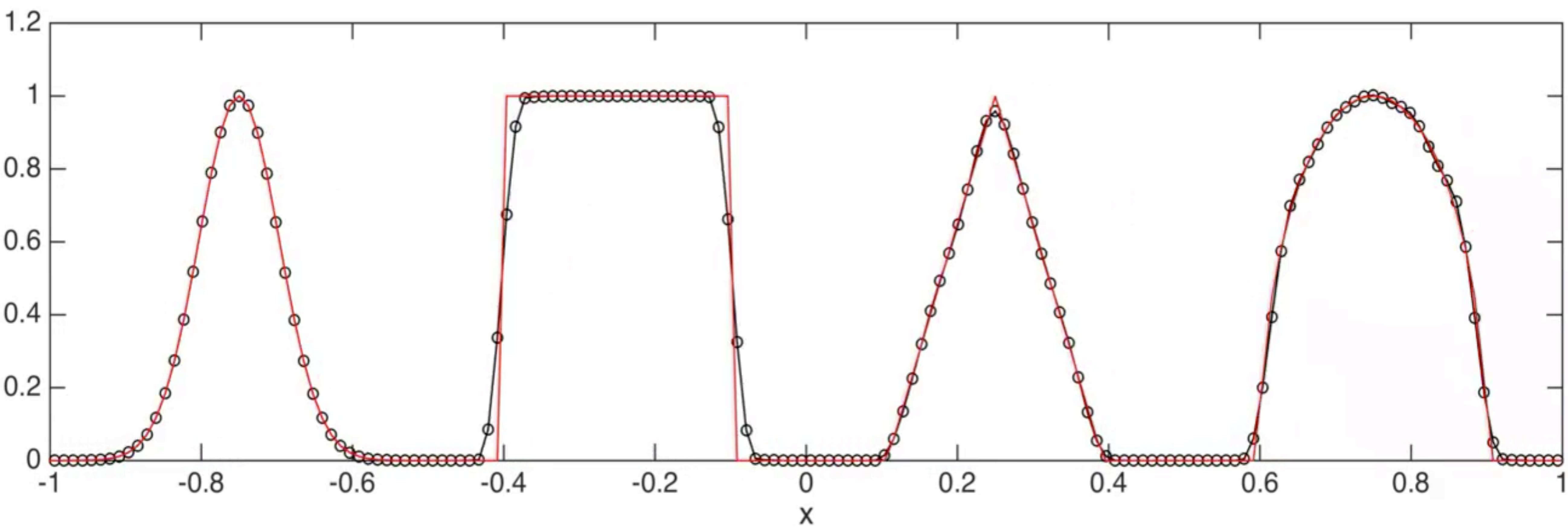
Topic 3

Numerical Solutions to the Advection-Diffusion Equations

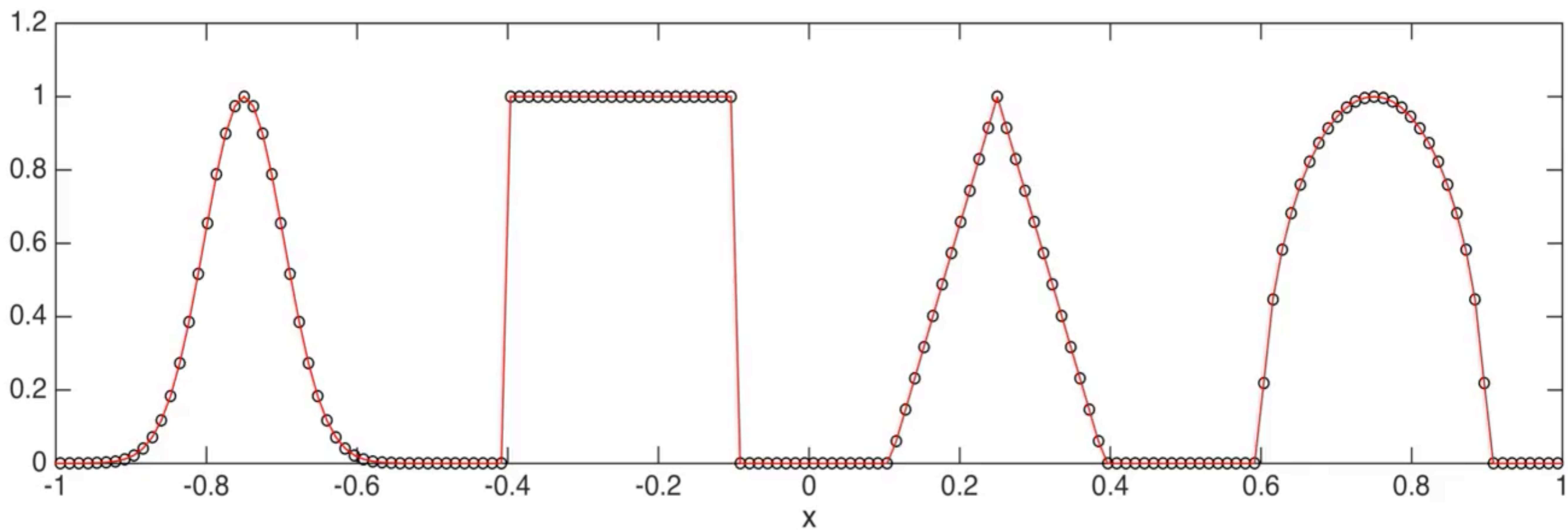


Outline

- Review of Finite Volume method
- Reconstruction, Flux and Slope Limiting
- Introduction to the idea of Flux limiting
- The Partial Donor Cell method (flux correct transport)
- Extending to non-linear problems
- Extending to multi-dimensional problems

Advection of four shapes in 1D

$$\frac{\partial Q}{\partial t} + u_0 \frac{\partial Q}{\partial x} = 0$$

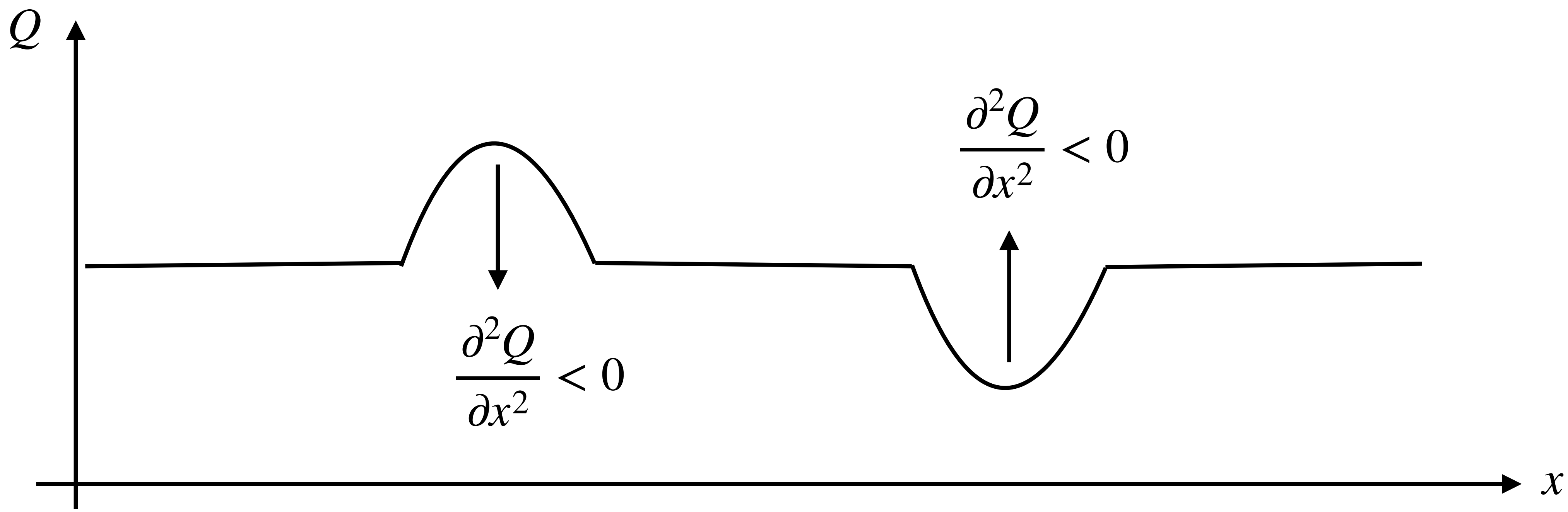


Gaussian	**Square**	**Triangle**	**Half-circle**
Smooth	Discontinuity	Discontinuity in 1st derivative	Steep changes in derivatives

A few things about the modified equation

Upwind method

Original Equation	Numerical Approximation	Modified Equation
$\frac{\partial Q}{\partial t} + u_0 \frac{\partial Q}{\partial x} = 0$	$Q_i^{n+1} = Q_i^n - \frac{u_0 \Delta t}{\Delta x}(Q_i^n - Q_{i-1}^n)$	$\frac{\partial q}{\partial t} + u_0 \frac{\partial q}{\partial x} \approx \beta_{xx} \frac{\partial^2 q}{\partial x^2}$



$$\frac{\partial q}{\partial t} + u_0 \frac{\partial q}{\partial x} \approx \beta_{xx} \frac{\partial^2 q}{\partial x^2}$$

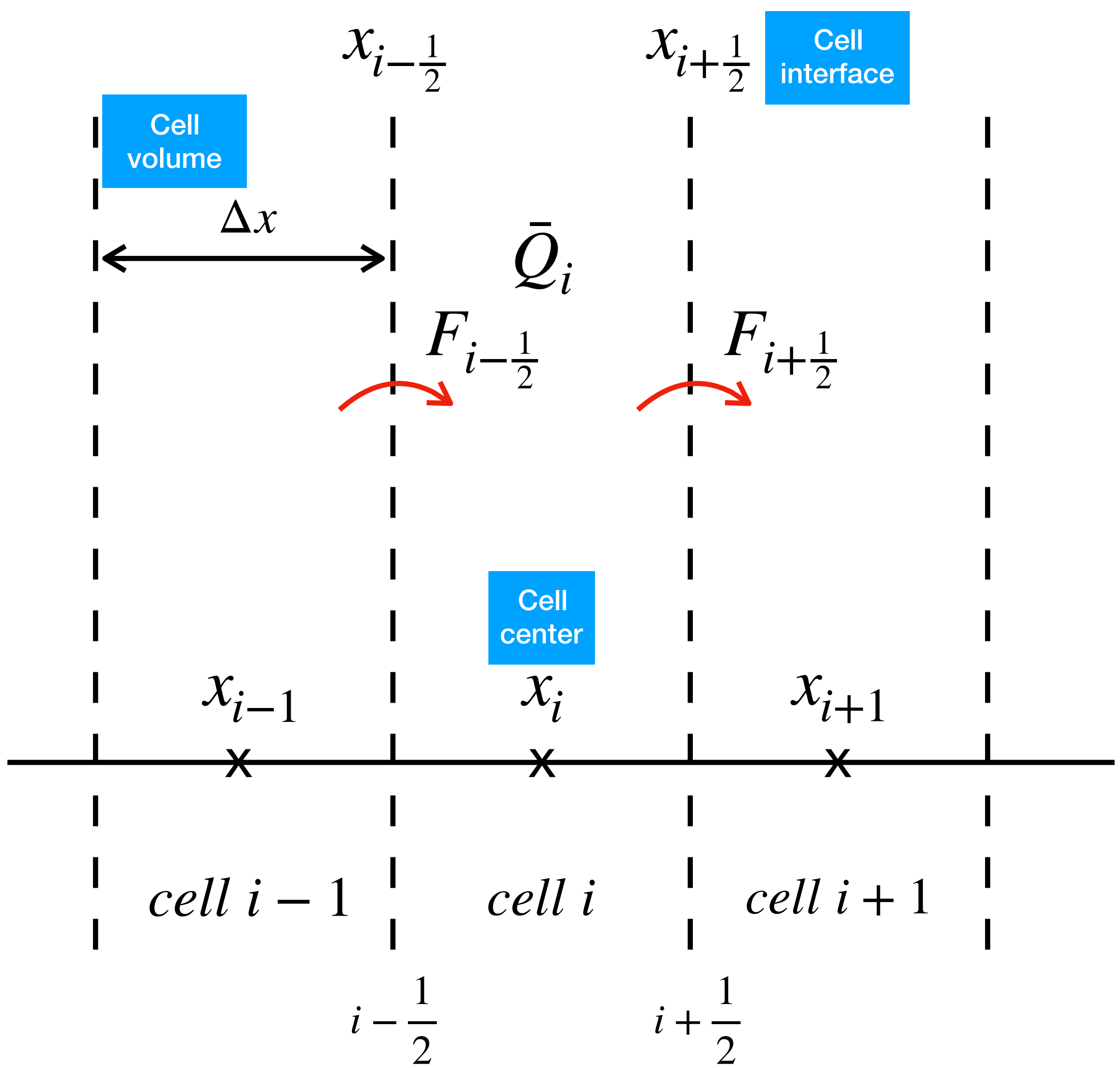
Analytical solution goes like $$q(x,t) = \int_{-\infty}^{+\infty} f(\xi - u_0 t) e^{-\frac{(\xi - x)^2}{\beta_{xx}}} d\xi$$

Finite Volume Methods

Basic form

$$\frac{\partial Q}{\partial t} + u_0 \frac{\partial Q}{\partial x} = 0 \xrightarrow{\text{re-write}} \frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0 \qquad F(Q) = u_0 Q \quad \text{for linear advection}$$

Let's discretize the solution domain:



Integrate the PDE in cell i

$$\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} dx \left(\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} \right) = 0$$

$$\longrightarrow \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial Q}{\partial t} dx = - \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} \frac{\partial F(Q)}{\partial x} dx$$

$$\longrightarrow \frac{\partial}{\partial t} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} Q dx = - F(Q) \Big|_{i-\frac{1}{2}}^{i+\frac{1}{2}}$$

$$\longrightarrow \frac{\partial}{\partial t} \bar{Q} \Delta x = - F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}$$

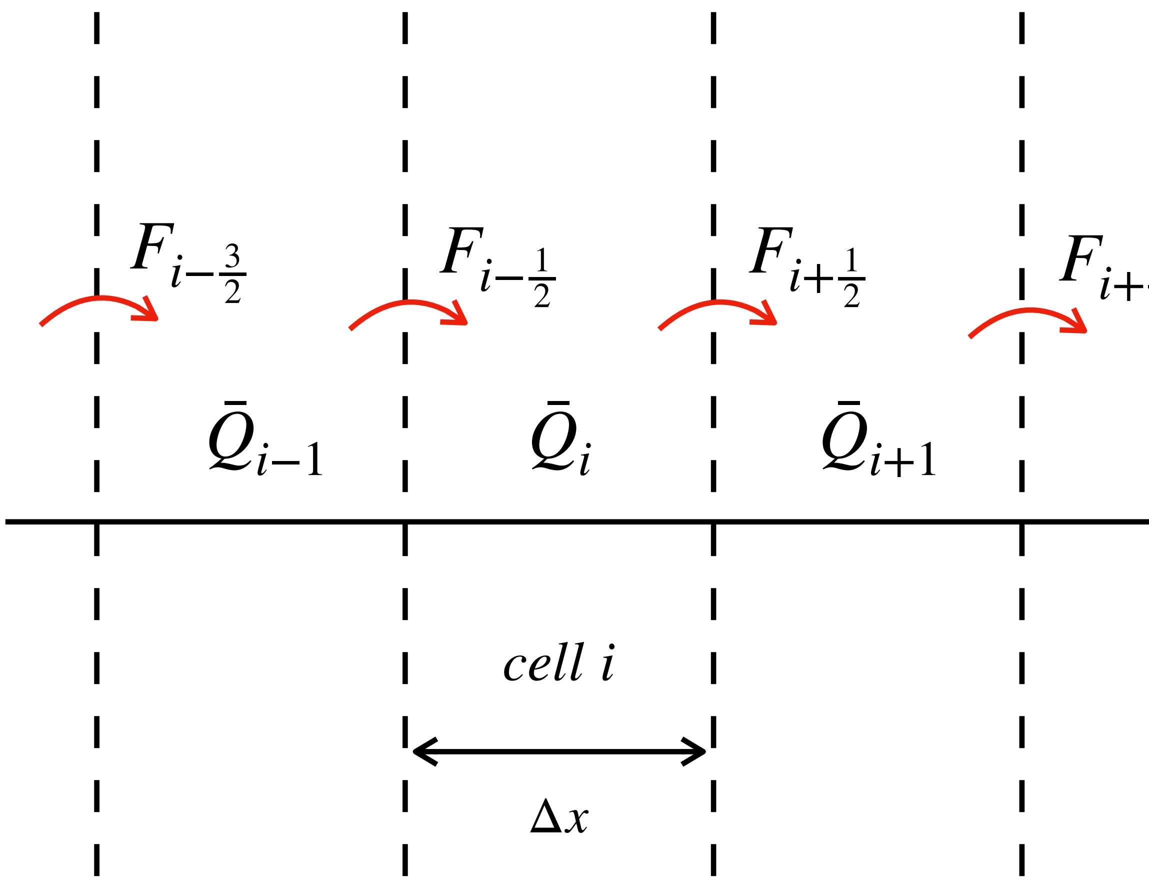
Rate of mass change

Flux in & out of cell i

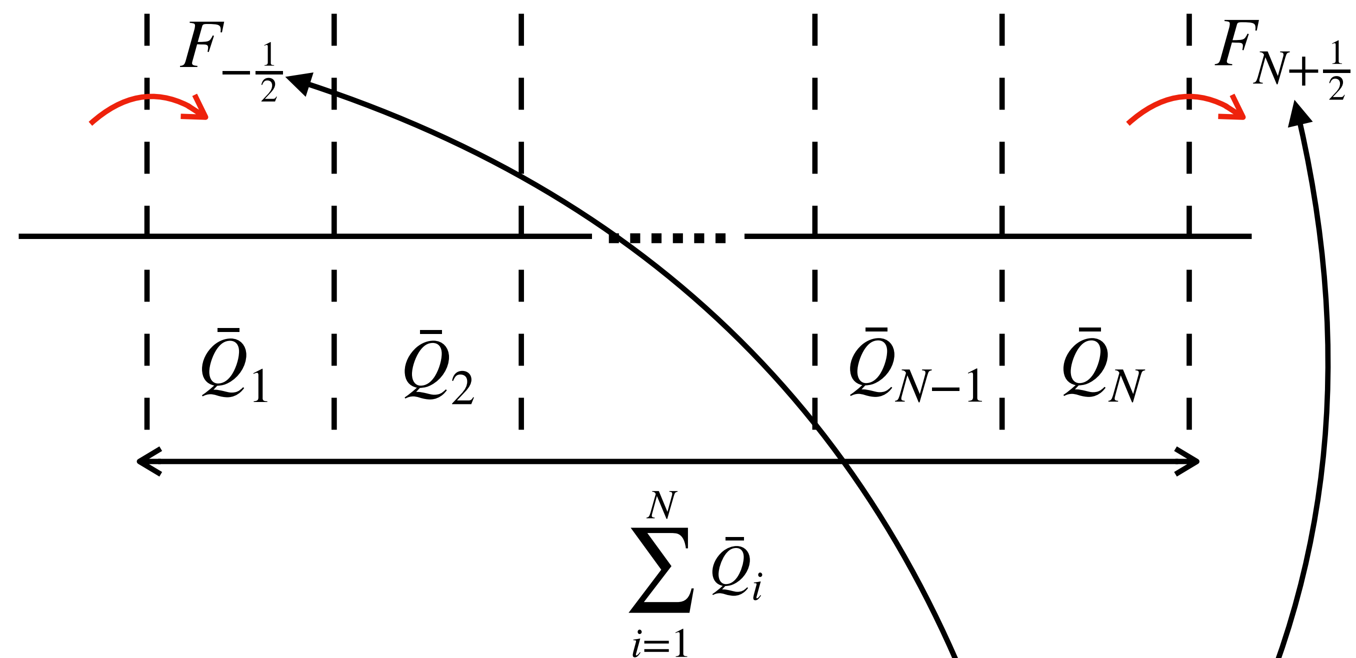
Finite Volume Methods

Conservation nature

$$\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$



Sum over all the cells:



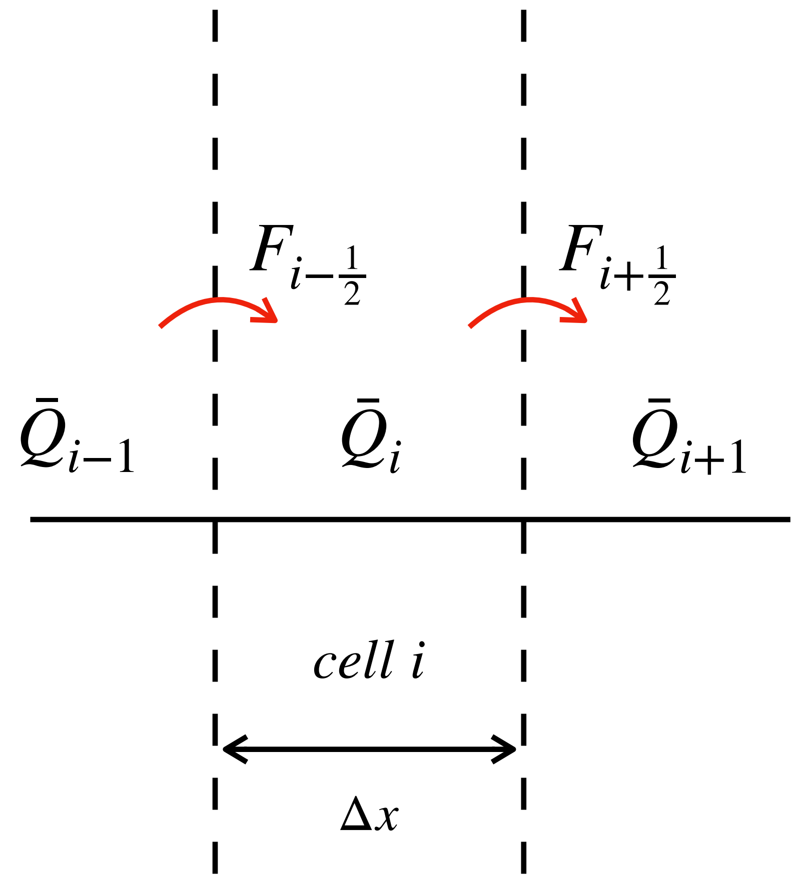
$$\sum_{i=1}^{N}\frac{\partial}{\partial t}\bar{Q}_i = \frac{\partial}{\partial t}\sum_{i=1}^{N}\bar{Q}_i = \frac{1}{\Delta x}\sum_{i=1}^{N}(F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}})$$

$$= \frac{1}{\Delta x}\left[(F_{-\frac{1}{2}} - \cancel{F_{\frac{1}{2}}}) + (\cancel{F_{\frac{1}{2}}} - \cancel{F_{\frac{3}{2}}}) + (\cancel{F_{\frac{3}{2}}} - \cancel{F_{\frac{5}{2}}}) + \ldots(\cancel{F_{N-\frac{1}{2}}} - F_{N+\frac{1}{2}})\right] = \frac{1}{\Delta x}(\boxed{F_{-\frac{1}{2}}} - \boxed{F_{N+\frac{1}{2}}})$$

If the flux at i=-1/2 and i=N+1/2 is zero, then the total mass is not going to change

The 1st-order upwind scheme is horrible

Flux of the 1st-order upwind method



FV form:

$$\frac{\partial}{\partial t}Q_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

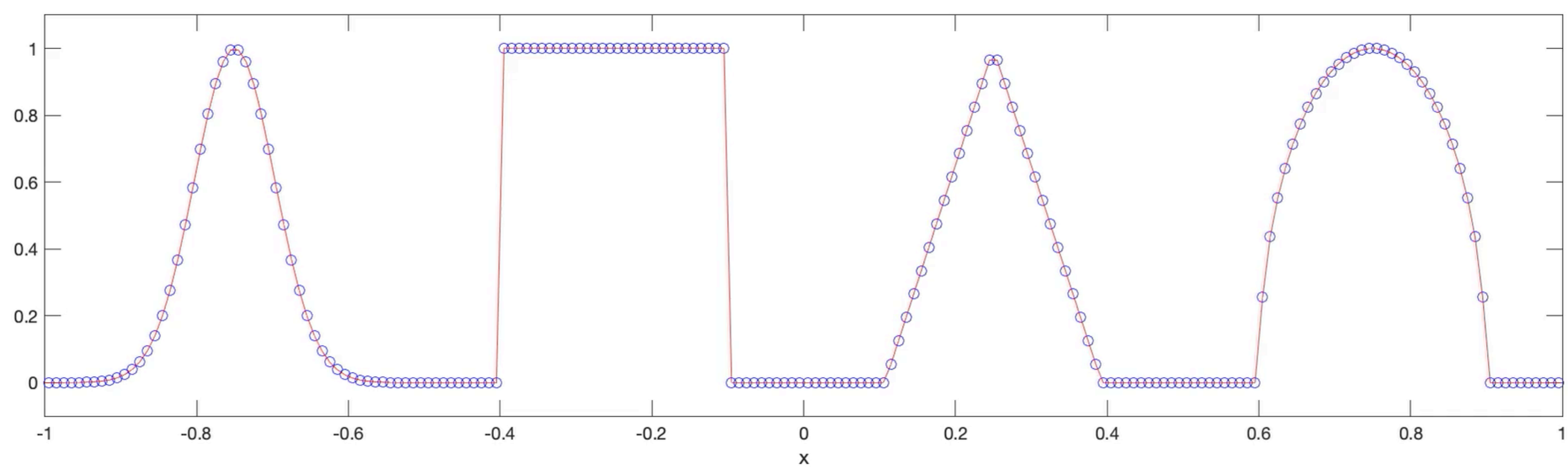
Interface Flux:

$$F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \frac{1}{2}(u_0 Q_i + u_0 Q_{i+1}) - \frac{1}{2}|u_0|(Q_{i+1} - Q_i)$$

Alternatively:

$F(Q) = u_0 Q$

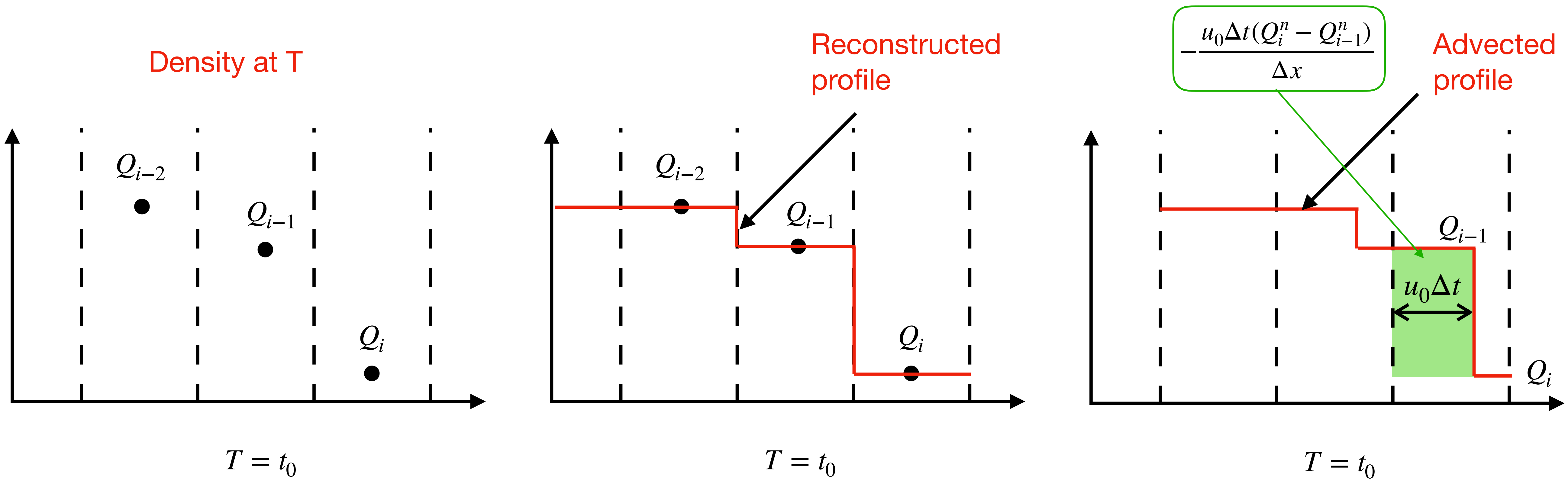
$$F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \frac{1}{2}(F(Q_i) + F(Q_{i+1})) - \frac{1}{2}|u_0|(Q_{i+1} - Q_i)$$



The Advection Nature of the equation

The REA framework

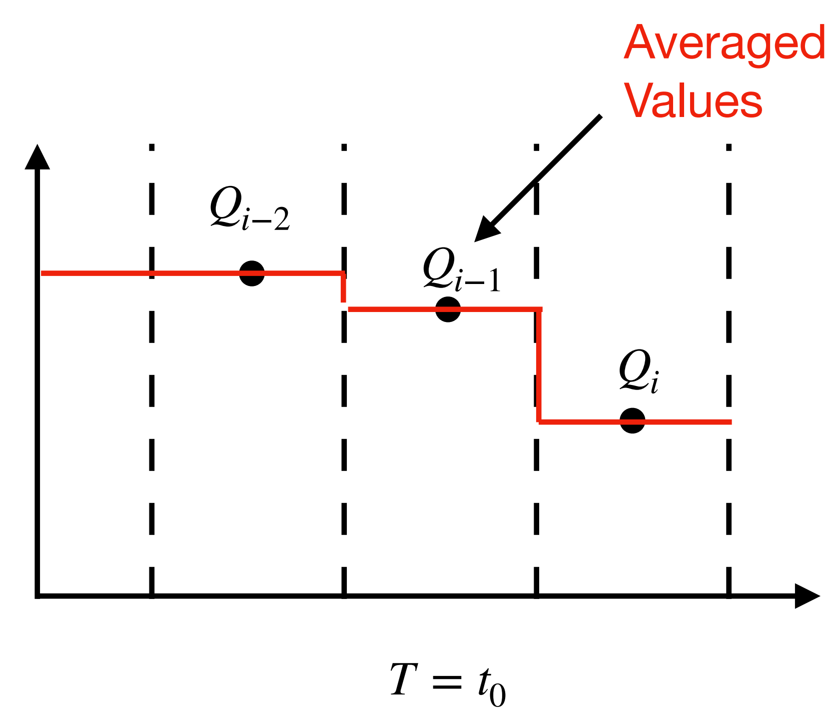
So the upwind scheme is basically an advection



Here's what happened in the upwind method:

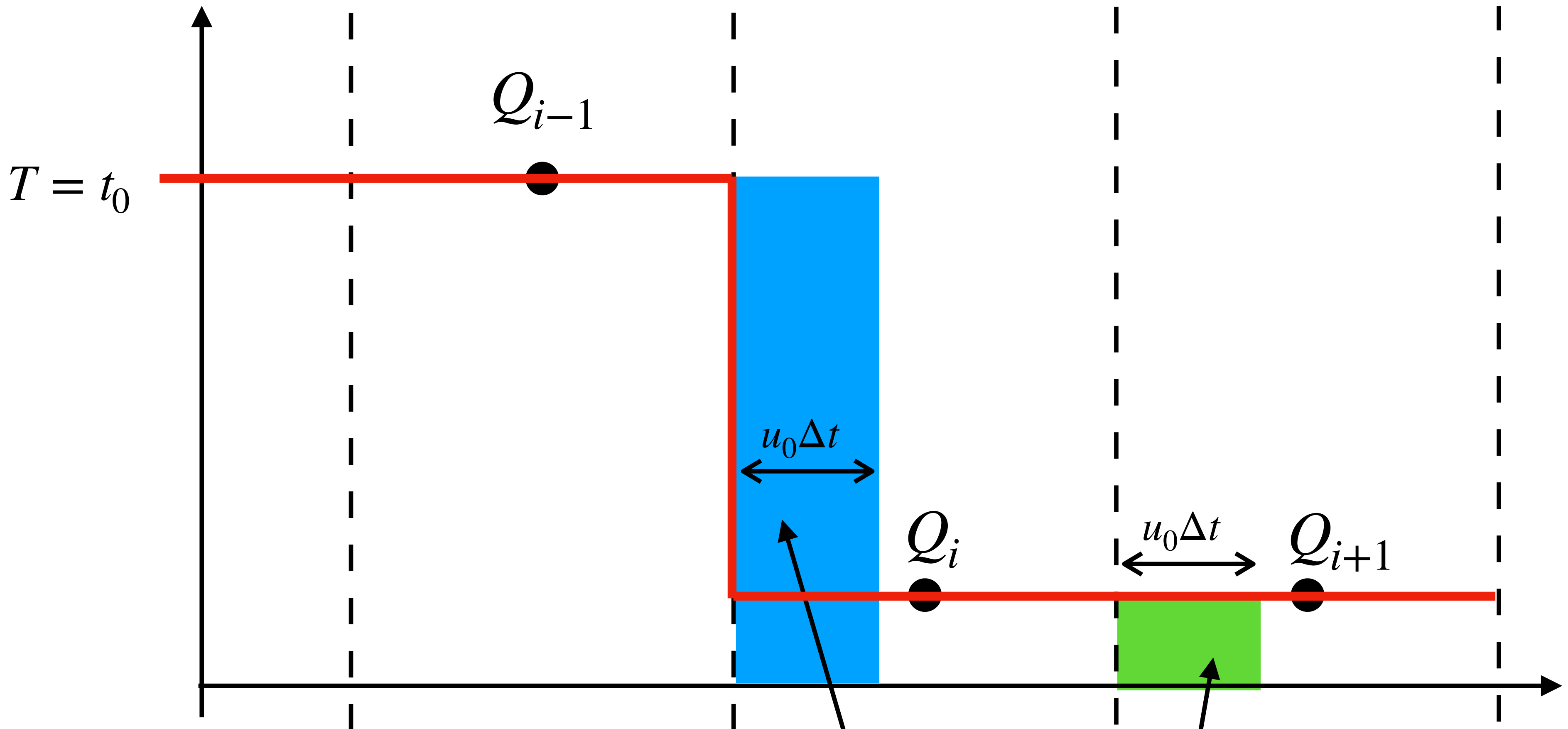
1. From Q_i, do a piecewise-constant reconstruction;
2. Move the reconstructed profile by u*delta t
3. Average the shifted profile in each cell to get new Q_i

Reconstruct - ***Evolve*** - ***Average*** (REA framework)



The Transport Nature of the equation

The flux balance interpretation



$$Q_i^{n+1} = Q_i^n - \frac{u_0 \Delta t (Q_i^n - Q_{i-1}^n)}{\Delta x} = Q_i^n + \left(\boxed{\frac{u_0 \Delta t}{\Delta x} Q_{i-1}^n} - \boxed{\frac{u_0 \Delta t}{\Delta x} Q_i^n} \right) \in (Q_{i-1}, Q_i)$$

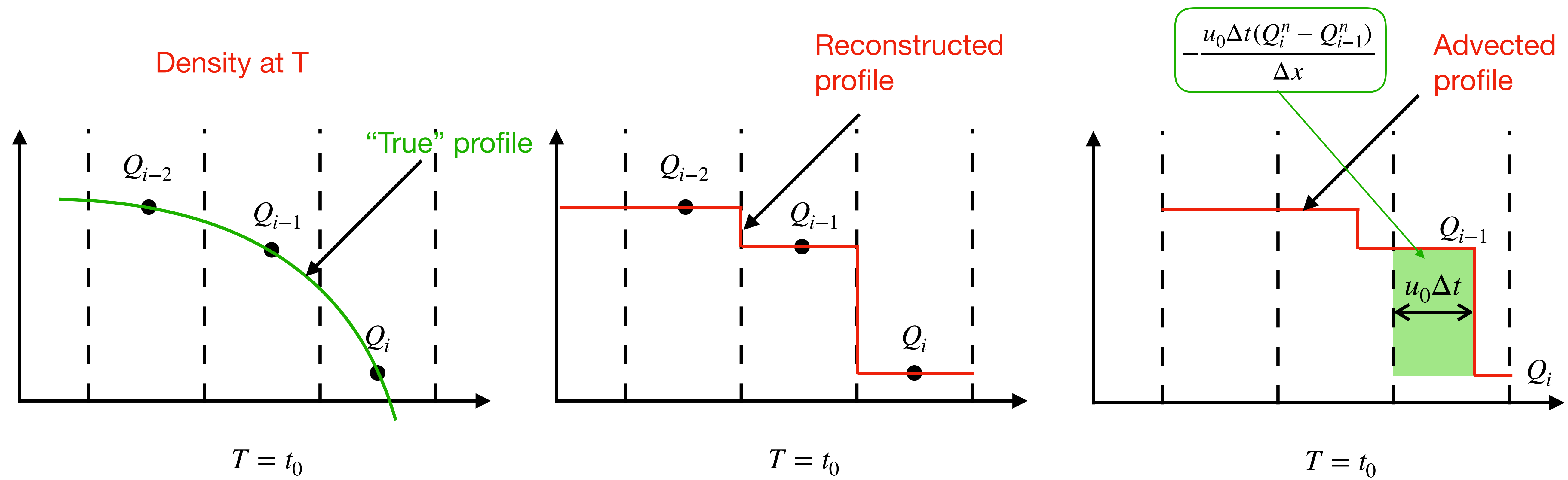
Mass entering cell i

Mass leaving cell i

Finite Volume Methods

REA w/ the 1st-order upwind method

The upwind method is basically a piecewise constant reconstruction scheme



Which means we're assuming the density profile within cell i is constant:

$$q(x) = Q_i, \;\; x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$$

By doing this, the modified equation has an explicit diffusion term $\frac{\partial q}{\partial t} + u_0 \frac{\partial q}{\partial x} \approx \beta_{xx} \frac{\partial^2 q}{\partial x^2}$

This is because we are treating the initial profile Q(x) as step functions - possible very far away from the "true" solution unless delta x is really small.

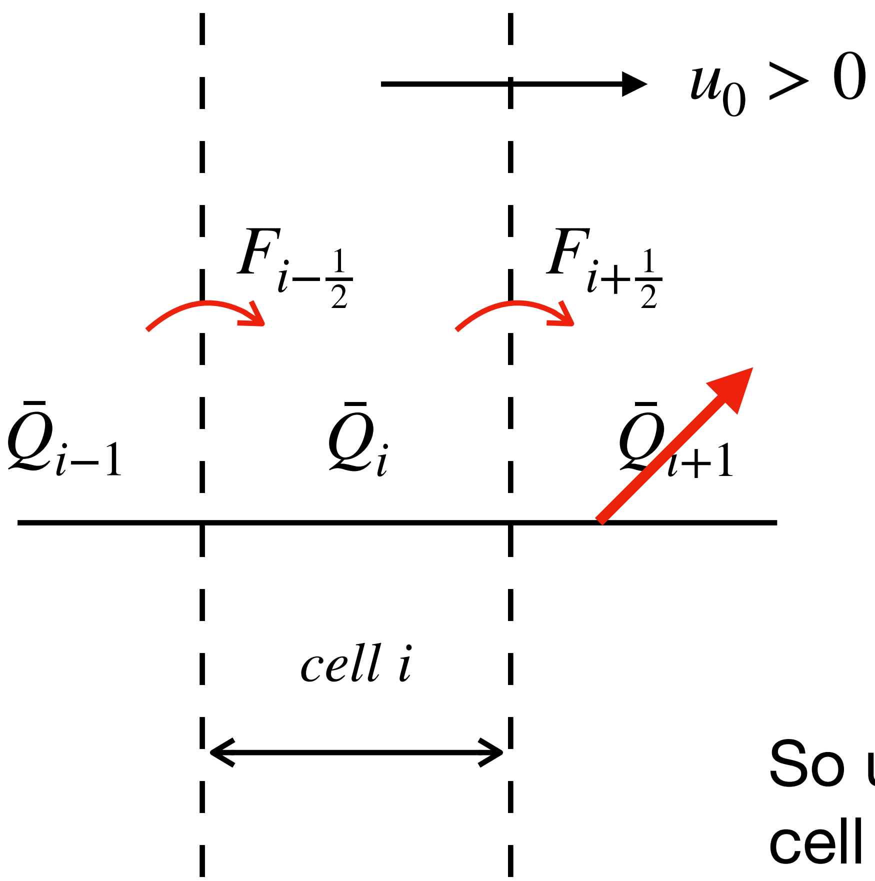
Upwind flux versus Central flux

We now know that the upwind scheme can be written as a combination of a second-order flux with a diffusion term:

$$F(Q) = u_0 Q$$

$$F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \underbrace{\frac{1}{2}(F(Q_i) + F(Q_{i+1}))}_{\textbf{2nd-order flux}} - \underbrace{\frac{1}{2}|u_0|(Q_{i+1} - Q_i)}_{\textbf{Diffusion term}}$$

why?



