \mathbf{B}}{B^2}\left(\frac{\partial \mathbf{B}}{\partial t}\right)_\perp - \mathbf{j} \times \mathbf{B}}$$

The drift force Fd is written as

$$\mathbf{F}_\alpha^d = -\frac{\rho_\alpha q_\alpha}{m_\alpha}\mathbf{u}_\alpha^d \times \mathbf{B} = \frac{\rho_\alpha}{\rho}\left[\left(\sum_\beta \rho_\beta \mathbf{u}_\beta \cdot \nabla \mathbf{u}_\beta + \sum_i \nabla P_i\right)_\perp - \mathbf{j} \times \mathbf{B}\right]$$

$$-\left(\rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla P_\alpha\right)_\perp + \frac{\rho_\alpha \left(\mathbf{u} - \mathbf{u}_\alpha\right) \cdot \mathbf{B}}{B^2}\left(\frac{\partial \mathbf{B}}{\partial t}\right)_\perp$$

Here's the new form of individual momentum eqns

$$
\begin{aligned}
\rho_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t} = & -\frac{\mathbf{BB}}{B^2} \cdot (\rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla P_\alpha + \frac{n_\alpha q_\alpha}{ne} \nabla P_e) - \\
& \frac{\rho_\alpha}{\rho B^2} \mathbf{B} \times \{ \sum_\beta \rho_\beta \mathbf{u}_\beta \cdot \nabla \mathbf{u}_\beta + \sum_i \nabla P_i - \mathbf{j} \times \mathbf{B} - \frac{\rho(\mathbf{u}_M - \mathbf{u}_\alpha) \cdot \mathbf{B}}{B^2} (\frac{\partial \mathbf{B}}{\partial t}) \} \times \mathbf{B}
\end{aligned}
$$

Or in a more conservative

$$
\begin{aligned}
\frac{\partial \rho_\alpha \mathbf{u}_\alpha}{\partial t} = & -\frac{\mathbf{BB}}{B^2} \cdot (\nabla \cdot \rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \nabla P_\alpha + \frac{n_\alpha q_\alpha}{ne} \nabla P_e) + \mathbf{u}_\perp (\frac{\partial \rho_\alpha}{\partial t} - \frac{\rho_\alpha}{\rho} \frac{\partial \rho}{\partial t}) \\
& \frac{\rho_\alpha}{\rho B^2} \mathbf{B} \times \{ \sum_\beta \nabla \cdot \rho_\beta \mathbf{u}_\beta \mathbf{u}_\beta + \sum_i \nabla P_i - \mathbf{j} \times \mathbf{B} - \frac{\rho(\mathbf{u}_M - \mathbf{u}_\alpha) \cdot \mathbf{B}}{B^2} (\frac{\partial \mathbf{B}}{\partial t}) \} \times \mathbf{B}
\end{aligned}
$$

What the hack does these crazy equations do?

- **ALL the species move (couple) at the same speed (ExB) in the perp direction**
- **Adiabatic expansion for each species in the para direction (key for ion outflow)**
- **Ambipolar E field couples ion species in the para direction**
- **Summing over all the species gives exactly the total momentum equation**
- **The dB/dt term comes from the change in the direction of the magnetic field.**

Parallel and Perpendicular Momentum Eqns

$$\left(\rho_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t}\right)_{\parallel} = -\frac{\mathbf{BB}}{B^2} \cdot \left(\rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla p_\alpha + \frac{n_\alpha q_\alpha}{ne} \nabla p_e\right)$$

Hydrodynamics along B

$$\left(\frac{\partial \mathbf{u}_\alpha}{\partial t}\right)_{\perp} = -\frac{\mathbf{B}}{\rho B^2} \times \left[\boxed{\sum_\beta \rho_\beta \mathbf{u}_\beta \cdot \nabla \mathbf{u}_\beta} + \boxed{\sum_i p_i} - \boxed{\mathbf{j} \times \mathbf{B}} - \boxed{\frac{\rho(\mathbf{u} - \mathbf{u}_\alpha) \cdot \mathbf{B}}{B^2} \left(\frac{\partial \mathbf{B}}{\partial t}\right)}\right] \times \mathbf{B}$$

Bulk advection | Total Pressure | Lorentz | Magnetic Rotation

- In the parallel direction, every species follows its own force balance through the hydrodynamic equation plus the am bipolar electric field term, e.g., ionospheric outflow fluid can move freely along the field lines (if grad Pe = 0)
- In the perpendicular direction, every species follows exactly the same equation for the perpendicular velocity, i.e., the ExB drift which is determined by the magnetohydrodynamic force balance of the bulk fluid - everyone sees a portion of the Lorentz force
- The magnetic rotation term operates when the species velocity deviates to the bulk in the parallel direction, which looks like a collision term, leading to a transfer of momentum among species.

