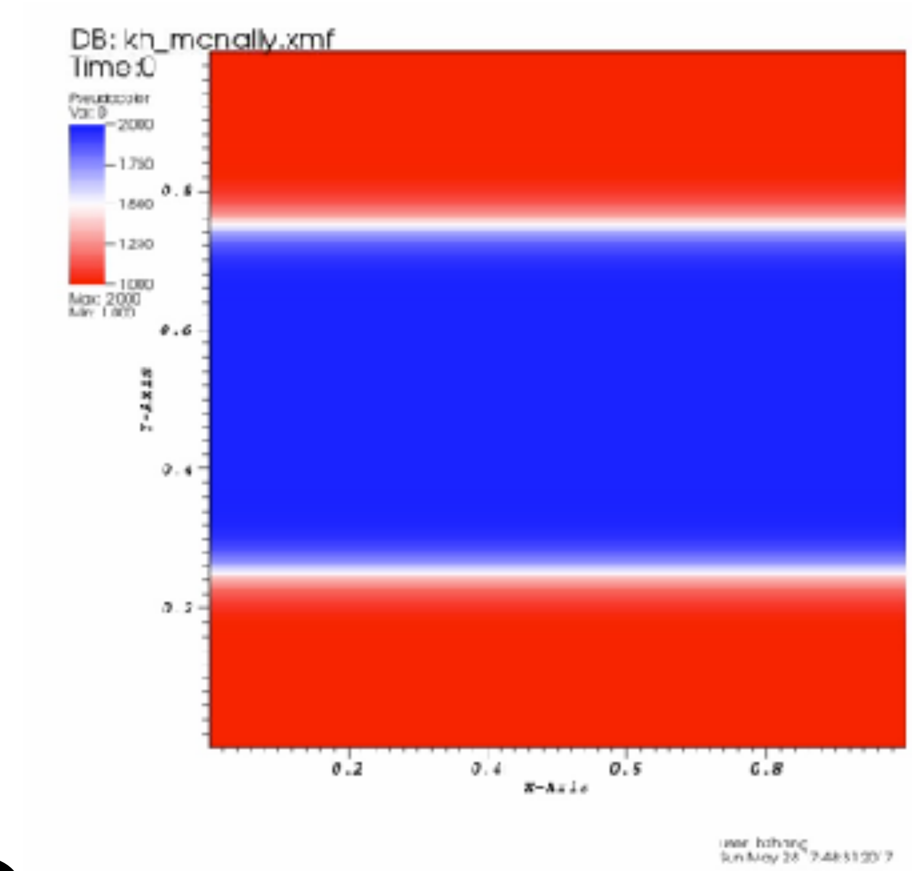
Equations of MHD 1

From Microscopic to Macroscopic

Euler's Equations



Mass

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) = 0$$

Energy

$$\frac{\partial}{\partial t} E_P + \nabla \cdot ((E_P + p)\mathbf{u}) = 0$$

Ideal MHD

Mass

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum

Hydrodynamic force

Lorentz force

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) = \mathbf{J} \times \mathbf{B} = \nabla \cdot \left(\frac{1}{2} \frac{B^2}{\mu_0} \bar{\mathbf{I}} - \frac{1}{\mu_0} \mathbf{B}\mathbf{B} \right)$$

Reynolds Stress

Maxwell Stress

Energy

$$\frac{\partial}{\partial t} E_P + \nabla \cdot ((E_P + p)\mathbf{u}) = \mathbf{u} \cdot \mathbf{J} \times \mathbf{B}$$

Elements of Plasma Kinetic Theory

Concepts

What is it?

A plasma is a system containing a very large number of interacting charged particles, so that for its analysis it is appropriate and convenient to use a statistical approach - microscope

Key concepts

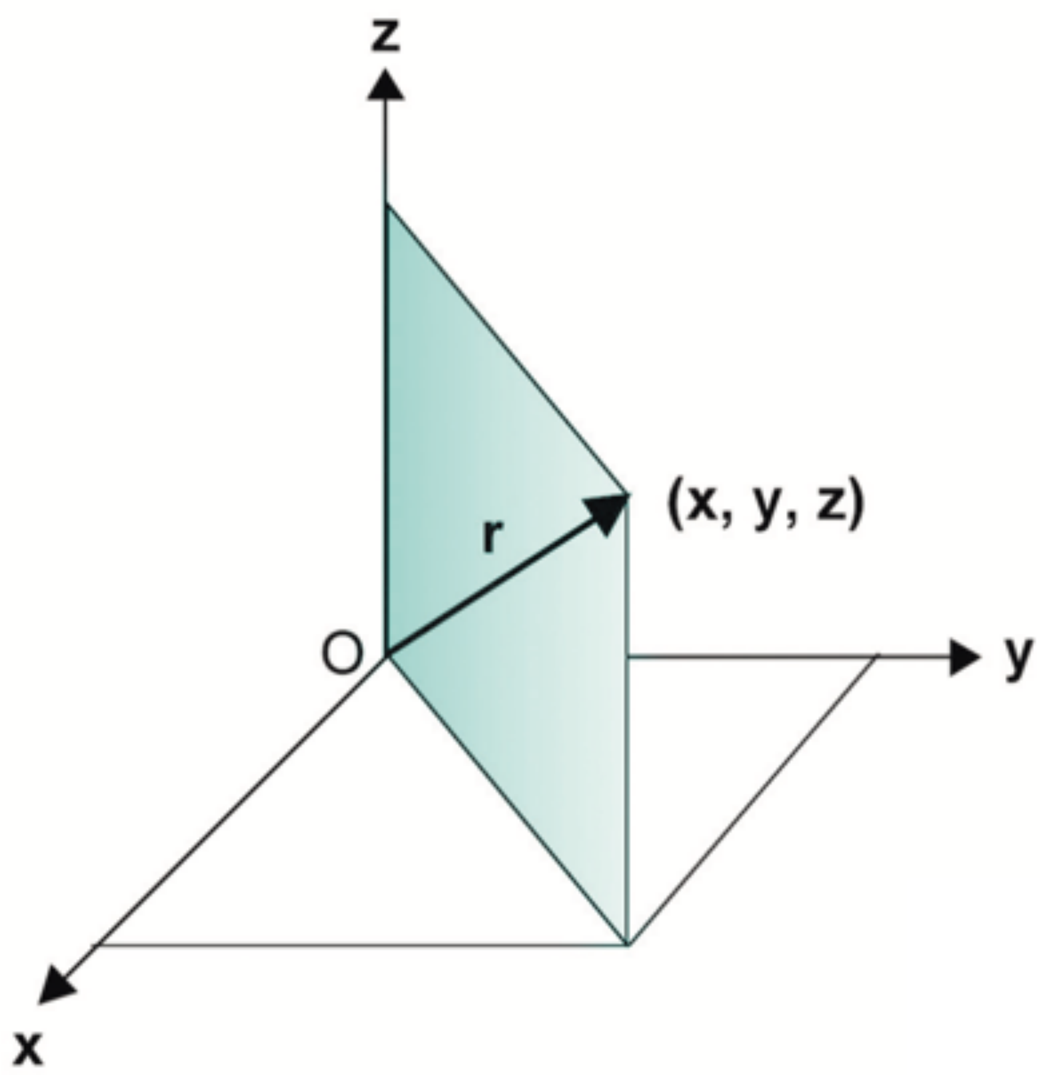
- Phase space
- Distribution function
- Moment integrals

Governing equations

- The Boltzmann equation
- The Vlasov Equation
- Maxwell's equations
- General transport equation (Macroscopic)

Elements of Plasma Kinetic Theory

Configuration Space



At any instant of time each particle in the plasma can be localized by a position vector **r** drawn from the origin of a coordinate system to the center of mass of the particle. In a Cartesian frame of reference, we have

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

Unit vectors

This is also known as the ***configuration space***. The linear velocity of the center of mass of the particle is

