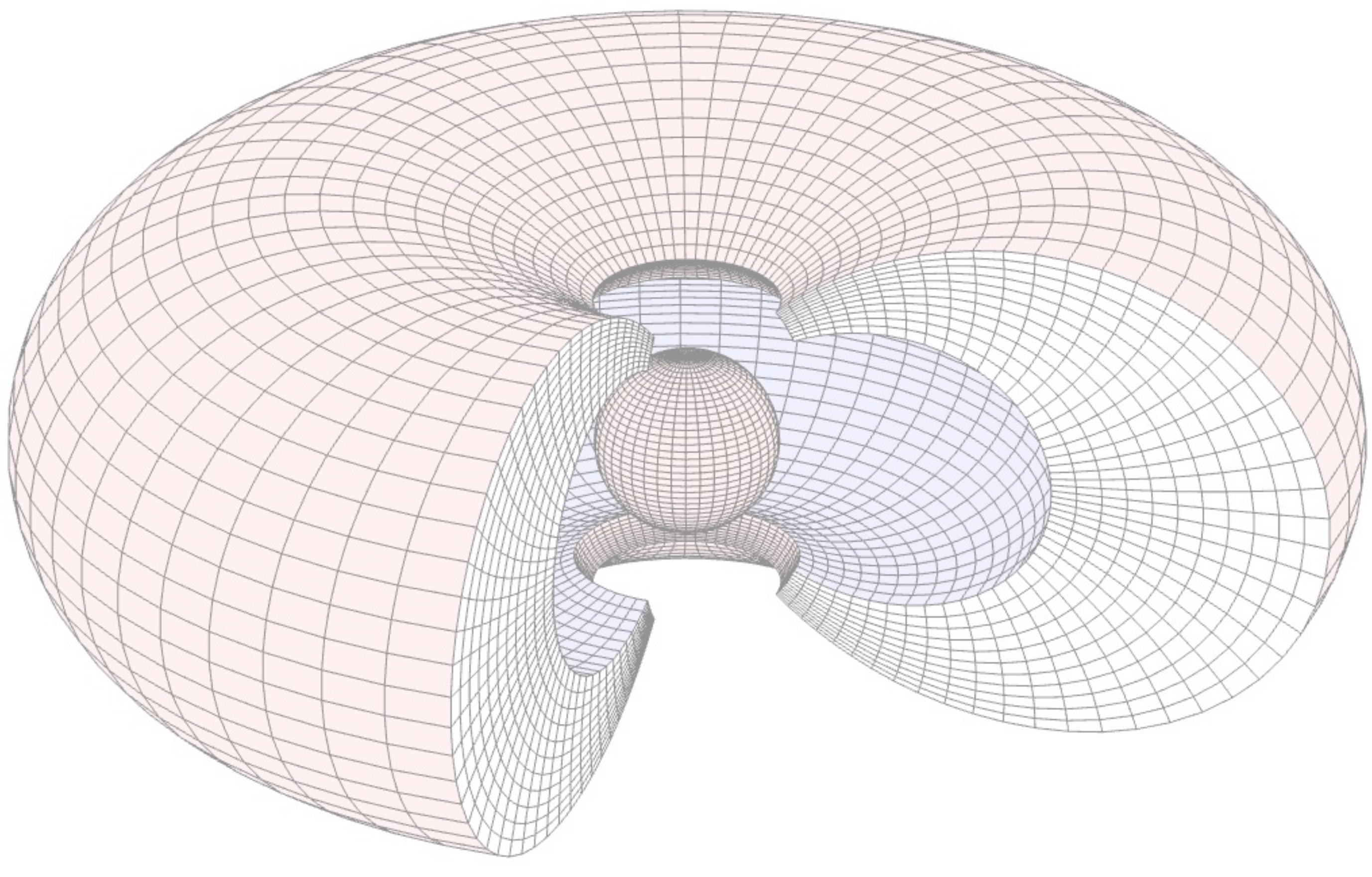
Topic 2

How to Simulate 1-D MHD Flow in Curvilinear Geometry?



Outline

- Review of MHD equations
- Coordinate Systems
- Scaling Factors
- Dispersion Relation
- Put everything together

The full set of Normalized MHD equations

Dimensionless Equations

Mass Equation
$$\frac{\partial}{\partial \tilde{t}}\tilde{\rho} = -\tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}})$$

Velocity Equation
$$\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = -\tilde{\nabla}\tilde{p} + \tilde{\mathbf{J}}\times\tilde{\mathbf{B}}$$

Pressure Equation
$$\tilde{p} = \frac{\beta}{2}\tilde{\rho}^{\gamma}$$

Faraday's Law
$$\frac{\partial\tilde{\mathbf{B}}}{\partial \tilde{t}} = -\tilde{\nabla}\times\tilde{\mathbf{E}}$$

Ohm's Law
$$\tilde{\mathbf{E}} = -\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}$$

Ampere's Law
$$\tilde{\mathbf{J}} = \tilde{\nabla}\times\tilde{\mathbf{B}}$$

Normalization Relations

$$L_0 = T_0U_0 \qquad U_0 = B_0/\sqrt{\mu_0\rho_0}$$

$$E_0 = U_0B_0 \qquad J_0 = \frac{B_0}{\mu_0L_0}$$

Interpretations:

- this set of normalized equation is called "dimensionless", e.g., u = 1 means the magnitude of the velocity is one Alfvén speed - very convenient in understanding the solutions
- We ignore the ***tilde*** in the following analysis and use the following set of MHD equations:

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\rho\mathbf{u}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J}\times\mathbf{B}$$

$$p = \frac{\beta_0}{2}\rho^{\gamma}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B}$$

$$\mathbf{J} = \nabla\times\mathbf{B}$$

Dimensionless Ideal MHD equations

This is the set of equations solved in mhd.m

1-D MHD equations

Plasma variables

$$\rho, u_x, u_y, p$$

Field variables

$$B_y, E_z, J_z$$

Put the equations together:

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot \rho \mathbf{u}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

$$p = \frac{\beta_0}{2}\rho^\gamma$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u_x)$$

$$\frac{\partial u_x}{\partial t} = -u_x\frac{\partial u_x}{\partial x} + \frac{1}{\rho}\left(-\frac{\partial p}{\partial x} - J_z B_y\right)$$

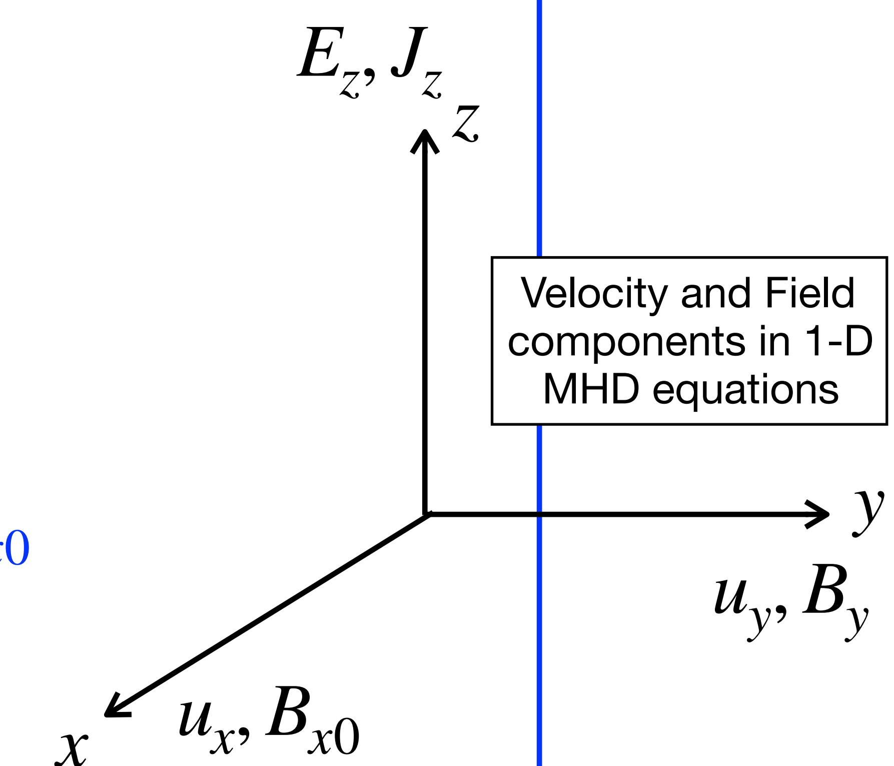
$$\frac{\partial u_y}{\partial t} = -u_x\frac{\partial u_y}{\partial x} + \frac{1}{\rho}J_z B_{x0}$$

$$p = \frac{\beta_0}{2}\rho^\gamma$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$E_z = -u_x B_y + u_y B_{x0}$$

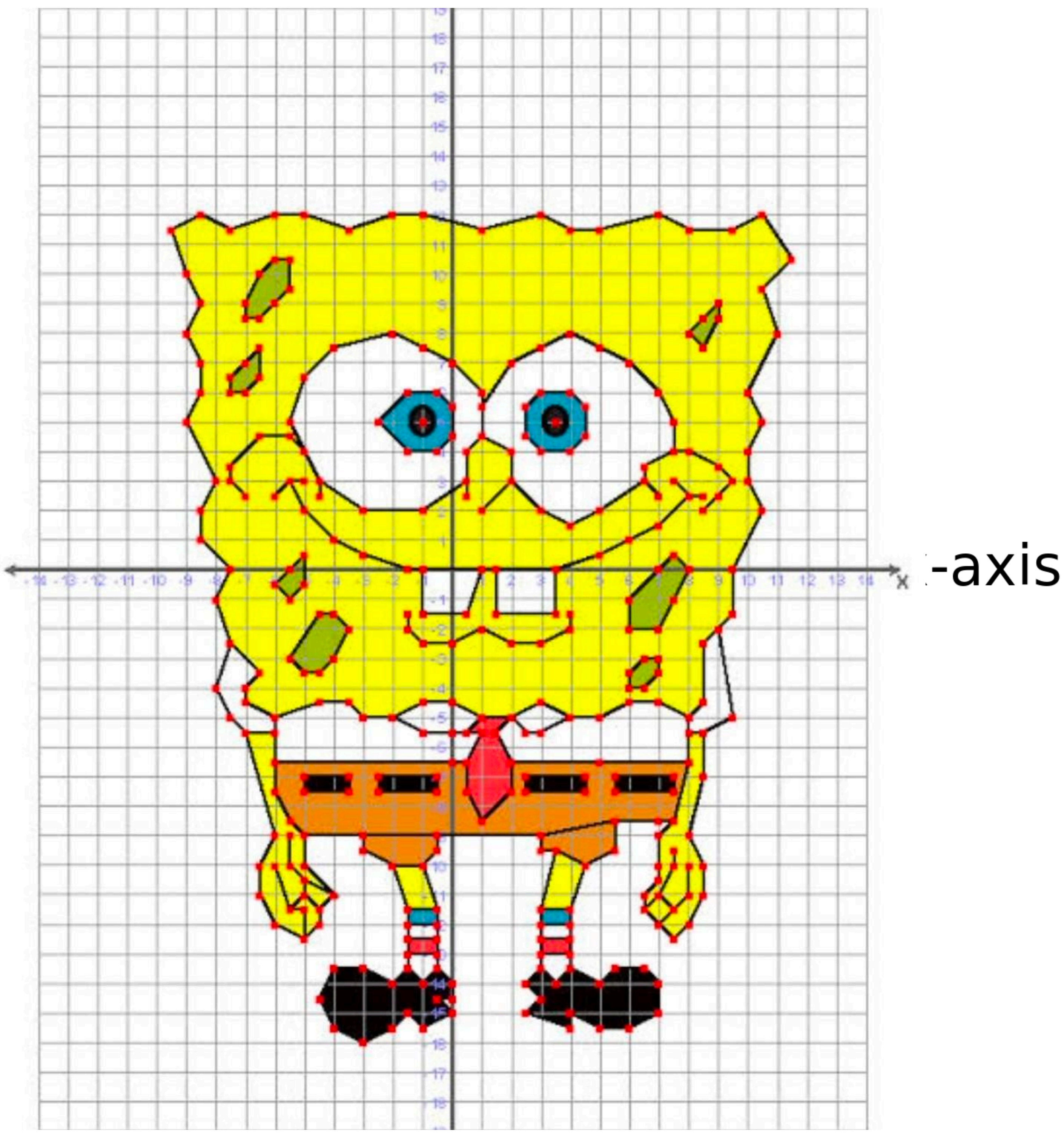
$$J_z = \frac{\partial B_y}{\partial x}$$