Let's difference the flux F_i+1/2:

$$F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \sim -\frac{1}{2}|u_0|\underbrace{(Q_{i+1} - 2Q_i + Q_{i-1})}_{\sim \frac{\partial^2 Q}{\partial x^2}}$$

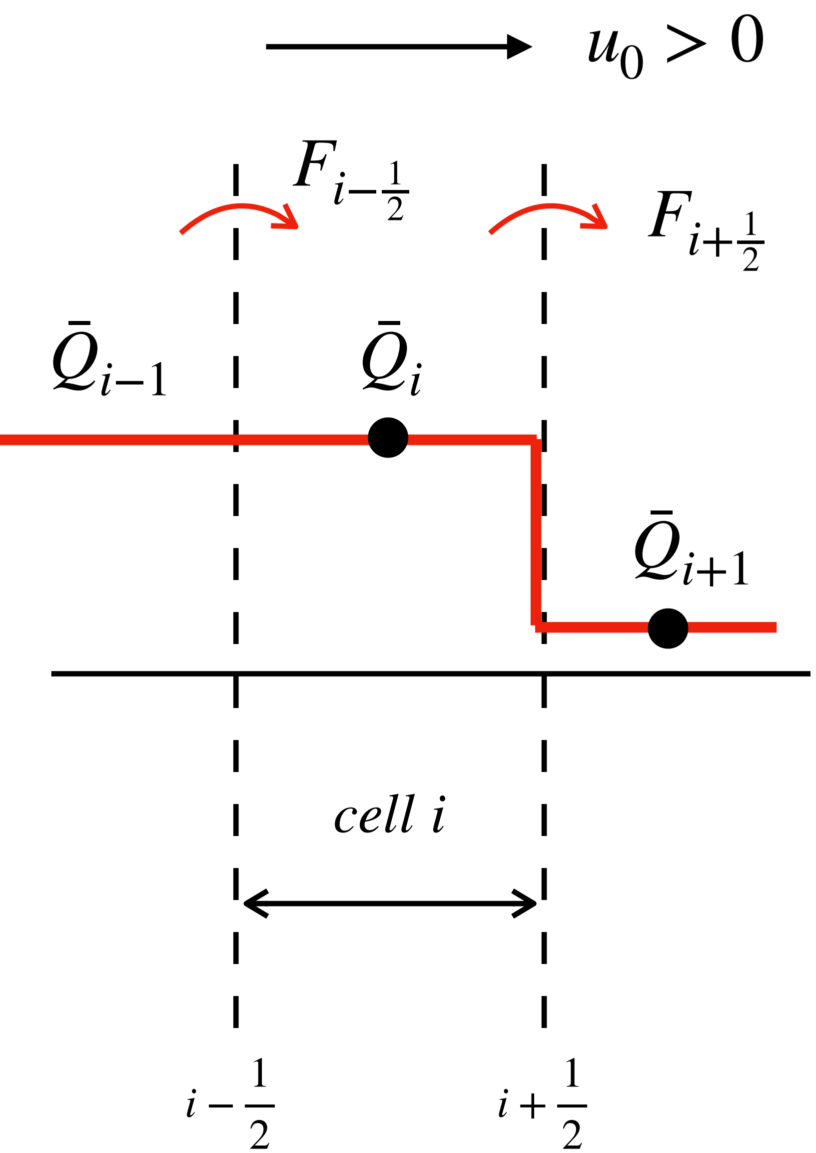
So upwind scheme basically cancels out the furthest downwind cell and make the flux one-sided

This leads to the so-called family of **central schemes**

Which means as long as we know the speed of the waves, we don't need to care about the direction of the propagation anymore

Origin of Numerical diffusion in Upwind

Upwind: $F_{i+\frac{1}{2}}(Q_i, Q_{i+1}) = \frac{1}{2}(F(Q_i) + F(Q_{i+1})) - \frac{1}{2}|u_0|(Q_{i+1} - Q_i)$



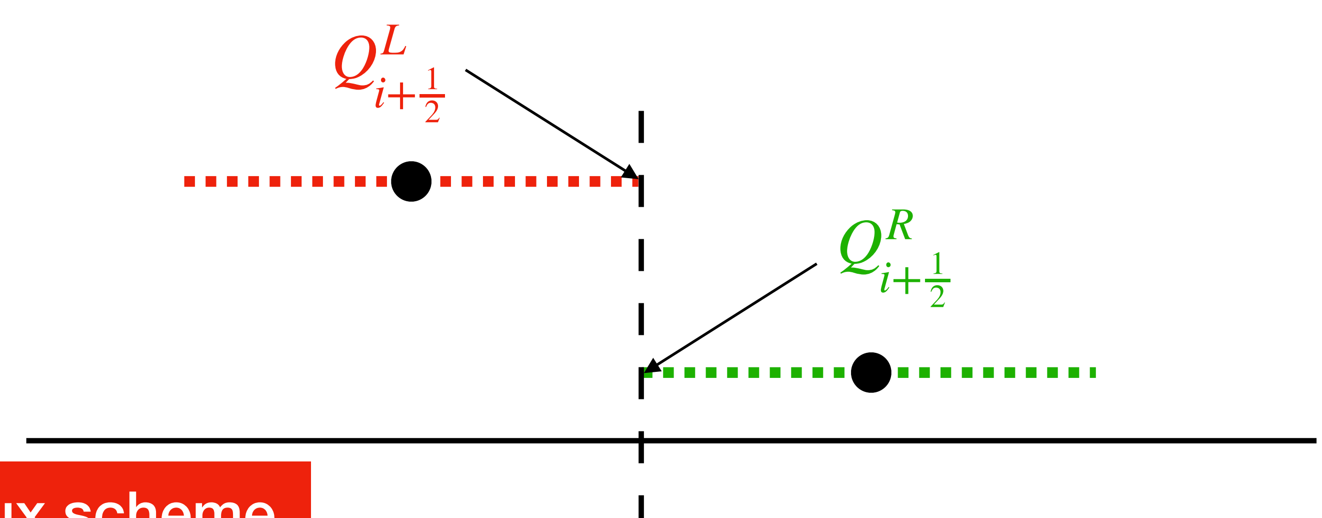
So as long as Q_i is not equal to Q_i-1, numerical diffusion is introduced, which is described by the modified equation as

$$\boxed{\frac{\partial q}{\partial t} + u_0\frac{\partial q}{\partial x} \approx \beta_{xx}\frac{\partial^2 q}{\partial x^2}}$$

Numerical diffusion

Modified equation

If we regard Qi and Q_i+1 as zeroth-order reconstruction:



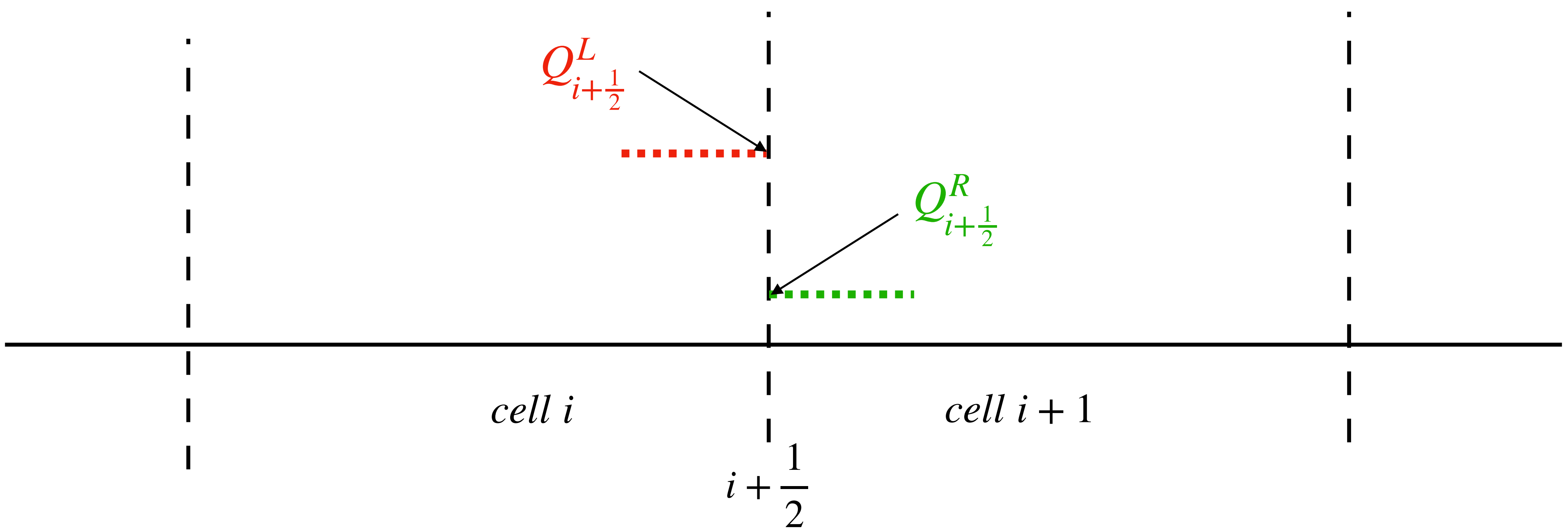
Central flux scheme

Upwind Flux is basically: $F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_0|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$

Central Schemes

Interface states and wave speed

The use of upwind schemes require the knowledge of the wave speed and direction of the propagation - not always available in non-linear problems. So central schemes are convenient which does not require the information of the wave propagation:



Lax-Friedrichs $$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{\Delta x}{2\Delta t}(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

Rusanov $$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

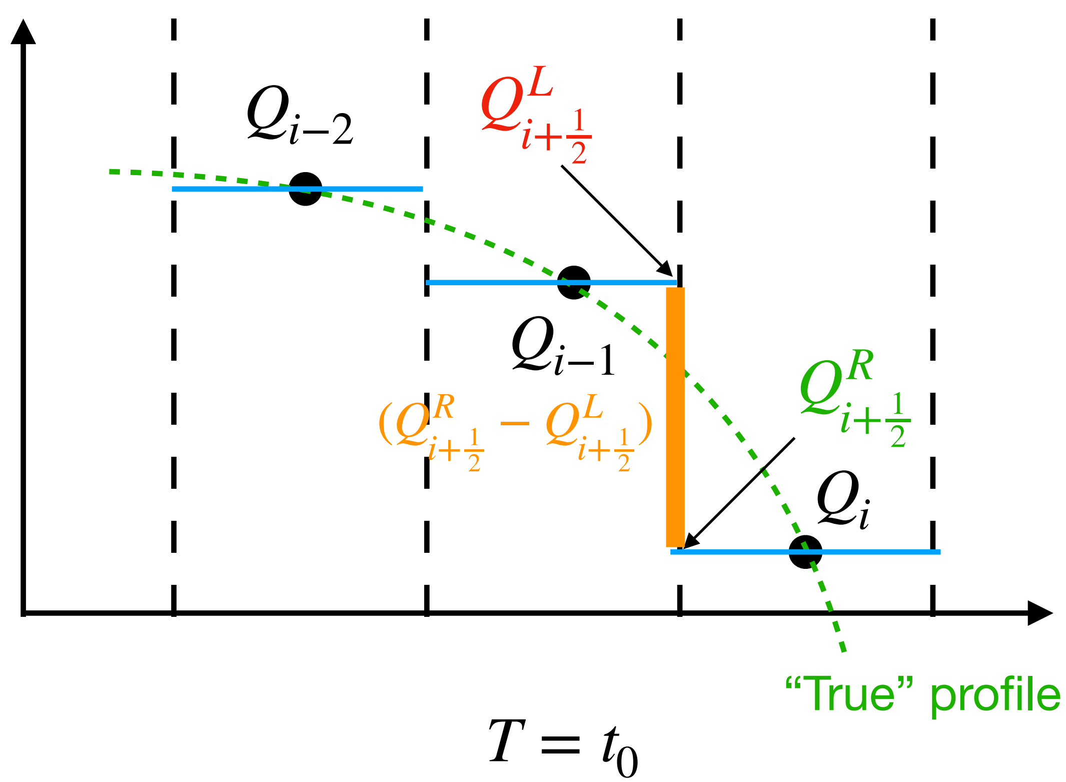
So the numerical diffusion is basically from Q_R - Q_L, how to reduce that?

Finite Volume Methods

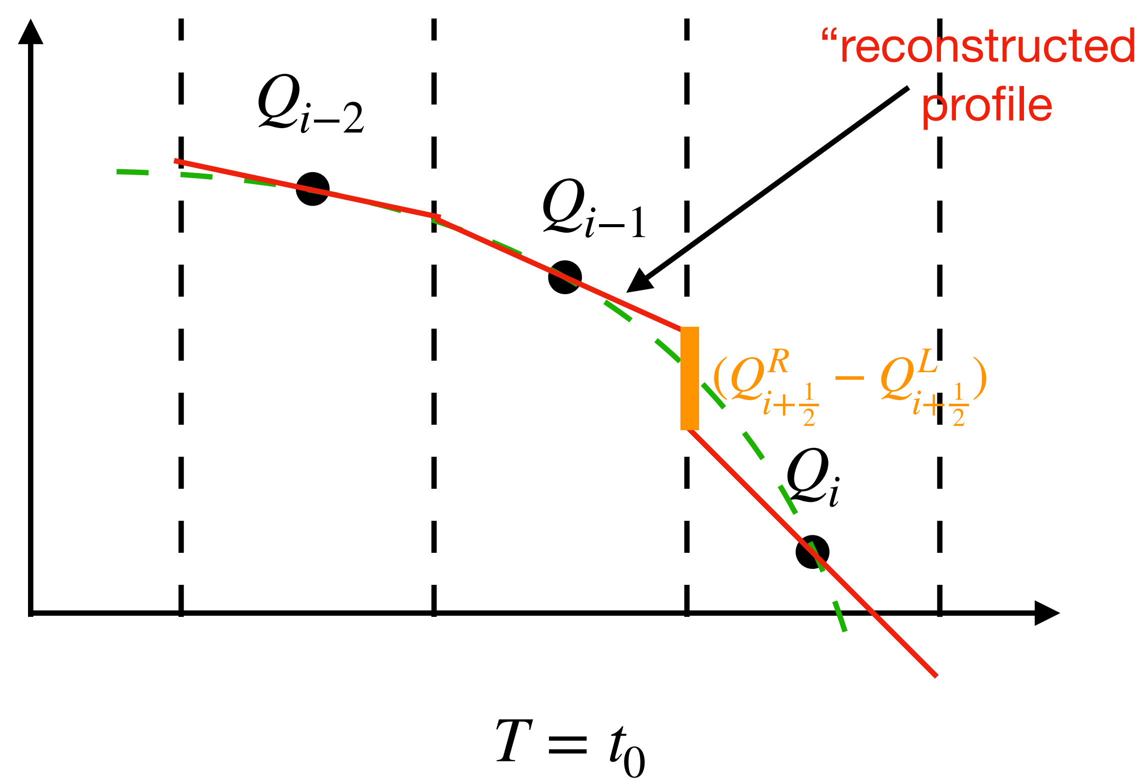
REA with Second-order methods

Now to improve the solution, using a more accurate reconstruction for the profile is needed:

Zeroth-order Reconstruction



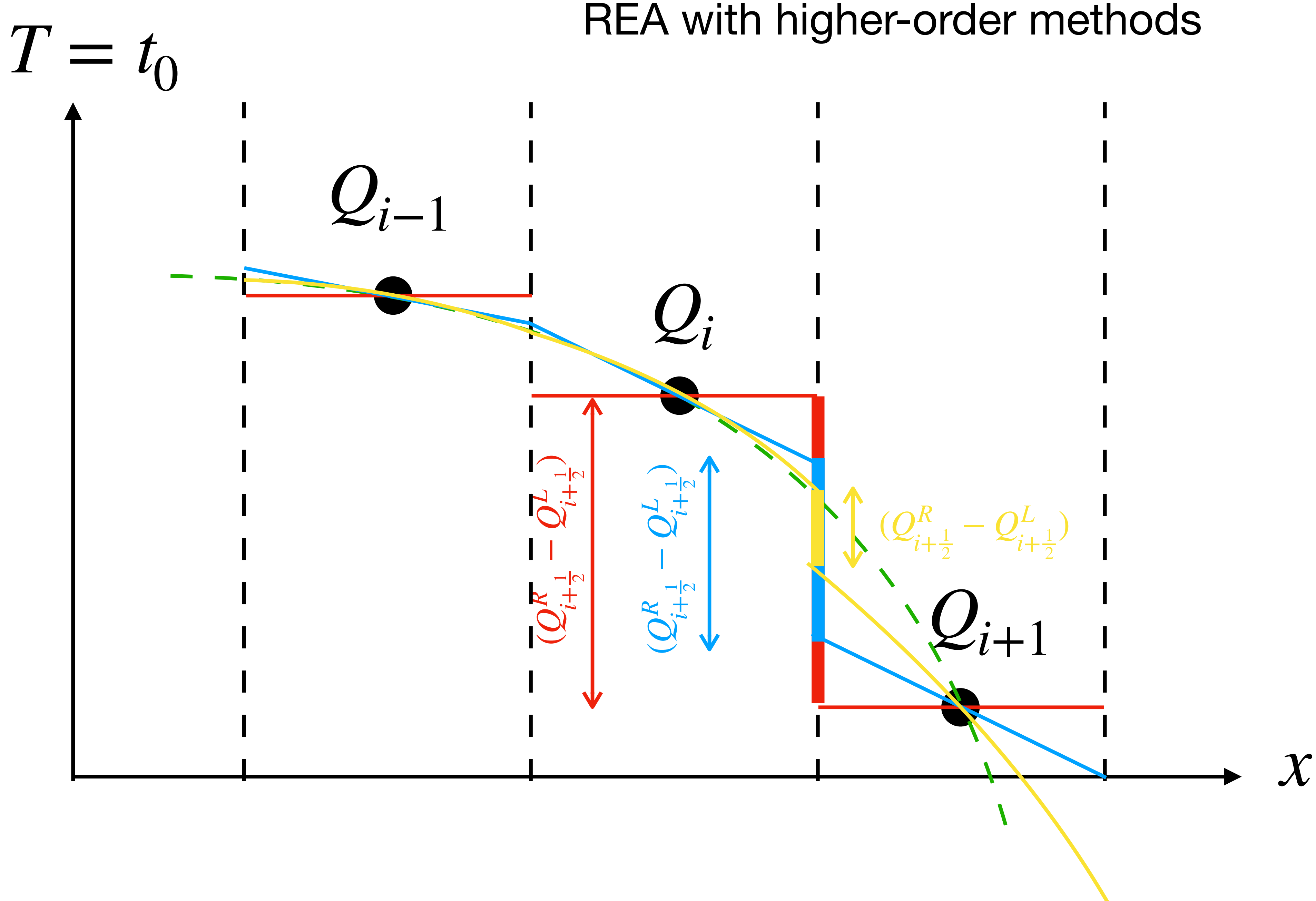
1st-order Reconstruction



Flux is basically: $F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \underbrace{\frac{1}{2}|u_0|\boxed{(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})}}_{\text{Diffusion term}}$

Finite Volume Methods

REA with higher-order methods



High-order reconstruction

↓

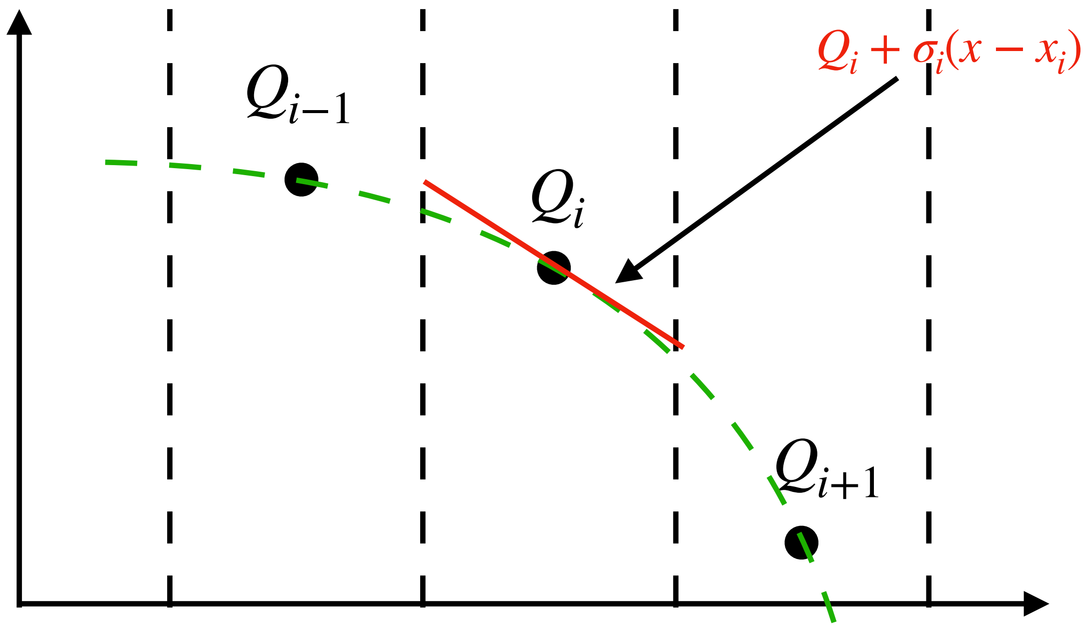
Low diffusion

$$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \underbrace{\frac{1}{2}\,|\,u_0\,|\,\boxed{(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})}}_{\text{Diffusion term}}$$

Does that mean we can go arbitrary high order reconstruction for QL and QR?

