Magnetohydrodynamics

origin, principles and applications in space physics

Topics/Schedule

Date	Lecture	Topic	Applications
	1	What is MHD?	
	2	Electrodynamics for MHD	ES/Inductive M-I Coupling
	3	Fluid Dynamics for MHD	LLBL
	4	MHD equations 1	Field-aligned currents
	5	MHD equations 2	Multi-fluid Gamera
	6	MHD waves 1	BBF
	7	MHD waves 2	Auroral physics
	8	MHD instability 1	Planetary KHI
	9	MHD instability 2	Interchange/ballooning
	10	MHD shocks	BS and MP
	11	MHD reconnection	GEM reconnection
	12	TBD	

What is MHD?

Magnetohydrodynamics (MHD) is the study of the interaction between magnetic fields and moving, conducting fluids.

MHD = Magnetic field + Hydrodynamics

Maxwell's equations

Fluid Dynamics

Who cares about MHD?

MHD

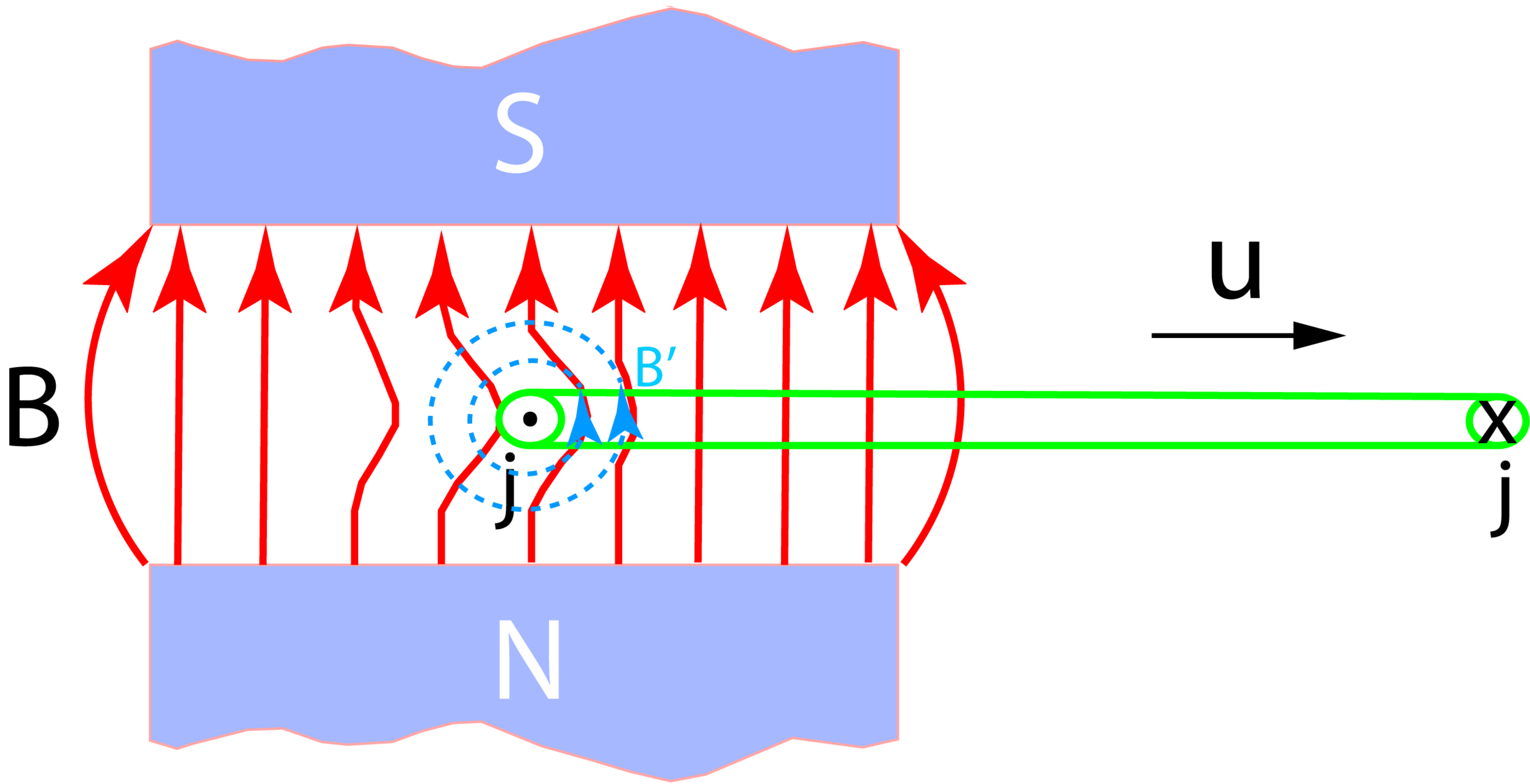
Engineering

Liquid Matel
engineering pump

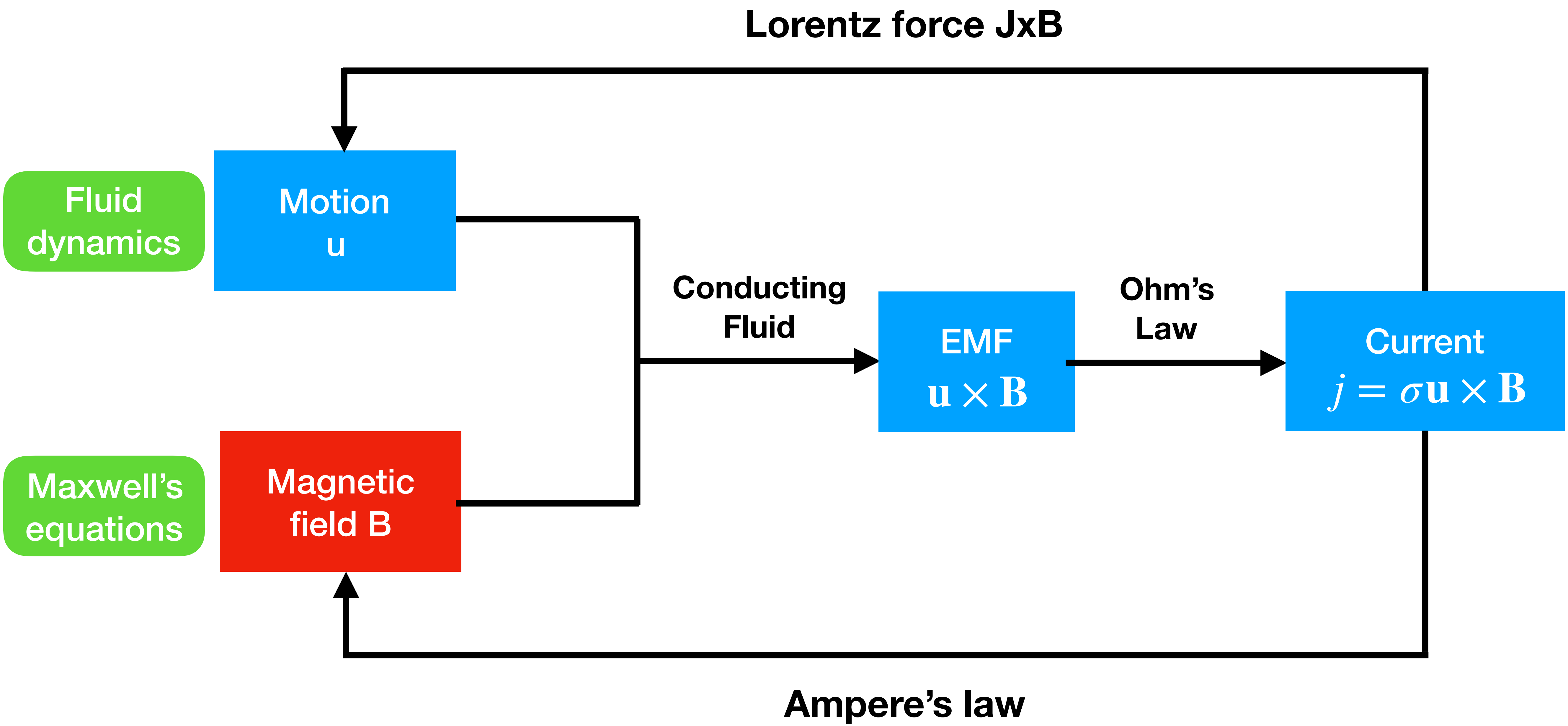
Science

Fusion plasma
Space plasma
Astrophysics

What is MHD?



What is MHD?



Lorentz force JxB
Fluid dynamics