Velocity and Field components in 1-D MHD equations

Coordinate Systems

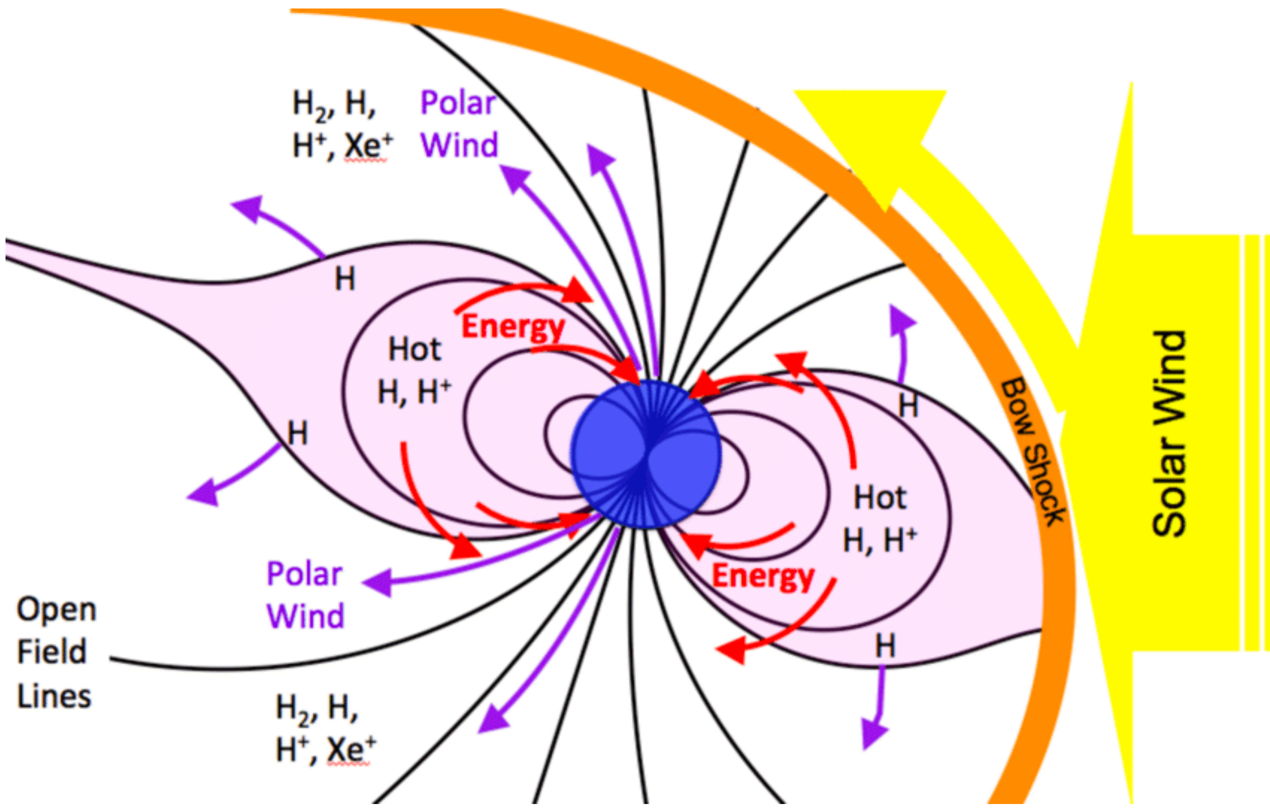
Cartesian



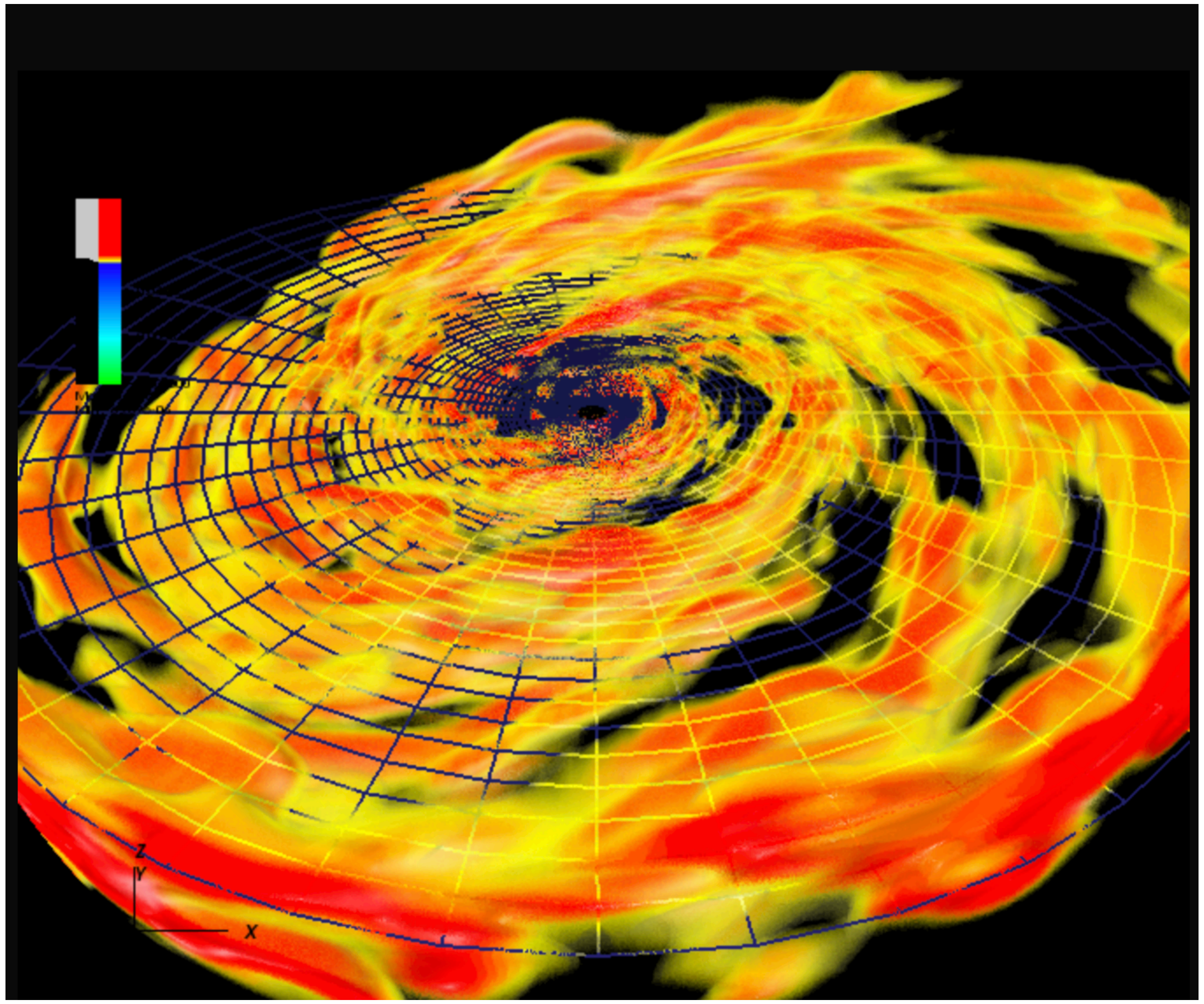
- We’ve learned how to develop a 1-D MHD code in **Cartesian** geometry.
- The 1-D MHD code can be extended to multi-dimensions (topic 3)
- But in space physics most of the problems are not suitable for the Cartesian geometry
- That’s why curvilinear systems are useful in a lot of space physics problems

Coordinate Systems

curvilinear systems in space plasmas



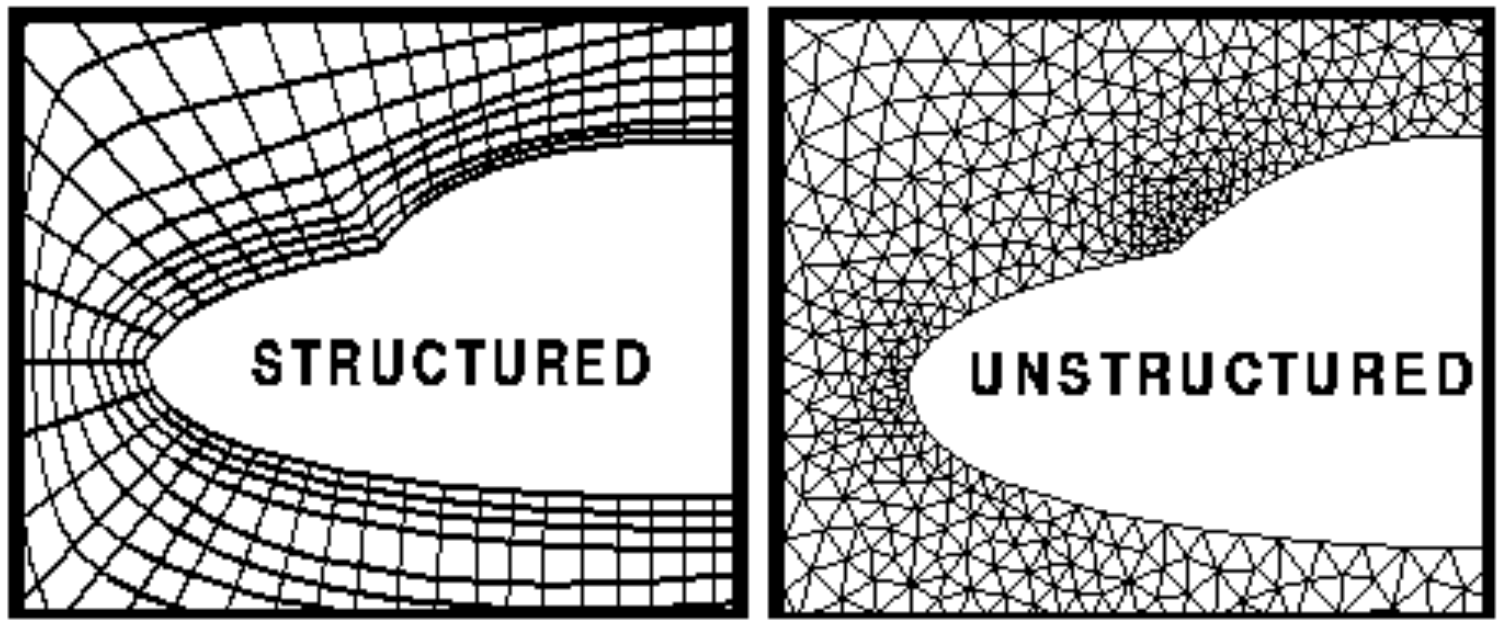
Dipole system
(Magnetospheres)



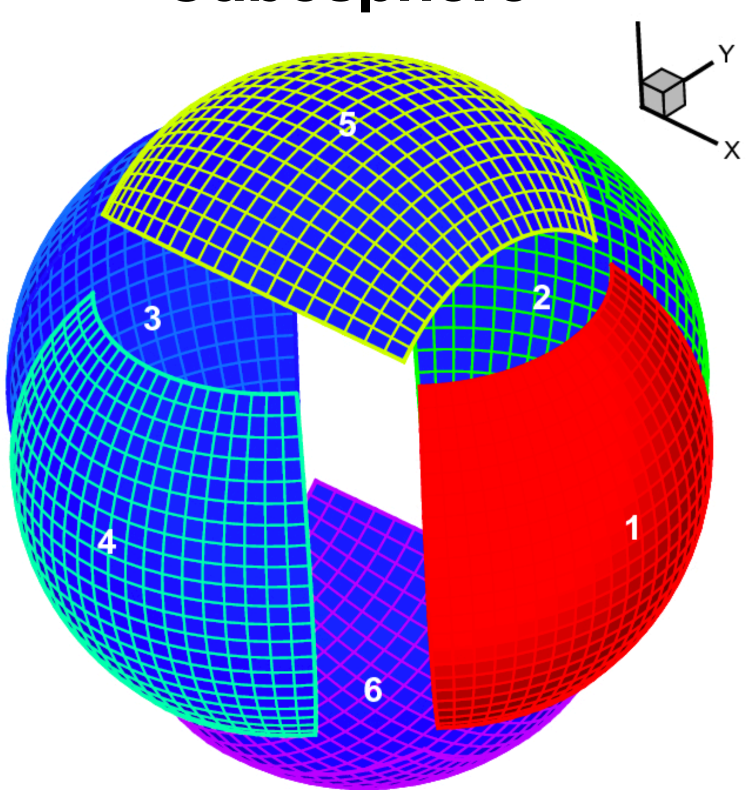
Cylindrical system
(Accretion discs)