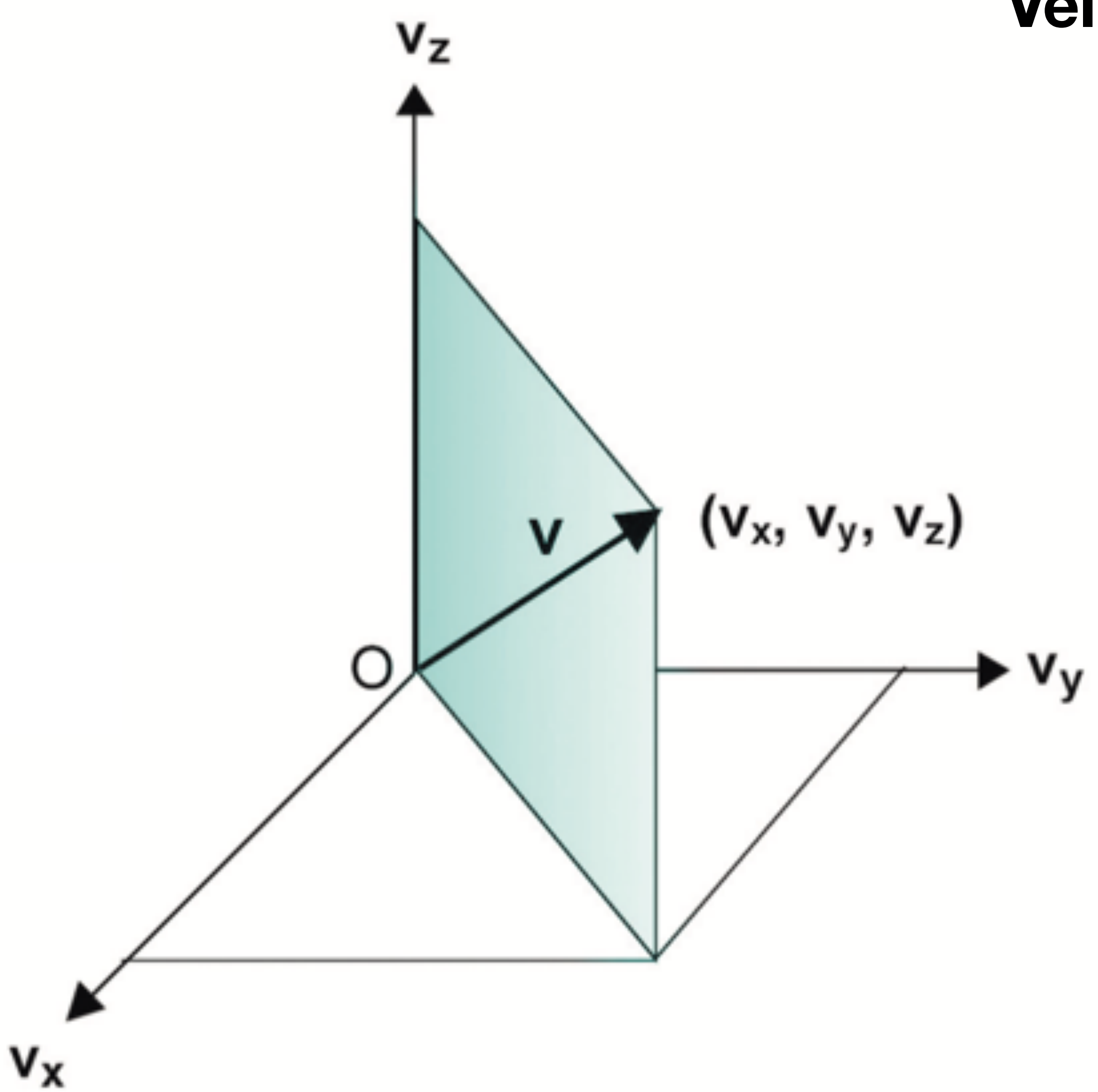
$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$$

Where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

Elements of Plasma Kinetic Theory

Velocity Space



In analogy with the configuration space defined by the position coordinates (x, y, z), it is convenient to introduce the **velocity space** defined by the **velocity coordinates** (v_x,v_y,v_z). In this space the velocity vector v can be viewed as a position vector drawn from the origin of the coordinate system (v_x,v_y,v_z) to the center of mass of the particle

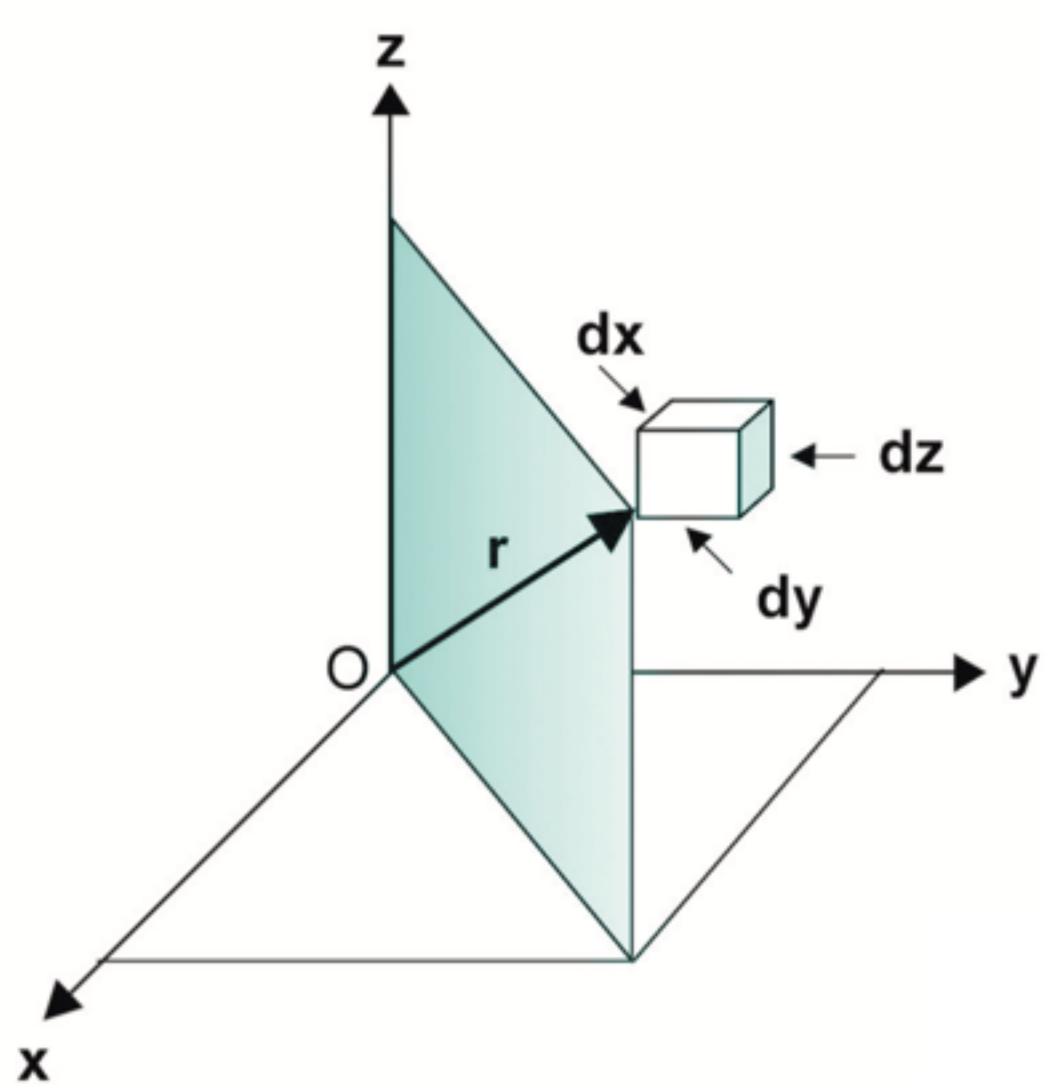
Phase Space

- From the point of view of classical mechanics the instantaneous dynamic state of each particle can be specified by its position and velocity vectors. It is convenient, therefore, to consider the phase space defined by the six coordinates (x,y,z,v_x,v_y,v_z).
- The coordinates (r, v) of the representative point give the position and velocity of the particle. When the particle moves, its representative point describes a trajectory in phase space.