Motion u
Conducting Fluid
EMF $\mathbf{u} \times \mathbf{B}$
Ohm's Law
Current $j = \sigma \mathbf{u} \times \mathbf{B}$
Maxwell's equations
Magnetic field B
Ampere's law

Several Key Parameters to note

1. $\mathbf{u}, \mathbf{B}$: **fluid velocity and magnetic field**

2. σ: **conductivity determines the property of the field-fluid interaction**

3. $\dfrac{\Delta B}{B}$ **ratio**

Since $\Delta \mathbf{B} \sim \oint \mathbf{E} \cdot \mathbf{dl} \sim \sigma |\mathbf{u} \times \mathbf{B}| l = \sigma u B l$

So $\dfrac{\Delta B}{B} \sim \sigma u l$ (This is called the Magnetic Reynolds number Rm)

Two limits: $R_m \gg 1$ **Magnetospheric plasma**

$R_m \ll 1$ **Ionospheric plasma?**

From Electromagnetism to MHD

A few more important parameters

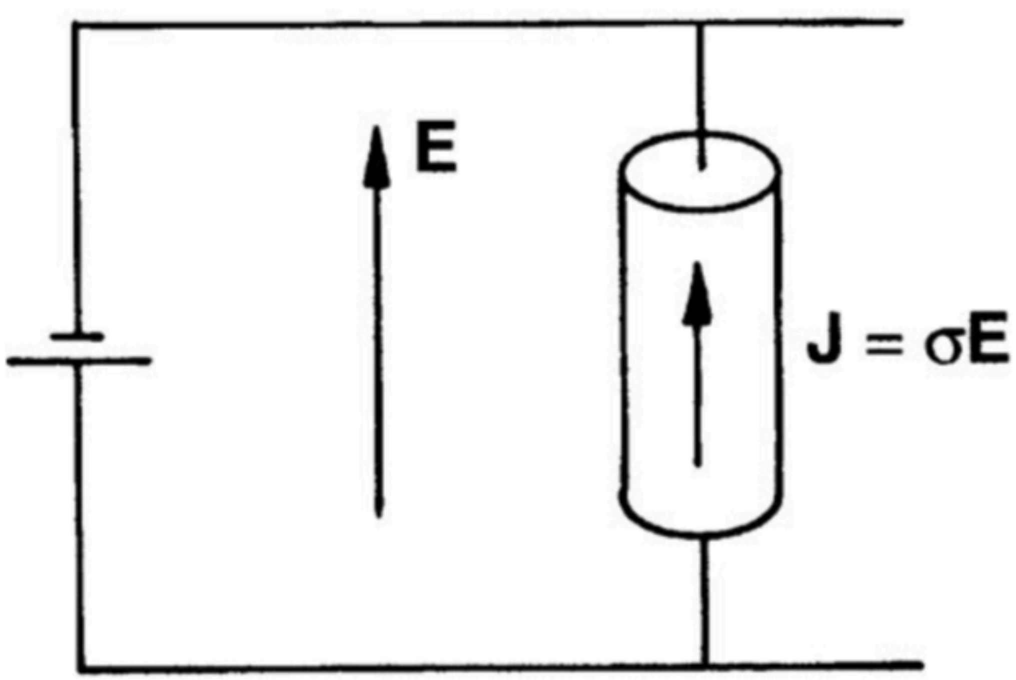
- Magnetic Reynolds number $R_m = \mu \sigma u l$
- Magnetic damping time $\tau = \left(\frac{\sigma B^2}{\rho} \right)^{-1}$
- Alfvén speed $v_A = \frac{B}{\sqrt{\mu \rho}}$