Why 2nd-order method oscillates?

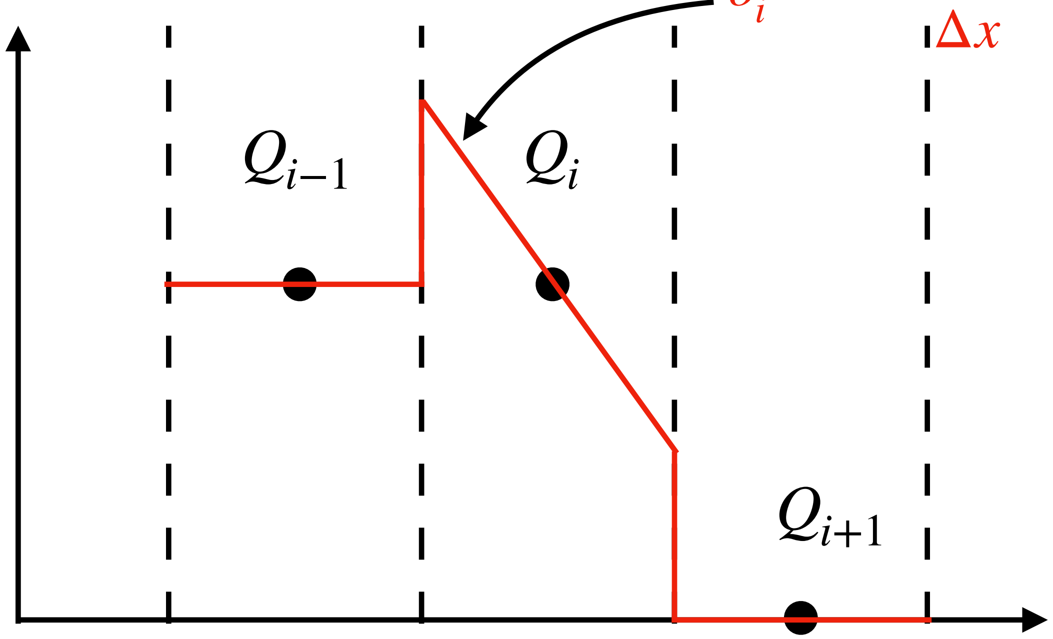
Lax-Wendroff



$T = t_0$

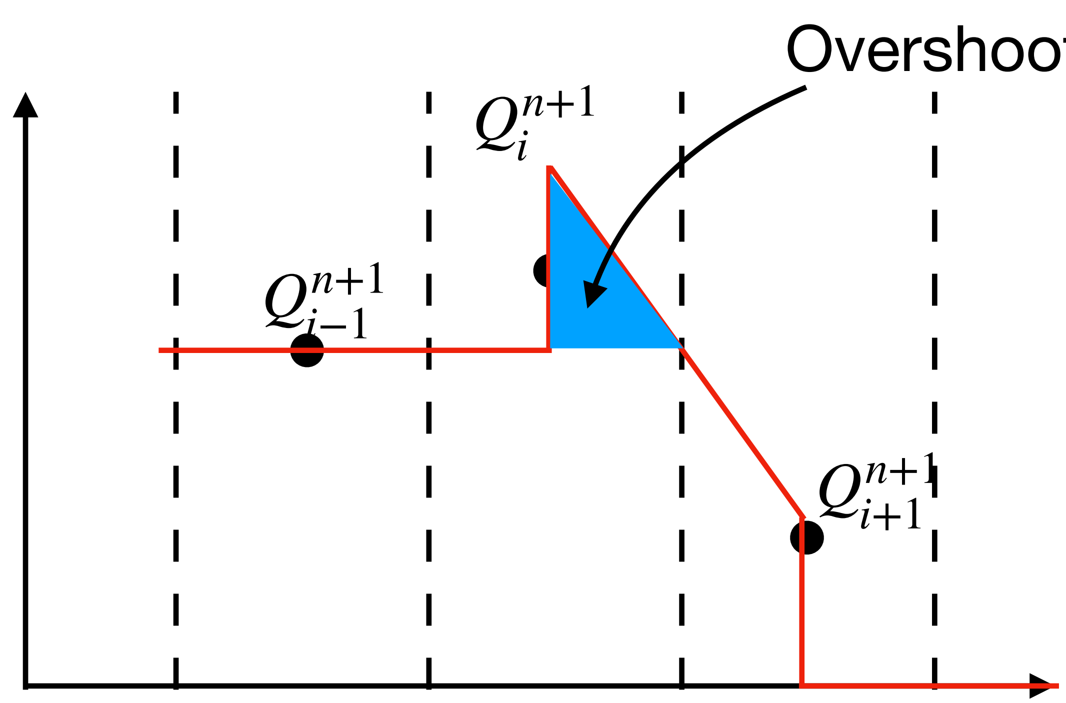
Use this slope in reconstruction

$$\sigma_i^n = \frac{Q_{i+1}^n - Q_i^n}{\Delta x}$$ (Lax Wendroff method)



$T = t_0$

Advect



$T = t_0 + \Delta t$

Total Variation

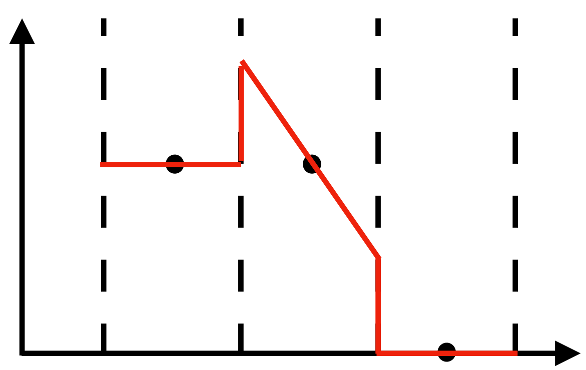
Definition: $TV \equiv \sum_{i=-\infty}^{+\infty} |Q_{i+1}^n - Q_i^n|$ For the linear advection equation, TV(Q) is a constant

Definition of TVD: a two-step method is TVD if $TV(Q^{n+1}) \leq TV(Q^n)$

So if a method is TVD, then if the initial profile is “monotonic”: $Q^{n+1} \geq Q^n$ for all i

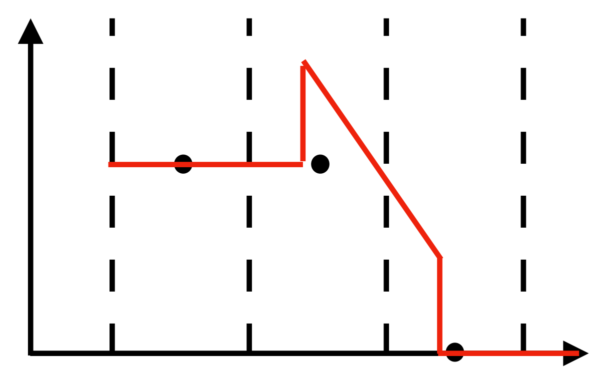
It will ***remain monotonic*** in all the future time steps (monotonicity preserving)

Reconstruct



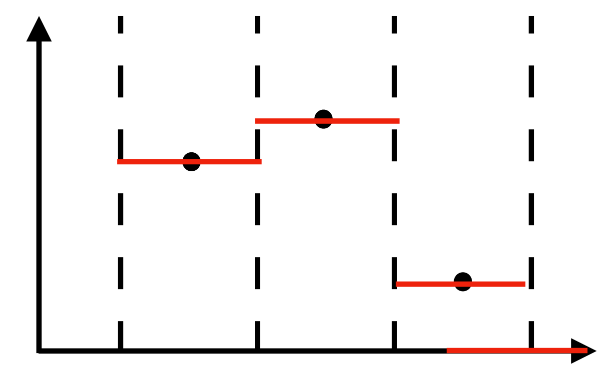
TV?

Evolve



$TV(q(x, t_{n+1}) = TV(q(x, t_n))$

Average

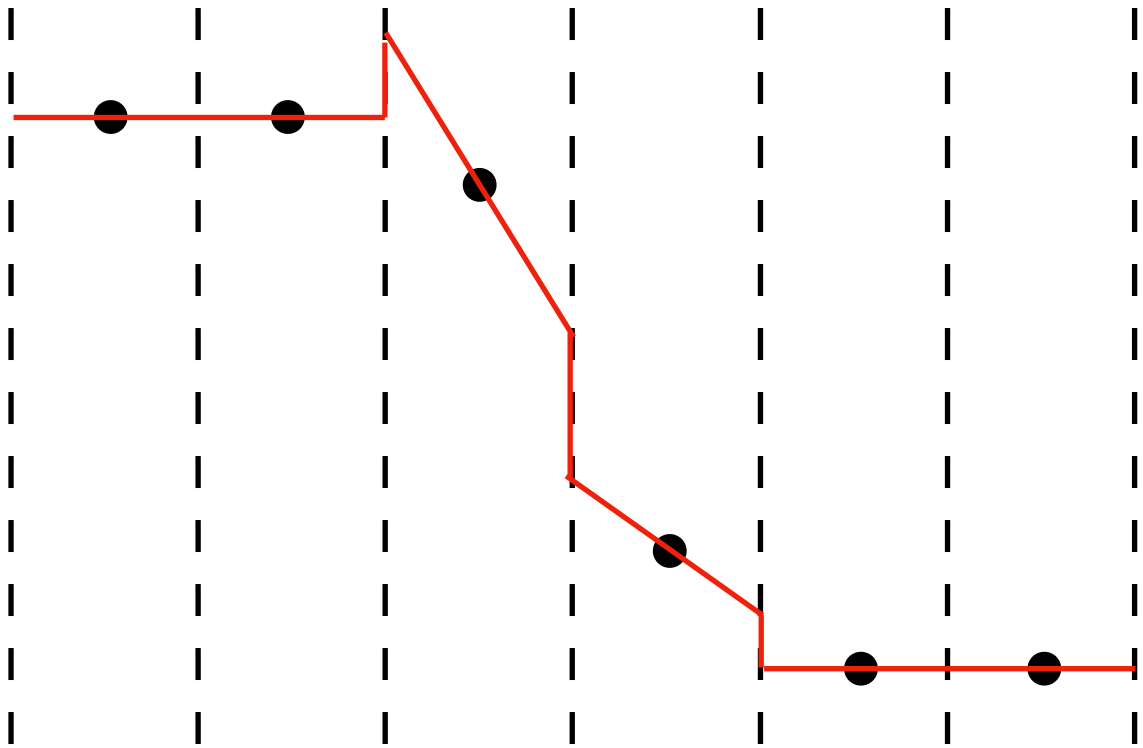


$TV(q(x, t_{n+1}) = TV(Q^{n+1})$

So it is the reconstruction step determines whether a scheme is TVD or not

Slope limiters for TVD solutions

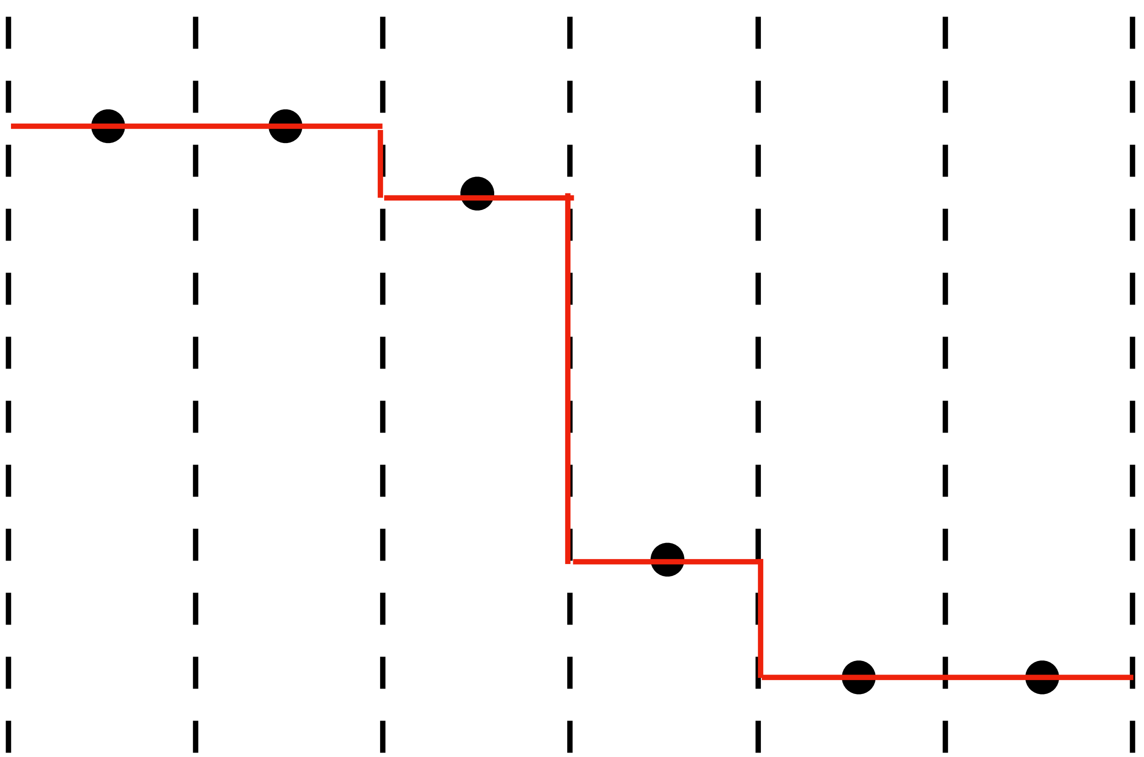
non-TVD reconstruction



$$\sigma_i^n = \frac{Q_{i+1}^n - Q_i^n}{\Delta x}$$ (Lax Wendroff method)

It's second-order but apparently non-TVD!

TVD reconstruction (PCM)

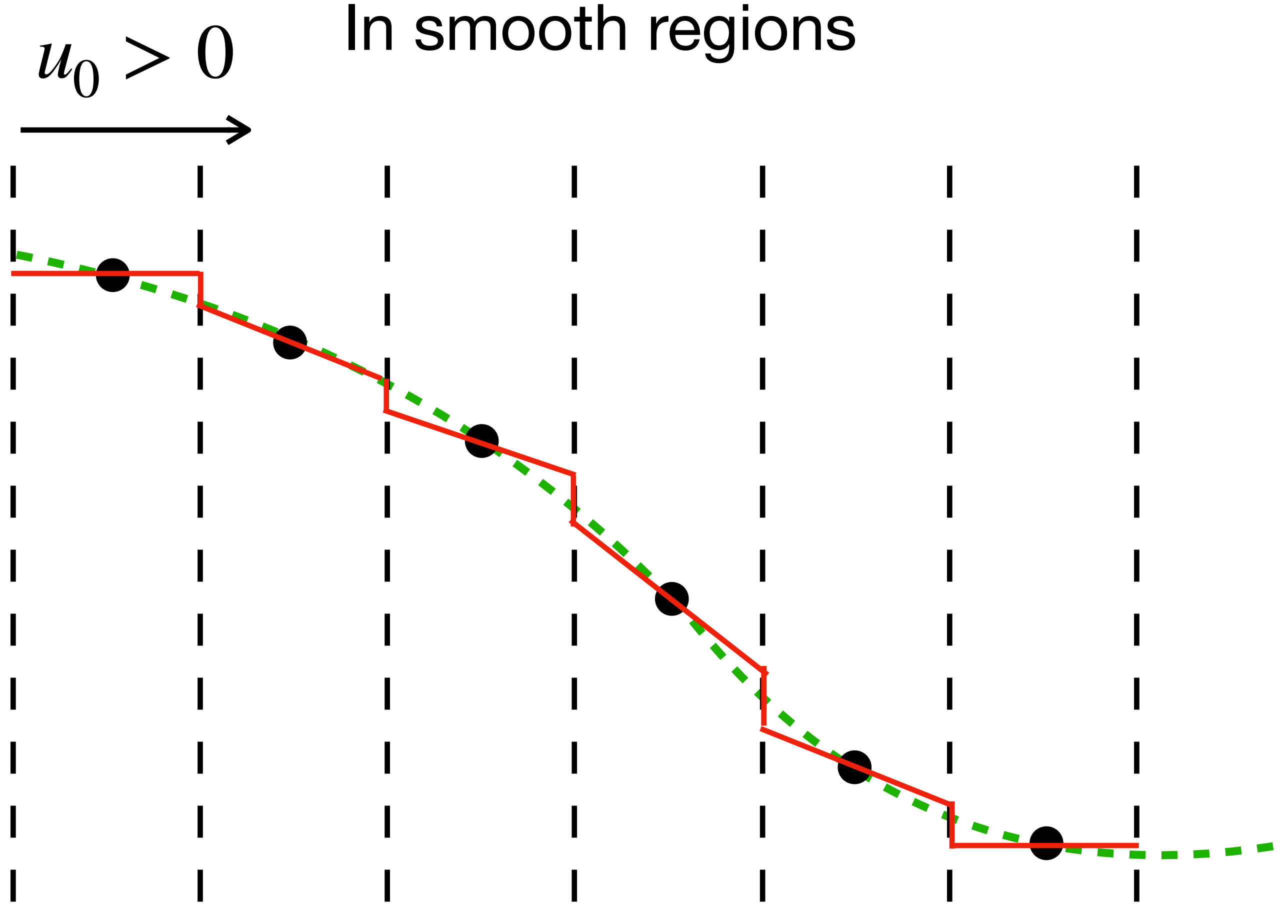


$$\sigma_i^n = 0$$ (first-order upwind)

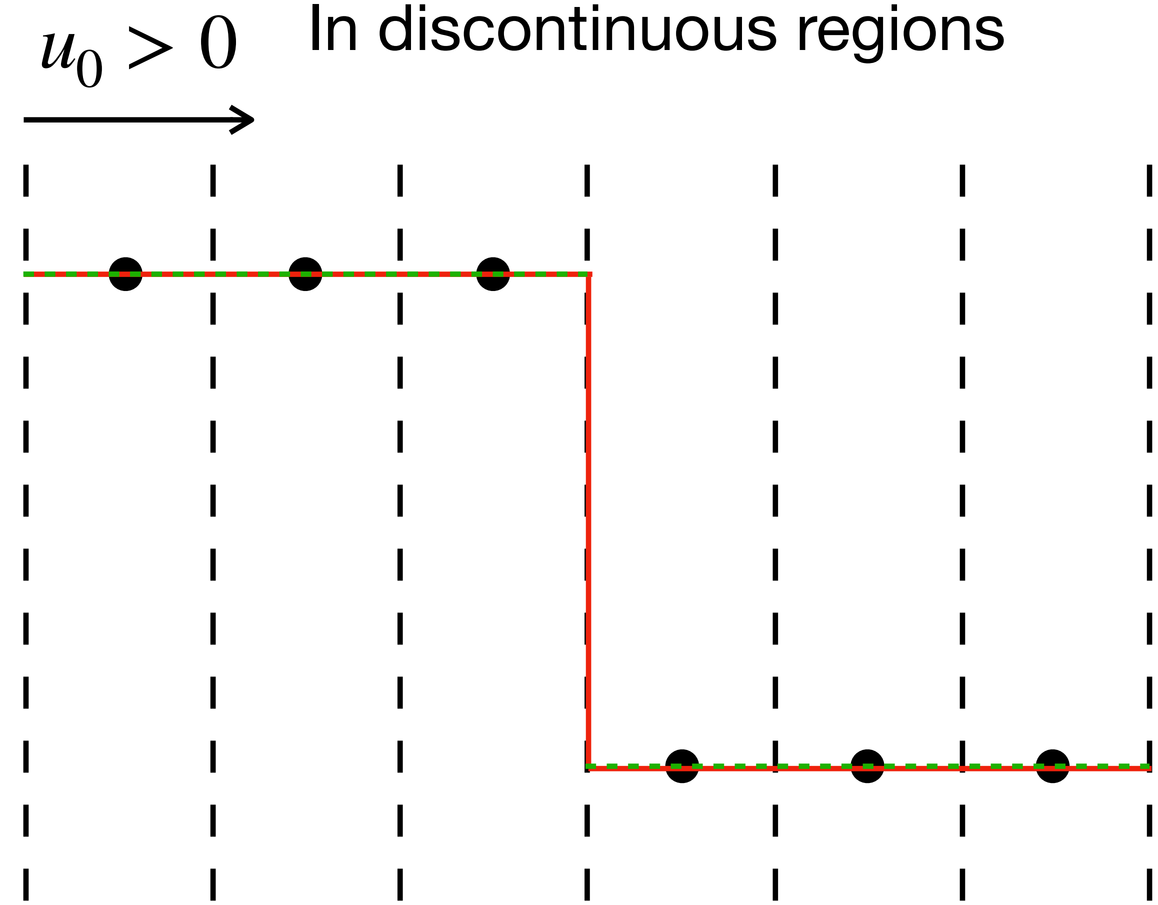
It's TVD but apparently first-order

What does slope limiters do?