Coordinate Systems

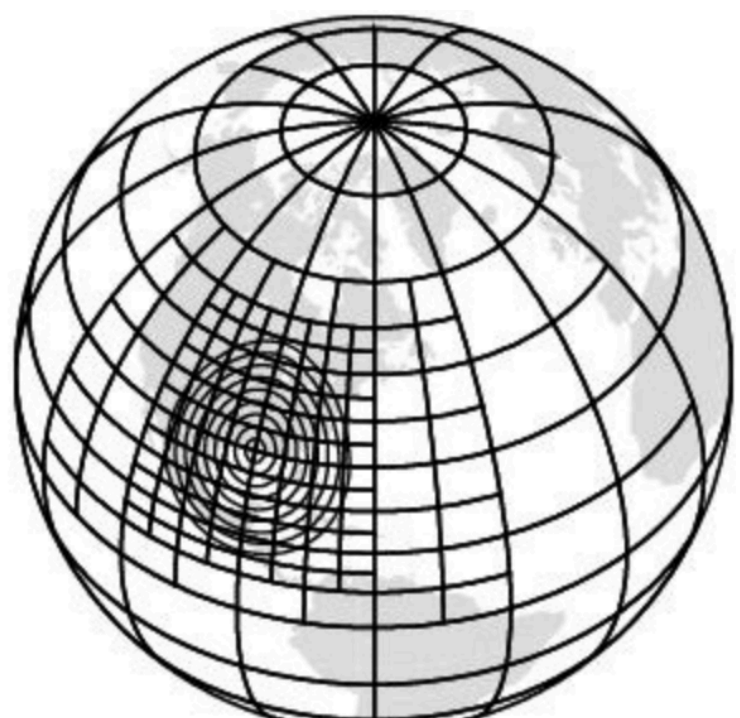
Unstructured grids



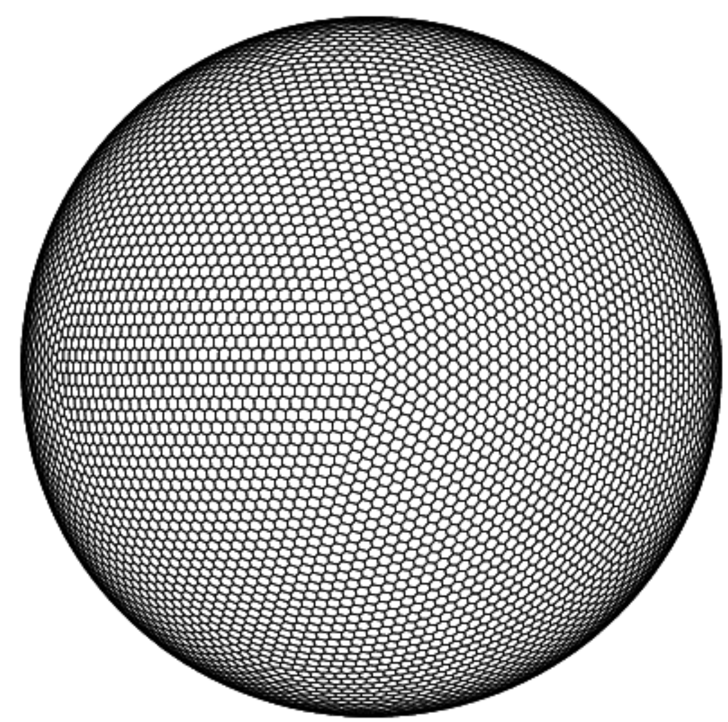
Cubesphere



Block grids

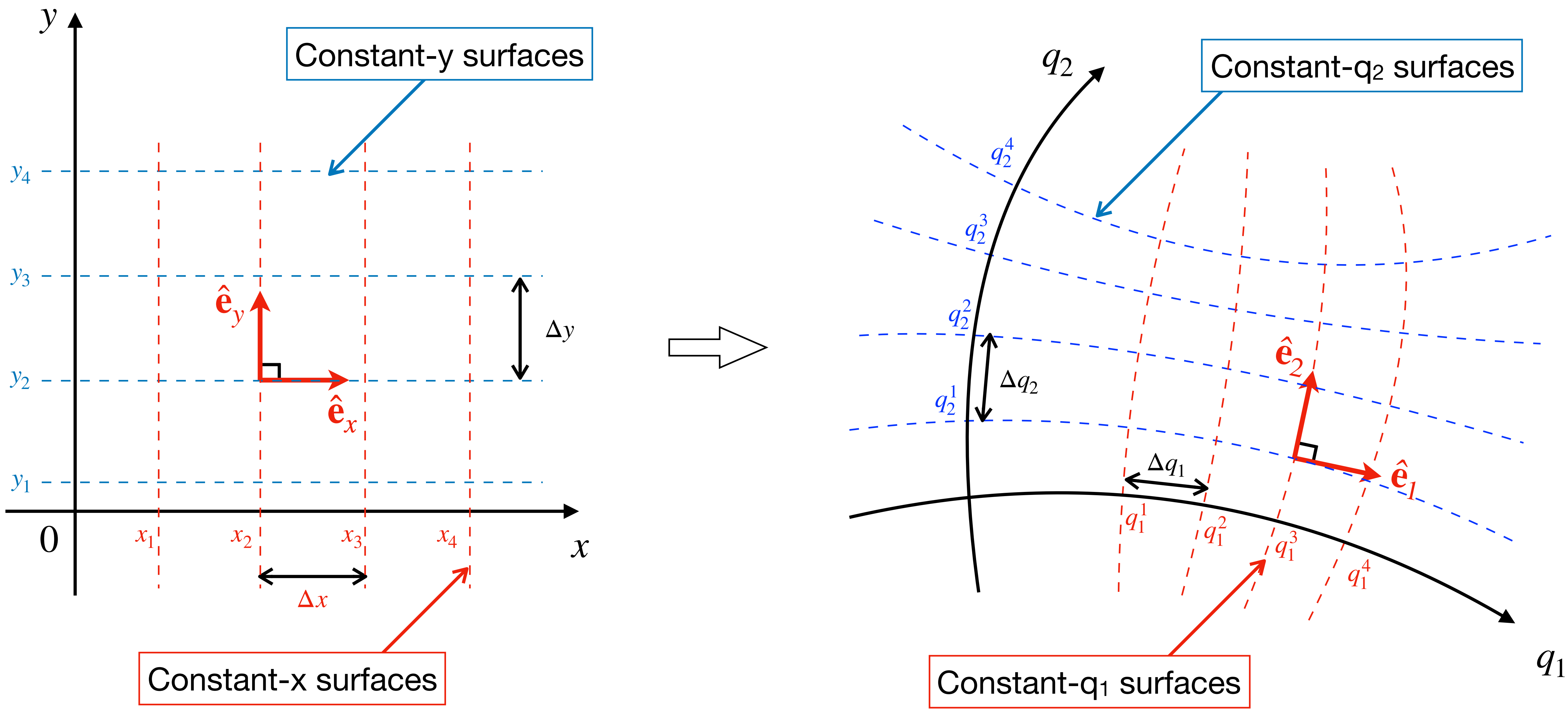


Voronoi grids



Coordinate Systems

From Cartesian to Curvilinear Coordinates

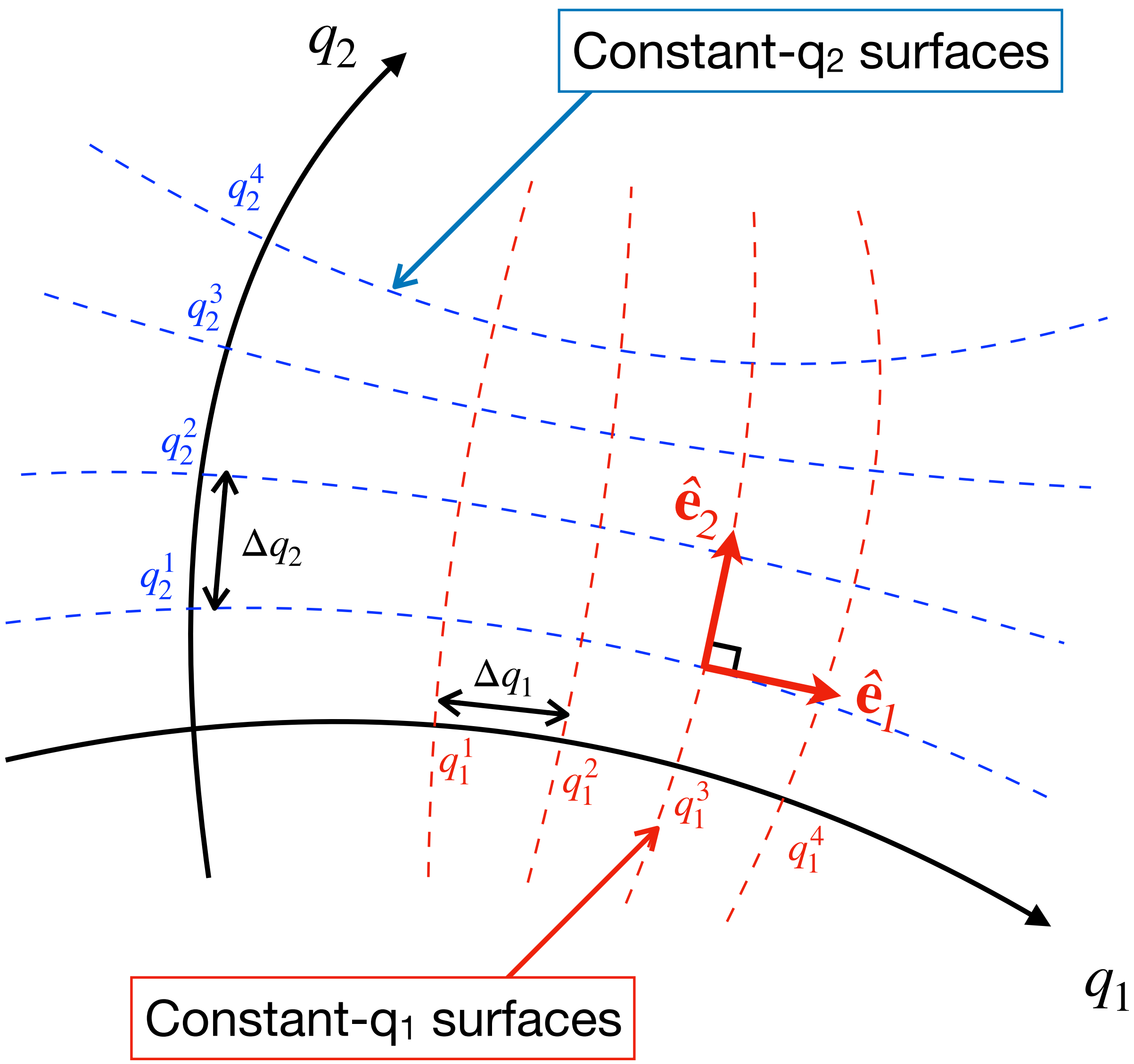


$\Delta S = |\Delta x|$

$\Delta s = h_1 \Delta q_1$ **h1 is** the scaling factor for length

Coordinate Systems

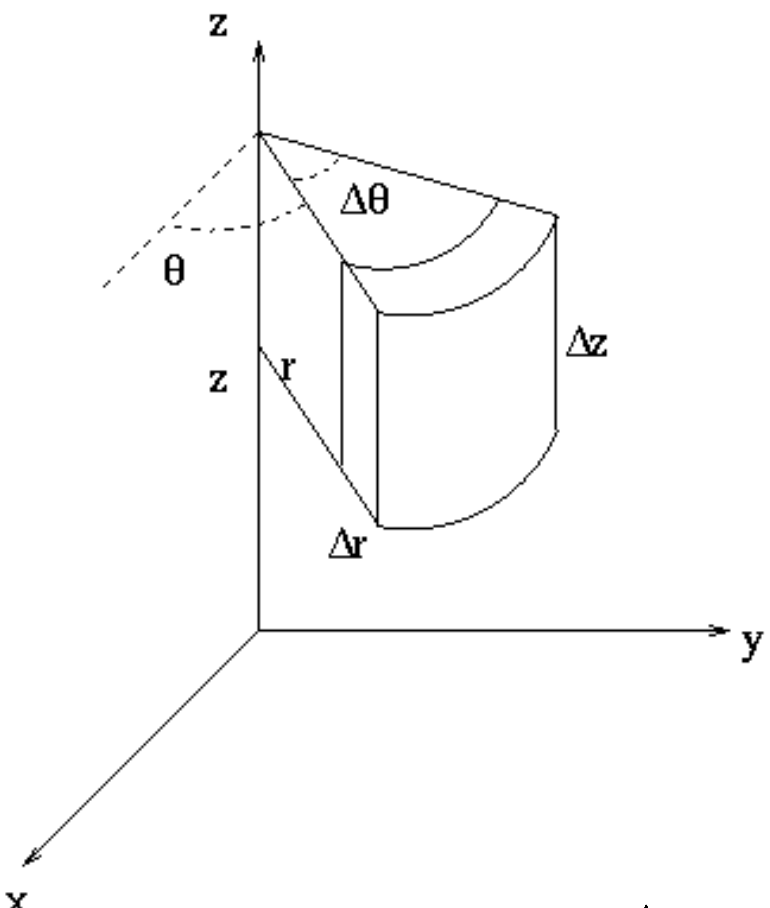
Scaling factors



$\Delta s = h_1 \Delta q_1$ **h1 is** the scaling factor for length

For example:

- Cylindrical geometry (r, θ, z)



$h_r = 1,\ h_\theta = r \longrightarrow$

$\Delta s_r = \Delta r$

$\Delta s_\theta = r\Delta\theta$

Operations in Coordinate Systems

Gradient, div, curl

The Jacobian is defined as the product of the scale factors:

$$J = h_1 h_2 h_3$$

And the gradient of a scalar variable is $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{\mathbf{e}}_3$

Divergence of a vector is now $\nabla \cdot \mathbf{u} = \frac{1}{J}\left[\frac{\partial}{\partial q_1}(u_1 h_2 h_3) + \frac{\partial}{\partial q_2}(u_2 h_3 h_1) + \frac{\partial}{\partial q_3}(u_3 h_1 h_2)\right]$

The curl of a vector is slightly more complicated

$$\nabla \times \mathbf{u} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(u_3 h_3) - \frac{\partial}{\partial q_3}(u_2 h_2)\right]\hat{\mathbf{e}}_1 + \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial q_3}(u_1 h_1) - \frac{\partial}{\partial q_1}(u_3 h_3)\right]\hat{\mathbf{e}}_2 + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(u_2 h_2) - \frac{\partial}{\partial q_2}(u_1 h_1)\right]\hat{\mathbf{e}}_3$$

For Cartesian geometry $h_1 = h_2 = h_3 \equiv 1$

So we get the familiar forms of the operators

Operations in Coordinate Systems

The curl and Laplacian operator

$$\nabla \times \mathbf{u} = \begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial q_2}(u_3h_3) - \frac{\partial}{\partial q_3}(u_2h_2)\right]\hat{\mathbf{e}}_1 + \frac{1}{h_3h_1}\left[\frac{\partial}{\partial q_3}(u_1h_1) - \frac{\partial}{\partial q_1}(u_3h_3)\right]\hat{\mathbf{e}}_2$$

$$+\frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}(u_2h_2) - \frac{\partial}{\partial q_2}(u_1h_1)\right]\hat{\mathbf{e}}_3$$

$$\nabla^2 f = \nabla\cdot(\nabla f) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

Scale Factors in Coordinate Systems

Examples

Cartesian coordinates: $(q_1, q_2, q_3) \to (x, y, z)$

$$h_x = 1 \quad h_y = 1 \quad h_z = 1$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial f}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\mathbf{e}}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\hat{\mathbf{e}}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\mathbf{e}}_z$$

Cylindrical coordinates: $(q_1, q_2, q_3) \to (\rho, \phi, z)$

$$h_\rho = 1 \quad h_\phi = r \quad h_z = 1$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}$$

$$\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \left(\frac{1}{r}\frac{\partial (r u_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi}\right)\hat{\mathbf{e}}_z$$

Spherical coordinates: $(q_1, q_2, q_3) \to (r, \theta, \phi)$

$$h_r = 1 \quad h_\theta = r \quad h_\phi = r\sin\theta$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta u_\phi)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{u} = \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta u_\phi)}{\partial \theta} - \frac{\partial u_\theta}{\partial \phi}\right)\hat{\mathbf{e}}_r + \frac{1}{r\sin\theta}\left(\frac{\partial u_r}{\partial \phi} - \frac{\partial (r\sin\theta u_\phi)}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{e}}_\phi$$

What about Dipole coordinates?

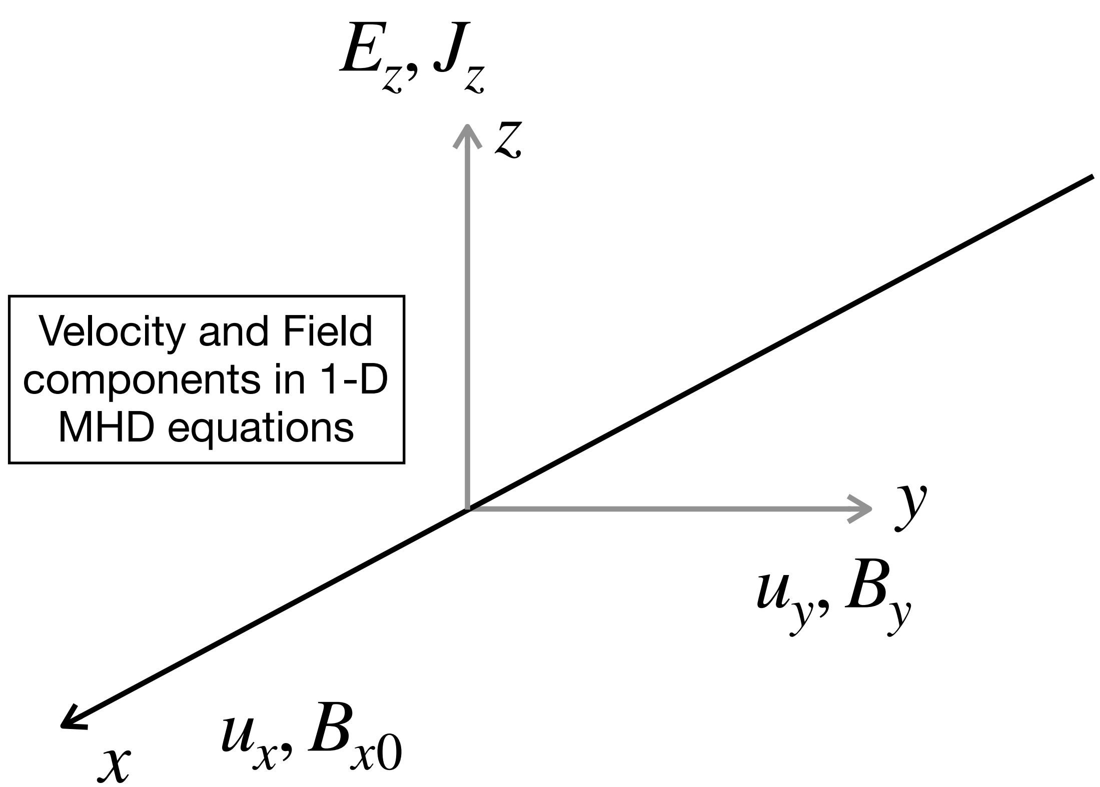
1-D MHD equations in Coordinate Systems

Let's do 1-D

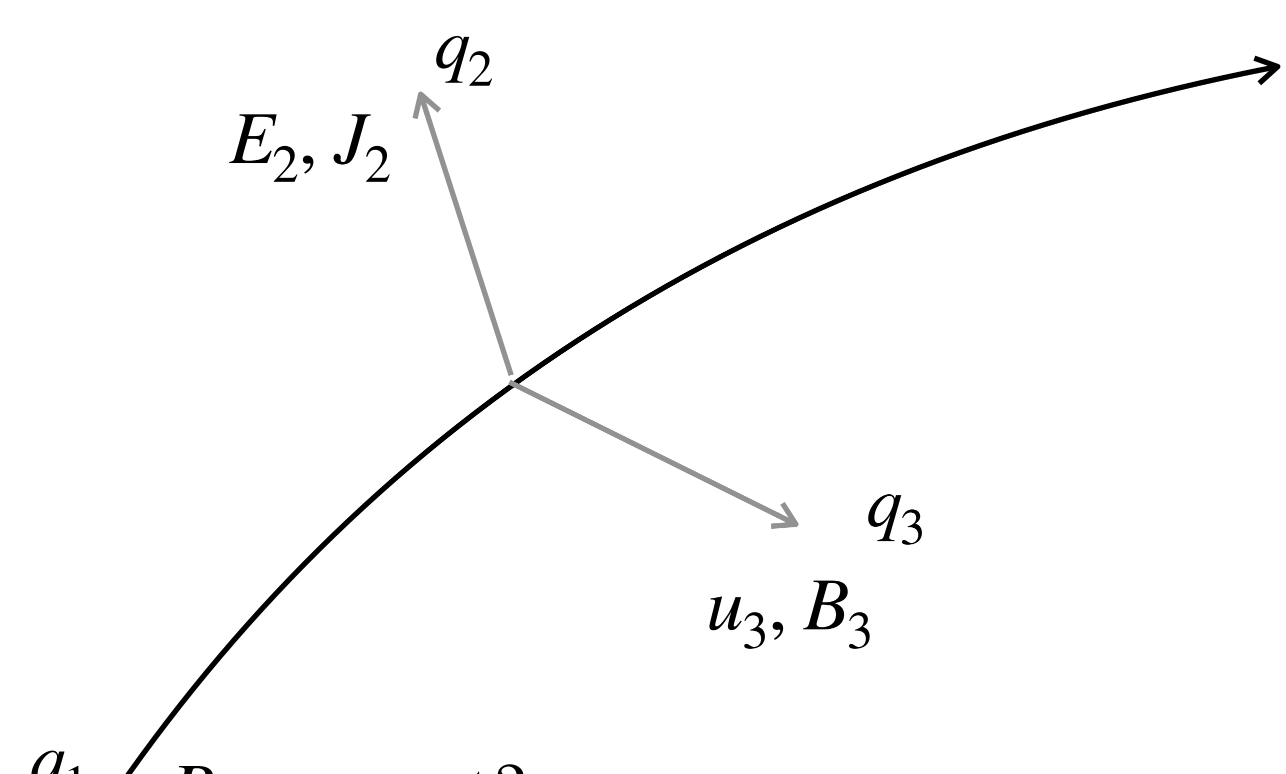
$$J = h_1 h_2 h_3 \qquad \nabla \cdot \mathbf{u} = \frac{1}{J}\frac{\partial}{\partial q_1}(u_1 h_2 h_3)$$

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{\mathbf{e}}_1 \qquad \nabla \times \mathbf{u} = +\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(u_2 h_2) - \frac{\partial}{\partial q_2}(u_1 h_1)\right]$$

Recall in the 1-D MHD code with cartesian geometry:



Let's do something similar with q1

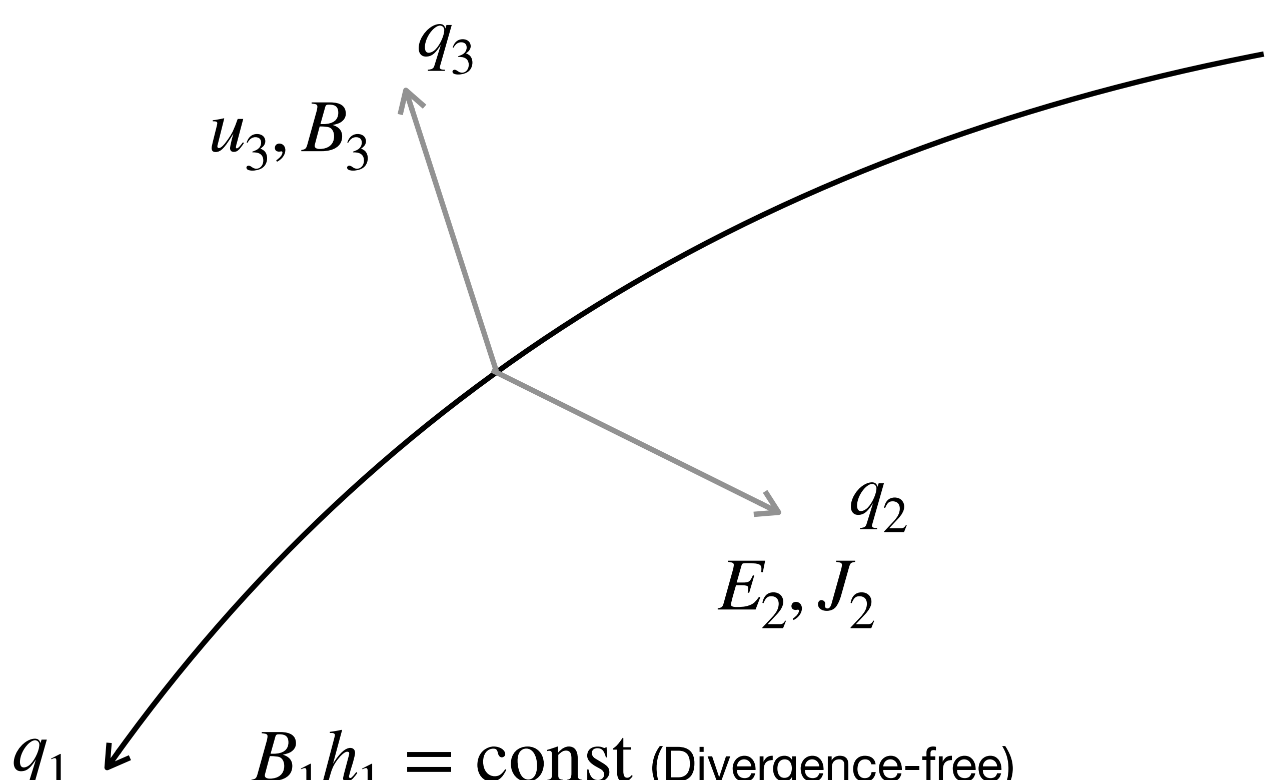


$B_1 = \text{const}?$

$u_1 = 0\ (why?)$

MHD equations in Dipole Coordinates

With this configuration, there are only a few equations left in 1-D MHD



$$B_1 h_1 = \text{const} \quad \text{(Divergence-free)}$$

$$u_1 = 0 \quad \text{(Alfvén-mode)}$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} \approx -(u_2 \hat{\mathbf{e}}_2) \times (B_1 \hat{\mathbf{e}}_1) = u_2 B_1^0 \hat{\mathbf{e}}_3$$

$$\mathbf{J} = \nabla \times \mathbf{B} = \frac{1}{h_1 h_2} \frac{\partial}{\partial q_1} (B_2 h_2) \hat{\mathbf{e}}_3$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = -\frac{1}{h_3 h_1} \frac{\partial}{\partial q_1} (E_3 h_3) \hat{\mathbf{e}}_2$$

$$= -\frac{1}{h_3 h_1} \frac{\partial}{\partial q_1} (u_2 B_1^0 h_3) \hat{\mathbf{e}}_2$$

Here we assume small perturbations

The momentum equation becomes:

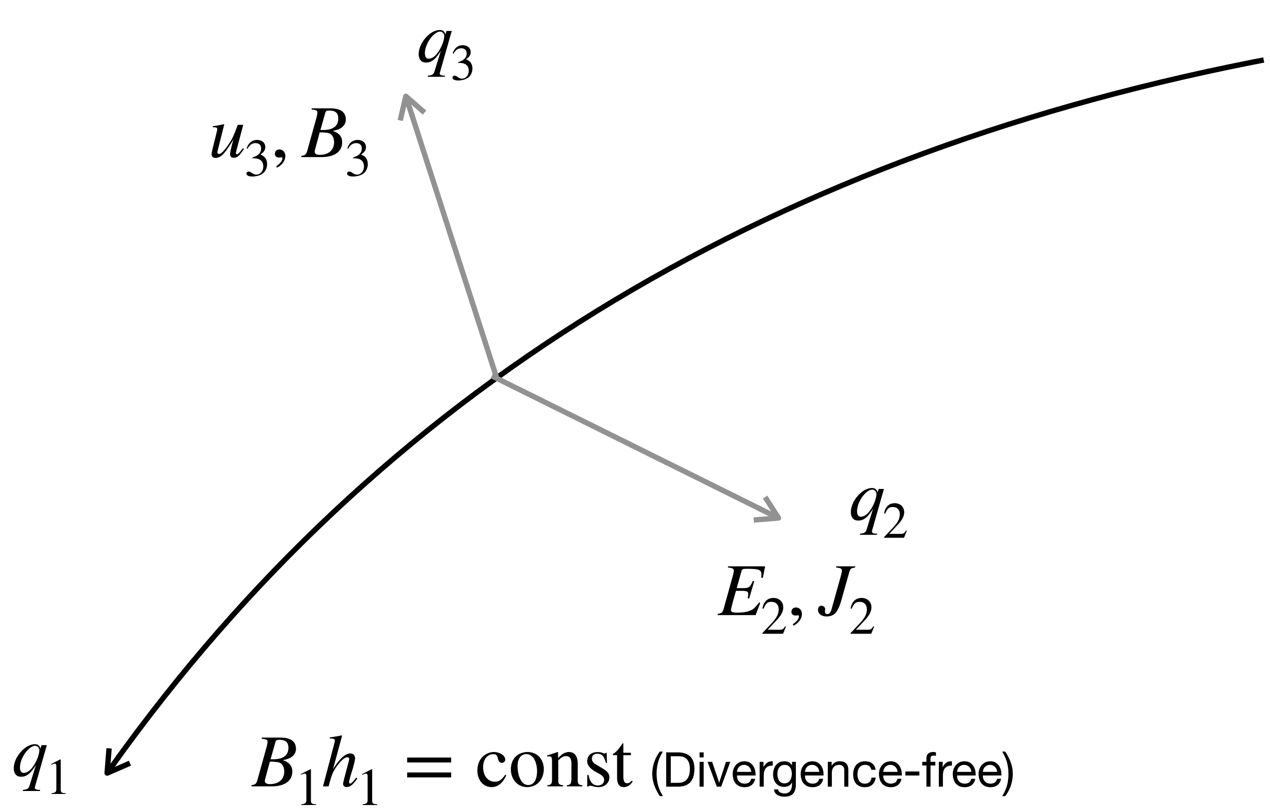
Cold plasma

$$\underbrace{\rho \frac{D\mathbf{u}}{Dt}}_{\sim \rho_0 \frac{\partial u_2}{\partial t}} = -\cancel{\nabla p} + \mathbf{J} \times \mathbf{B} \longrightarrow \rho_0 \frac{\partial u_3}{\partial t} = -J_2 B_1^0$$

Linear perturbations

MHD equations in Dipole Coordinates

With this configuration, there are only a few equations left in 1-D MHD



$B_1 h_1 = \text{const}$ (Divergence-free)

$u_1 = 0$ (Alfvén-mode)

Mass equation - not needed (why?)

Pressure equation - not needed (why?)

Velocity equation - q_2 component

$$\rho_0 \frac{\partial u_3}{\partial t} = - J_2 B_1^0$$

Faraday's law - q2 component

$$\frac{\partial B_3}{\partial t} = \frac{1}{h_1 h_2} \frac{\partial}{\partial q_1} (E_2 h_2) \hat{\mathbf{e}}_3$$

Ohm's law - q3 component

$$E_2 = - u_3 B_1^0$$

Ampere's - q3 component

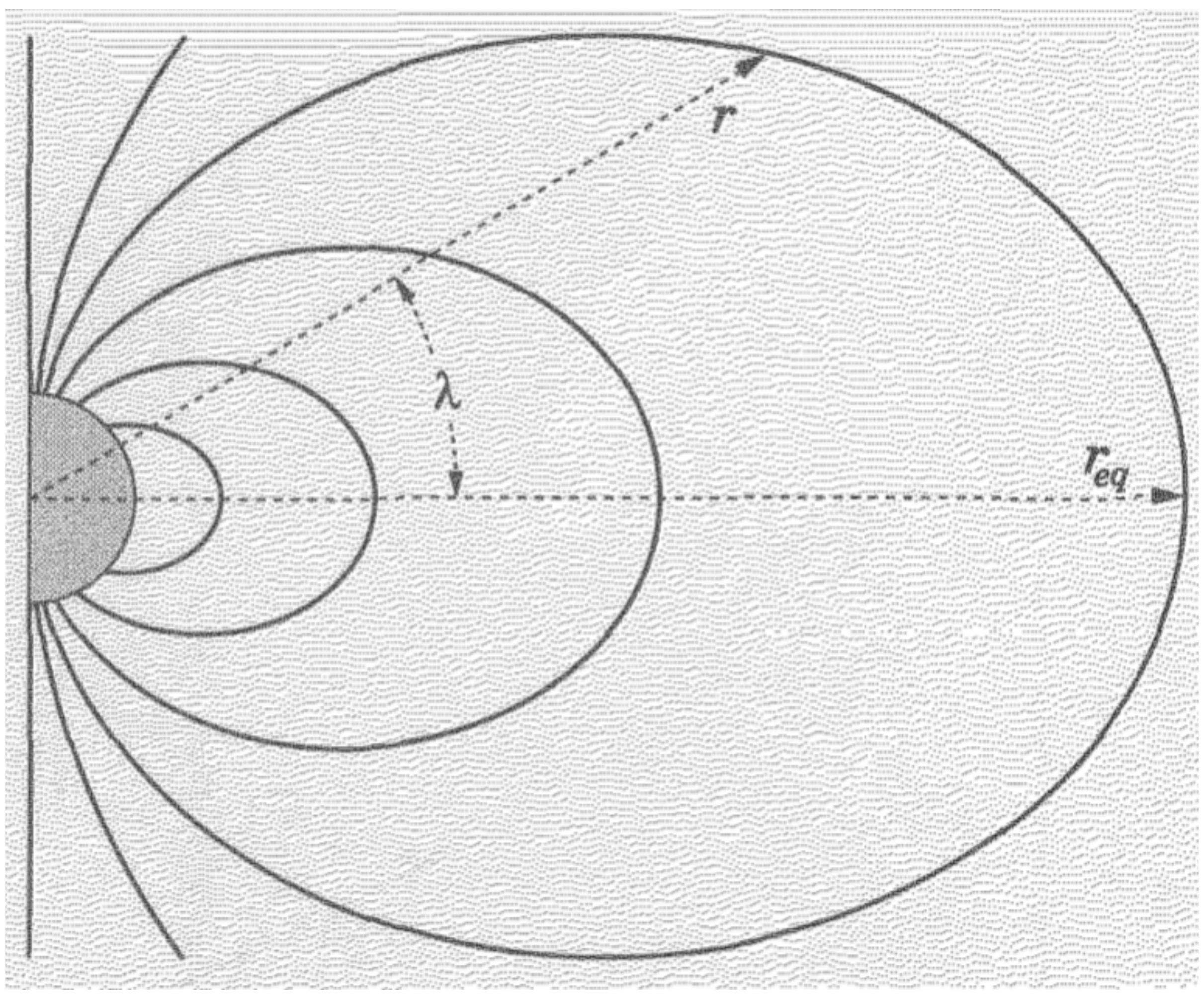
$$J_2 = - \frac{1}{h_3 h_1} \frac{\partial}{\partial q_1} (B_3 h_3)$$

Dipole Coordinates

Dipole field - the concept of L

At distance not too far from the Earth's surface, the magnetic field can be approximated by a ***dipole*** field. Introducing the concept of Earth's dipole moment, Me = 8.05e22 Am^2, and choosing a spherical (r, theta, phi) coordinate system: we have

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{M_e}{r^3}\left(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda\right)$$



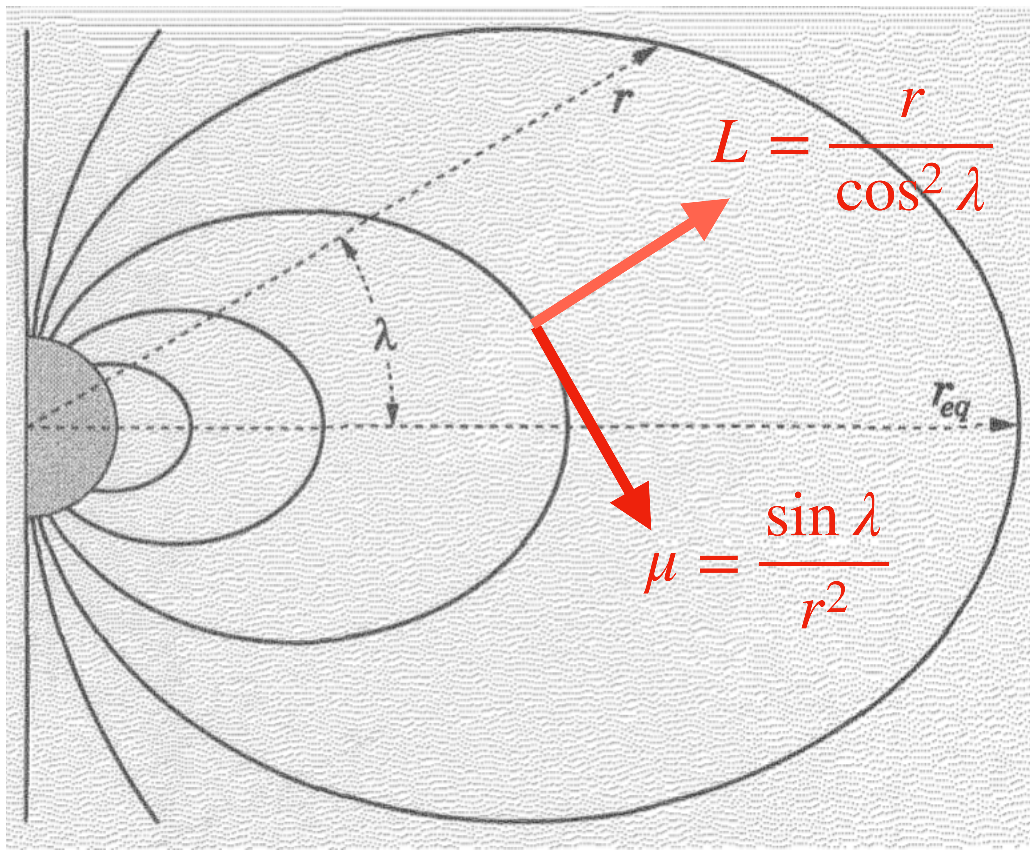
The strength of the geomagnetic field is

$$B = \frac{\mu_0}{4\pi}\frac{M_e}{r^3}\sqrt{1 + 3\sin^3\lambda}$$

(dipole component)

Dipole Coordinates

the concept of L



In order to get the field line equation $r = f(\lambda)$

The line of force equation is $d\mathbf{s} \times \mathbf{B} = \mathbf{0}$ i.e., ds is along B

We get the following equation:

$$(dr, rd\lambda) \times (B_r, B_\lambda) = 0$$

$$\longrightarrow \frac{dr}{B_r} = \frac{rd\lambda}{B_\lambda}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{M_e}{r^3}\left(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda\right)$$