	R_m	v_A	τ
Solar Corona			
Solar Wind			
Magnetosphere			
Ionosphere			

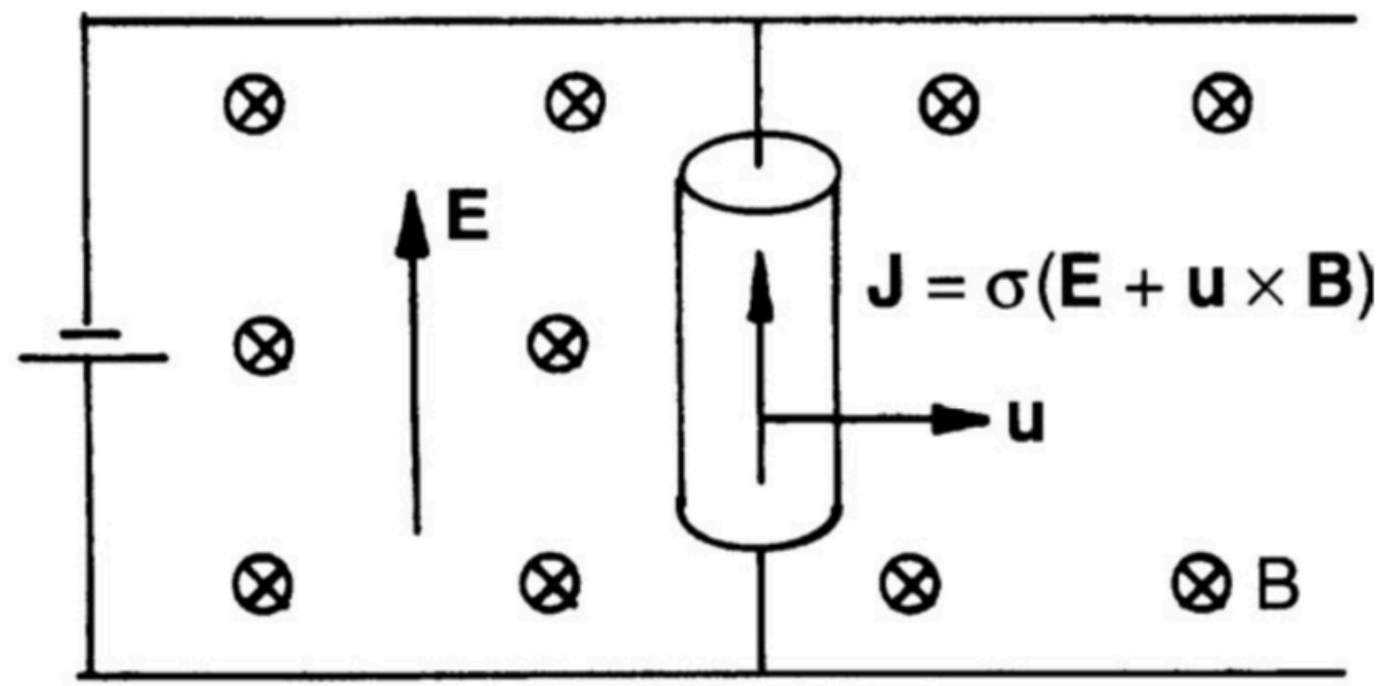
From Electromagnetism to MHD

A few more important laws

1). Ohm's Law



Stationary conductor



Moving conductor

$$j = \sigma \mathbf{E}$$

$$j = \sigma\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

External electric field

Motion

B field

Effective Electric field

$$\mathbf{j} = \sigma \mathbf{E} \qquad\qquad \mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

External electric field　Motion　B field

The quantity E + u × B, which is the total electromagnetic force per unit charge, arises frequently in electrodynamics and it is convenient to give it a label Er:

$$\mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) = \sigma \mathbf{E}_r$$

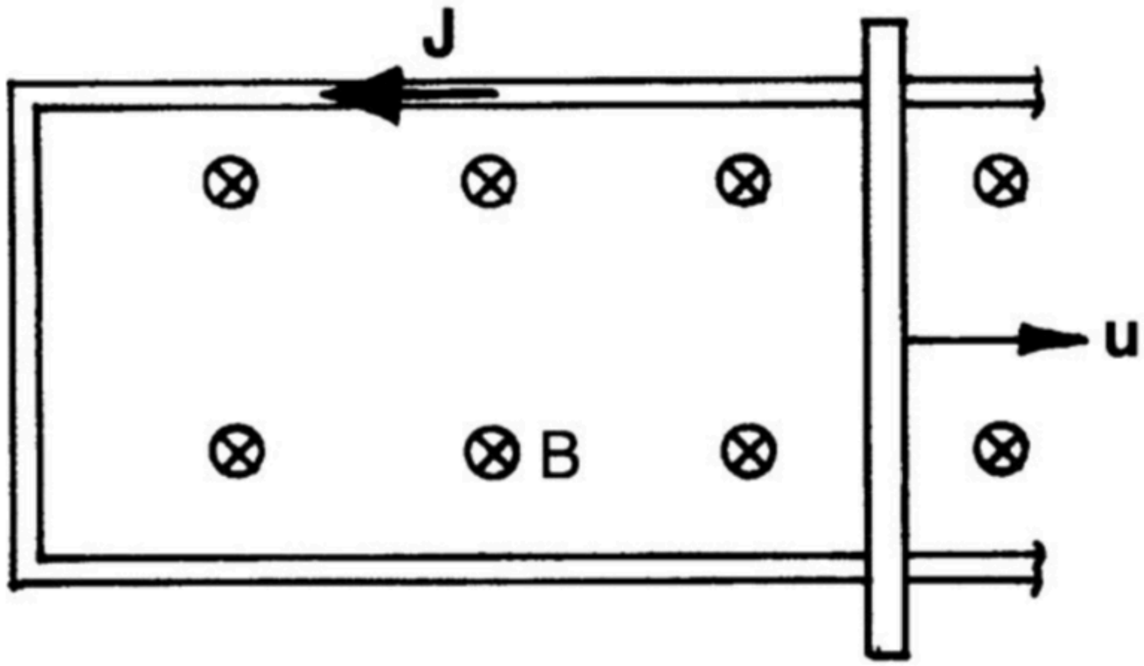
E_r is the electric field measured in a frame of reference moving with velocity u relative to the laboratory frame

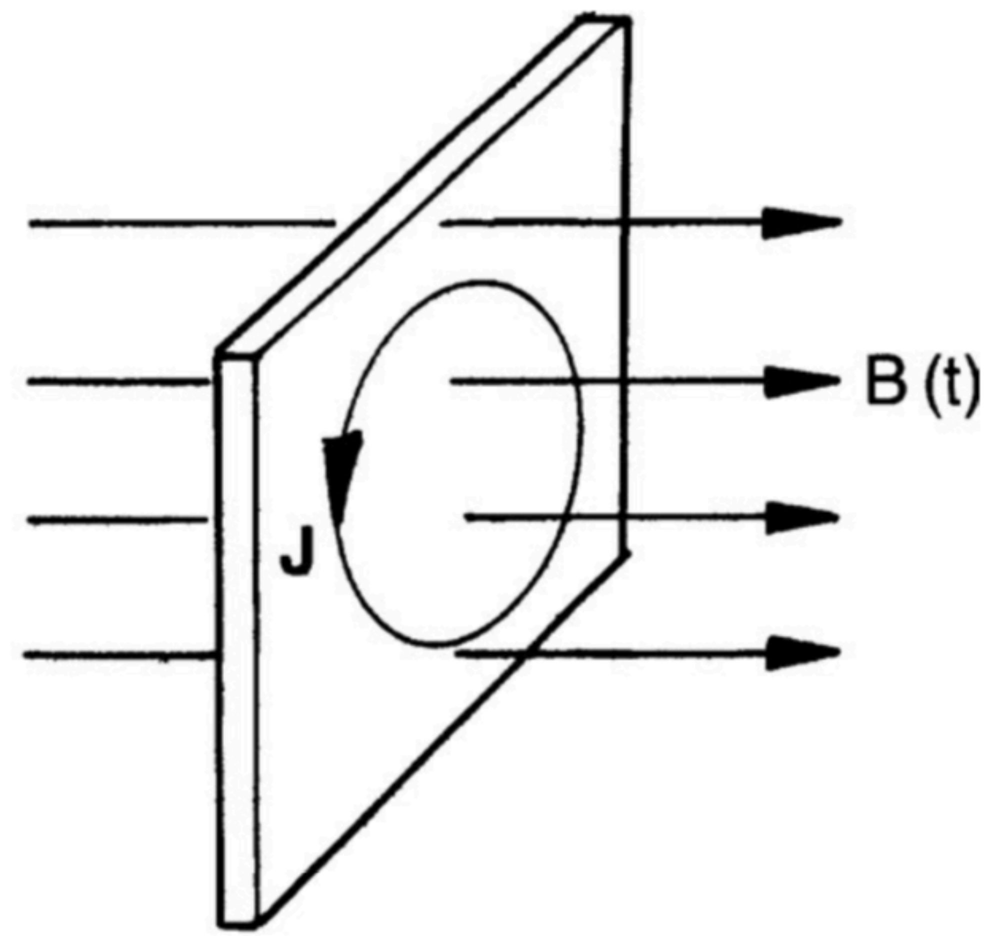
$$\mathbf{E}_r = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