In smooth regions



In discontinuous regions



$$\sigma_i^n = \text{minmod}\left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x}, \frac{Q_i^n - Q_{i-1}^n}{\Delta x}\right)$$

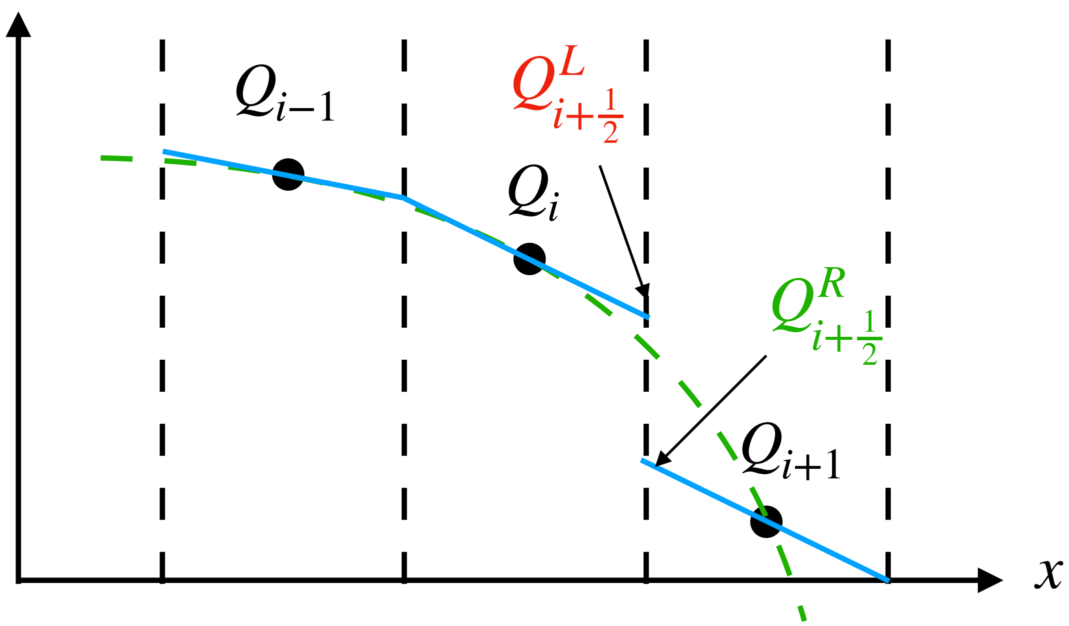
$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0, \\ b & \text{if } |b| < |a| \text{ and } ab > 0, \\ 0 & \text{if } ab \leq 0. \end{cases}$$

- In smooth regions, the minmod slope limiter gives a profile that approximates the true profile with piecewise linear functions (2nd-order accuracy)
- In discontinuous regions, the minmod slop limiter chooses the smaller slope which is degenerated to the 1st-order upwind method (guaranteed TVD)

Summary of the Finite Volume framework

Step 1: Interface Reconstruction

purpose: get interface values



INPUT:	Q_i	
OUTPUT:	$Q^L_{i+\frac{1}{2}}$	$Q^R_{i+\frac{1}{2}}$

Step 2: Flux calculation

$$F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$$

INPUT:	$Q^L_{i+\frac{1}{2}}$	$Q^R_{i+\frac{1}{2}}$
OUTPUT:	$F_{i+\frac{1}{2}}$	

Step 3: Finite Volume Update

$$\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

INPUT:	Q_i^n	$F_{i+\frac{1}{2}}$
OUTPUT:	Q_i^{n+1}	

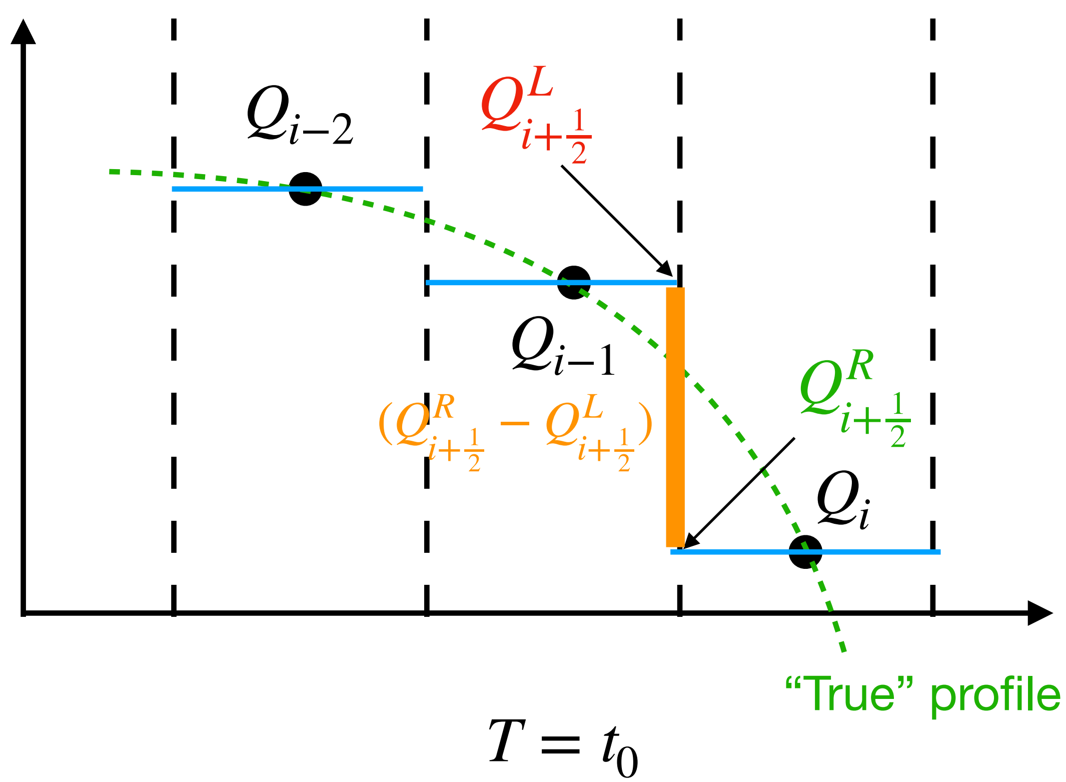
$$Q_i^n \xrightarrow{\text{Reconstruct}} \begin{matrix} Q^L_{i+\frac{1}{2}} \\ Q^R_{i+\frac{1}{2}} \end{matrix} \xrightarrow{\text{Flux calculation}} F_{i+\frac{1}{2}} \xrightarrow{\text{F-V update}} Q_i^{n+1}$$

Slope limiting methods - gets tedious

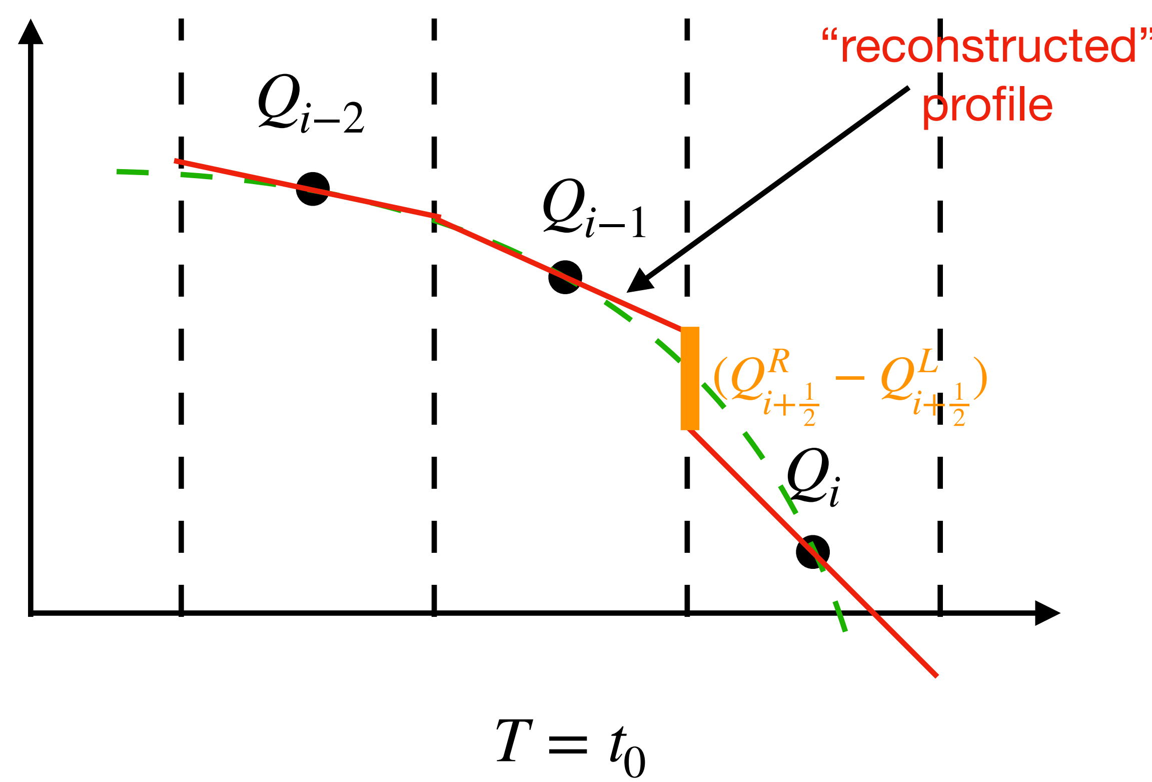
REA with higher-order methods

Now to improve the solution, using a more accurate reconstruction for the profile is needed:

Zeroth-order Reconstruction



1st-order Reconstruction

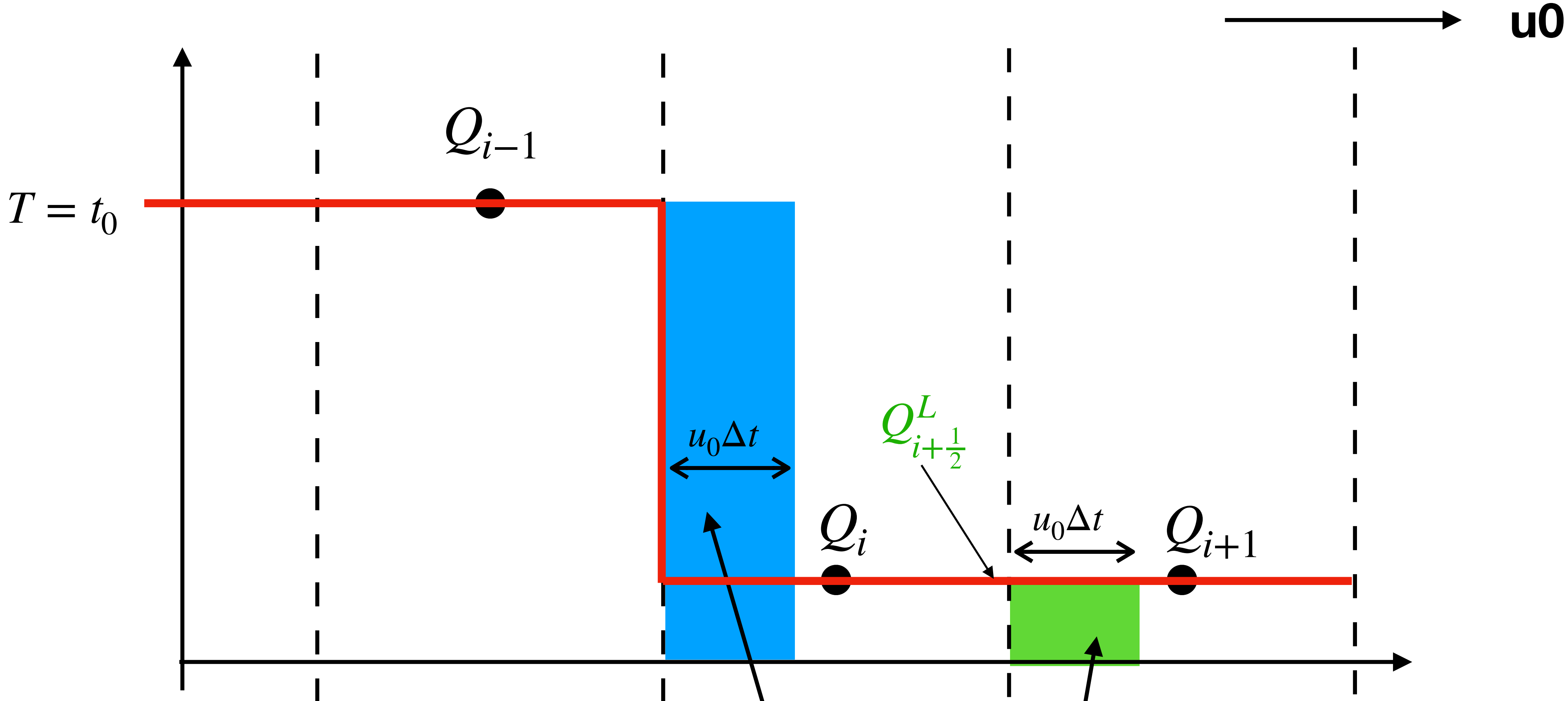


Flux is basically: $F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \underbrace{\frac{1}{2}|u_0|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})}_{\textbf{Diffusion term}}$

Depending on the reconstructed profile q(x)

The Donor Cell Method

Recall the flux balance interpretation **Upwind**



$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) \quad = Q_i^n + \frac{\Delta t}{\Delta x}\left(\boxed{u_0 Q_{i-1}^n} - \boxed{u_0 Q_i^n}\right) \quad \in (Q_{i-1}, Q_i)$$

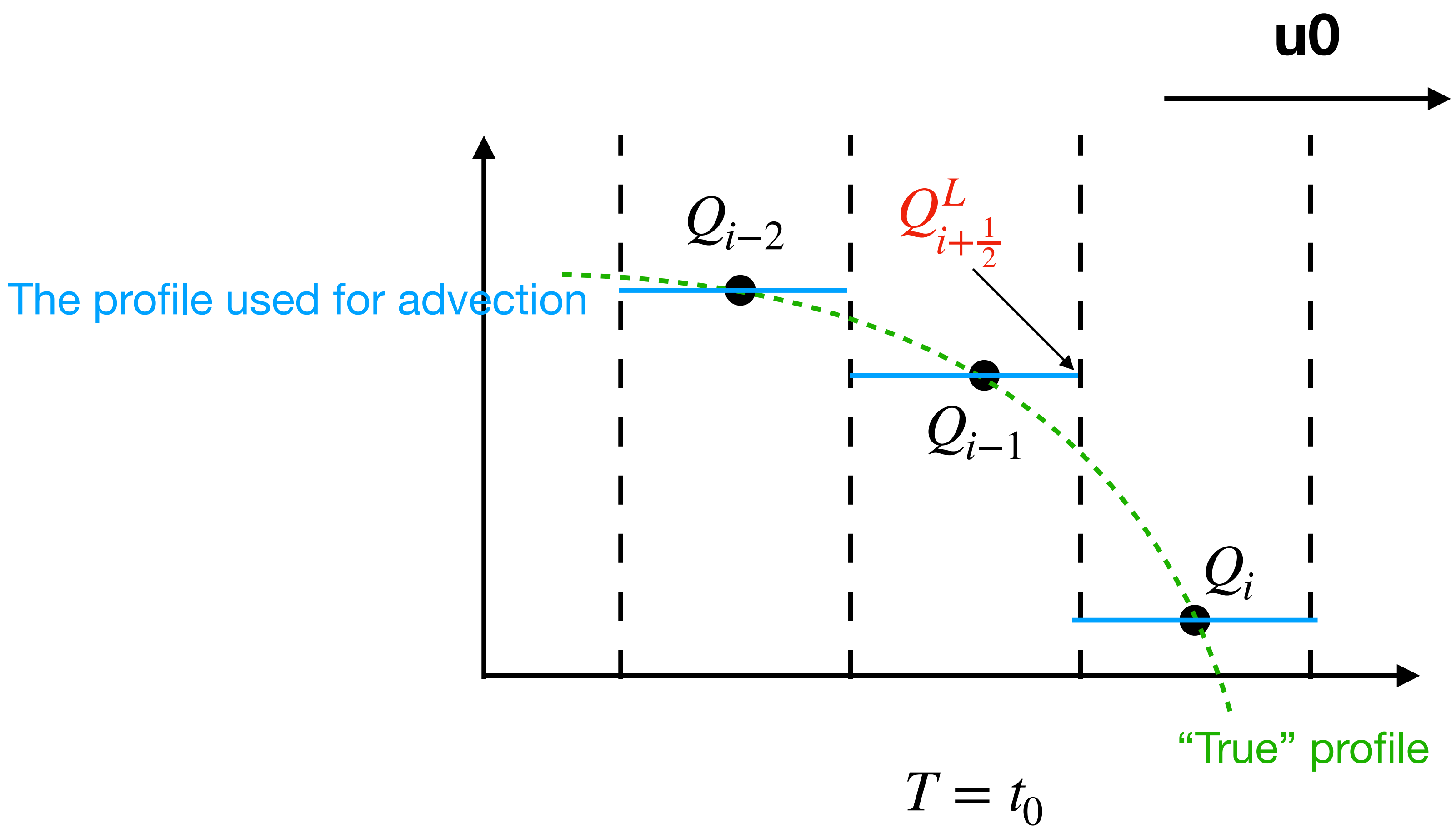
Flux entering cell i

Flux leaving cell i

$F_{i-\frac{1}{2}}$ $\quad$ $F_{i+\frac{1}{2}}$

Improve the Donor Cell Method

The donor cell method is extremely diffusive because:

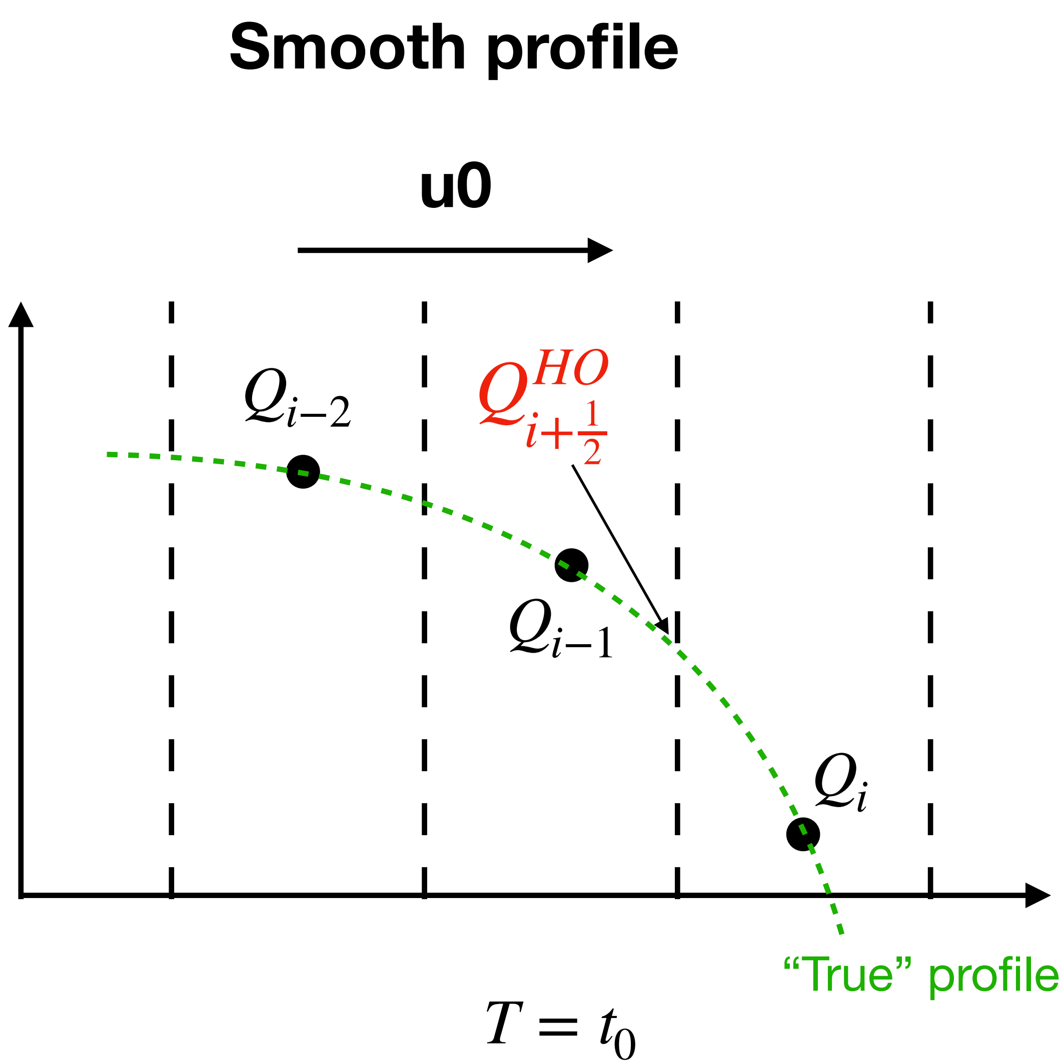


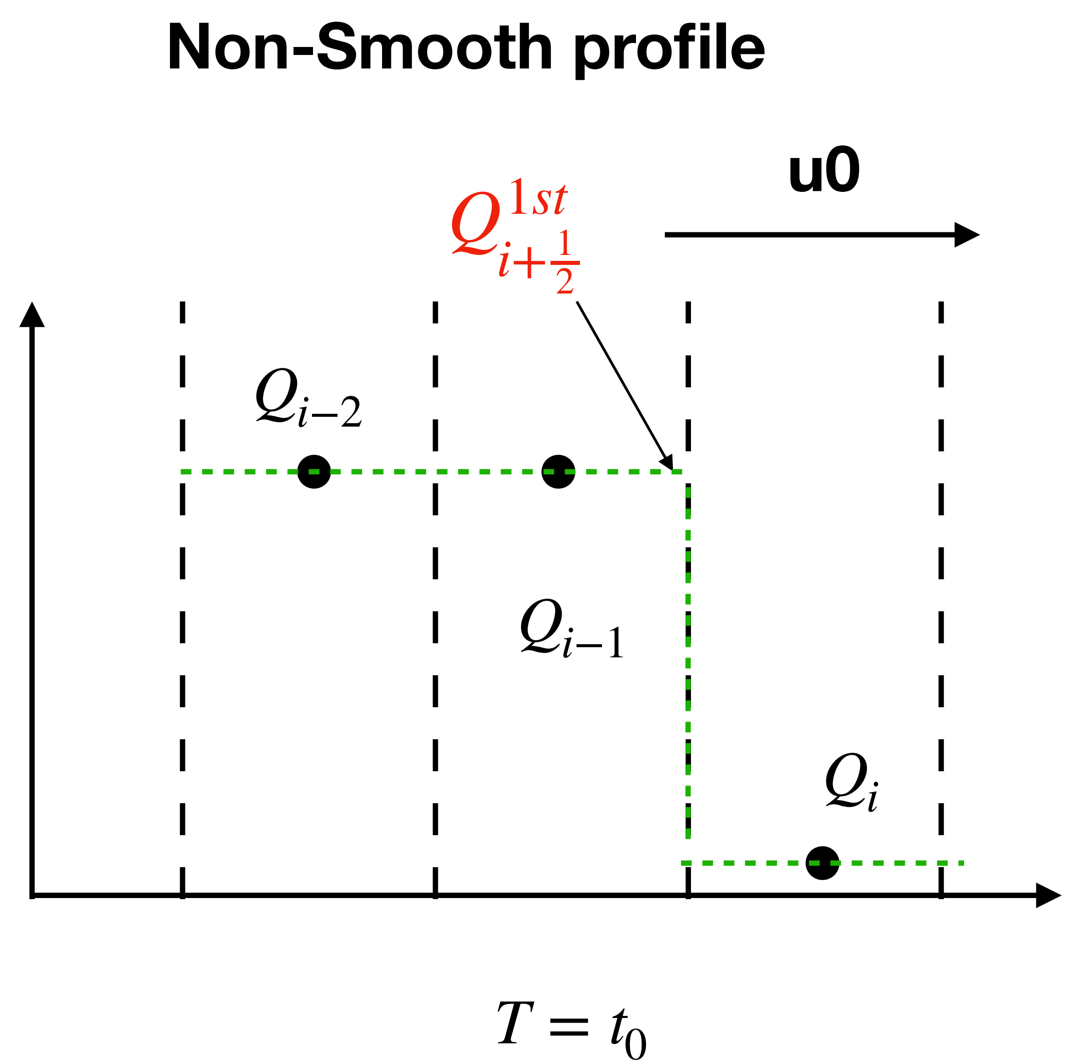
We are always using the cell from the upwind direction (donor), which makes the flux exactly the 1st-order upwind method - very diffusive but stable and TVD

To improve the donor cell method, let's think about this way - smooth and non-smooth profiles

Improve the Donor Cell Method

The donor cell method is extremely diffusive because:



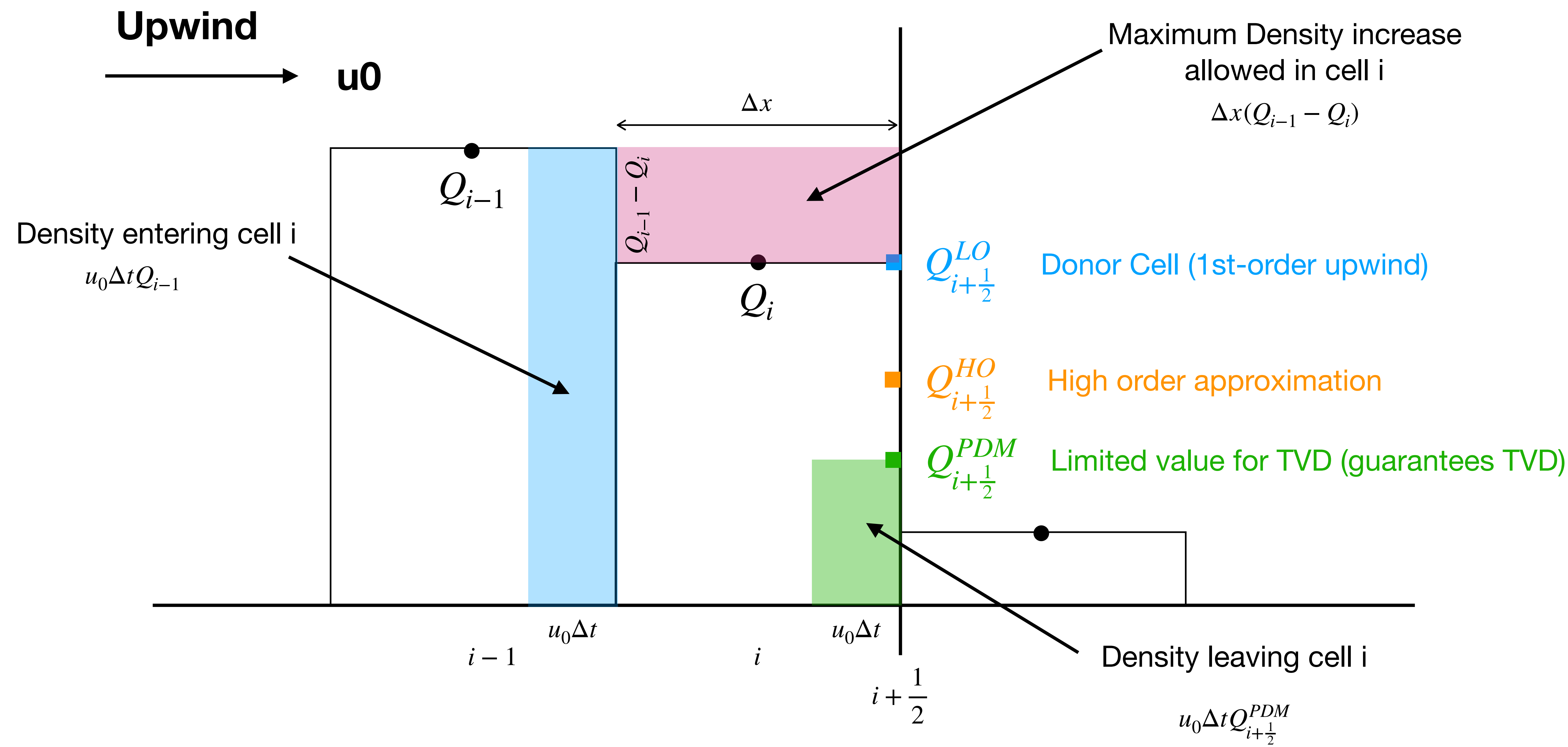


Idea: in smooth structure region, use a high-order approximation for accuracy

in non-smooth structure region, use a low-order upwind value for stability (TVD)

The Partial Donor Cell Method

How to "correct" the left interface state $Q^L_{i+\frac{1}{2}}$



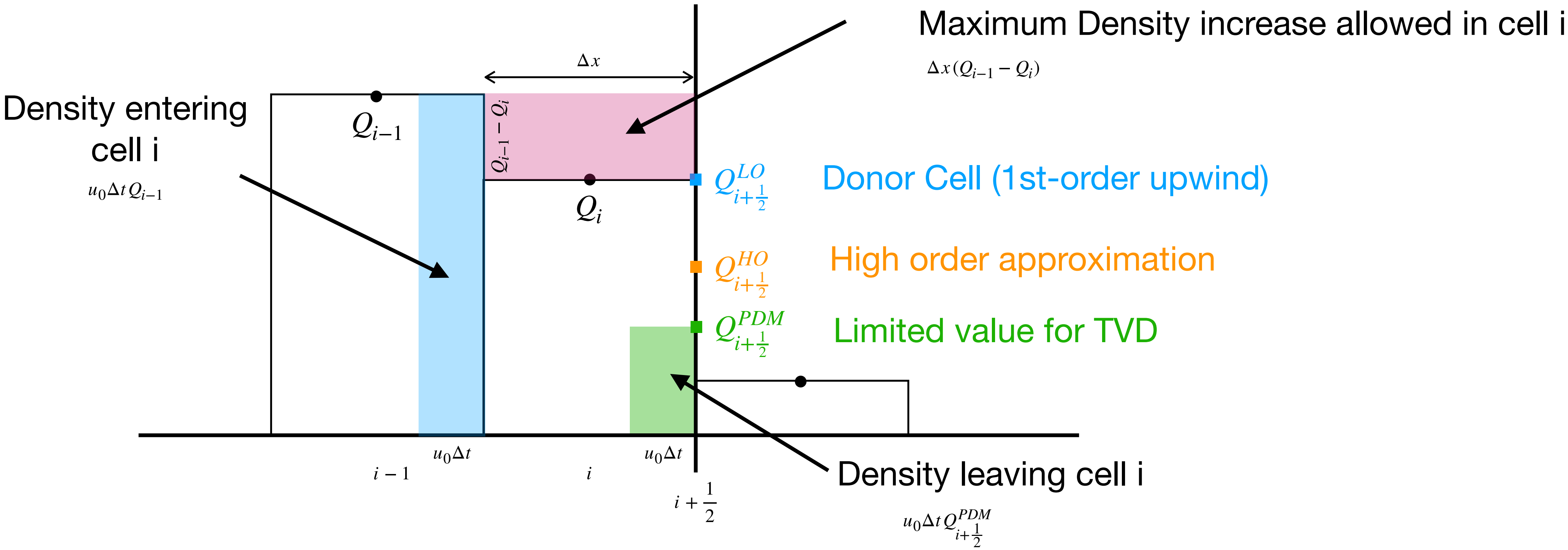
Now we can find the $Q^{PDM}_{i+\frac{1}{2}}$, which is the "limited" value for TVD:

$$u_0 \Delta t Q_{i-1} - u_0 \Delta t Q^{PDM}_{i+\frac{1}{2}} = \Delta x (Q_{i-1} - Q_i) \longrightarrow Q^{PDM}_{i+\frac{1}{2}} = \frac{1}{\epsilon} Q_i + (1 - \frac{1}{\epsilon}) Q_{i-1}$$

If we use any interface value < $Q^{PDM}_{i+\frac{1}{2}}$, cell i goes overshoot (or undershoot)

The Partial Donor Cell Method

How to “correct” the interface flux



Now we have three candidates for interface values at i+1/2: $Q^{PDM}_{i+\frac{1}{2}}$ $Q^{HO}_{i+\frac{1}{2}}$ $Q^{LO}_{i+\frac{1}{2}}$

Which one to use?

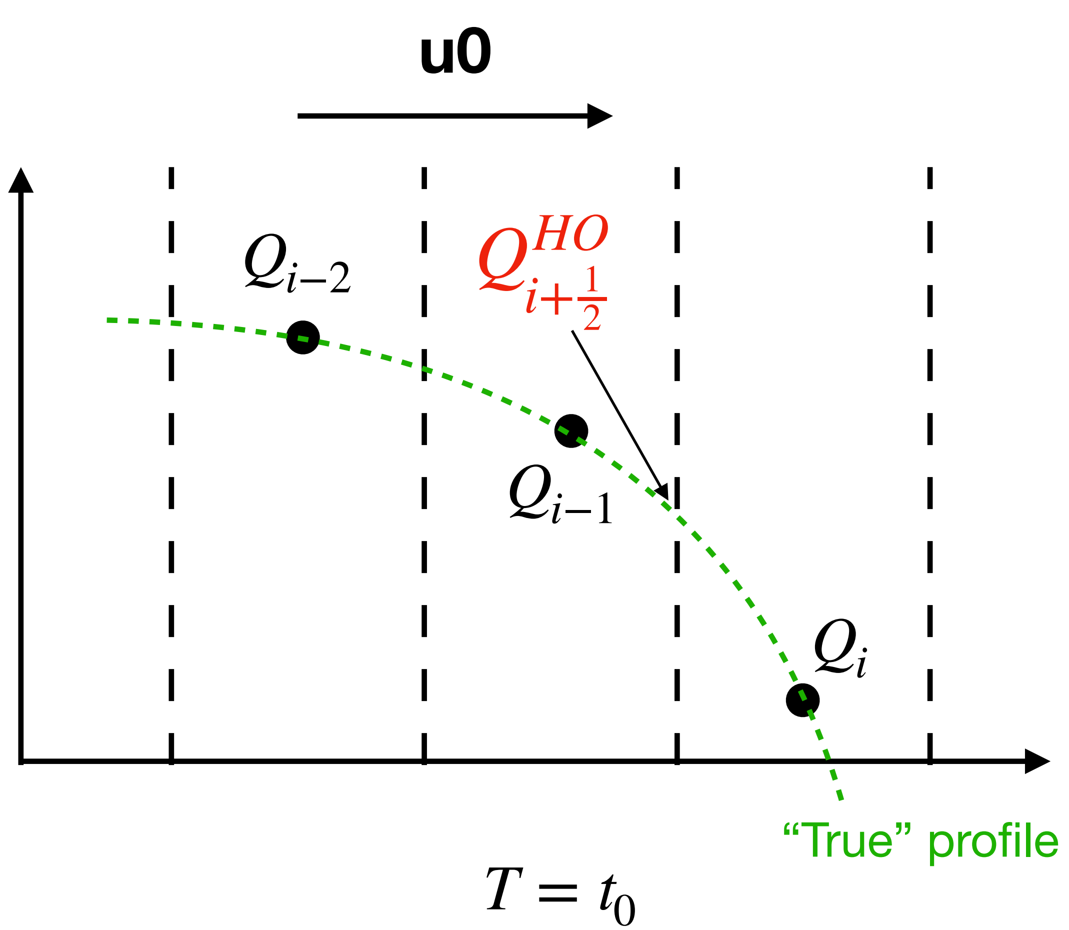
The most intuitive choice is to always use the value in the middle:

$$Q^{L}_{i+\frac{1}{2}} = \mathrm{median}(Q^{LO}_{i+\frac{1}{2}}, Q^{HO}_{i+\frac{1}{2}}, Q^{PDM}_{i+\frac{1}{2}})$$

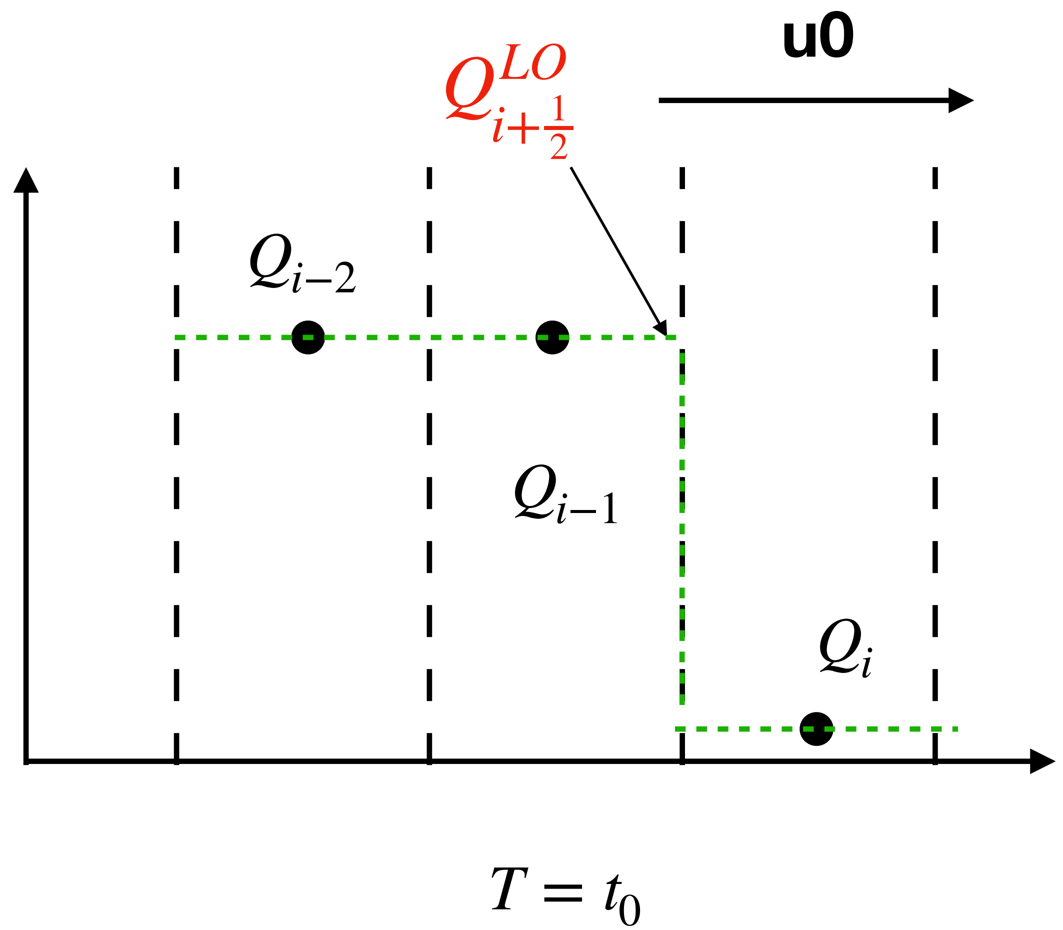
Improve the Donor Cell Method

The partial donor cell method is much less diffusive because it tries to use the high-order approximation whenever possible:

Smooth profile



Non-Smooth profile



Idea: in smooth structure region, use a high-order approximation for accuracy

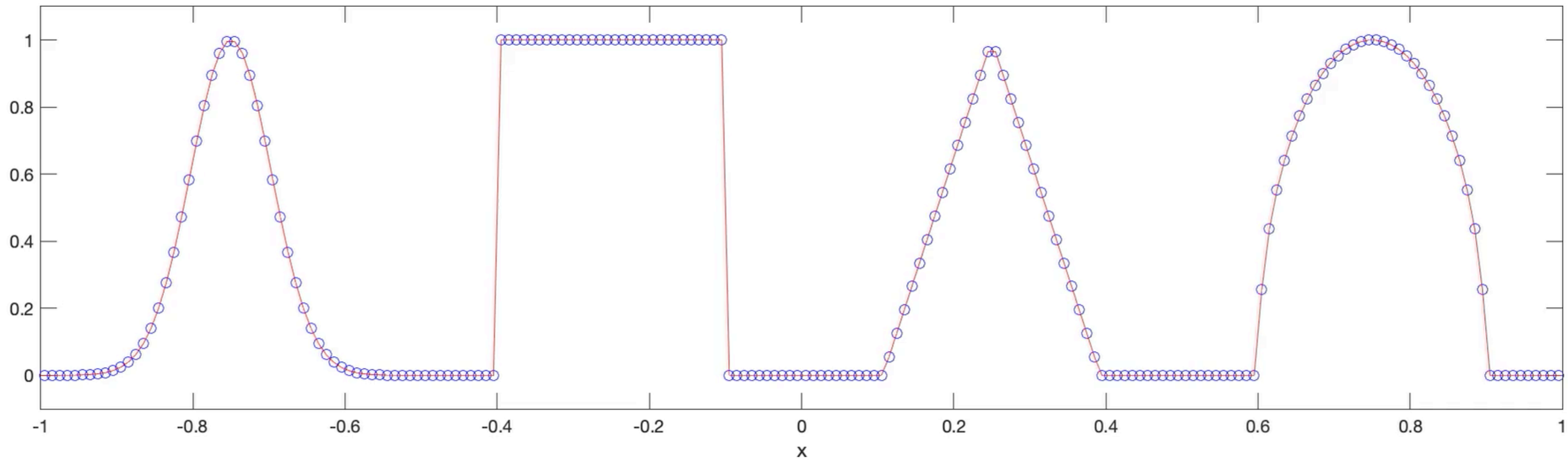
$Q^{HO}_{i+\frac{1}{2}}$ (arbitrary high order)

in non-smooth structure region, use a low-order upwind value for stability (TVD)

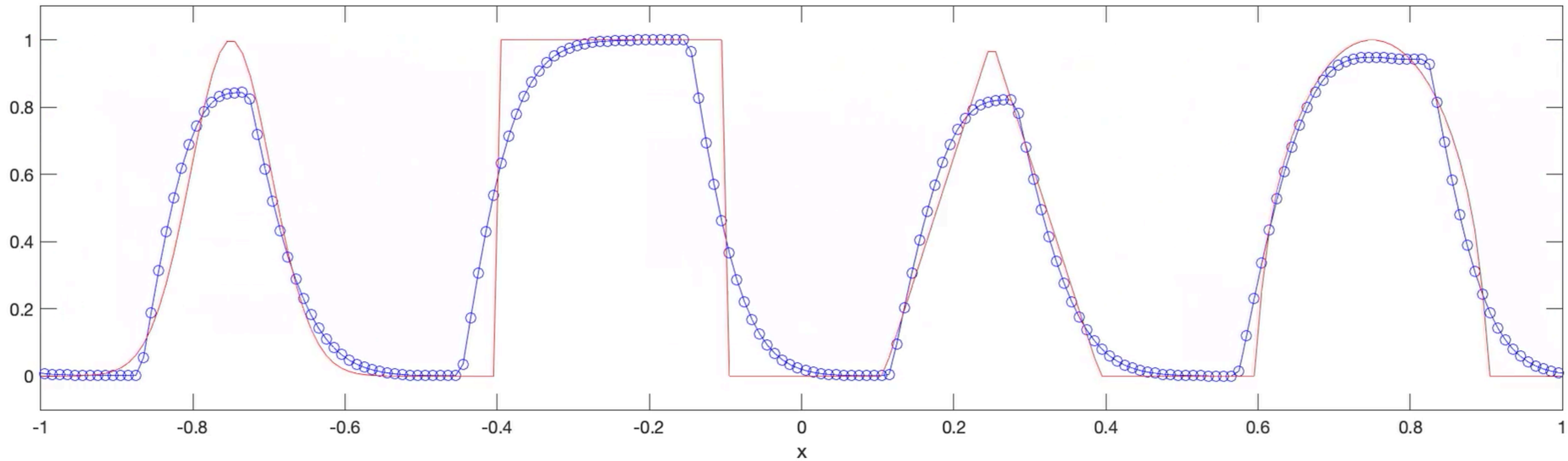
The Partial Donor Cell Method

Test Results - 1st and 2nd order

1st-order (donor cell, or upwind)

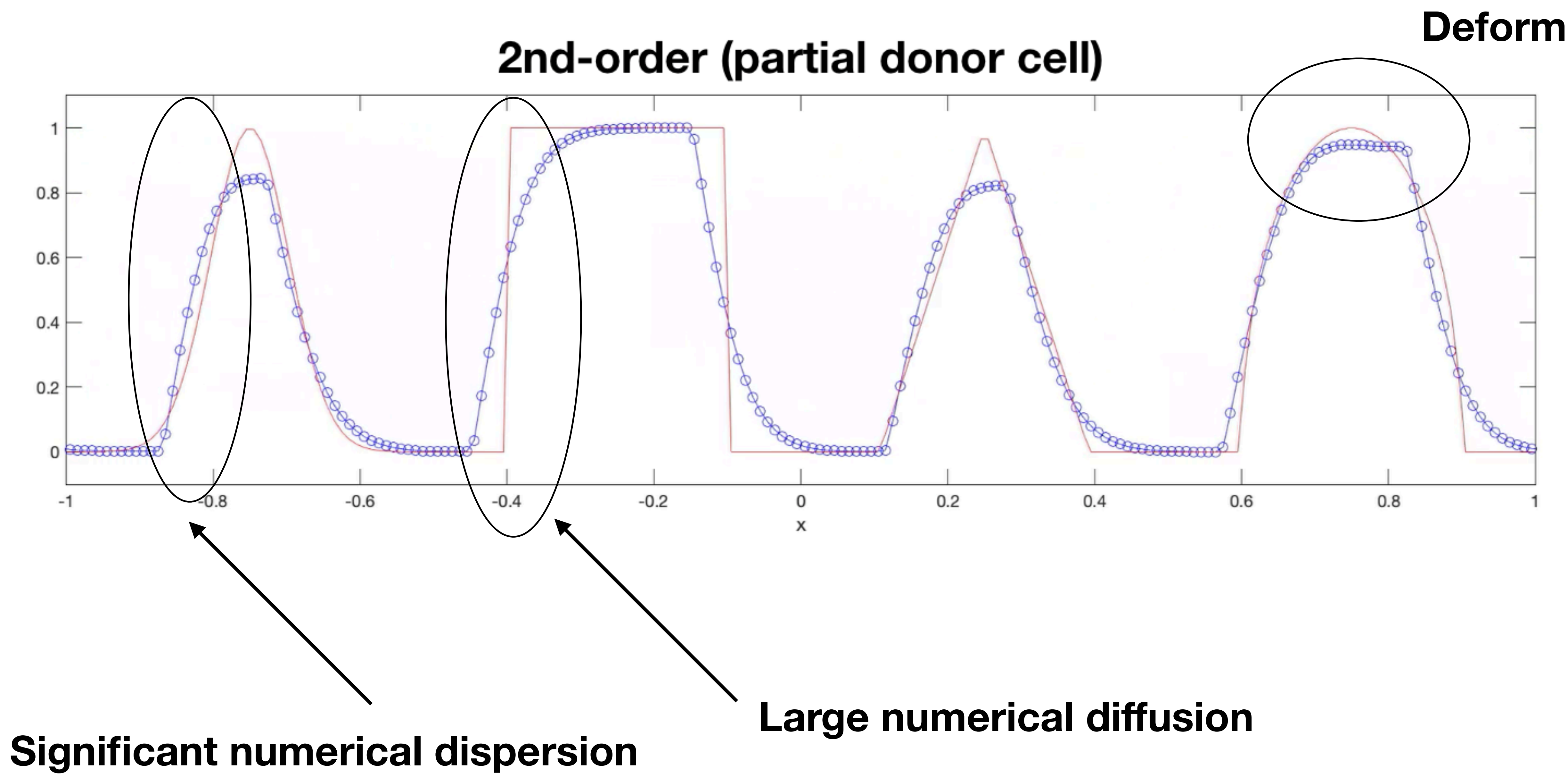


2nd-order (partial donor cell)



The Partial Donor Cell Method

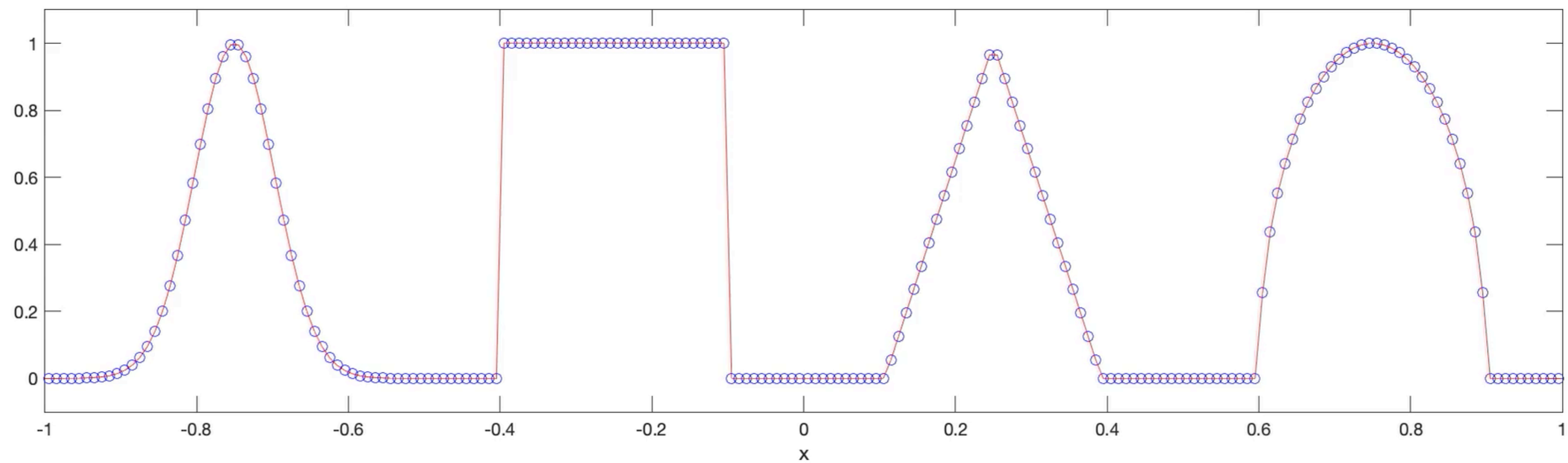
2nd order isn't good enough



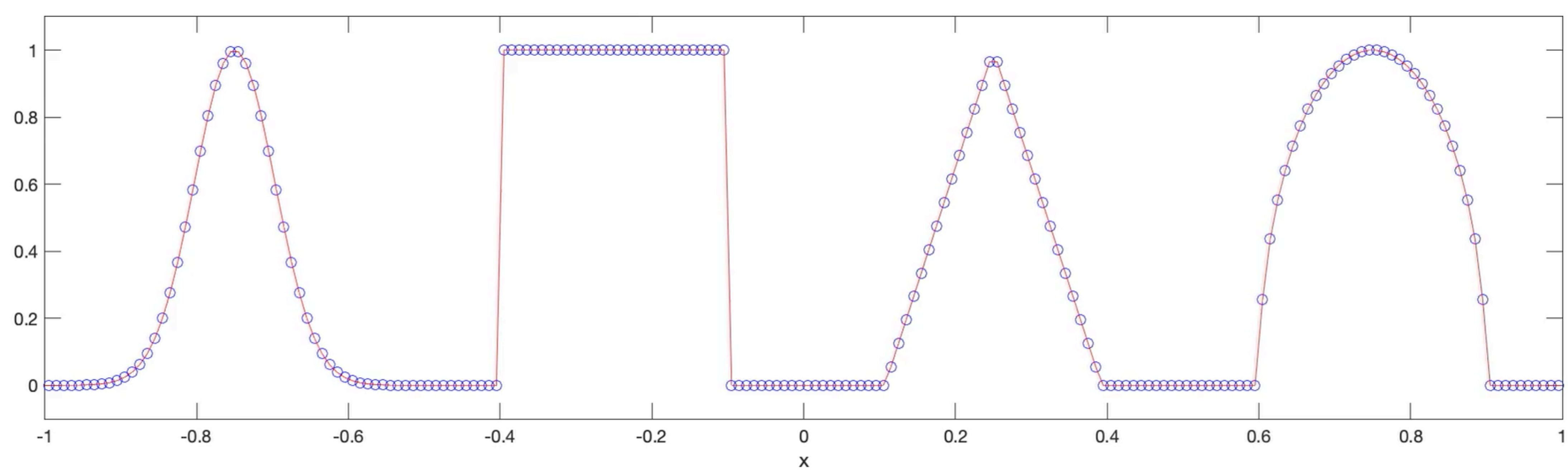
The Partial Donor Cell Method

Test Results - 1st and 2nd order

3rd-order



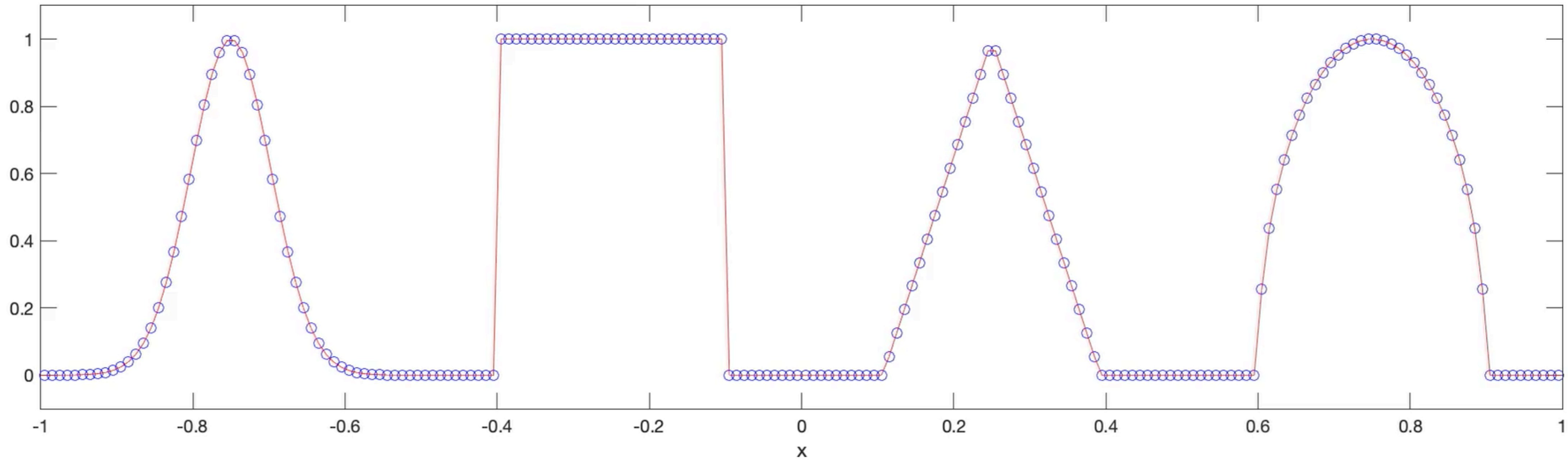
5th-order



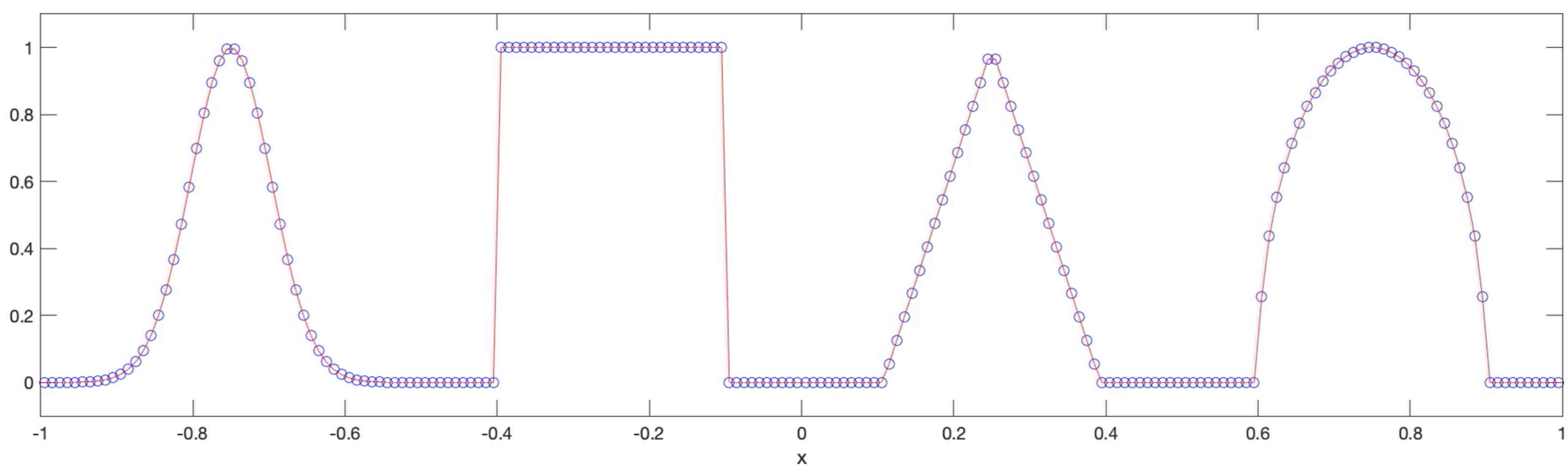
The Partial Donor Cell Method

Test Results - beyond 2nd-order

7th-order



9th-order



Extending to Non-Linear Equations

$$Q_i^n \xrightarrow{\text{Reconstruct}} \begin{matrix} Q^L_{i+\frac{1}{2}} \\ Q^R_{i+\frac{1}{2}} \end{matrix} \xrightarrow{\text{Flux calculation}} F_{i+\frac{1}{2}} \xrightarrow{\text{F-V update}} Q_i^{n+1}$$

Extending the above steps to a non-linear equation is very straightforward

For example, Burger's equation:

$$\frac{\partial Q}{\partial t} + Q\frac{\partial Q}{\partial x} = 0 \longrightarrow \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}Q^2\right) = 0 \xrightarrow{F(Q) = \frac{1}{2}Q^2} \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}F(Q) = 0$$