Compare the Multi-fluid MHD equations

Multi-fluid BATSRUS

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{u_i} = 0$$

$$\frac{\partial n_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i \mathbf{u_i} \mathbf{u_i} + \nabla p_i - \frac{n_i q_i}{m_i} (\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

$$\frac{\partial \mathscr{E}_i}{\partial t} + \nabla \cdot \mathbf{u_i} \left(\mathscr{E}_i + p_i \right) - n_i q_i \mathbf{u}_i \cdot (\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E}$$

$$\mathbf{E} = - \mathbf{u}_j \times \mathbf{B} - \frac{1}{ne} \left(\mathbf{j} \times \mathbf{B} - \nabla p_e \right)$$

Mean velocity of the ion current

$$\mathbf{u}_j = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha / \sum_\alpha n_\alpha q_\alpha$$

Allows individual drift around the bulk speed

Multi-fluid LFM/GAMERA

$$\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \mathbf{u_i} = 0$$

$$\frac{\partial n_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i \mathbf{u_i} \mathbf{u_i} + \nabla p_i - \mathbf{F}_\alpha^d = 0$$

$$\frac{\partial \mathscr{E}_i}{\partial t} + \nabla \cdot \mathbf{u_i} \left(\mathscr{E}_i + p_i \right) - \mathbf{u}_\alpha \cdot \mathbf{F}_\alpha^d = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E}$$

$$\mathbf{E} = - \mathbf{u}_{BULK} \times \mathbf{B}$$

$$\mathbf{F}_\alpha^d = - \frac{\rho_\alpha q_\alpha}{m_\alpha} \mathbf{u}_\alpha^d \times \mathbf{B} = \frac{\rho_\alpha}{\rho} \left[\left(\sum_\beta \rho_\beta \mathbf{u}_\beta \cdot \nabla \mathbf{u}_\beta + \sum_i \nabla P_i \right)_\perp - \mathbf{j} \times \mathbf{B} \right] - \left(\rho_\alpha \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha + \nabla P_\alpha \right)_\perp + \frac{\rho_\alpha \left(\mathbf{u} - \mathbf{u}_\alpha \right) \cdot \mathbf{B}}{B^2} \left(\frac{\partial \mathbf{B}}{\partial t} \right)_\perp$$

Allows only ExB drift at the bulk speed

Single-fluid MHD equations

Electron mass equation:

$$\frac{\partial}{\partial t}(n_e m_e) + \nabla \cdot (n_e m_e \mathbf{u}_e) = 0$$

Ion mass equations:

$$\frac{\partial}{\partial t}(n_i m_i) + \nabla \cdot (n_i m_i \mathbf{u}_i) = 0$$

$$+ \xrightarrow{\text{ignore Source}} \frac{\partial}{\partial t}\underbrace{(n_e m_e + n_i m_i)}_{\rho} + \nabla \cdot \underbrace{(n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i)}_{\rho\mathbf{u}} = 0$$

Bulk mass **Bulk momentum**

Mass conservation $$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$m_e \approx 0$ $n_e = n_i$ $n_e e \mathbf{u}_e - n_i e \mathbf{u}_i = -\mathbf{J}$

Electron momentum:

$$\frac{\partial n_e m_e \mathbf{u_e}}{\partial t} + \nabla \cdot n_e m_e \mathbf{u_e} \mathbf{u_e} + \nabla p_e + n_e e(\mathbf{E} + \mathbf{u_e} \times \mathbf{B}) = 0$$

$\mathbf{u}_i \approx \mathbf{u} + O(\frac{m_e}{m_i})$

Ion momentum: **+**

$$\frac{\partial n_i m_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i m_i \mathbf{u_i} \mathbf{u_i} + \nabla p_i - n_i e(\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} \quad + \nabla \cdot \rho \mathbf{u}\mathbf{u} \quad + \nabla p \quad - \mathbf{J} \times \mathbf{B}$$