From Electromagnetism to MHD

A few more important laws

2). Faraday's Law





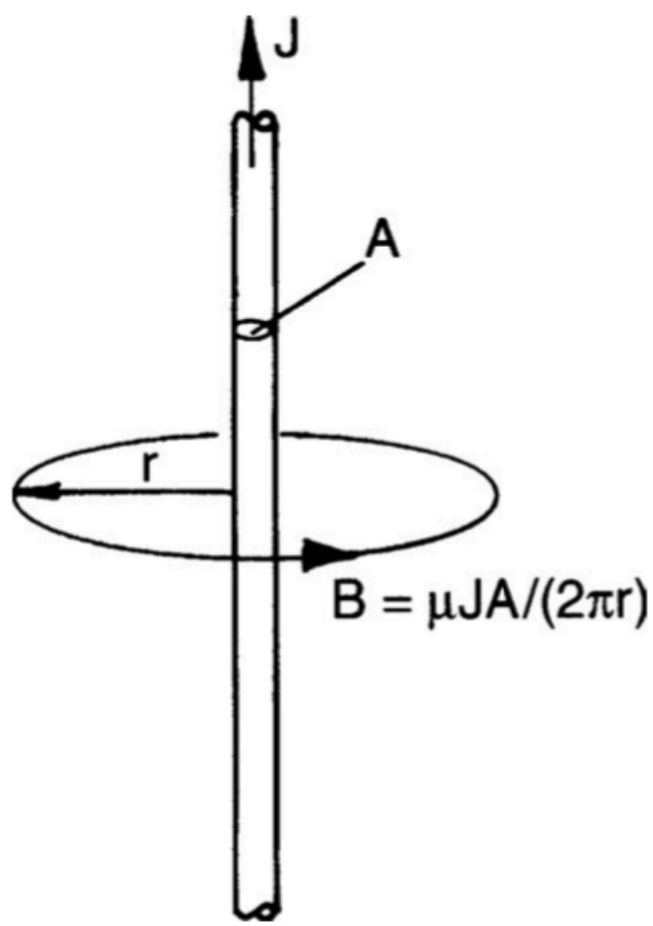
EMF (electromotive force)

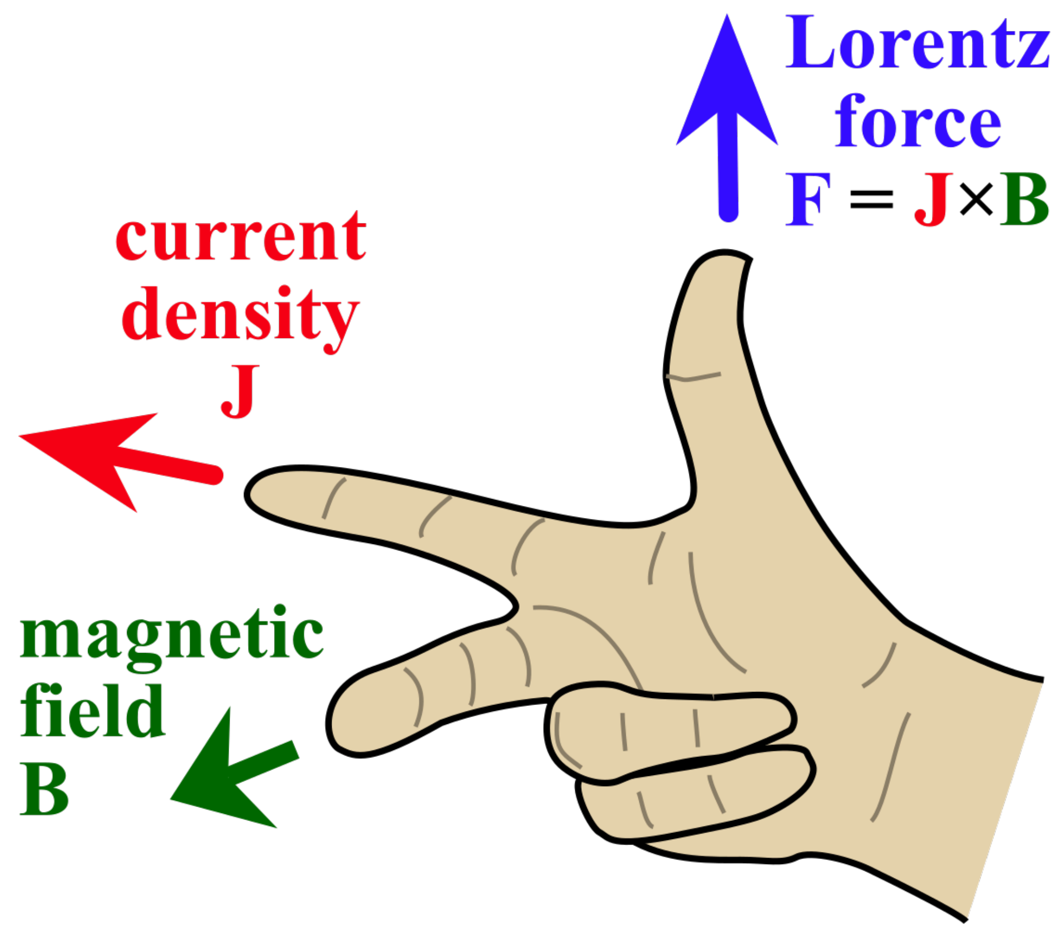
$$EMF = \oint_C \mathbf{E}_r \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} = -\frac{d\Phi}{dt}$$