Elements of Plasma Kinetic Theory

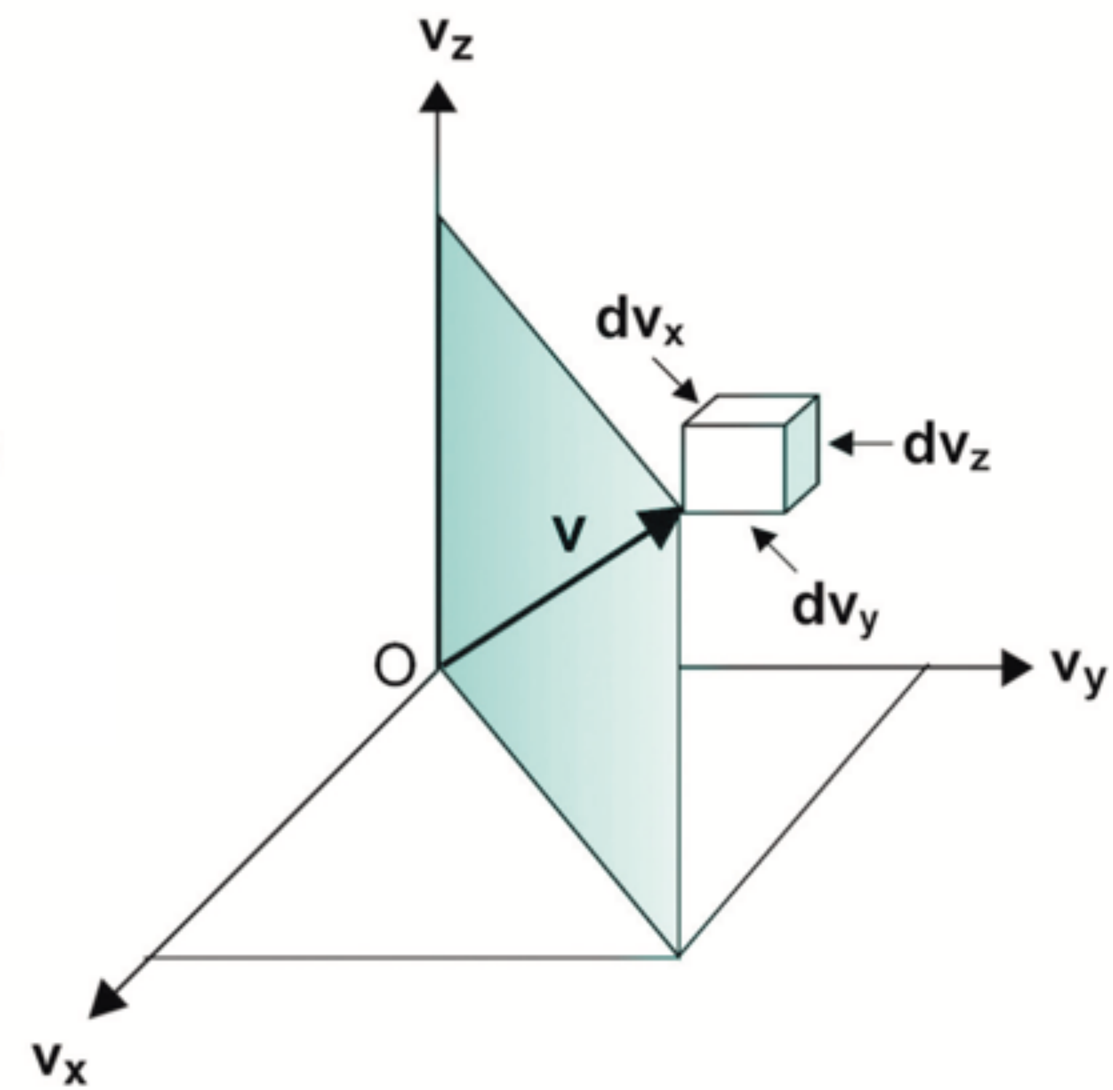
Volume Elements

In Configuration space: $d^3r = dx\,dy\,dz$



This differential element of volume should be a finite element of volume, sufficiently large to contain a very large number of particles, yet sufficiently small in comparison with the characteristic lengths associated with the spatial variation of physical parameters of interest

Similarly In Velocity space:



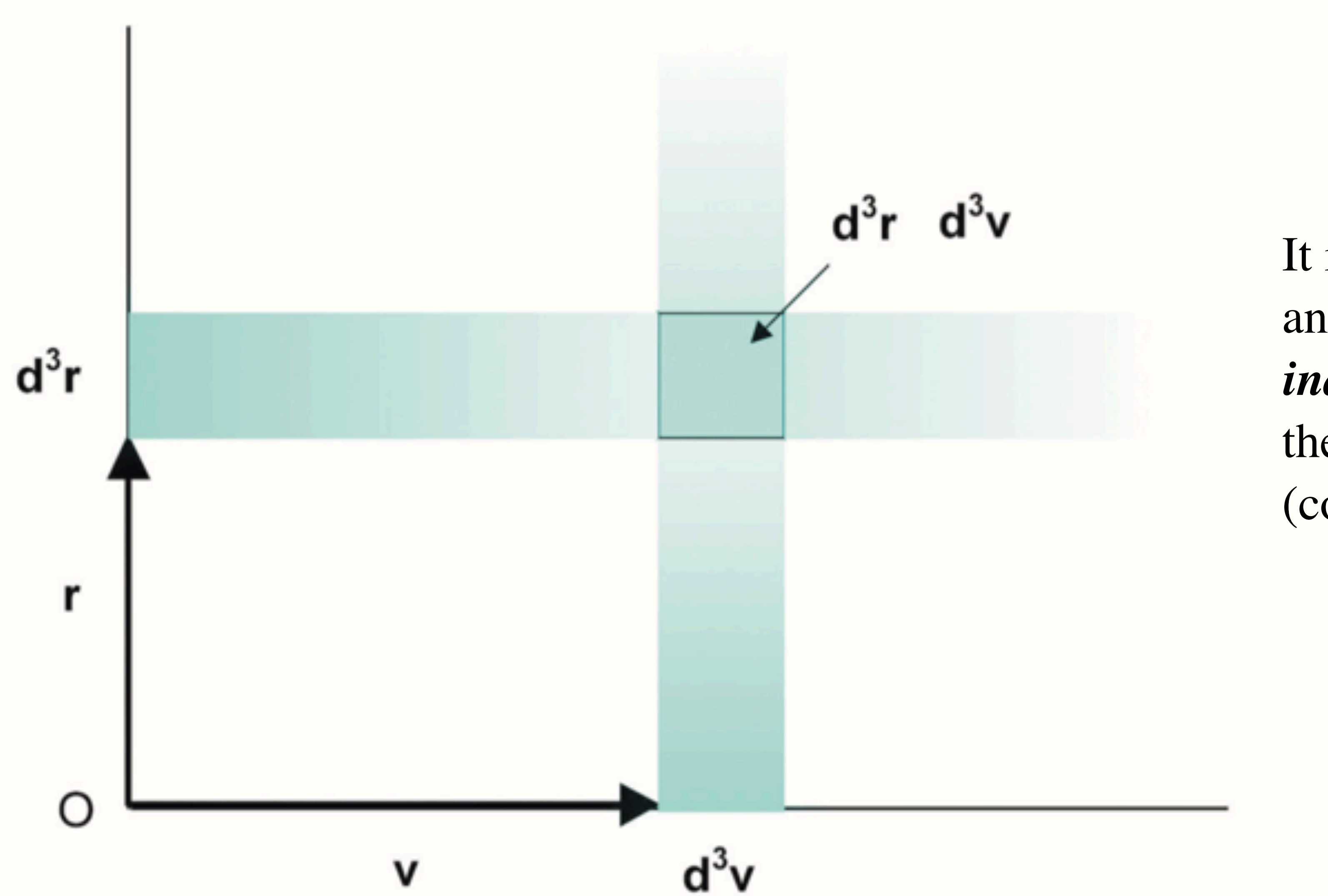
$$d^3v = dv_x\,dv_y\,dv_z$$

Elements of Plasma Kinetic Theory

Volume Elements

In phase space (μ-space) a differential element of volume may be imagined as a six-dimensional cube, represented by

$$d^3r\, d^3v = dx\, dy\, dz\, dv_x\, dv_y\, dv_z$$



It is important to note that the coordinates r and v of phase space are considered to be ***independent*** variables, since they represent the position of individual volume elements (containing many particles) in phase space.

$$\frac{dv_i}{dx_j} \equiv 0$$

Elements of Plasma Kinetic Theory

Distribution Functions

A plasma distribution function is basically the density of the plasma in phase space

# of particles
Inside a volume

$$f_\alpha(\mathbf{r}, \mathbf{v}, t) = \frac{d^6 N_\alpha(\mathbf{r}, \mathbf{v}, t)}{d^3 r d^3 v}$$

Distribution function

Species Index

Config Space

Velocity Space

Differential volume

Properties of distribution function $f_\alpha(\mathbf{r}, \mathbf{v}, t)$

- Finite and positive
- As v approaches infinity, $f_\alpha(\mathbf{r}, \mathbf{v}, t) -> 0$
- Summing over the distribution function gives the total # of particles
- If f is not a function of r, it's called homogenous
- If f is only depends on the magnitude of the velocity: isotropic

Thermal equilibrium: **homogeneous**, **isotropic** and **time independent**: f(v)

The Evolution of a distribution function

Boltzmann Equation

How does a distribution function vary as a function of **r**, **v** and t? - collisionless case

Recall: $d^6 N_\alpha(\mathbf{r}, \mathbf{v}, t) = f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3 r d^3 v$

represents the number of particles of type α that, at the instant t, are situated within the volume element $d^3r\ d^3v$ of phase space, about the coordinates (r,v).