Re-arrange

$$\longrightarrow \frac{dr}{r} = \frac{2\sin\lambda d\lambda}{\cos\lambda} = \frac{2d(\cos\lambda)}{\cos\lambda}$$

Integrate:

$$\longrightarrow \boxed{r(\lambda) = r_{eq}\cos^2\lambda}$$

Field line equation

Define:

$$r_{eq} = \frac{r}{\cos^2\lambda} \equiv L$$

i.e., for a given field line, L is constant

Dipole coordinates

$\mu = \frac{\sin\lambda}{r^2}$ (along a field line)

$L = \frac{r}{\cos^2\lambda}$ (across field lines)

ϕ (azimuthal)

Orthogonality of the Dipole Coordinates

$$\nabla\mu \cdot \nabla L \equiv 0$$

$$\mu = \frac{\sin\lambda}{r^2}$$
$$L = \frac{r}{\cos^2\lambda}$$

Since $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{3}\frac{\partial f}{\partial \lambda}\hat{\mathbf{e}}_\lambda$

$$\longrightarrow \quad \nabla\mu = \frac{\partial}{\partial r}\left(\frac{\sin\lambda}{r^2}\right)\hat{\mathbf{e}}_r + \frac{1}{3}\frac{\partial}{\partial\lambda}\left(\frac{\sin\lambda}{r^2}\right)\hat{\mathbf{e}}_\lambda$$

$$= -\frac{2\sin\lambda}{r^3}\hat{\mathbf{e}}_r + \frac{\cos\lambda}{r^3}\hat{\mathbf{e}}_\lambda$$

$$= \frac{1}{r^3}\left(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda\right)$$

$$\longrightarrow \quad \nabla L = \frac{\partial}{\partial r}\left(\frac{r}{\cos^2\lambda}\right)\hat{\mathbf{e}}_r + \frac{1}{3}\frac{\partial}{\partial\lambda}\left(\frac{r}{\cos^2\lambda}\right)\hat{\mathbf{e}}_\lambda$$

$$= \frac{1}{\cos^3\lambda}\left(\cos\lambda\hat{\mathbf{e}}_r + 2\sin\lambda\hat{\mathbf{e}}_\lambda\right)$$

$$\longrightarrow \nabla\mu \cdot \nabla L \equiv 0$$

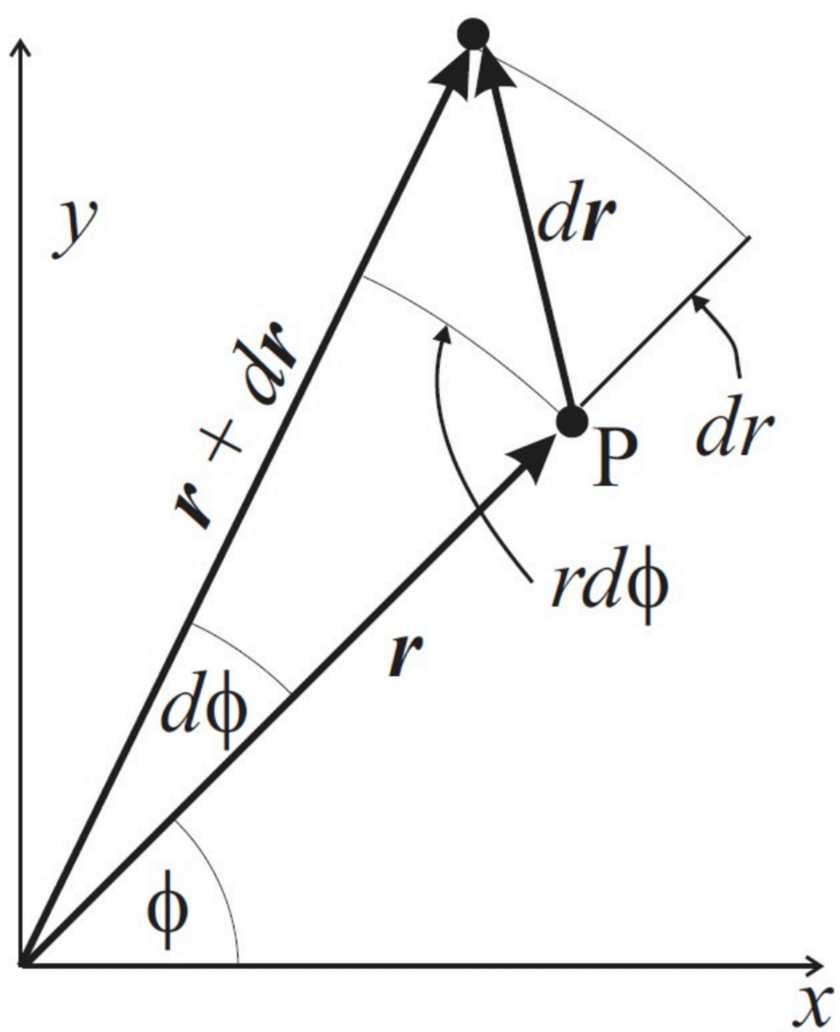
So the dipole coordinates are orthogonal! **What about phi?**

Scale Factors of the Dipole Coordinates

$$h_\mu \, , h_L \, , h_\phi$$

In a general curvilinear coordinate system (q_1, q_2, q_3)

The scaling factors are defined as $\mathbf{h}_i = \dfrac{\partial s}{\partial q_i}$ which means the “correction” to the length of a differential arc along coordinate q_i



- When theta is constant, an increase in r gives the increase in the arc length dr
- When r is constant, an increase in theta gives the increase in the arc length r*dtheta
- So the total increase in arc length would be

$$d\mathbf{s} = dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta$$

$$\longrightarrow \quad h_r = \left| \frac{\partial s}{\partial r} \right| = 1 \qquad h_\theta = \left| \frac{\partial \mathbf{s}}{\partial \theta} \right| = r$$

Scale Factors of the Dipole Coordinates

How to calculate h_μ , h_L

So in the dipole coordinates, the scale factors are calculated as $h_\mu^2 = \left(\frac{d\mathbf{s}}{d\mu}\right)^2$

In Cartesian geometry, $d\mathbf{s} = dx\hat{\mathbf{e}}_x + dy\hat{\mathbf{e}}_y + dz\hat{\mathbf{e}}_z$

So you would think about calculating the scale factor as

$$h_\mu^2 = \left(\frac{d\mathbf{s}}{d\mu}\right)^2 = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2$$

But the problem is that the dipole coordinate mu is much easier in the r, lambda system:

$$\mu = \frac{\sin\lambda}{r^2}$$

So let’s use the expression of ds in the r, lambda system and calculate h_mu as:

$$d\mathbf{s} = dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta \longrightarrow h_\mu^2 = \left(\frac{d\mathbf{s}}{d\mu}\right)^2 = \left(\frac{\partial r}{\partial \mu}\right)^2 + \left(\frac{r\partial\lambda}{\partial \mu}\right)^2$$

$$\boxed{\begin{aligned} \mu &= \frac{\sin\lambda}{r^2} \\ L &= \frac{r}{\cos^2\lambda} \end{aligned}}$$

Scale Factors of the Dipole Coordinates

How to calculate h_μ , h_L

Now we have $h_\mu^2 = \left(\frac{\partial r}{\partial \mu}\right)^2 + \left(\frac{r\partial \lambda}{\partial \mu}\right)^2$ the task is to compute $\frac{\partial r}{\partial \mu}$ and $\frac{\partial \lambda}{\partial \mu}$

Let's start with the definition of the dipole coordinates: $\mu = \frac{\sin\lambda}{r^2}$ $L = \frac{r}{\cos^2\lambda}$

$$L = \frac{r}{\cos^2\lambda} = \frac{r}{1-\sin^2\lambda} \xrightarrow{\mu = \frac{\sin\lambda}{r^2}} L = \frac{r}{1-\mu^2 r^4} \longrightarrow \frac{r}{L} = 1-\mu^2 r^4$$

Take the partial derivative of the r/L equation:

$$\frac{r}{L} = 1-\mu^2 r^4 \xrightarrow{\frac{\partial}{\partial \mu}} \frac{1}{L}\frac{\partial r}{\partial \mu} = -\left(2\mu r^4 + 4\mu^2 r^3 \frac{\partial r}{\partial \mu}\right) \longrightarrow \frac{\partial r}{\partial \mu} = -\frac{2\mu r^4}{1/L + 4\mu^2 r^3}$$

$$\xrightarrow[L = \frac{r}{\cos^2\lambda}]{\mu = \frac{\sin\lambda}{r^2}} \frac{\partial r}{\partial \mu} = -\frac{2r^4\frac{\sin\lambda}{r}}{\frac{\cos^2\lambda}{r} + 4r^3\frac{\sin^2\lambda}{r^4}} = -\frac{2r^3\sin\lambda}{1+3\sin^2\lambda}$$

Scale Factors of the Dipole Coordinates

How to calculate h_μ , h_L

Similarly, we have $\dfrac{\partial\lambda}{\partial\mu} = \dfrac{r^2\cos\lambda}{1+3\sin^2\lambda}$ *(HW exercises)*

Now, the first scale factor h_mu is calculated as

$$h_\mu^2 = \left(\frac{\partial r}{\partial\mu}\right)^2 + \left(\frac{r\partial\lambda}{\partial\mu}\right)^2 = \left(\frac{2r^3\sin\lambda}{1+3\sin^2\lambda}\right)^2 + \left(\frac{r^2\cos\lambda}{1+3\sin^2\lambda}\right)^2 = \frac{r^6}{1+3\sin^2\lambda}$$

$$\longrightarrow \boxed{h_\mu = \frac{r^3}{\sqrt{1+3\sin^2\lambda}}}$$

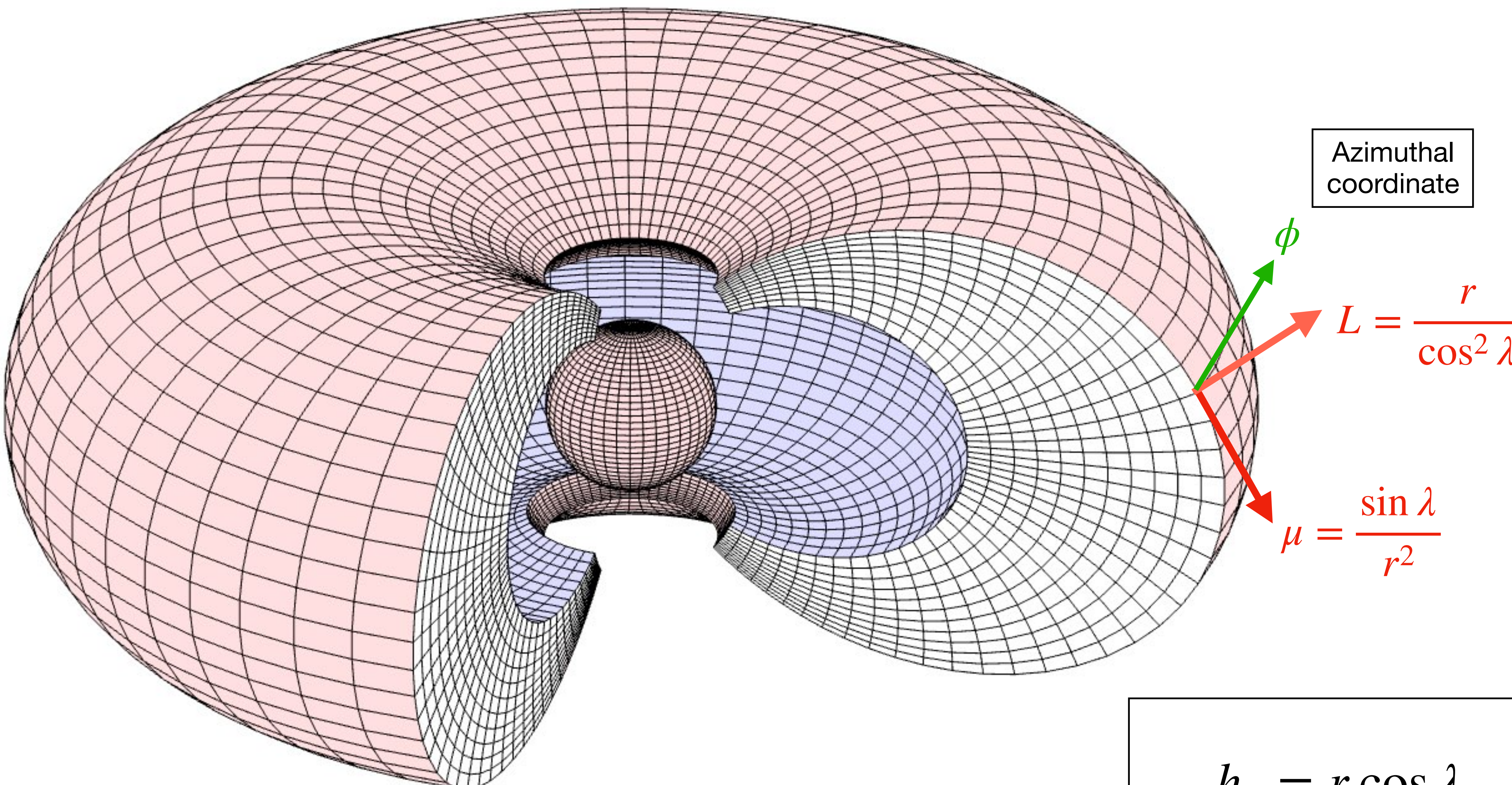
With a similar amount of algebra (freshman level calculus - leave as HW exercises)

$$\longrightarrow \boxed{h_L = \frac{\cos\lambda^3}{\sqrt{1+3\sin^2\lambda}}}$$

What about the phi component?

Scale Factors of the Dipole Coordinates

What about h_ϕ



3-D view of dipole geometry

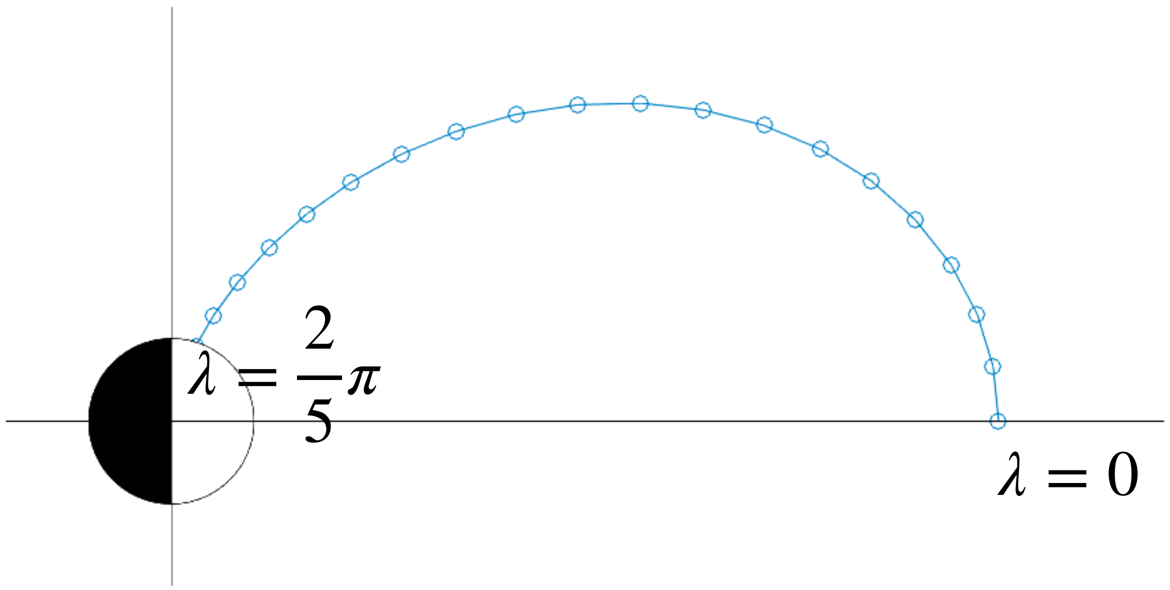
$$\boxed{h_\phi = r \cos \lambda}$$

The same as in spherical coordinates!