From Electromagnetism to MHD

A few more important laws

2). Ampere's Law and Lorentz force



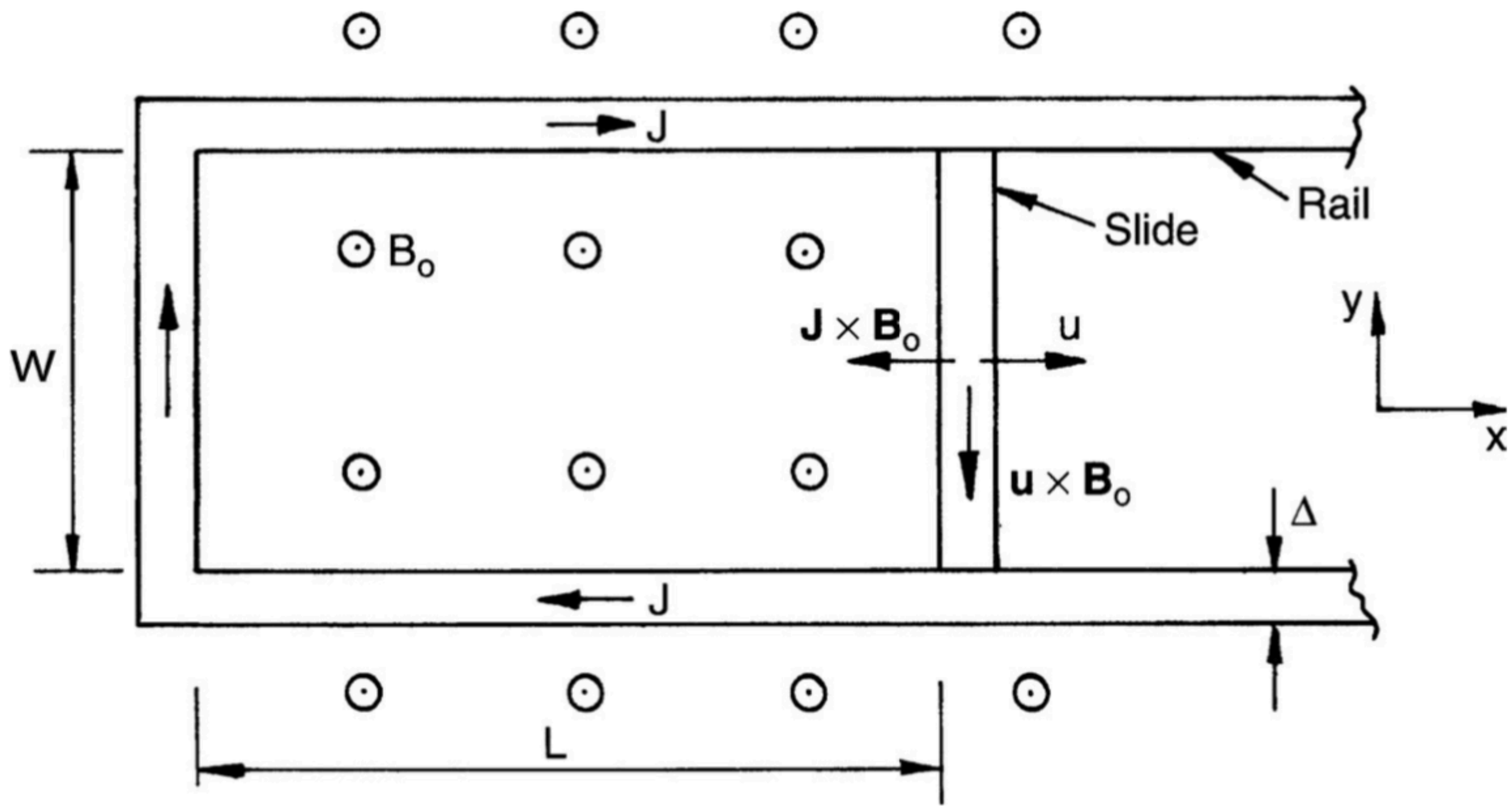


$$\oint_C \mathbf{B}_r \cdot \mathbf{dl} = \mu \int_S \mathbf{j} \cdot \mathbf{dS}$$

$$\mathbf{F} = \mathbf{j} \times \mathbf{B}$$

From Electromagnetism to MHD

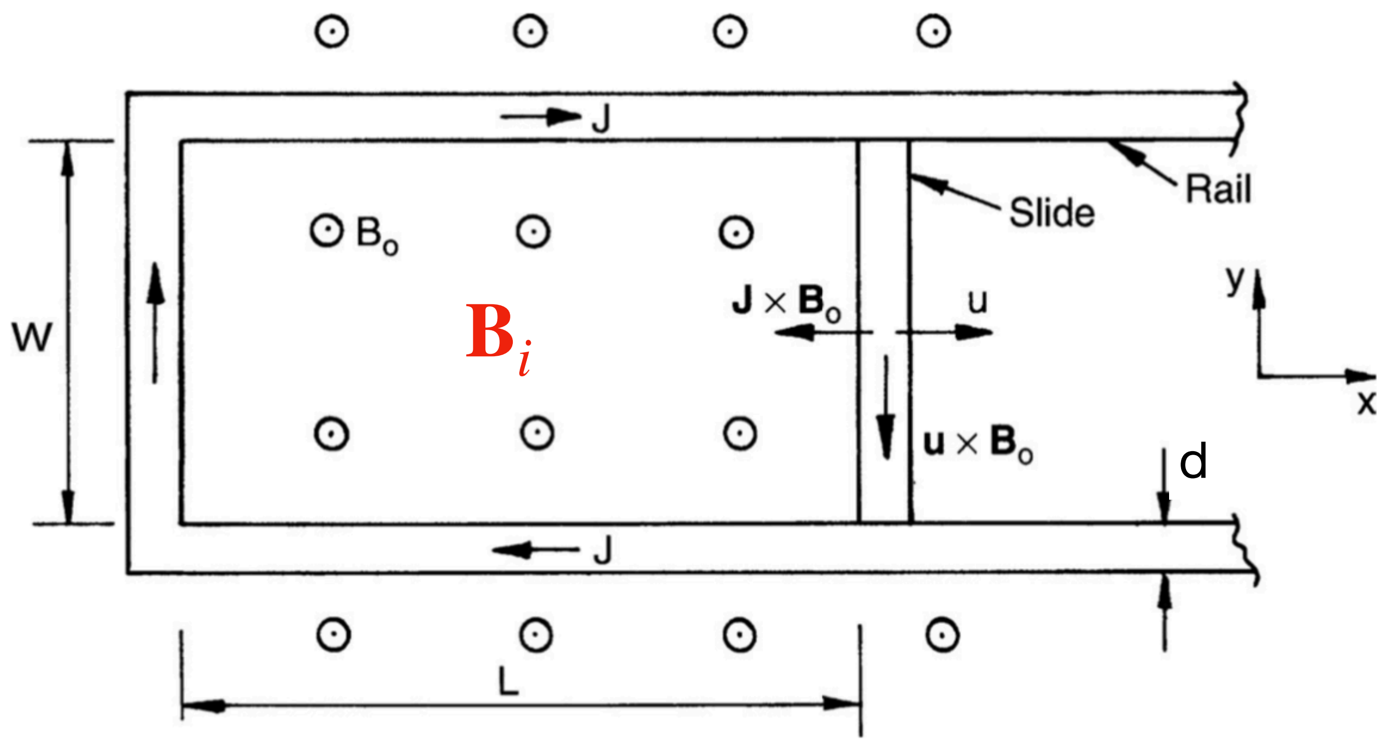
A familiar high-school experiment (高考题)