Suppose that each particle is subjected to an external force F. In the absence of particle interactions, a particle of type α with coordinates about (r,v) in phase space, at the instant t, will be found after a time interval dt about the new coordinates (r',v') such that

$$\mathbf{r}'(t + dt) = \mathbf{r} + \mathbf{v}\, dt$$

$$\mathbf{v}'(t + dt) = \mathbf{v} + \mathbf{a}\, dt$$

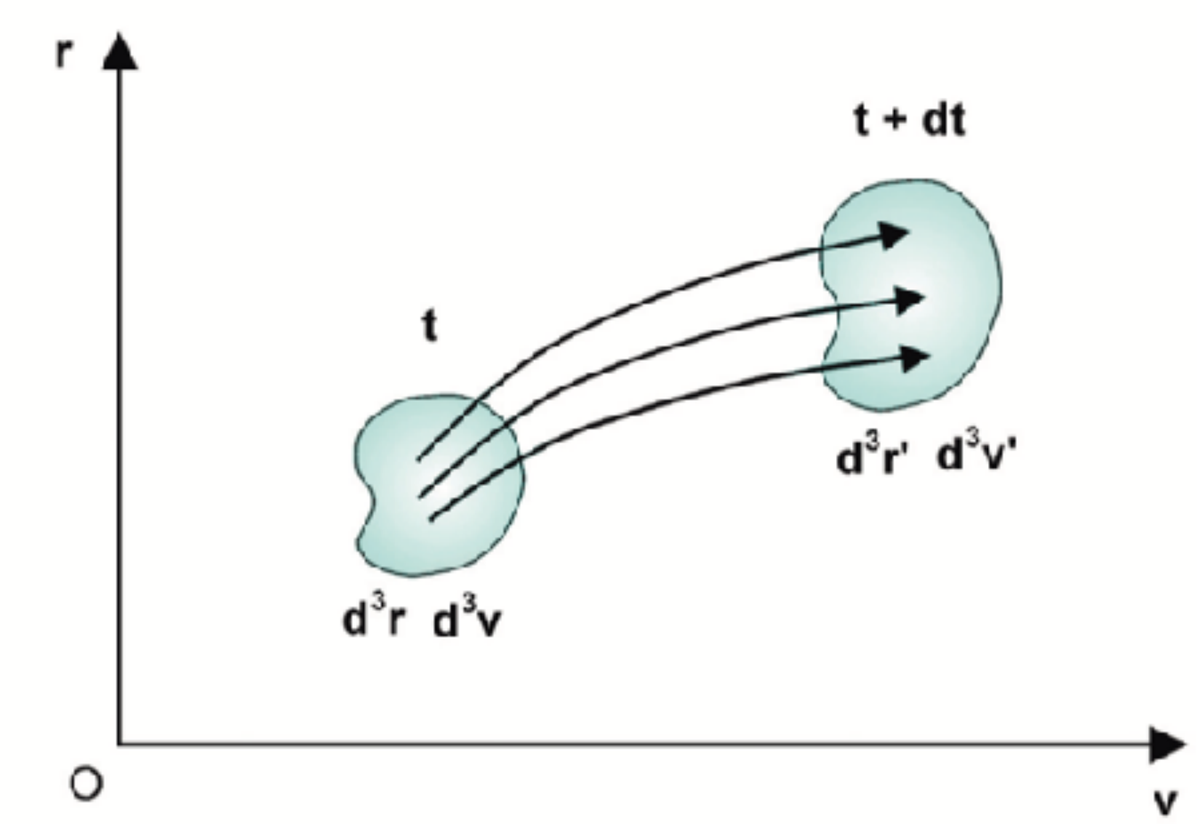
Where $\mathbf{a} = \mathbf{F}/m_\alpha$ is the acceleration

The Evolution of a distribution function

Boltzmann Equation

Thus, all particles of species α inside the volume element $d^3r\, d^3v$ of phase space, about (**r**,**v**) at the instant **t**, will occupy a new volume element $d^3r'\, d^3v'$, about (r', v') after the interval dt (see Fig. 4). Since we are considering the same particles at t and at t + dt, we must have

$$d^6N_\alpha(\mathbf{r},\mathbf{v},t) \equiv f_\alpha(\mathbf{r},\mathbf{v},t)d^3r d^3v = f_\alpha(\mathbf{r}',\mathbf{v}',t+dt)d^3r'd^3v'$$



It is shown that $d^3r d^3v = |J| d^3r'd^3v'$

Jacobin $|J| = 1$

We get

$$f_\alpha(\mathbf{r}',\mathbf{v}',t+dt) - f_\alpha(\mathbf{r},\mathbf{v},t) = 0$$

Density in phase space remains constant in time

$$\lim_{dt\to 0} \frac{f_\alpha(\mathbf{r}+\mathbf{v}dt, \mathbf{v}+\mathbf{a}dt, t+dt) - f_\alpha(\mathbf{r},\mathbf{v},t)}{dt} = \frac{df_\alpha}{dt} = 0$$

Liouville Theorem

The Evolution of a distribution function

Vlasov Equation $\frac{df_\alpha(\mathbf{r}, \mathbf{v}, t)}{dt} = 0$

Where $\frac{d}{dt}$ is the total derivative: $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \mathbf{a} \cdot \nabla_v$

With $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$ and $\nabla_v = \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} + \frac{\partial}{\partial v_z}\hat{z}$

Substituting:

$$\frac{\partial f_\alpha(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha(\mathbf{r}, \mathbf{v}, t) + \mathbf{a} \cdot \nabla_v f_\alpha(\mathbf{r}, \mathbf{v}, t) = 0$$

Collisionless Boltzmann Equation

Recall Navier-Stokes equation   Applying Newton's law to fluid element

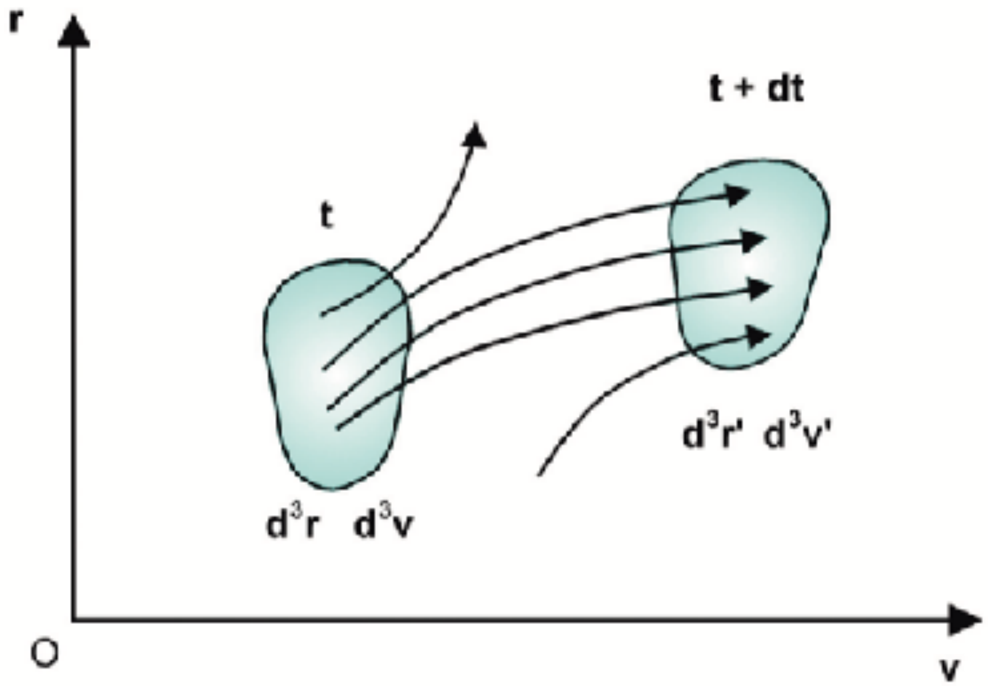
N-S equation

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{F} \xrightarrow[\text{External forces}]{\text{Pressure gradient force, viscous force}} \boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_{ext}}$$