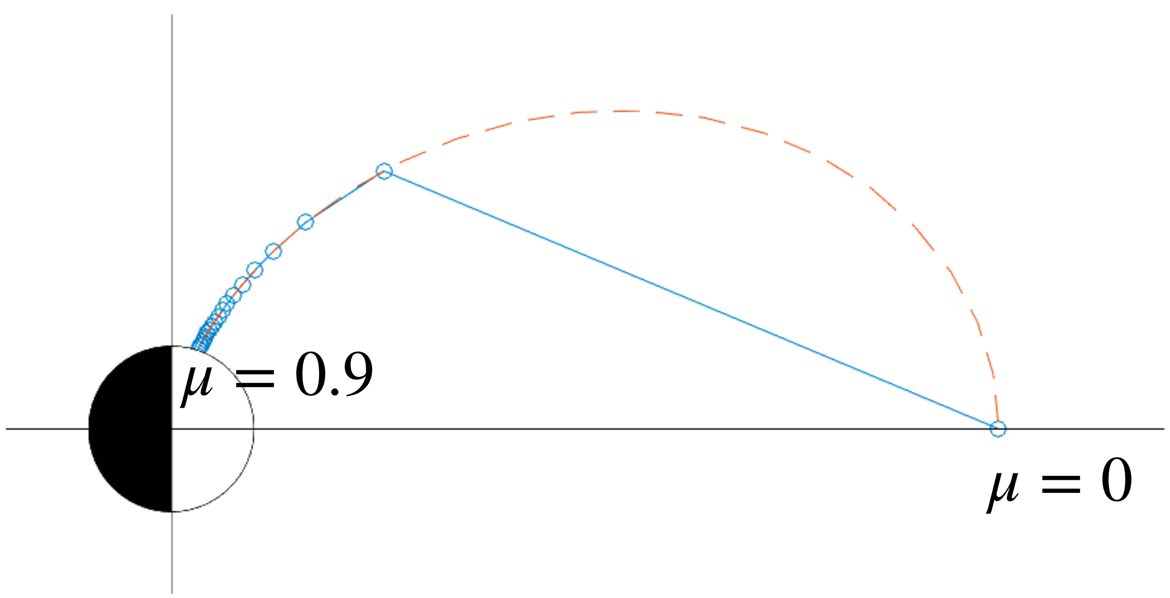
Discretization of the Dipole Coordinates

The problem with constant mu

Uniform discretization in theta

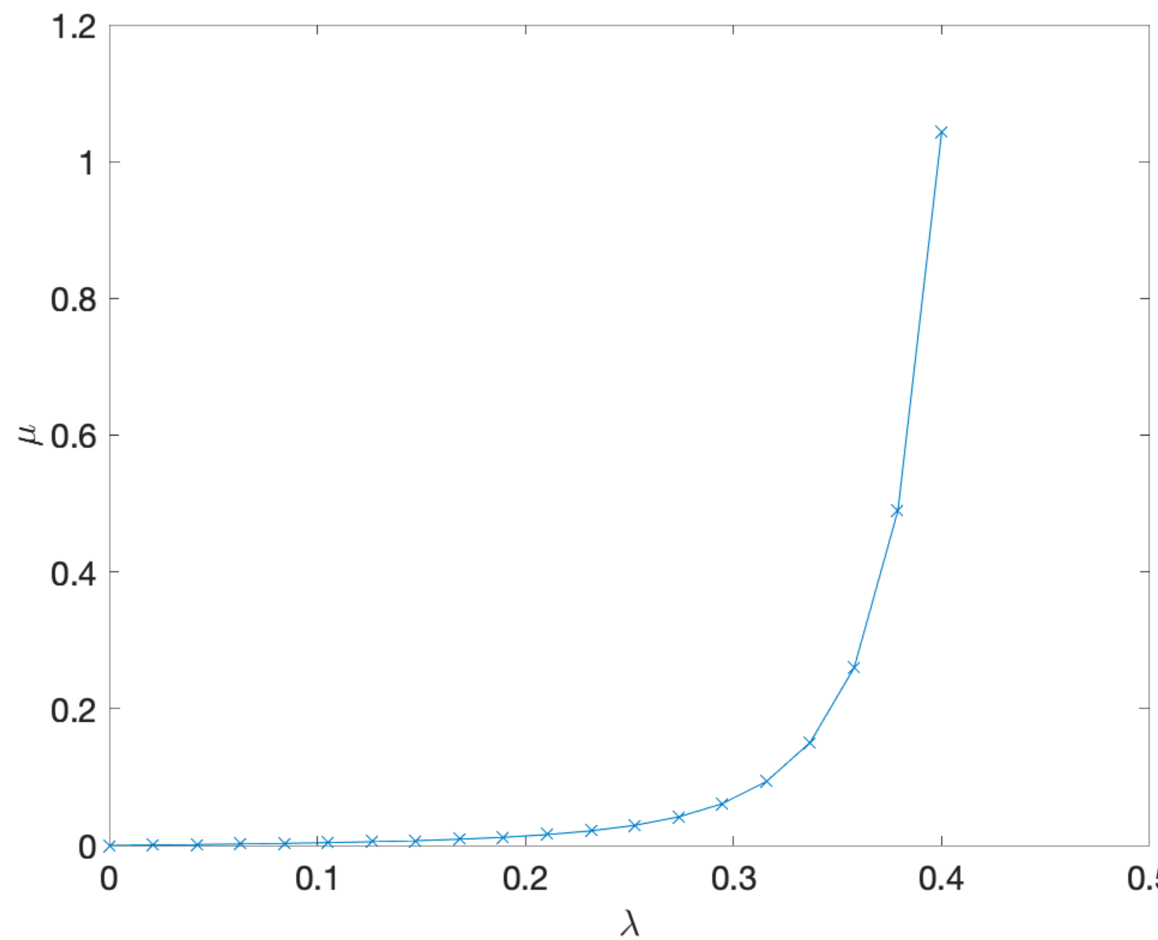


Uniform discretization in mu



The problem in the uniform-mu discretization is due to the fact that the mu dependence is highly non-linear

$$\mu = \frac{\sin\lambda}{r^2} \xrightarrow{\text{uniform in } \mu} \text{non uniform in } \lambda$$



Discretization of the Dipole Coordinates

The problem with constant mu

So when using the dipole coordinates, we shift to another parallel coordinate instead of mu

$$\boxed{\zeta = \sin\lambda} \longrightarrow \mu = \frac{\zeta}{r^2} \qquad (1)$$

Now the new dipole coordinates becomes (ζ, L, ϕ):

We need to compute a new scale factor

$$h_\zeta = \left| \frac{d\mathbf{s}}{d\mu} \cdot \frac{d\mu}{d\zeta} \right| \xrightarrow{\text{Need}} \mu(\zeta, L)$$

Since $L = \dfrac{r}{\cos^2\lambda} = \dfrac{r}{1 - \sin^2\lambda} = \dfrac{r}{1 - \zeta^2} \longrightarrow r = L(1 - \zeta^2) \qquad (2)$

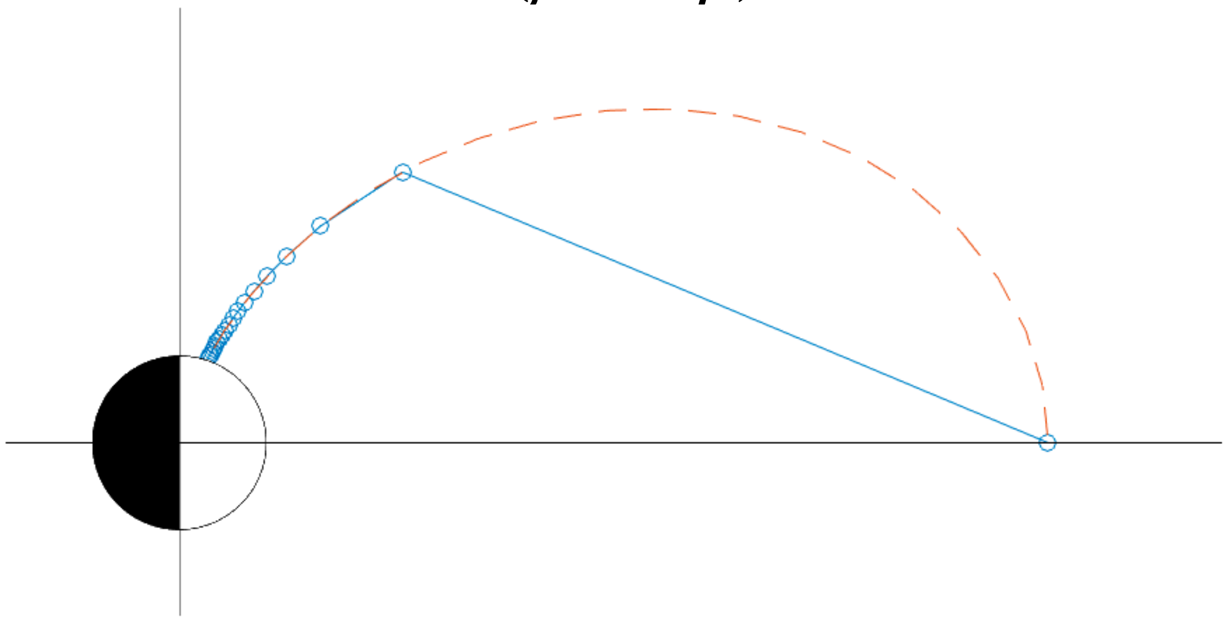
Use 1 and 2: $\mu = \dfrac{\zeta}{L^2(1 - \zeta^2)^2} \longrightarrow \dfrac{\partial \mu}{\partial \zeta} = \dfrac{1 + 3\zeta^2}{L^2(1 - \zeta^2)^3}$

So $h_\zeta = h_\mu \dfrac{\partial \mu}{\partial \zeta} = \dfrac{r^3}{L^3} \dfrac{\sqrt{1 + 3\zeta^2}}{(1 - \zeta^2)^3} \xrightarrow{L = \frac{r}{1 - \zeta^2}} \boxed{h_\zeta = L\sqrt{1 + 3\zeta^2}}$

Summary of the Dipole Coordinates

Standard dipole system

(μ, L, ϕ)

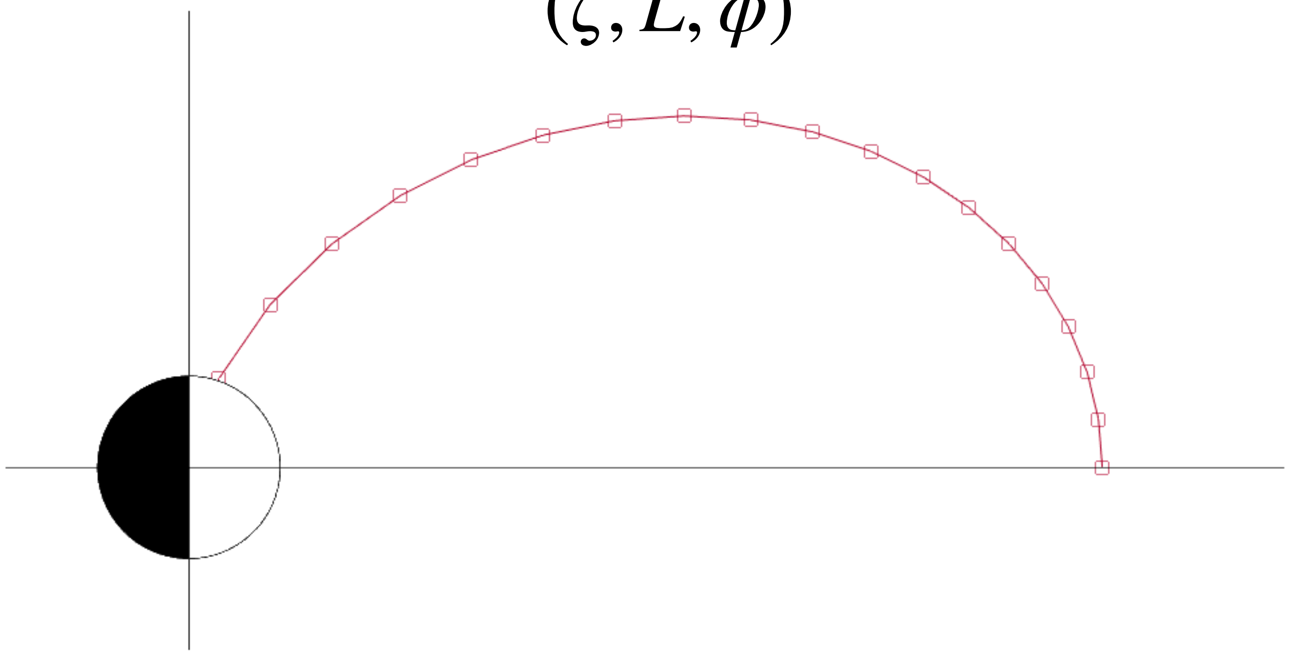


$$\mu = \frac{\sin\lambda}{r^2} \qquad h_\mu = \frac{r^3}{\sqrt{1 + 3\sin^2\lambda}}$$

$$L = \frac{r}{\cos^2\lambda} \qquad h_L = \frac{\cos\lambda^3}{\sqrt{1 + 3\sin^2\lambda}}$$

Modified dipole system

(ζ, L, ϕ)



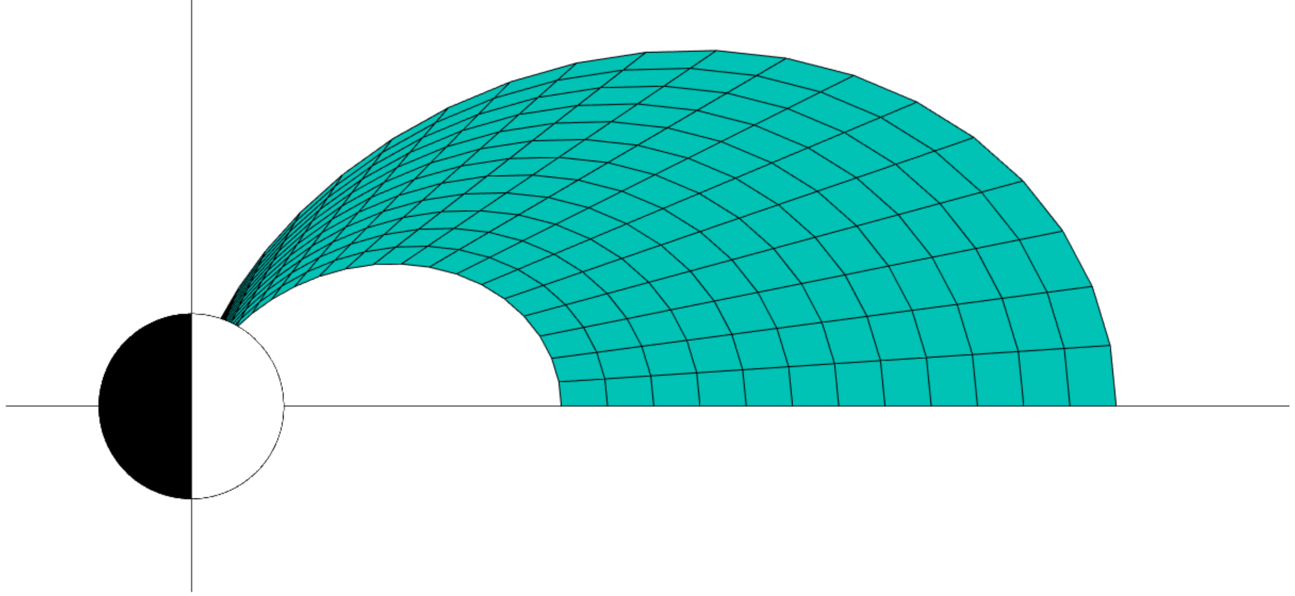
$$\zeta = \sin\lambda \qquad h_\zeta = L\sqrt{1 + 3\zeta^2}$$

$$L = \frac{r}{1 - \zeta^2} \qquad h_L = \frac{\cos\lambda^3}{\sqrt{1 + 3\sin^2\lambda}}$$