Width: W
Distance: L
Slide width:
Magnetic field: B0
Conductivity: sigma

Question: when the motion of the slide is given a tap, what's the response?

From Electromagnetism to MHD



Width: W
Distance: L
Slide width: d
Magnetic field: B0
Conductivity: sigma

At t = 0, the slide is given a speed u

Ampere's law: $\mathbf{B}_i = -(\mu dJ)\hat{\mathbf{z}}$ Note that the direction of B_i is such as to try to maintain a constant flux in the current loop

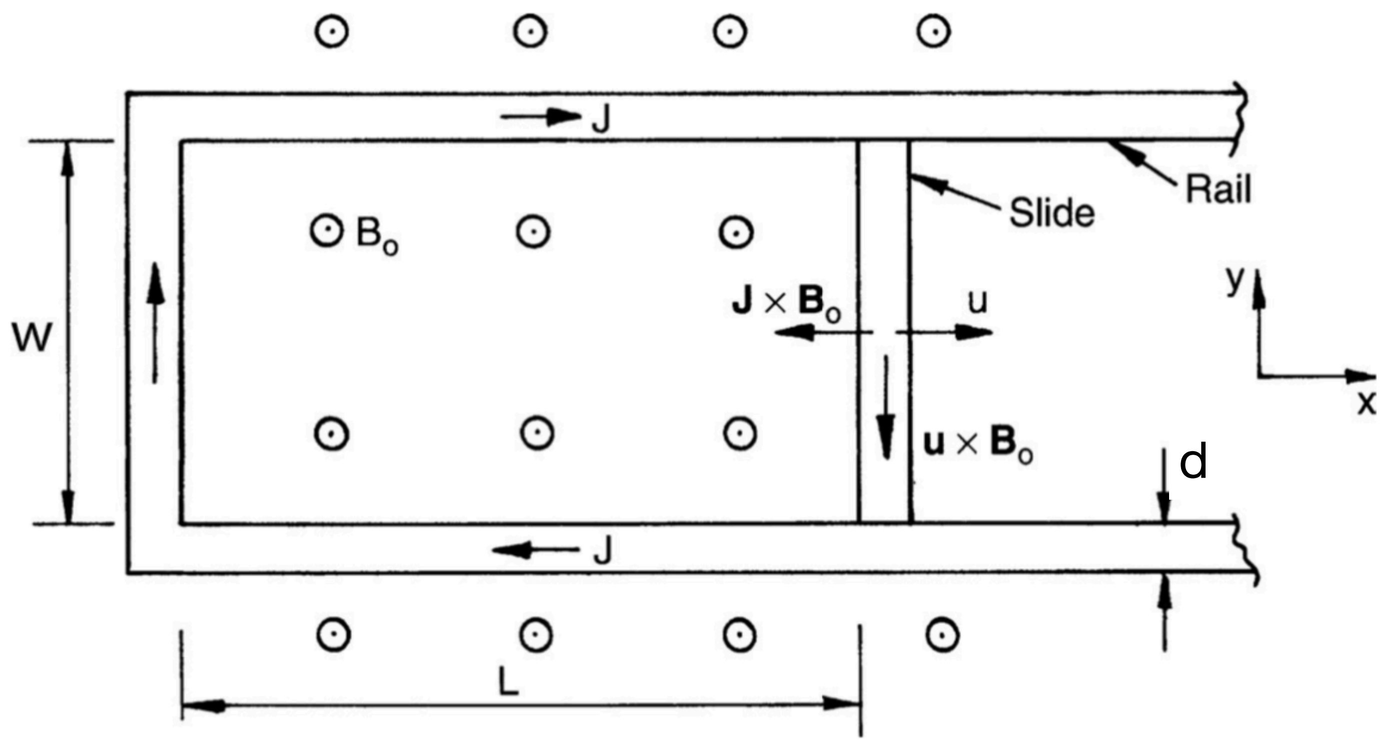
Faraday's law+Ohm's law: $$\frac{1}{\sigma}\oint_C \mathbf{j}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

Use Ampere's law: $\longrightarrow$ $$\frac{d\Phi}{dt} = \frac{d}{dt}\left[LW\left(B_0 - \mu dJ\right)\right] = 2\left(L+W\right)\frac{J}{\sigma}$$

Lorentz force $$\mathbf{F} = -J\left(B_0 - \frac{1}{2}\mu dJ\right) dW\hat{\mathbf{x}}$$

Newton's law: $$\rho\frac{d^2L}{dt^2} = \rho\frac{du}{dt} = -J\left(B_0 - \mu dJ/2\right)$$

From Electromagnetism to MHD



A couple of parameters:

$$B_i = \mu d J \quad T = \mu \sigma d W$$

$$l = dW/L \quad R_m = \mu \sigma u l = uT/L$$

$$\frac{d\Phi}{dt} = \frac{d}{dt}\left[LW\left(B_0 - \mu d J\right)\right] = 2\left(L+W\right)\frac{J}{\sigma} \longrightarrow \frac{d}{dt}\left[L\left(B_0 - B_i\right)\right] = \frac{2\left(L+W\right)B_i}{T}$$

$$\rho\frac{d^2L}{dt^2} = \rho\frac{du}{dt} = -J\left(B_0 - \mu d J/2\right) \longrightarrow \rho d\frac{d^2L}{dt^2} = \rho d\frac{du}{dt} = \frac{\left(B_0 - B_i\right)^2}{2\mu} - \frac{B_i^2}{2\mu}$$

Now we got two non-linear PDEs, it's time to make assumptions based on Rm

$R_m \gg 1$ $R_m \ll 1$

The perfect conducting case $R_m \gg 1$

In this case, $R_m = \frac{u}{L} \gg T$, **or** $R_m = \mu\sigma u l \gg 1$,

$$\frac{d}{dt}\left[L\left(B_0 - B_i\right)\right] = \cancelto{0}{\frac{2\left(L+W\right)B_i}{T}} \approx 0$$

the flux Φ linking the current path is conserved during the motion: $\Phi = LWB_0$