$\frac{D\mathbf{u}}{Dt}$ is expressed as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ Total derivative

The Evolution of a distribution function

Boltzmann Equation $\dfrac{df_\alpha(\mathbf{r},\mathbf{v},t)}{dt} = collision$



If we denote this net gain or loss of particles of type α, as a result of collisions during the interval dt, in the volume element $d^3r\ d^3v$,

$$\left[\frac{\delta f_\alpha(\mathbf{r},\mathbf{v},t)}{\delta t}\right]_{coll} d^3r d^3v dt$$

Then the Liouville mapping in the phase space becomes

$$[f_\alpha(\mathbf{r}',\mathbf{v}',t+dt) - f_\alpha(\mathbf{r},\mathbf{v},t)]d^3rd^3v = \left[\frac{\delta f_\alpha(\mathbf{r},\mathbf{v},t)}{\delta t}\right]_{coll} d^3rd^3vdt$$

$$\longrightarrow \quad \frac{f_\alpha(\mathbf{r}+\mathbf{v}dt, \mathbf{v}+\mathbf{a}dt, t+dt) - f_\alpha(\mathbf{r},\mathbf{v},t)}{dt} = \left[\frac{\delta f_\alpha(\mathbf{r},\mathbf{v},t)}{\delta t}\right]_{coll}$$

$$\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha(\mathbf{r},\mathbf{v},t) + \mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r},\mathbf{v},t) = \underbrace{\left[\frac{\delta f_\alpha(\mathbf{r},\mathbf{v},t)}{\delta t}\right]_{coll}}_{\text{Collision operator}}$$

Collisional Boltzmann Equation

Linking microscopic and macroscopic

Number density and bulk velocity

The number density, $n_\alpha(r,t)$, is a macroscopic variable defined in configuration space as the number of particles of type α, per unit volume, irrespective of velocity.

$$n_\alpha(\mathbf{r},t) = \frac{1}{d^3r}\int_v d^6N_\alpha(\mathbf{r},\mathbf{v},t) = \int_v f_\alpha(\mathbf{r},\mathbf{v},t)d^3v$$

Definition

$$f_\alpha(\mathbf{r},\mathbf{v},t)d^3rd^3v$$

Integrating over velocity space

The average velocity of the particles of type α can be obtained as follows. First we multiply $d^6N_\alpha(r, v, t)$ by the particle velocity **v**, next we integrate over all possible velocities, and finally we divide the result by the total number of type α particles contained in d^3r, irrespective of velocity.

$$\mathbf{u}_\alpha(\mathbf{r},t) = \frac{1}{n_\alpha(\mathbf{r},t)d^3r}\int_v \mathbf{v}\, d^6N_\alpha(\mathbf{r},\mathbf{v},t) = \frac{1}{n_\alpha(\mathbf{r},t)}\int_v \mathbf{v}\, f_\alpha(\mathbf{r},\mathbf{v},t)d^3v$$

Multiple by v

Integrating over velocity space

* Note that both $n_\alpha(r, t)$ and $u_\alpha(r, t)$ are **macroscopic** variables that depend only upon the coordinates r and t.

Average value and Macroscopic variables

Moments of a distribution function

The macroscopic variables, such as number density, flow velocity, kinetic pressure, thermal energy flux, and so on, can be considered as average values of physical quantities, involving the collective behavior of a large number of particles.

To each particle in the plasma we can associate some molecular property, $\chi(\mathbf{r},\mathbf{v},t)$, the total value of $\chi(\mathbf{r},\mathbf{v},t)$ for all the particles of type α inside $d^3r\, d^3v$ is given by

$$\chi(\mathbf{r}, \mathbf{v}, t)\, d^6N_\alpha(\mathbf{r}, \mathbf{v}, t) = \chi(\mathbf{r}, \mathbf{v}, t)\, f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3r\, d^3v$$

The total value of $\chi(r, v, t)$ for all the particles of type α inside the volume element d^3r of configuration space, irrespective of velocity:

$$d^3r \int_v \chi(\mathbf{r}, \mathbf{v}, t)\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v$$

The average value of $\chi(r,v,t)$ can now be obtained by dividing (1.2) by the number of particles of type α inside d^3r about r, at the instant t, i.e., by $n_\alpha(r,t)\, d^3r$

$$< \chi(\mathbf{r}, \mathbf{v}, t) >_\alpha = \frac{1}{n_\alpha(\mathbf{r}, \mathbf{v}, t)} \int_v \chi(\mathbf{r}, \mathbf{v}, t)\, f_\alpha(\mathbf{r}, \mathbf{v}, t)\, d^3v$$

The symbol $< >_\alpha$ denotes the average value with respect to velocity space for the particles of type α.

Also called “moment integrals”

$$< 1 >_\alpha = 1 \qquad\qquad < \mathbf{v} >_\alpha = \mathbf{u}_\alpha(\mathbf{r}, t)$$

Average value and Macroscopic variables

Bulk velocity and peculiar velocity

Average velocity $\mathbf{u}_\alpha(\mathbf{r},t) = <\mathbf{v}>_\alpha = \dfrac{1}{n_\alpha(\mathbf{r},t)}\displaystyle\int_v \mathbf{v} f_\alpha(\mathbf{r},\mathbf{v},t)d^3v$ (Bulk velocity)

Since $<\mathbf{u}_\alpha>_\alpha = \dfrac{1}{n_\alpha(\mathbf{r},t)}\displaystyle\int_v \mathbf{u}_\alpha f_\alpha(\mathbf{r},\mathbf{v},t)d^3v \equiv \mathbf{u}_\alpha$

The **peculiar** velocity or random velocity c_α is defined as the velocity of a type α particle relative to the average velocity $u_\alpha(r,t)$,

$$\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha$$

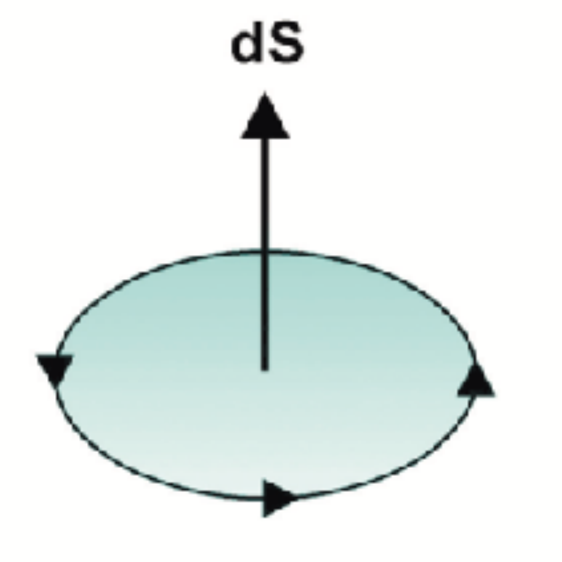
Therefore,

$$<\mathbf{c}_\alpha> = <\mathbf{v} - \mathbf{u}_\alpha> = <\mathbf{v}> - <\mathbf{u}_\alpha> = 0$$

$\mathbf{c}_\alpha$ is the one associated with the ***random*** or ***thermal*** motions of the particles.

$\mathbf{u}_\alpha$ is the one associated with the ***bulk*** motions of the particles.