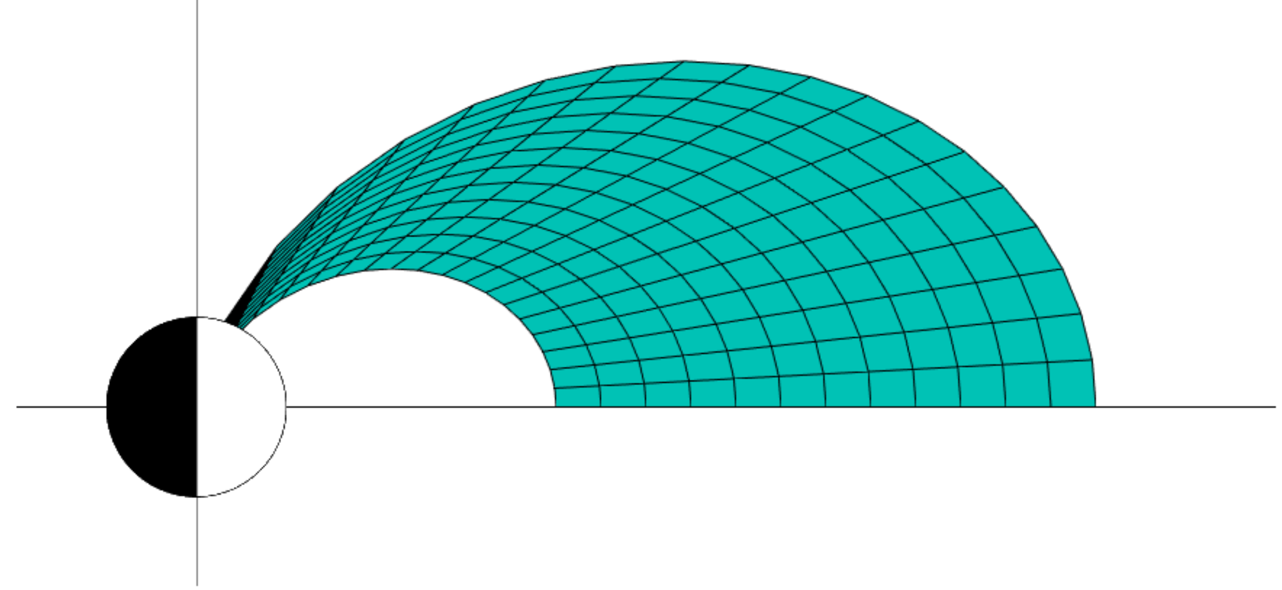
Caution in 2D Dipole Coordinates

In a 1-D dipole grid, we're free to map from mu to zeta without problems, but in 2-D, it's not the case!

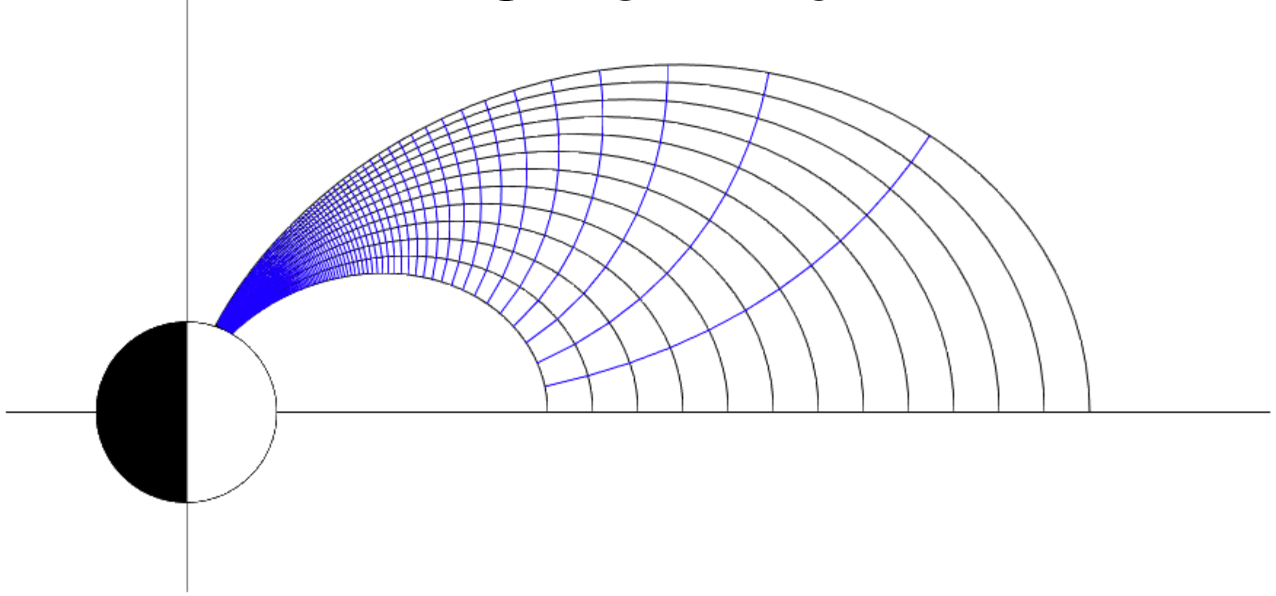
Uniform lambda



Uniform zeta



Uniform mu



To do a 2-D orthogonal dipole grid, you need to be more careful!

Back to 1-D Dipole Coordinates

For a 1-D dipole grid, we dan use zeta as our coordinate and don't have to worry about the orthogonality of the grid by mapping constant mu to other L shells

So let's get back to the 1-D MHD equations in dipole coordinates

So far we've got

$$q_1 = \zeta = \sin\lambda$$

$$q_2 = L = \frac{r}{1-\zeta^2}$$

$$q_3 = \phi \text{ (eastward azimuthal angle)}$$

And scale factors

$$h_1 = h_\zeta = L\sqrt{1+3\zeta^2}$$

$$h_2 = h_L = \frac{\cos\lambda^3}{\sqrt{1+3\sin^2\lambda}} = \frac{(1-\zeta^2)^{3/2}}{\sqrt{1+3\zeta^2}}$$

$$h_3 = h_\phi = r\cos\lambda = L(1-\zeta^2)^{3/2}$$

Comment on Dipole Coordinates

Recall in the 1-D MHD equations in Cartesian geometry, we have

$$\nabla \cdot \mathbf{B} = 0 \longrightarrow \frac{\partial B_x}{\partial x} = 0 \longrightarrow B_x \text{ const}$$

What about the dipole magnetic field? Does this hold in dipole geometry given that B_mu is not a constant along mu?

First let's take a look at the dipole magnetic field equation:

$$\mathbf{B}_{dipole} = B_\mu \hat{\mathbf{e}}_\mu \longrightarrow = B_{eq} \left(\frac{L}{r}\right)^3 \sqrt{1 + 3\sin^2\lambda}\ \hat{\mathbf{e}}_\mu$$

Let's normalize the B field with Beq and remove the r and theta dependence:

$$\underbrace{\frac{\mathbf{B}_{dipole}}{B_{eq}}}_{B_\zeta} = \left(\frac{L}{r}\right)^3 \sqrt{1 + 3\sin^2\lambda}\ \hat{\mathbf{e}}_\mu \xrightarrow[\zeta = \sin\lambda]{L = \frac{r}{1-\zeta^2}} B_\zeta = \frac{\sqrt{1+3\zeta^2}}{(1-\zeta^2)^3}$$

So it is clear that the dipole B field B_zeta is a function of zeta, does that mean div B isn't going to be zero?

Comment on Dipole Coordinates

Recall in the divergence operator in curvilinear geometry:

$$\nabla \cdot \mathbf{B} = \frac{1}{J}\frac{\partial}{\partial q_1}(B_1 h_2 h_3)$$

Although B1 is no longer constant along q1, but if B1 ~ 1/h2h3, the div B can still be zero

$$B_1 = \frac{\sqrt{1+3\zeta^2}}{(1-\zeta^2)^3}$$

$$h_1 = h_\zeta = L\sqrt{1+3\zeta^2}$$

$$h_2 = h_L = \frac{(1-\zeta^2)^{3/2}}{\sqrt{1+3\zeta^2}}$$

$$h_3 = h_\phi = L(1-\zeta^2)^{3/2}$$

$$\longrightarrow h_2 h_3 = L\frac{(1-\zeta^2)^3}{\sqrt{1+3\zeta^2}}$$

$$\longrightarrow B_1 h_2 h_3 = L$$

Not a function of zeta

$$\nabla \cdot \mathbf{B} = 0$$

So the divergence of B is zero in the dipole geometry

Normalized B and Rho

The normalized B field in the zeta coordinate is written as

$$\boxed{B_0 = \frac{\sqrt{1+3\zeta^2}}{(1-\zeta^2)^3}}$$

Background
Magnetic field

For solving the MHD equations, we also need an expression for the density rho_0 in the zeta coordinates

For the geospace, we can assume a power-law dependence for rho_0 as a function of r

$$\rho_0 = \rho_{eq}\left(\frac{L}{r}\right)^{\alpha}$$

Background
mass density

At the magnetic equator, $\lambda = 0 \ \rightarrow \ \zeta = 0 \longrightarrow \rho_0 = \rho_{eq}$

Let's normalize rho=1 at the magnetic equator, we have the background mass density as

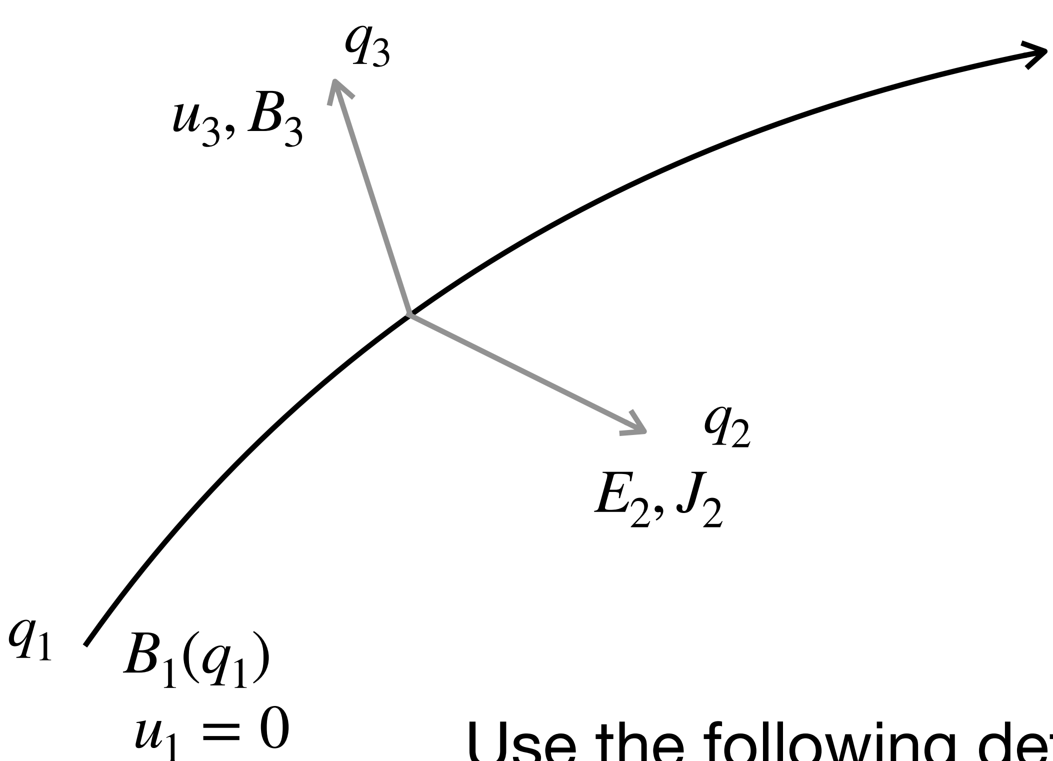
$$\boxed{\rho_0 = \left(\frac{L}{r}\right)^{\alpha} = \left(1-\zeta^2\right)^{-\alpha}}$$

Physics

Dispersion Relation in Dipole

Start from the velocity equation

$$\frac{\partial u_3}{\partial t} = -\frac{1}{\rho_0} J_2 B_1^0 \xrightarrow{J_2 = -\frac{1}{h_3 h_1}\frac{\partial}{\partial q_1}(B_3 h_3)} \frac{\partial u_3}{\partial t} = \frac{B_0}{\rho_0 h_1 h_3}\frac{\partial B_3 h_3}{\partial \zeta} \longrightarrow \frac{\partial^2 u_3}{\partial t^t} = \frac{B_0}{\rho_0 h_1 h_3}\frac{\partial}{\partial \zeta} h_3 \frac{\partial B_3}{\partial t}$$



Now use the Faraday's law

$$\frac{\partial B_3}{\partial t} = -\frac{1}{h_1 h_2}\frac{\partial}{\partial q_1}(E_2 h_2) = \frac{1}{h_1 h_2}\frac{\partial}{\partial q_1}(u_3 B_0 h_2)$$

We have

$$\frac{\partial^2 u_3}{\partial t^t} = \frac{B_0}{\rho_0 h_1 h_3}\frac{\partial}{\partial \zeta}\left(\frac{h_3}{h_1 h_2}\frac{\partial}{\partial \zeta} u_3 B_0 h_2\right)$$

Use the following definitions of the scale factor

$$h_1 = h_\zeta = L\sqrt{1 + 3\zeta^2}$$

$$h_2 = h_L = \frac{(1 - \zeta^2)^{3/2}}{\sqrt{1 + 3\zeta^2}}$$

$$h_3 = h_\phi = L(1 - \zeta^2)^{3/2}$$

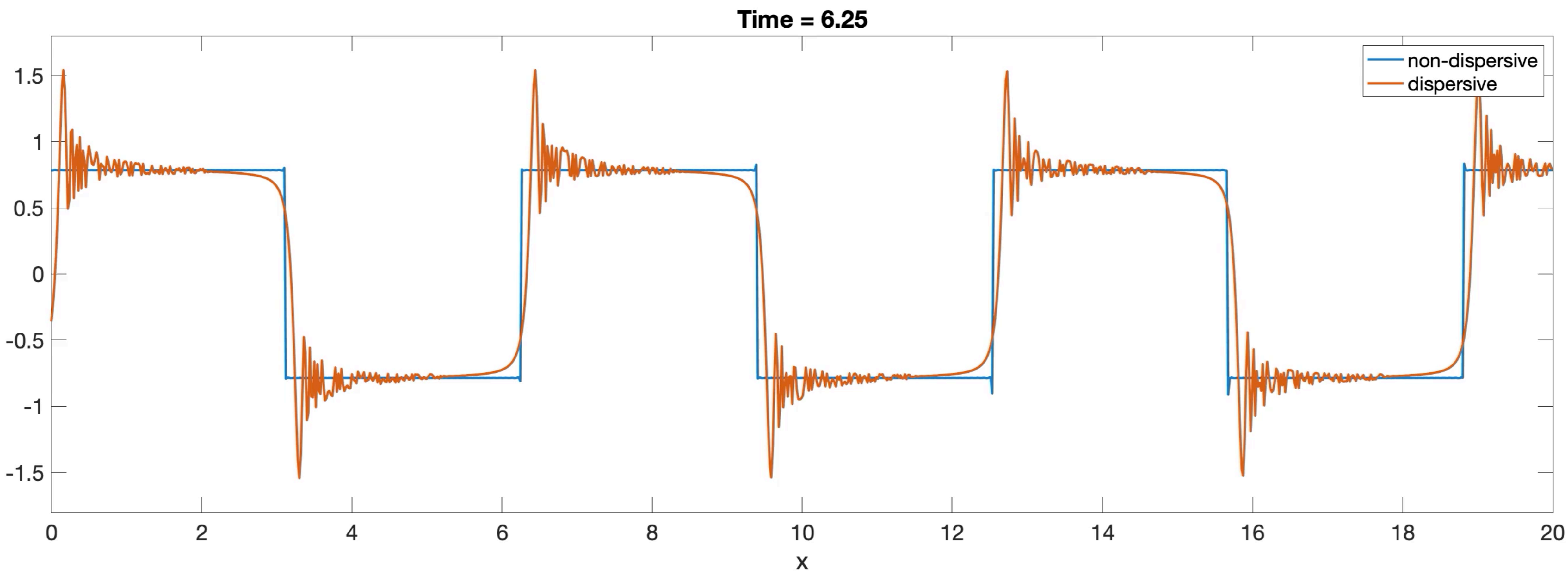
With

$$B_0 = \frac{\sqrt{1 + 3\zeta^2}}{(1 - \zeta^2)^3} \qquad \rho_0 = \left(\frac{L}{r}\right)^\alpha = \left(1 - \zeta^2\right)^{-\alpha}$$