Let's expand the distance parameter: $L = L_0 + \eta = \frac{\Phi}{B_0 W} + \eta$

substitute

$$\rho d\frac{d^2L}{dt^2} = \frac{\left(B_0 - B_i\right)^2}{2\mu} - \frac{B_i^2}{2\mu} \xrightarrow[\text{leading order}]{\text{Keep the}} \boxed{\frac{d^2\eta}{dt^2} + \frac{B_0^2}{\rho\mu d L_0}\eta = 0}$$

Wave equation

Physical meaning:

When the magnetic Reynolds number is high, the slide **oscillates** in an elastic manner
With an angular frequency

$$\omega = \frac{v_A}{\sqrt{dL_0}}$$

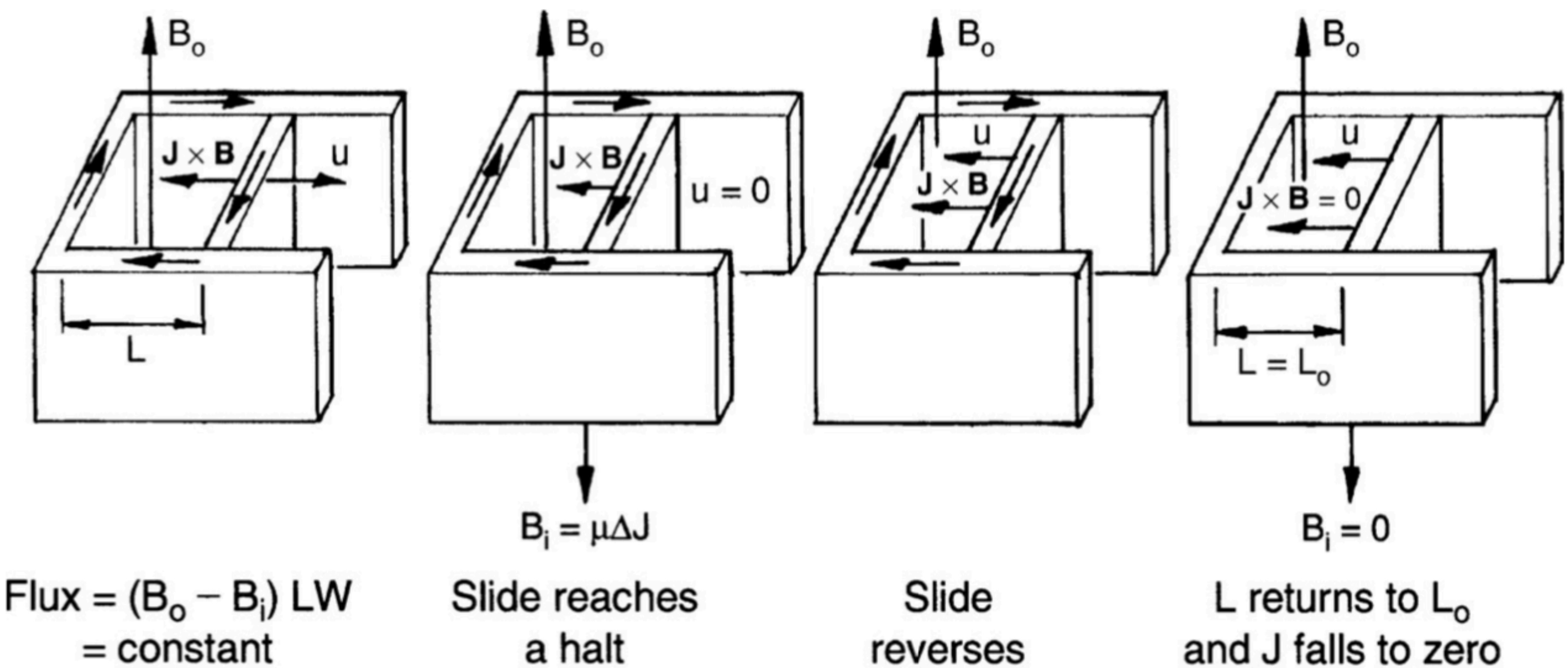
The perfect conducting case $R_m \gg 1$

motion equation

$$\boxed{d\frac{d^2\eta}{dt^2} + \frac{B_0^2}{\rho\mu d L_0}\eta = 0}$$

The key equation?

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\frac{1}{\sigma}\oint_C \mathbf{j}\cdot d\mathbf{l}$$



The poor conducting case $R_m \ll 1$

In this case, $R_m = \frac{u}{L} \ll \frac{1}{T}$, **or** $R_m = \mu\sigma u l \ll 1$,

The induction equation tells that $B_i \ll B_0$

$$\frac{d}{dt}\left[L\left(B_0 - \cancelto{0}{B_i}\right)\right] = \frac{2\left(L+W\right)B_i}{T} \approx \frac{dLB_0}{dt} = uB_0 \longrightarrow B_i = \frac{uT}{2(L+W)}B_0$$

substitute

$$\rho d\frac{du}{dt} = \frac{\left(B_0 - B_i\right)^2}{2\mu} - \frac{B_i^2}{2\mu} \longrightarrow \boxed{\frac{du}{dt} + \frac{W}{2(L+W)}\left(\frac{\sigma B_0^2}{\rho}\right)u = 0}$$

Exponential decay equation

Physical meaning:

When the magnetic Reynolds number is low, the slide velocity **decays** exponentially on a time scale defined as

$$\tau = \left(\frac{\sigma B_0^2}{\rho}\right)^{-1}$$

The poor conducting case $R_m \ll 1$

It is easy to show that the time rate of change in the kinetic energy is:

$$\frac{dE_{kin}}{dt} = -\int \frac{J^2}{\sigma} dV$$

The magnetic field now appears to play a dissipative role. Thus the mechanical energy of the slide is lost to heat via **Ohmic (Joule)** dissipation.