Average value and Macroscopic variables



Flux

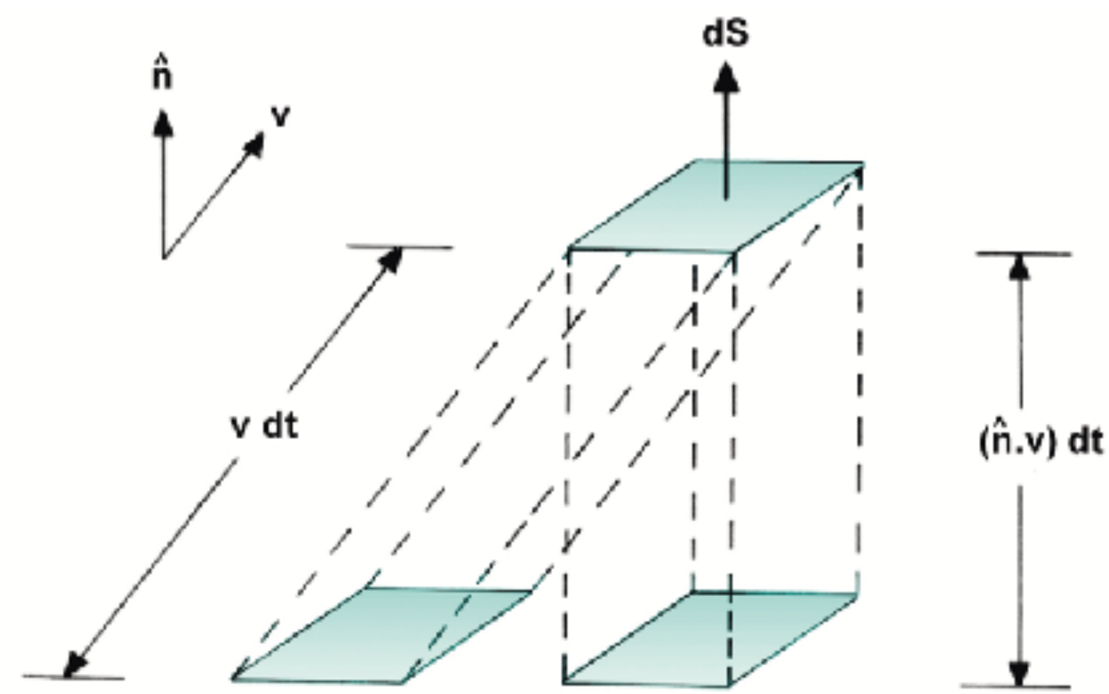
purpose: to calculate the number of particles of species α that move across d**S** during the time interval dt.

method: to identify a volume for the moment integrals

$$d^3r = d\mathbf{S}\cdot\mathbf{v}dt = \hat{n}\cdot\mathbf{v}\,dS\,dt$$

Integrate over volume: $d^3r\int_v \chi(\mathbf{r},\mathbf{v},t)\,f_\alpha(\mathbf{r},\mathbf{v},t)\,d^3v$

$$\int_v \chi(\mathbf{r},\mathbf{v},t)\,f_\alpha(\mathbf{r},\mathbf{v},t)\hat{n}\cdot\mathbf{v}\,d^3v\,dt\,dS$$



Dividing by *dSdt*, we get the net flux passing through the surface in unit time:

$$\Phi_{\alpha n}(\chi) = \int_v \chi(\mathbf{r},\mathbf{v},t)\,f_\alpha(\mathbf{r},\mathbf{v},t)\hat{n}\cdot\mathbf{v}\,d^3v$$

Scalar χ

$$\Phi_{\alpha n}(\chi) = \hat{n}\cdot\mathbf{\Phi}_\alpha(\chi) = \hat{n}\cdot(n_\alpha < \chi\mathbf{v} >_\alpha)$$

Vector $\mathbf{X}$

$$\Phi_{\alpha n}(\mathbf{X}) = \hat{n}\cdot\mathbf{\Phi}_\alpha(\mathbf{X}) = \hat{n}\cdot(n_\alpha < \mathbf{X}\mathbf{v} >_\alpha)$$

Average value and Macroscopic variables

Momentum Flow Tensor

This quantity is defined as the net momentum transported per unit area and time through some surface element **n** dS. Take χ(r,v,t) as the component of momentum of the type α particles along some direction specified by the unit vector **j**

$$\chi_j = m_\alpha \mathbf{v} \cdot \mathbf{j} = m_\alpha v_j$$

The **j-momentum** transported through direction **n** is calculated as

$$\Pi_{\alpha jn}(\mathbf{r}, t) = n_\alpha < m_\alpha(\mathbf{j} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{n}) > = n_\alpha m_\alpha < v_j v_n >$$

Use the peculiar velocity $\mathbf{c}_\alpha = \mathbf{v} - \mathbf{u}_\alpha$

$$\Pi_{\alpha jn}(\mathbf{r}, t) = = n_\alpha m_\alpha < c_{\alpha j} c_{\alpha n} > + n_\alpha m_\alpha u_{\alpha j} u_{\alpha n}$$

In a tensor form

$$\mathbf{\Pi}_\alpha(\mathbf{r}, t) = n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha > + n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha$$

$$\begin{aligned} \mathit{\Pi}_\alpha &= \widehat{\mathbf{x}}\widehat{\mathbf{x}}\Pi_{\alpha xx} + \widehat{\mathbf{x}}\widehat{\mathbf{y}}\Pi_{\alpha xy} + \widehat{\mathbf{x}}\widehat{\mathbf{z}}\Pi_{\alpha xz} \\ &+ \widehat{\mathbf{y}}\widehat{\mathbf{x}}\Pi_{\alpha yx} + \widehat{\mathbf{y}}\widehat{\mathbf{y}}\Pi_{\alpha yy} + \widehat{\mathbf{y}}\widehat{\mathbf{z}}\Pi_{\alpha yz} \\ &+ \widehat{\mathbf{z}}\widehat{\mathbf{x}}\Pi_{\alpha zx} + \widehat{\mathbf{z}}\widehat{\mathbf{y}}\Pi_{\alpha zy} + \widehat{\mathbf{z}}\widehat{\mathbf{z}}\Pi_{\alpha zz} \end{aligned}$$

Average value and Macroscopic variables

Pressure

Concept: The pressure of a gas is usually defined as the force per unit area exerted by the gas molecules through collisions with the walls of the containing vessel. This force is equal to the rate of transfer of molecular momentum to the walls of the container.

How to calculate: The pressure on *d**S*** is defined as the rate of transport of molecular momentum per unit area, that is, the flux of momentum across dS due to the **random** particle motions.

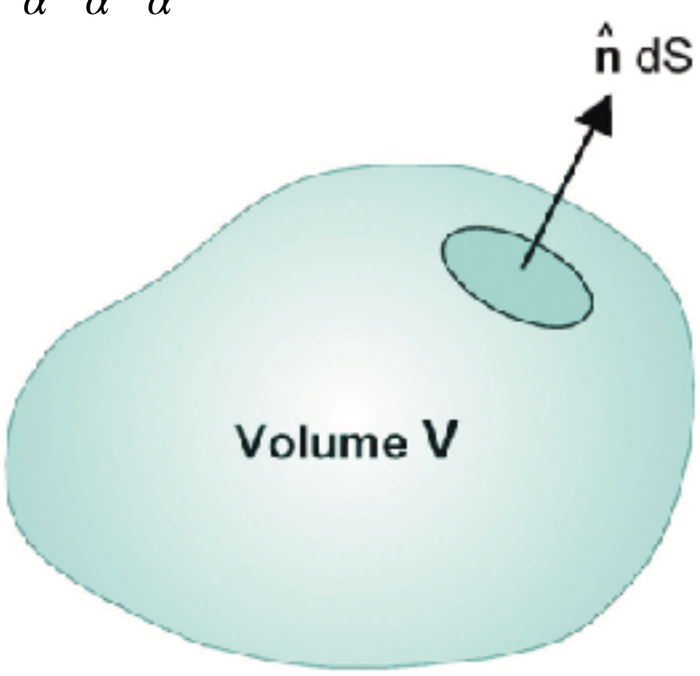
$$P_{\alpha jn} = n_\alpha m_\alpha < c_{\alpha j} c_{\alpha n} >$$

This is also known as the pressure tensor

$$\mathbf{P}_\alpha = n_\alpha m_\alpha < \mathbf{c}_\alpha \mathbf{c}_\alpha > = \mathbf{\Pi}_\alpha(\mathbf{r}, t) - n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha$$

The force per unit area, f_α, acting on the area element **n** dS, as the result of the random particle motions:

$$\mathbf{f}_\alpha = -\mathbf{P}_\alpha \cdot \hat{n} = -n_\alpha m_\alpha < \mathbf{c}_\alpha(\mathbf{c}_\alpha \cdot \hat{n}) >$$



Forces per unit area and volume

In general, for any arbitrary distribution of individual velocities, the vector quantity

$$-n_\alpha m_\alpha < \mathbf{c}_\alpha(\mathbf{c}_\alpha \cdot \hat{n}) > = -\mathbf{P}_\alpha \cdot \hat{n} dS$$

represents the rate of change of the plasma momentum within the closed surface S, due to the exchange of species α through the surface element n dS, and the force per unit area is simply

$$-\mathbf{P}_\alpha \cdot \hat{n}$$

Force per unit area

The force per unit volume inside the plasma, due to the random particle motions, can be obtained by integrating the surface force over the closed surface S bounding the volume element V , dividing the result by V , and then taking the limit as V tends to zero

$$-\lim_{V \to 0} \left[\frac{1}{V} \oint \mathbf{P}_\alpha \cdot \hat{n} dS \right] = -\nabla \cdot \mathbf{P}_\alpha$$