What is Dispersion

Non-dispersive : waves with different k travels at the same speed

dispersive : waves with different k travels at different speed



$$f(x - vt) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)(x - vt)]$$ Non-dispersive — "waveform" kept the same

$$= \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)(x - v(n)t)]$$ dispersive — "waveform" oscillatory

$$v(n) \sim \sqrt{2n+1}$$

Physics

Dispersion Relation in Dipole

We get a second-order PDE for the velocity $\dfrac{\partial^2 u_3}{\partial t^t} = \dfrac{B_0}{\rho_0 h_1 h_3}\dfrac{\partial}{\partial \zeta}\left(\dfrac{h_3}{h_1 h_2}\dfrac{\partial}{\partial \zeta} u_3 B_0 h_2\right)$

Let's define $\omega = \sqrt{1 + 3\zeta^2}$

Using the equations for B0, rho0, h1, h2, h3, we have

$$\frac{\partial^2 u_3}{\partial t^t} = \frac{\omega}{(1-\zeta^2)^3}\frac{1}{(1-\zeta^2)^{-\alpha}}\left(\frac{1}{L\omega}\frac{1}{L(1-\zeta^2)^{3/2}}\right)\cdot$$

$$\frac{\partial}{\partial \zeta}\left[L(1-\zeta^2)^{3/2}\left(\frac{1}{L\omega}\frac{\omega}{(1-\zeta^2)^{3/2}}\right)\frac{\partial}{\partial \zeta}\left(u_3\frac{\omega}{(1-\zeta^2)^3}\frac{(1-\zeta^2)^{3/2}}{\omega}\right)\right]$$

With a small amount of algebra, we get

$$\frac{\partial^2 u_3}{\partial t^2} = \frac{1}{L^2(1-\zeta^2)^{\frac{9}{2}-\alpha}}\frac{\partial^2}{\partial \zeta^2}\left[\boxed{\frac{u_3}{(1-\zeta^2)^{\frac{3}{2}}}}\right]$$

Define as u_3'

Physics

Dispersion Relation in Dipole

The PDE for u_3: $$\frac{\partial^2 u_3}{\partial t^2} = \frac{1}{L^2(1-\zeta^2)^{\frac{9}{2}-\alpha}} \frac{\partial^2}{\partial \zeta^2}\left[\frac{u_3}{(1-\zeta^2)^{\frac{3}{2}}}\right]$$

Now we have $$u_3' = \frac{u_3}{(1-\zeta^2)^{\frac{3}{2}}} \xrightarrow{\zeta,\, t \text{ independent}} \frac{\partial^2 u_3}{\partial t^2} = (1-\zeta^2)^{\frac{3}{2}} \frac{\partial^2 u_3'}{\partial t^2}$$

Simplified as $$\Rightarrow \quad \frac{\partial^2 u_3'}{\partial t^2} = \frac{1}{L^2(1-\zeta^2)^{6-\alpha}} \frac{\partial^2 u_3'}{\partial \zeta^2}$$

Assume harmonic wave solutions: $$u_3' = \tilde{u}(\zeta)e^{-i\omega t} \longrightarrow \frac{\partial^2 u_3}{\partial t^2} = -\omega^2(1-\zeta^2)^{\frac{3}{2}} u_3'$$

Substitute in eq 1 $$\longrightarrow \boxed{\left[\frac{\partial^2}{\partial \zeta^2} + \omega^2 L^2 (1-\zeta^2)^{6-\alpha}\right]\tilde{u} = 0}$$ ODE for dispersion

If alpha = 6, the (wave) equation can be solved analytically

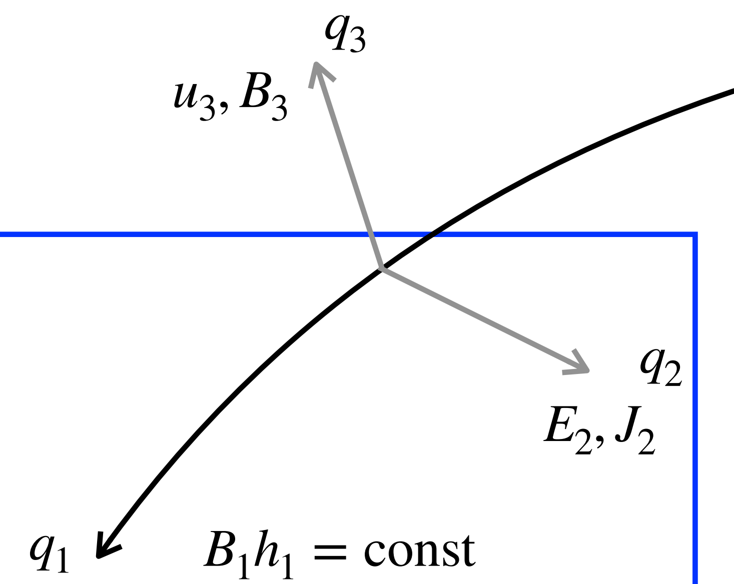
Math

Solve for the Dispersion Relation

2n-order ODE for alpha = 6

Put Everything Together

1-D MHD equations in dipole geometry



$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u_x)$$

$$\frac{\partial u_x}{\partial t} = -u_x\frac{\partial u_x}{\partial x} + \frac{1}{\rho}\left(-\frac{\partial p}{\partial x} - J_z B_y\right)$$

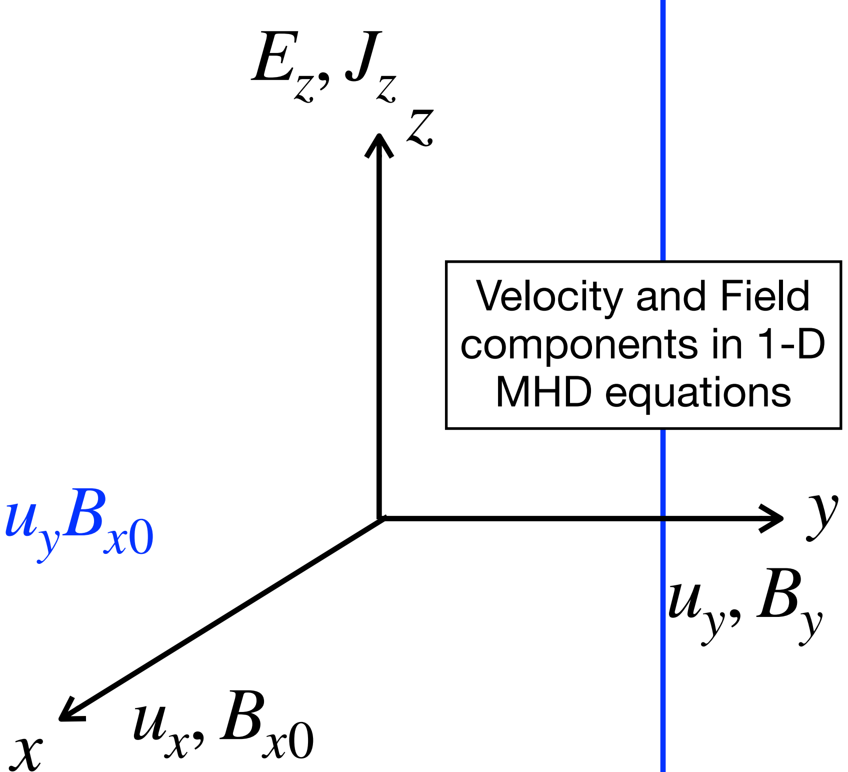
$$\frac{\partial u_y}{\partial t} = -u_x\frac{\partial u_y}{\partial x} + \frac{1}{\rho}J_z B_{x0}$$

$$p = \frac{\beta_0}{2}\rho^\gamma$$

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_z}{\partial x}$$

$$E_z = -u_x B_y + u_y B_{x0}$$

$$J_z = \frac{\partial B_y}{\partial x}$$



$$\rho = \rho_0$$

$$u_1 = 0$$

$$\rho_0\frac{\partial u_3}{\partial t} = -J_2 B_1^0$$

$$p = p_0$$

$$\frac{\partial B_3}{\partial t} = -\frac{1}{h_1 h_2}\frac{\partial}{\partial q_1}(E_2 h_2)\hat{\mathbf{e}}_3$$

$$E_2 = -u_3 B_1^0$$

$$J_2 = -\frac{1}{h_3 h_1}\frac{\partial}{\partial q_1}(B_3 h_3)$$

$$B_1 h_1 = \text{const}$$
$$u_1 = 0$$

$$h_1 = h_\zeta = L\sqrt{1+3\zeta^2}$$

$$h_2 = h_L = \frac{(1-\zeta^2)^{3/2}}{\sqrt{1+3\zeta^2}}$$

$$h_3 = h_\phi = L(1-\zeta^2)^{3/2}$$

Put Everything Together

Implement FD 1-D MHD solving in dipole geometry

Step 1: choose equally spaced ζ_i between $-\zeta_{ion}$ and $+\zeta_{ion}$

$$\Delta\zeta = 2\zeta_{ion}/\mathrm{nid}$$

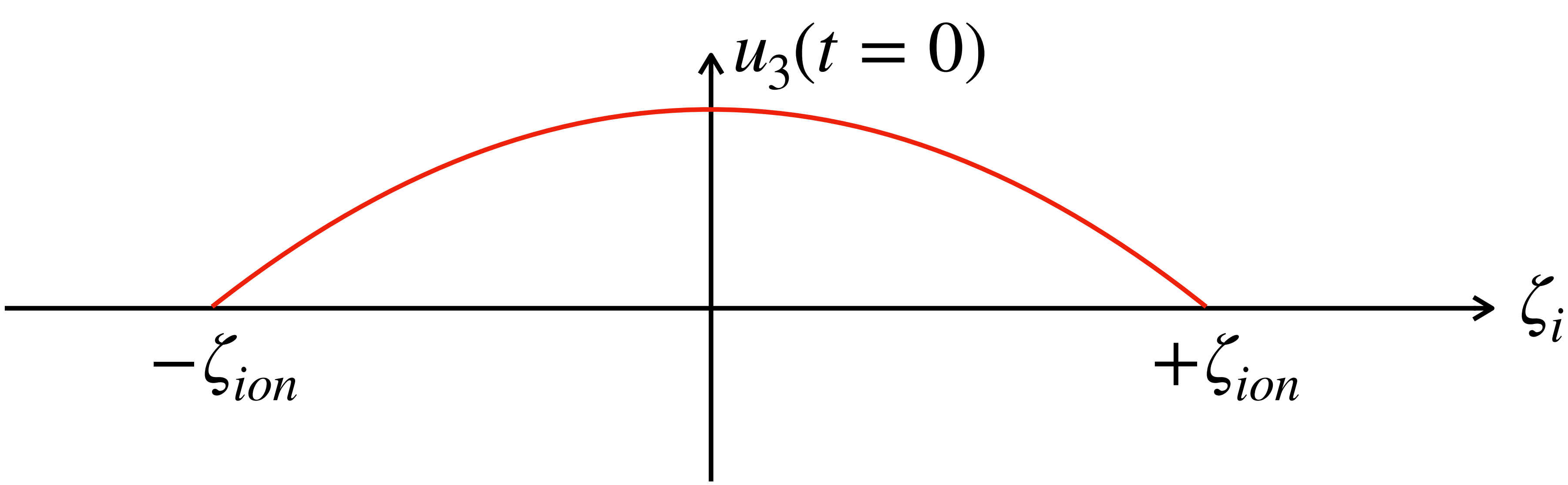
$$\mathrm{ni} = \mathrm{nid} + 2$$

$$\zeta = ((1:\mathrm{ni}) - 1.5) \cdot \Delta\zeta - \zeta_{\mathrm{ion}}$$

Step 2: Define constant values and scale factors

$$B_0^{\mu}(\zeta) \quad \rho_0^{\mu}(\zeta) \quad h_{\zeta}(\zeta_i) \equiv h_1 \quad h_L(\zeta_i) \equiv h_2 \quad h_{\phi}(\zeta_i) \equiv h_3$$

Step 3: setup initial perturbations of u_3 (u_phi):



$$u_3 = \cos\left(\frac{\pi}{2}\frac{\zeta}{\zeta_{ion}}\right)$$

$$B_3 = 0$$

Put Everything Together

Implement FD 1-D MHD solving in dipole geometry

Step 4: Use the leapfrog trapezoidal time stepping method to evolve the 1-D linear MHD equations in dipole geometry:

$$\frac{\partial u_3}{\partial t} = -\frac{J_2(\zeta)B_0(\zeta)}{\rho_0(\zeta)}$$

$$\frac{\partial B_3}{\partial t} = -\frac{1}{h_1(\zeta)h_2(\zeta)}\frac{\partial}{\partial \zeta}\left(E_2(\zeta)h_2(\zeta)\right)$$

You also need

$$J_2(\zeta) = -\frac{1}{h_3(\zeta)h_1(\zeta)}\frac{\partial}{\partial \zeta}(B_3(\zeta)h_3(\zeta))$$

$$E_2(\zeta) = -u_3(\zeta)B_0(\zeta)$$

Step 5: finite difference approximation for spatial derivatives

$$\frac{\partial}{\partial \zeta}\left(fh\right) = \frac{1}{2\Delta\zeta}(f_{i+1}h_{i+1} - f_{i-1}h_{i-1})$$

Put Everything Together

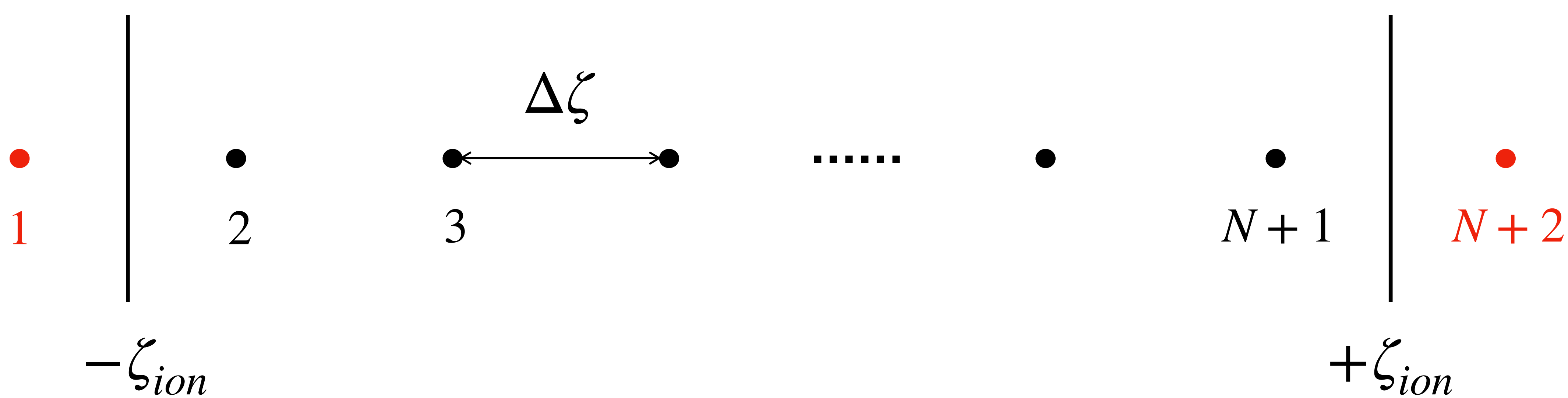
Boundary conditions in dipole geometry

Step 6: Use perfect conducting boundary conditions for energy conservation, which is

$$B_0, \rho_0, h_1, h_2, h_3, B_3 : \quad (+)$$

$$u_3, E_2 . J_2 : \quad (-)$$

at the boundaries



HW assignment # 3