In such a low conductivity (high resistivity) case, the induced magnetic field is small, then the Faraday's law becomes:

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \approx 0$$

E is potential field

Now the Ohm's law and the Lorentz force are written as

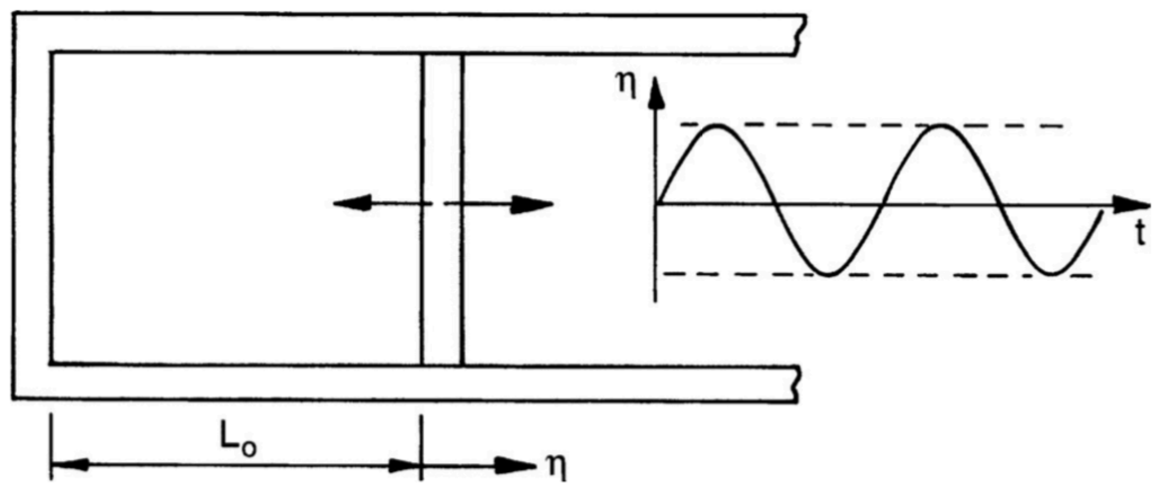
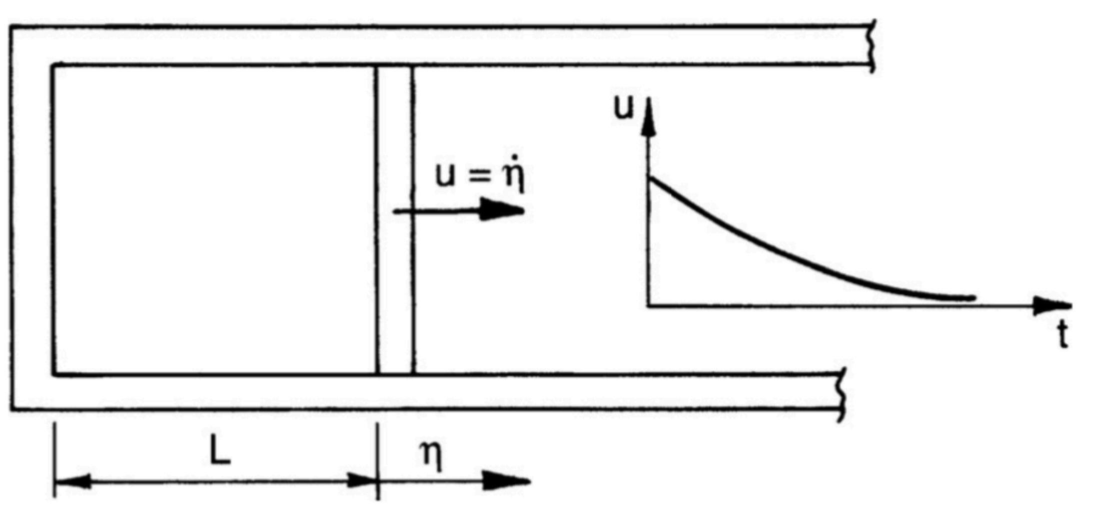
$$\mathbf{J} = \sigma\left(-\nabla V + \mathbf{u} \times \mathbf{B}_0\right) \qquad \mathbf{F} = \mathbf{J} \times \mathbf{B}_0$$

Electric potential

Constant Magnetic field

These two equations are the hallmarkOf low-Rm MHD

Summary of the Electromagnetism case

	Physical Picture	Interpretation
$R_m \gg 1$		Induced B field comparable to B0 Magnetic flux is conserved JxB force is non dissipative
$R_m \ll 1$		Induced B field negligible Magnetic flux is not conserved JxB force is dissipative

Now When is MHD applicable?

validity of the MHD assumption

- **Slow, Non-relativistic: u<<c**
- **Fluid assumption**

Scale analysis:

$$\frac{\partial}{\partial t} f + A \cdot f = S \xrightarrow[\frac{\partial}{\partial t} \sim \omega \sim \frac{1}{\tau_\omega}]{f \sim e^{i\omega t}} \cancel{\frac{1}{\tau_\omega}} f + \cancel{A} \cdot f = S$$

$$0, \; if \; \frac{1}{\tau_\omega} \ll A$$

$$0, \; if \; \frac{1}{\tau_\omega} \gg A$$

This is used extensively in the theory of MHD (kinetic as well)

Now When is MHD applicable?

What does "Slow" mean?

- **Slow, Non-relativistic: u<<c**

In MHD "slow" means evolution on time scales longer than those on which individual particles are important, or on which the electrons and ions might evolve independently of one another.

$$\text{slow} \iff \frac{\partial}{\partial t} \ll \begin{cases} \omega_{pe} \equiv \sqrt{\dfrac{4\pi e^2 n_e}{m_e}} & \text{plasma frequency} \\ \Omega_i \equiv \dfrac{eB}{cm_p} & \text{Ion gyro-frequency} \\ \nu_{ei} \equiv \dfrac{\pi n_e e^4 \ln\Lambda}{\sqrt{m_e}(k_{\rm B}T)^{3/2}} & \text{collision frequency} \end{cases}$$

				frequencies rad/sec			
	$n_e\,[\mathrm{cm}^{-3}]$	$T\,[\mathrm{K}]$	$B\,[\mathrm{G}]$	ω_{pe}	Ω_i	ν_{ei}	MHD
magnetotail	1	10^7	10^{-4}	10^5	0.1	10^{-9}	$\Delta t \gg 10$ sec† 此处忽略碰撞
Solar corona (AR)	10^9	10^6	10^2	10^9	10^6	10^2	$\Delta t \gg 10$ ms
Solar interior (200 Mm)	10^{23}	10^6	10^5	10^{16}	10^9	10^{16}	$\Delta t \gg 10$ ns
Galaxy	1	10^4	10^{-6}	10^5	10^{-2}	10^{-6}	$\Delta t \gg 10$ days
Ionosphere							

Now When is ideal MHD applicable?

validity of the ideal MHD assumption

- **Assume the plasma behaves as a fluid:**
  - highly collisional
  - Distribution is Maxwellian - in thermal equilibrium
  - Only consider macroscopic behavior (low frequency, long wavelength)
- **Assume the ion gyro radius is small: (what about electrons?)**
- **Classic physics - ignore the most significant physics advances since 1860**
  - No displacement current
  - Non relativistic (u<<c)
  - No quantum physics
- **Assume the plasma fully ionized:**
  - Limited applications in partially ionized plasmas like the photosphere, ionosphere
- **Ignore resistivity, viscosity, thermal conduction, radiation**

Question: is the ideal MHD assumption valid for solar wind, magnetospheric, ionospheric plasmas?