Force per unit volume

Scalar pressure and temperature

An important macroscopic variable is the ***scalar pressure,*** or mean hydrostatic pressure, p_α. It is defined as one-third the trace of the pressure tensor:

$$p_\alpha = \frac{1}{3}\left(P_{\alpha xx} + P_{\alpha yy} + P_{\alpha zz}\right) = \frac{1}{3} n_\alpha m_\alpha < c_{\alpha x}^2 + c_{\alpha y}^2 + c_{\alpha z}^2 > = \frac{1}{3} n_\alpha m_\alpha < c_\alpha^2 >$$

Another important parameter for a macroscopic description of a plasma is its temperature. The absolute temperature T_α, for the type α particles, is a measure of the mean kinetic energy of the **random** particle motions:

$$\frac{1}{2} k T_{\alpha i} = \frac{1}{2} m_\alpha < c_{\alpha i}^2 >$$

When the distribution of random velocities is **isotropic** $c_{\alpha x}^2 = c_{\alpha y}^2 = c_{\alpha z}^2 = \frac{1}{3} c_\alpha^2$

The scalar pressure becomes $p_\alpha = P_{\alpha xx} = P_{\alpha yy} = P_{\alpha zz} = n_\alpha m_\alpha < c_{\alpha i}^2 >$

Thus we have *the equation of state for* ***ideal gas***

$$p_\alpha = n_\alpha k T_\alpha$$

Maxwell-Boltzmann Distribution

Equilibrium State

$$f_\alpha(\mathbf{v}) = n_\alpha \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \exp\left(-\frac{m_\alpha(\mathbf{v} - \mathbf{u}_\alpha)^2}{2kT_\alpha} \right)$$

Note that the number density n and the temperature T are constants, independent of r and t. This distribution function represents the only permanent mode for the distribution of the particle velocities in the gas, for specified values of n and T .

Local Equilibrium

$$f_\alpha(\mathbf{r}, \mathbf{v}, t) = n_\alpha(\mathbf{r}, t) \left(\frac{m_\alpha}{2\pi T_\alpha(\mathbf{r}, t)} \right)^{3/2} \exp\left(-\frac{m_\alpha(\mathbf{v} - \mathbf{u}(\mathbf{r}, t))^2}{2kT_\alpha(\mathbf{r}, t)} \right)$$

Assumption: In many situations of interest we are dealing with a gas that, although not in equilibrium, is not very far from it. It is then a good approximation to say that in the neighborhood of any point in the gas, there is an equilibrium state described by the above local Maxwell-Boltzmann distribution function. This basically how the fluxes are computed in the LFM and Gamera MHD codes

General Transport Equation

Macroscopic equations

Starting from the Boltzmann equation for the type α particles in the general form:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} \qquad f_\alpha \sim f_\alpha(\mathbf{r}, \mathbf{v}, t)$$

multiply each term by χ(v) and integrate the resulting equation over all of velocity space to obtain

$$\int_v d^3v\, \chi \left[\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} \right]$$

$$\int_v \chi \frac{\partial f_\alpha}{\partial t} d^3v + \int_v \chi \mathbf{v} \cdot \nabla f_\alpha\, d^3v + \int_v \chi \mathbf{a} \cdot \nabla_v f_\alpha\, d^3v = \int_v \chi \left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} d^3v$$

This is the microscopic form of the general transport equation (but useless)

General Transport Equation

Macroscopic equations

$$\underbrace{\int_v \chi \frac{\partial f_\alpha}{\partial t} d^3v}_{1} + \underbrace{\int_v \chi \mathbf{v} \cdot \nabla f_\alpha \, d^3v}_{2} + \underbrace{\int_v \chi \mathbf{a} \cdot \nabla_v f_\alpha \, d^3v}_{3} = \int_v \chi \left(\frac{\delta f_\alpha}{\delta t} \right)_{coll} d^3v$$

1 $$\int_v \chi \frac{\partial}{\partial t} d^3v = \frac{\partial}{\partial t}\left(\int_v \chi f_\alpha d^3v \right) - \cancelto{0}{\int_v f_\alpha \frac{\partial \chi}{\partial t} d^3v} = \boxed{\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha)}$$

2 $$\int_v \chi \mathbf{v} \cdot \nabla f_\alpha \, d^3v = \nabla \cdot (\int_v \mathbf{v} \chi f_\alpha d^3v) - \cancelto{0}{\int_v f_\alpha \mathbf{v} \cdot \nabla \chi d^3v} - \cancelto{0}{\int_v f_\alpha \chi \nabla \cdot \mathbf{v} d^3v} = \boxed{\nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha)}$$

0 - f->0 as v-> inf

0 - force is independent

3 $$\int_v \chi \mathbf{a} \cdot \nabla_v f_\alpha \, d^3v = \cancelto{0}{\int_v \nabla_v \cdot (\mathbf{a} \chi f_\alpha) \, d^3v} - \int_v f_\alpha \mathbf{a} \nabla_v \chi \, d^3v - \cancelto{0}{\int_v f_\alpha \chi \nabla_v \cdot \mathbf{a} \, d^3v} = \boxed{- n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha}$$

Combine 1,2,3:

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha) \right]_{coll}$$

General transport equation

From Kinetic to Fluid Equation

Mass conservation

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll}$$

The equation of continuity, or of conservation of mass, can be obtained by taking $\chi = m_\alpha$:

$$< \chi >_\alpha = m_\alpha \qquad < \chi \mathbf{v} >_\alpha = m_\alpha \mathbf{u}_\alpha \qquad \nabla_v \chi = \nabla_v m_\alpha \equiv 0$$

Substitute into the general transport equation, we get the mass continuity equation:

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Where $S_\alpha = m_\alpha \int_v \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3v = \left(\frac{\delta m_\alpha n_\alpha}{\delta t}\right)_{coll}$ represents the rate per unit volume at which particles of type α (with mass m_α) are produced or lost as a result of collisions.

For collisionless plasma

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0$$

From Kinetic to Fluid Equation

Momentum equation

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll}$$

The equation of motion, or of conservation of momentum, can be obtained by taking $\chi = m_\alpha \mathbf{v}$:

$$< \chi >_\alpha = m_\alpha \mathbf{u}_\alpha \qquad < \chi \mathbf{v} >_\alpha = < m_\alpha \mathbf{v}\mathbf{v} >_\alpha \qquad < \mathbf{a} \cdot \nabla_v \chi >_\alpha = m_\alpha \mathbf{a}$$

The second term is basically the momentum and pressure tensor we've already seen:

$$< m_\alpha \mathbf{v}\mathbf{v} >_\alpha = < m_\alpha(\mathbf{u}_\alpha + \mathbf{c}_\alpha)(\mathbf{u}_\alpha + \mathbf{c}_\alpha) >_\alpha = \underbrace{< m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha >_\alpha}_{m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha} + \underbrace{< m_\alpha \cancel{\mathbf{c}_\alpha} \mathbf{u}_\alpha >_\alpha}_{0} + \underbrace{< m_\alpha \mathbf{u}_\alpha \cancel{\mathbf{c}_\alpha} >_\alpha}_{0} + \underbrace{< m_\alpha \mathbf{c}_\alpha \mathbf{c}_\alpha >_\alpha}_{\mathbf{P}_\alpha / n_\alpha}$$

Now we get the momentum equation

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha) - n_\alpha m_\alpha < \mathbf{a} >_\alpha = \mathbf{A}_\alpha$$

Where $\mathbf{A}_\alpha = m_\alpha \int_v \mathbf{v} \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3 v = \left(\frac{\delta(m_\alpha n_\alpha \mathbf{u}_\alpha)}{\delta t}\right)_{coll}$

From Kinetic to Fluid Equation

Equation of motion

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha) - n_\alpha m_\alpha < \mathbf{a} >_\alpha = \mathbf{A}_\alpha$$

1 2 3

1 $$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) = n_\alpha m_\alpha \frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \frac{\partial n_\alpha m_\alpha}{\partial t}$$

2 $$\nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha) = n_\alpha m_\alpha (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + \mathbf{u}_\alpha \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot \mathbf{P}_\alpha$$

3 $$-n_\alpha m_\alpha < \mathbf{a} >_\alpha = - n_\alpha \mathbf{F}$$

Use the mass equation

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Now we get the equation of motion in a generalized form

$$n_\alpha m_\alpha \left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha \right] + \nabla \cdot \mathbf{P}_\alpha - n_\alpha \mathbf{F} = \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

$$n_\alpha m_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = - \nabla \cdot \mathbf{P}_\alpha + n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + n_\alpha m_\alpha \mathbf{g} + \mathbf{A}_\alpha - \mathbf{u}_\alpha S_\alpha$$

Total derivative

PGF

Lorentz

Gravity

Collisions

∇p_α
If isotropic

$n_\alpha \mathbf{F}$
External force

From Kinetic to Fluid Equation

The collision term

The term $\mathbf{A}_\alpha$ denotes the rate of change of the mean momentum per unit volume, due to collisions.