Momentum conservation $$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

Single-fluid MHD equations

Electron velocity: $$\frac{\partial n_e m_e \mathbf{u_e}}{\partial t} + \nabla \cdot n_e m_e \mathbf{u_e u_e} + \nabla p_e + e n_e (\mathbf{E} + \mathbf{u_e} \times \mathbf{B}) = \mathbf{A}_e$$

Ion velocity: $$\frac{\partial n_i m_i \mathbf{u_i}}{\partial t} + \nabla \cdot n_i m_i \mathbf{u_i u_i} + \nabla p_i - e n_i (\mathbf{E} + \mathbf{u_i} \times \mathbf{B}) = -\mathbf{A}_e$$

Combine: $$\text{(electron momentum)} + \text{(ion momentum)} \times \frac{m_e}{m_i}$$

$$m_e \frac{\partial}{\partial t}(n_i \mathbf{u}_i - n_e \mathbf{u}_e) + m_e \nabla \cdot (n_i \mathbf{u}_i \mathbf{u}_i - n_e \mathbf{u}_e \mathbf{u}_e)$$

$$= \nabla (p_e - \frac{m_e}{m_i} p_i) + e[(n_e + \frac{m_e}{m_i})\mathbf{E} + (n_e \mathbf{u}_e + \frac{m_e}{m_i} n_i \mathbf{u}_i) \times \mathbf{B}] - (\mathbf{A}_e + \frac{m_e}{m_i} \mathbf{A}_e)$$

$$\xrightarrow[\substack{n_e e \mathbf{u}_e - n_i e \mathbf{u}_i = -\mathbf{J} \\ \mathbf{A}_e = n m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e)}]{m_e/m_i \approx 0 \quad n_e = n_i = n} \quad \frac{m_e}{e} \frac{\partial \mathbf{J}}{\partial t} + m_e \nabla \cdot (n \mathbf{u}_i \mathbf{u}_i - n \mathbf{u}_e \mathbf{u}_e) = \nabla p_e + ne(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \underbrace{n m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e)}_{\sim ne\eta \mathbf{J}}$$

From the multi-fluid Hall MHD equations we know that $\mathbf{u}_e \approx \mathbf{u} - \dfrac{\mathbf{J}}{ne}$ $\quad \mathbf{u}_i \approx \mathbf{u} + O(\dfrac{m_e}{m_i})$

Then $$\frac{m_e}{ne} \nabla \cdot (n \mathbf{u}_i \mathbf{u}_i - n \mathbf{u}_e \mathbf{u}_e) = \frac{m_e}{ne^2} \nabla \cdot (\mathbf{uJ} + \mathbf{Ju} - \frac{\mathbf{JJ}}{ne})$$

Generalized Ohm's law

$$\boxed{\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla p_e + \frac{m_e}{ne^2} \nabla \cdot (\mathbf{uJ} + \mathbf{Ju} - \frac{\mathbf{JJ}}{ne})}$$

Generalized Ohm's Law

recall: the simple Ohm's law in Lecture 1: $\mathbf{J} = \sigma \mathbf{E}$

Here we have a more complicated relationship between J and E

$$\underbrace{\mathbf{E} + \mathbf{u} \times \mathbf{B}}_{\text{Convective}} = \underbrace{\eta \mathbf{J}}_{\text{Resistive}} + \underbrace{\frac{1}{ne} \mathbf{J} \times \mathbf{B}}_{\text{Hall}} - \underbrace{\frac{1}{ne} \nabla p_e}_{\text{Ambipolar}} + \underbrace{\frac{m_e}{ne^2} \nabla \cdot (\mathbf{uJ} + \mathbf{Ju} - \frac{\mathbf{JJ}}{ne})}_{\text{Electron inertia}}$$

MHD scale

Collision scale

Ion gyro scale

Electron thermal scale

Electron intertial scale

Notes:

- **The terms on the RHS may be neglected based on the relationship between collision, gyro frequency and characteristic length scale**
- **The resistivity term is only valid in collision-dominant plasma**
- **The Hall term is not a resistivity**
- **The terms on the RHS may be all neglected if the uxB term dominates the generalized Ohm's law:** $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$

Single-fluid MHD equations

Ideal MHD

Done!

Mass conservation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

Energy conservation

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}$$

Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

Resistive MHD

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

Tearing mode

Hall MHD

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{ne}\mathbf{J} \times \mathbf{B}$$

(low-freq) Whistler mode

Now, which MHD equations to use?