As a consequence of conservation of the total momentum in an elastic collision, the change in the momentum of one of the particles must be equal and opposite to the change in momentum of the other particle participating in the collision event.

An expression often used for the term of momentum transfer by collision is

$$\mathbf{A}_\alpha = -n_\alpha m_\alpha \sum_\beta \nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta)$$

$\nu_{\alpha\beta}$ is the **collision frequency** (s^{-1}) for momentum transfer. Since the momentum must be conserved:

$$n_\alpha m_\alpha \nu_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta) + n_\beta m_\beta \nu_{\beta\alpha}(\mathbf{u}_\beta - \mathbf{u}_\alpha)$$

Note that this expression for collision is not generally valid, it only works for situations when the different between the bulk velocities of the particle species is relatively small, and each species has a Maxwellian velocity distribution

Question: Is the collision term valid for ionospheric plasmas?

From Kinetic to Fluid Equation

The energy equation

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha < \chi \mathbf{v} >_\alpha) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha)\right]_{coll}$$

The equation of energy can be obtained by taking $\chi = m_\alpha v^2/2$:

$$< \chi >_\alpha = \frac{1}{2} m_\alpha < c_\alpha^2 > + \frac{1}{2} m_\alpha u_\alpha^2 = \frac{1}{2}(3p_\alpha + m_\alpha u_\alpha^2)$$

$$\cdot \nabla_v \chi = \frac{1}{2} m_\alpha \nabla_v (\mathbf{v} \cdot \mathbf{v}) = m_\alpha (\mathbf{v} \cdot \nabla_v \mathbf{v}) = m_\alpha \mathbf{v}$$

$$\longrightarrow \quad -n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = -n_\alpha < \frac{\mathbf{F}}{m_\alpha} \cdot \nabla_v \chi > = -n_\alpha < \mathbf{F} \cdot \mathbf{v} >_\alpha$$

Now the energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_\alpha + \frac{1}{2} n_\alpha m_\alpha u_\alpha^2\right) + \nabla \cdot \left[\frac{1}{2} n_\alpha m_\alpha < v^2 \mathbf{v} >_\alpha\right] - n_\alpha < \mathbf{F} \cdot \mathbf{v} >_\alpha = M_\alpha$$

where M_α represents the rate of energy density change due to collisions

$$M_\alpha = \frac{1}{2} m_\alpha \int_v v^2 \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3 v = \left(\frac{\delta(\frac{1}{2} m_\alpha n_\alpha < v^2 >_\alpha)}{\delta t}\right)_{coll}$$

From Kinetic to Fluid Equation

The energy equation

The second term in the energy equation:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_\alpha + \frac{1}{2}n_\alpha m_\alpha u_\alpha^2\right) + \nabla \cdot \left[\frac{1}{2}n_\alpha m_\alpha < v^2\mathbf{v} >_\alpha\right] - n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = M_\alpha$$

is written as $< v^2\mathbf{v} >_\alpha = < [(\mathbf{u}_\alpha + \mathbf{c}_\alpha)\cdot(\mathbf{u}_\alpha + \mathbf{c}_\alpha)](\mathbf{u}_\alpha + \mathbf{c}_\alpha) > \;\; = < [(u_\alpha^2 + 2\mathbf{u}_\alpha\cdot\mathbf{c}_\alpha + c_\alpha^2)(\mathbf{u}_\alpha + \mathbf{c}_\alpha) >$

$$= u_\alpha^2\mathbf{u}_\alpha + < c_\alpha^2 > \mathbf{u}_\alpha + 2 < \mathbf{c}_\alpha\mathbf{c}_\alpha > \cdot\,\mathbf{u}_\alpha + < c_\alpha^2\mathbf{c}_\alpha >$$

Scalar Pressure $\sim 3p_\alpha$

Pressure tensor $\sim \mathbf{P}_\alpha$

Heat Flux $\sim \mathbf{q}_\alpha$

With proper scaling, the second term becomes

$$\nabla\cdot\left[\frac{1}{2}n_\alpha m_\alpha < v^2\mathbf{v} >_\alpha\right] = \nabla\cdot\left[\frac{1}{2}n_\alpha m_\alpha u_\alpha^2\mathbf{u}_\alpha + \frac{3}{2}p_\alpha\mathbf{u}_\alpha + \mathbf{P}_\alpha\cdot\mathbf{u}_\alpha + \mathbf{q}_\alpha\right]$$

Then the energy equation (plasma energy) is

$$\frac{\partial \epsilon_P}{\partial t} + \nabla\cdot\left[\epsilon_P\mathbf{u}_\alpha + \mathbf{P}_\alpha\cdot\mathbf{u}_\alpha + \mathbf{q}_\alpha\right] - n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = M_\alpha$$

If the pressure is isotropic:

$$\frac{\partial \epsilon_P}{\partial t} + \nabla\cdot\left[(\epsilon_P + p_\alpha)\mathbf{u}_\alpha\right] + \nabla\cdot\mathbf{q}_\alpha - n_\alpha < \mathbf{F}\cdot\mathbf{v} >_\alpha = M_\alpha$$

The issue of Closure

$$\frac{\partial}{\partial t}(n_\alpha < \chi >_\alpha) + \nabla \cdot (n_\alpha \boxed{< \chi \mathbf{v} >_\alpha}) - n_\alpha < \mathbf{a} \cdot \nabla_v \chi >_\alpha = \left[\frac{\delta}{\delta t}(n_\alpha < \chi >_\alpha) \right]_{coll}$$

*** This term always introduce a higher moment of the distribution function**

For example the mass equation is the *zeroth* moment $\chi = m_\alpha$

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot \boxed{(n_\alpha \mathbf{u}_\alpha)} = S_\alpha$$

$< m_\alpha \mathbf{v} >_\alpha = m_\alpha \mathbf{u}_\alpha$ *First* moment

the momentum equation is the *first* moment $\chi = m_\alpha \mathbf{v}$

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \boxed{\mathbf{P}_\alpha}) - n_\alpha m_\alpha < \mathbf{a} >_\alpha = \mathbf{A}_\alpha$$

$< m_\alpha \mathbf{v}\mathbf{v} >_\alpha = m_\alpha \mathbf{u}_\alpha$ *Second* moment

the energy equation is the *second* moment $\chi = \frac{1}{2} m_\alpha v^2$

$$\frac{\partial \epsilon_P}{\partial t} + \nabla \cdot \left[(\epsilon_P + p_\alpha) \mathbf{u}_\alpha \right] + \nabla \cdot \boxed{\mathbf{q}_\alpha} - n_\alpha < \mathbf{F} \cdot \mathbf{v} >_\alpha = M_\alpha$$

Then how to close the equations?

$< m_\alpha c^2 \mathbf{v} >_\alpha = \mathbf{q}_\alpha$ *Third* moment

The issue of Closure

If the distribution is Maxwellian, then problem solved!

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = S_\alpha$$

$$\frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha) - n_\alpha \mathbf{F} = \mathbf{A}_\alpha$$

$$\frac{\partial \epsilon_P}{\partial t} + \nabla \cdot \left[(\epsilon_P + p_\alpha)\mathbf{u}_\alpha\right] + \cancel{\nabla \cdot \mathbf{q}_\alpha} - n_\alpha \mathbf{F} \cdot \mathbf{u}_\alpha = M_\alpha$$

0 if Maxwellian

$$< m_\alpha c^2 \mathbf{v} >_\alpha = \mathbf{q}_\alpha$$

So in order to close the equations, assumptions are needed for higher moments

Does that mean high moment equations are more accurate?