Step 1: reconstruction $Q_i^n \longrightarrow Q^L_{i+\frac{1}{2}} \quad Q^R_{i+\frac{1}{2}}$

$$F(Q^L_{i+\frac{1}{2}}) = \frac{1}{2}(Q^L_{i+\frac{1}{2}})^2$$

Step 2: Flux Evaluation $Q^L_{i+\frac{1}{2}} \quad Q^R_{i+\frac{1}{2}} \longrightarrow F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$

Need to estimate wave speed

Step 3: Finite-Volume Update $\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$

Extending to Non-Linear Equations

Example: Burgers' equation

$$\frac{\partial Q}{\partial t} + Q\frac{\partial Q}{\partial x} = 0 \longrightarrow \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}Q^2\right) = 0$$

Step 1: reconstruction $Q_i^n \longrightarrow Q^L_{i+\frac{1}{2}} \quad Q^R_{i+\frac{1}{2}}$

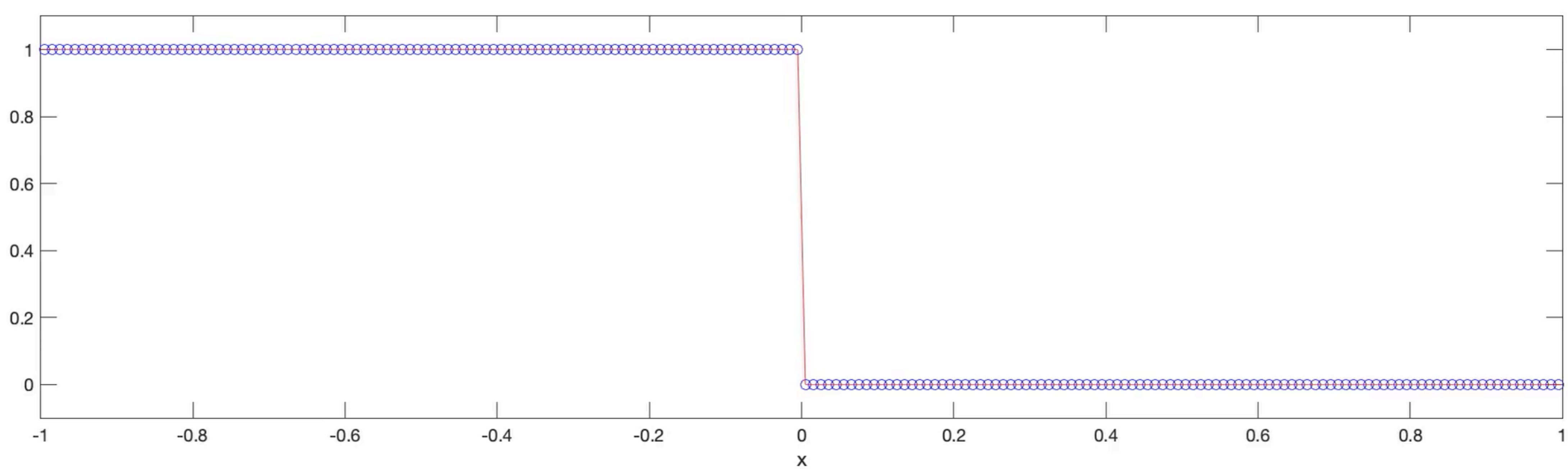
$F(Q^L_{i+\frac{1}{2}}) = \frac{1}{2}(Q^L_{i+\frac{1}{2}})^2$

Step 2: Flux Evaluation $Q^L_{i+\frac{1}{2}} \quad Q^R_{i+\frac{1}{2}} \longrightarrow F_{i+\frac{1}{2}}(Q^L_{i+\frac{1}{2}}, Q^R_{i+\frac{1}{2}}) = \frac{1}{2}(F(Q^L_{i+\frac{1}{2}}) + F(Q^R_{i+\frac{1}{2}})) - \frac{1}{2}|u_{max}|(Q^R_{i+\frac{1}{2}} - Q^L_{i+\frac{1}{2}})$

Step 3: Finite-Volume Update $\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$

Need to estimate wave speed

$$|u_{max}| = \max(|Q^L_{i+\frac{1}{2}}|, |Q^R_{i+\frac{1}{2}}|)$$



What if using the Finite-Difference Method?

$$\left(\frac{\partial u}{\partial t}\right)_i \approx \frac{u_i - u_{i-1}}{\Delta x}$$

Backward difference

Example: Burgers' equation

$$\frac{\partial Q}{\partial t} + Q\frac{\partial Q}{\partial x} = 0$$

u0 →

Forward difference

Backward difference

Euler Time-Stepping

$$\left.\frac{\partial Q}{\partial t}\right|_{t=t_n} = \frac{Q_i^{n+1} - Q_i^n}{\Delta t} + \mathcal{O}(\Delta t)$$

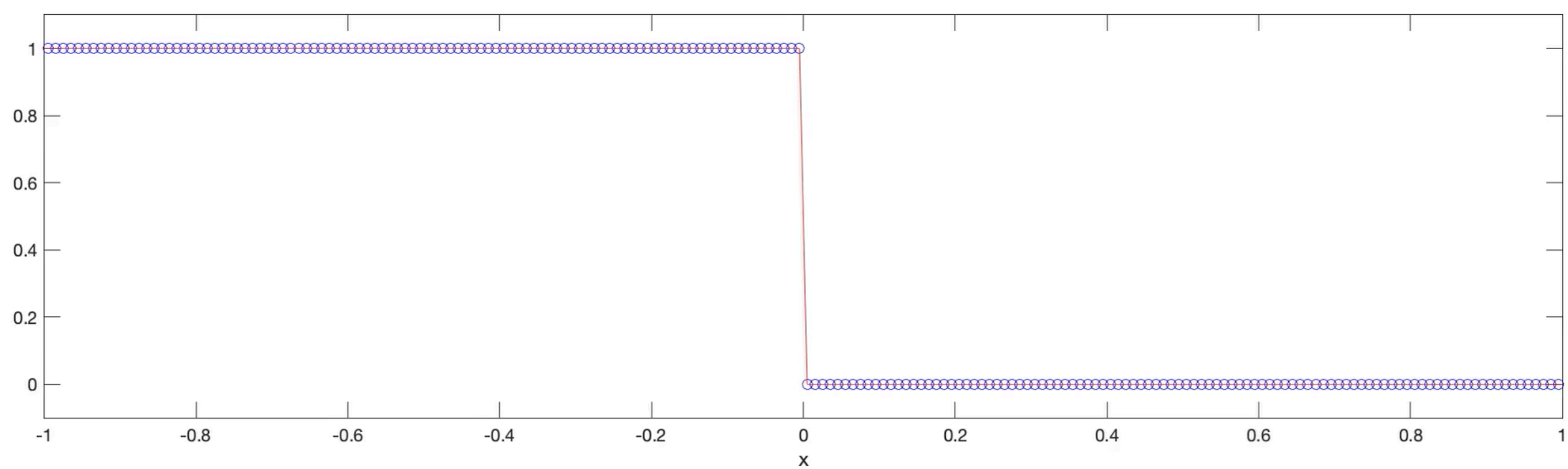
$$\left.\frac{\partial Q}{\partial x}\right|_{x=x_i} = \frac{Q_i^n - Q_{i-1}^n}{\Delta x} + \mathcal{O}(\Delta x)$$

Combine the two numerical derivatives

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} = -Q_i^n\frac{Q_i^n - Q_{i-1}^n}{\Delta x} + \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t) \longrightarrow$$

Forward Euler method

$$\boxed{Q_i^{n+1} \approx Q_i^n - \frac{\Delta t Q_i^n}{\Delta x}(Q_i^n - Q_{i-1}^n)}$$



The Importance of the Conservation Form

The numerical experiment of Burgers equation showed that

- Finite-difference form does not work in some non-linear cases - even though upwinding is enforced
- Finite-volume form does work in non-linear cases - regardless the form of the equations
- Primitive equations versus Conservative equations (tricky and subtle)

Which answer is correct?

Solving the non-conservative form of the MHD equations

Now here's a problem （那么问题来了）

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A global magnetohydrodynamic simulation of the Jovian magnetosphere

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Institute of Geophysics and Planetary Physics, University of California, Los Angeles

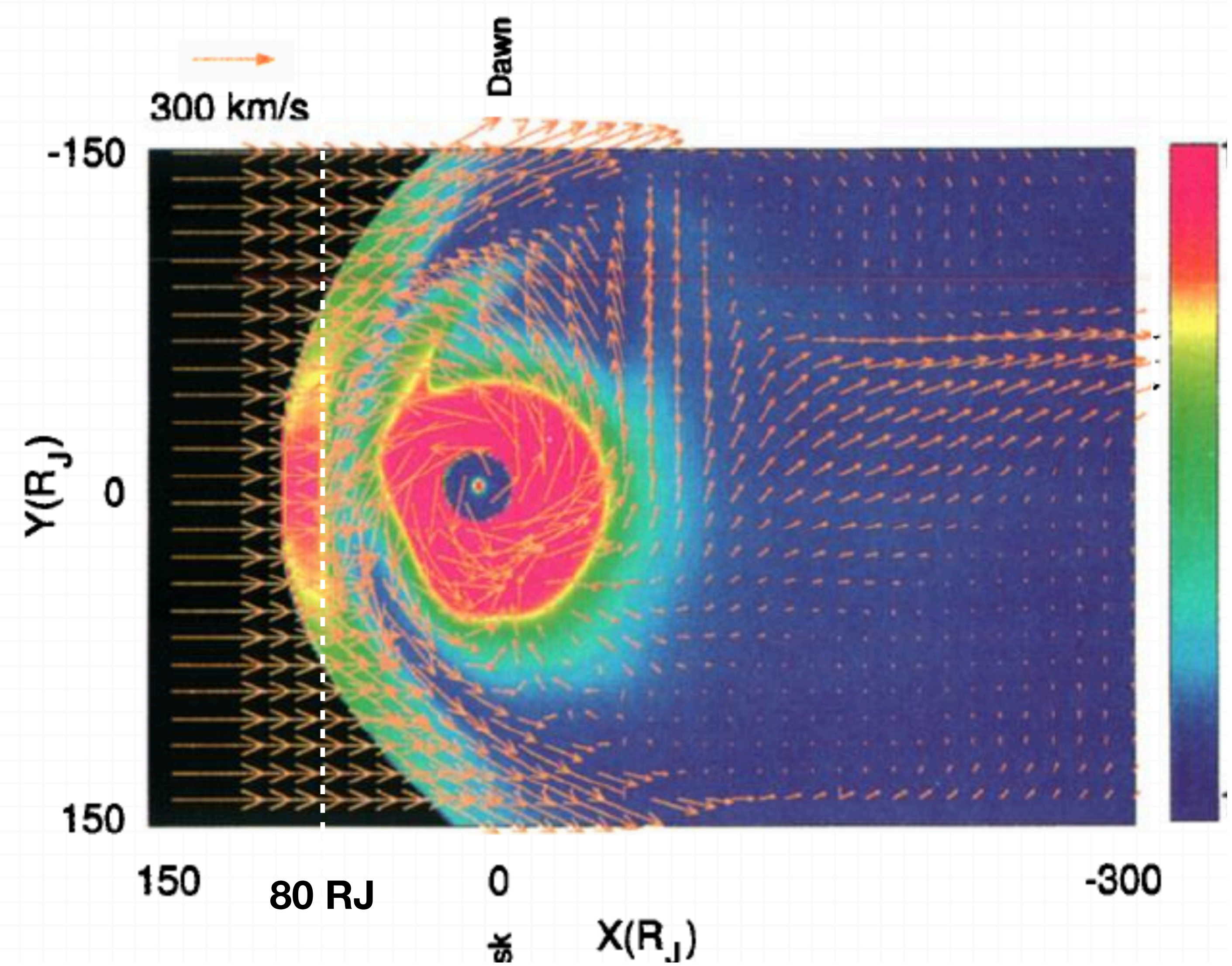
$$\partial \rho / \partial t = -\nabla \cdot (\mathbf{v}\rho) + D\nabla^2 \rho$$

$$\partial \mathbf{v} / \partial t = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla P/\rho + (\mathbf{J}\times\mathbf{B})/\rho + \mathbf{g} + \Phi/\rho$$

$$\partial P / \partial t = -(\mathbf{v}\cdot\nabla)P - \gamma P \nabla\cdot\mathbf{v} + D_p \nabla^2 P$$

$$\partial \mathbf{B} / \partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2 B$$

$$\mathbf{J} = \nabla\times(\mathbf{B} - \mathbf{B}_d)$$



The Ogino MHD model for the jovian magnetosphere gives 80 RJ as the stand-off distance - it's "consistent" with observations!

Extending to System of Linear Equations

The Eigen System

$$q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \dots \\ Q_m \end{bmatrix} \xrightarrow{\text{System of linear PDE}} \frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q = 0$$

Where

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \dots & A_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & A_{N3} & \dots & A_{Nm} \end{bmatrix}$$

For example, linear acoustics, consider mass and momentum equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_x)}{\partial x} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0$$

$$p = C\rho^\gamma$$

Linearize

$$\rho = \rho_0 + \tilde{\rho}$$
$$u = u_0 + \tilde{u}$$
$$p = u_0 + \tilde{p}$$

$$\frac{\partial \tilde{p}}{\partial t} + u_0 \frac{\partial \tilde{p}}{\partial x} + K_0 \frac{\partial \tilde{u}}{\partial x} = 0$$

$$\rho_0 \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{p}}{\partial x} + \rho_0 u_0 \frac{\partial \tilde{u}}{\partial x} = 0$$

Matrix form

$$\frac{\partial}{\partial t} \underbrace{\begin{bmatrix} \tilde{p} \\ \tilde{u} \end{bmatrix}}_{q} + \underbrace{\begin{bmatrix} u_0 & K_0 \\ 1/\rho_0 & u_0 \end{bmatrix}}_{A} \cdot \frac{\partial}{\partial x} \begin{bmatrix} \tilde{p} \\ \tilde{u} \end{bmatrix} = 0$$

Extending to System of Linear Equations

The Eigen System

$$q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \cdots \\ Q_m \end{bmatrix} \xrightarrow{\textbf{System of linear PDE}} \frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q = 0$$

Where $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \dots & A_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & A_{N3} & \dots & A_{Nm} \end{bmatrix}$

$m = N$ for most physical systems!

Diagonalize the matrix A by solving for the Eigenvectors: $Ar^p = \lambda^p r^p$

Define the transforming matrix as: $R = [r^1 \,|\, r^2 \,|\, \dots \,|\, r^N]$

We can show that the matrix R is nonsingular and has an inverse matrix R^{-1}

$$\longrightarrow R^{-1} A R = \Lambda = \begin{bmatrix} \lambda^1 & & & & \\ & \lambda^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda^N \end{bmatrix}$$

The linear equations become:

$$\frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q = 0 \longrightarrow R^{-1} \left(\frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q \right) = 0 \longrightarrow \frac{\partial}{\partial t}(R^{-1} q) + (R^{-1} A R) \frac{\partial}{\partial x} (R^{-1} q) = 0$$

$$\xrightarrow{R^{-1} q = w} \frac{\partial}{\partial t} w + \Lambda \frac{\partial}{\partial x} w = 0$$

Extending to System of Linear Equations

The Eigen System

$$q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \dots \\ Q_m \end{bmatrix} \xrightarrow{\text{System of linear PDE}} \frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q = 0$$

Where

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \dots & A_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ A_{N1} & A_{N2} & A_{N3} & \dots & A_{Nm} \end{bmatrix}$$

$m = N$ for most physical systems!

Which means the system of the equations is transformed in to a bunch of linear advection equations:

$$\frac{\partial}{\partial t} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} + \begin{bmatrix} \lambda^1 & & & \\ & \lambda^2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \lambda^N \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

e.g.,

$$\frac{\partial}{\partial t} w_1 + \lambda^1 \frac{\partial}{\partial w_1} = 0$$

$$\frac{\partial}{\partial t} w_2 + \lambda^2 \frac{\partial}{\partial w_2} = 0 \quad \xrightarrow[R^{-1} q = w]{\textbf{Solve for w}} \quad q = Rw$$

$$\dots$$

Where $R = [r^1 \,|\, r^2 \,|\, \dots \,|\, r^N]$

Extending to System of Nonlinear Equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u})$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} = -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \bar{\boldsymbol{I}} P) - \nabla \cdot \left(\bar{\boldsymbol{I}} \frac{B^2}{2} - \boldsymbol{B}\boldsymbol{B} \right)$$

$$\frac{\partial E_P}{\partial t} = -\nabla \cdot [\boldsymbol{u}(E_P + P)] - \boldsymbol{u} \cdot \nabla \cdot \left(\frac{B^2}{2} \bar{\boldsymbol{I}} - \boldsymbol{B}\boldsymbol{B} \right)$$

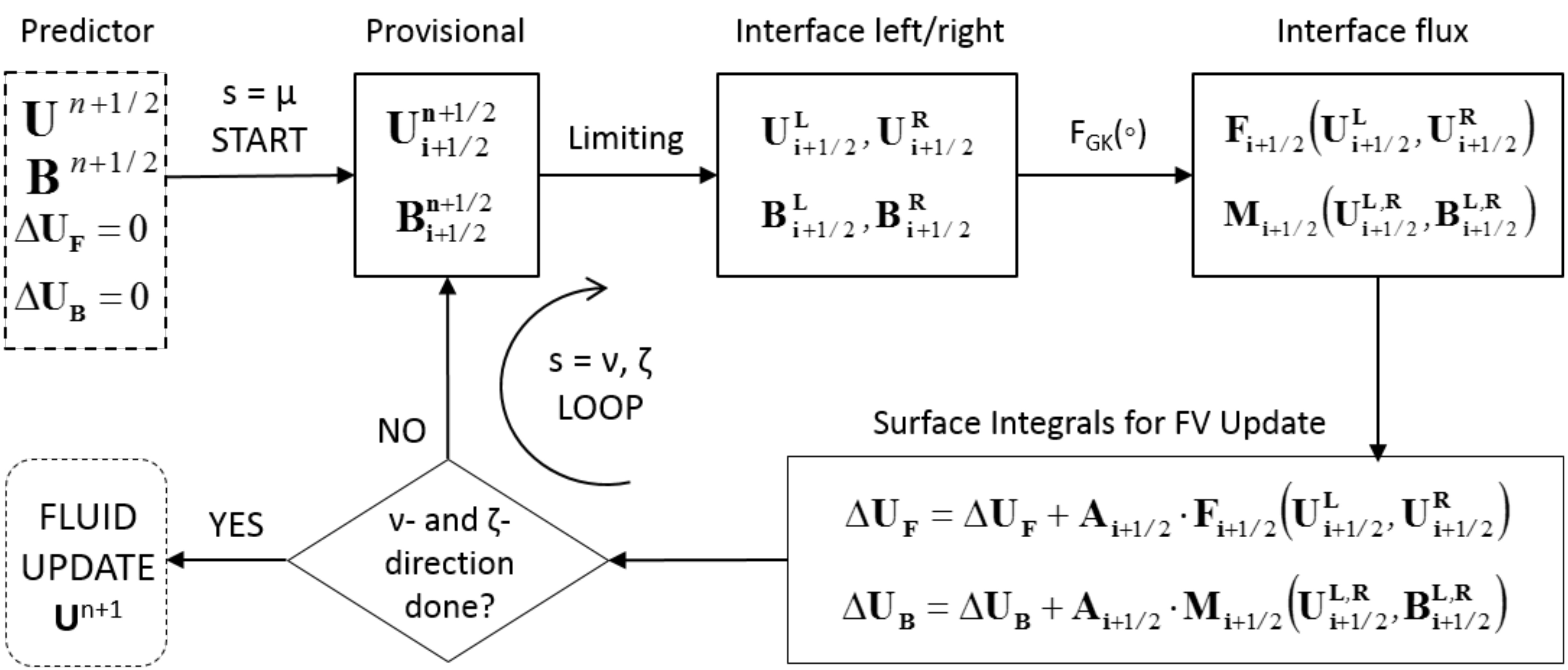
Vector form

$$\xrightarrow{\text{Vector form}} \quad \frac{\partial \boldsymbol{U}^C}{\partial t} = \nabla \cdot \boldsymbol{F}(\boldsymbol{U}^C) + \boldsymbol{M}(\boldsymbol{U}^C),$$

$$\boldsymbol{U}^C = \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \\ E_P \end{pmatrix}.$$

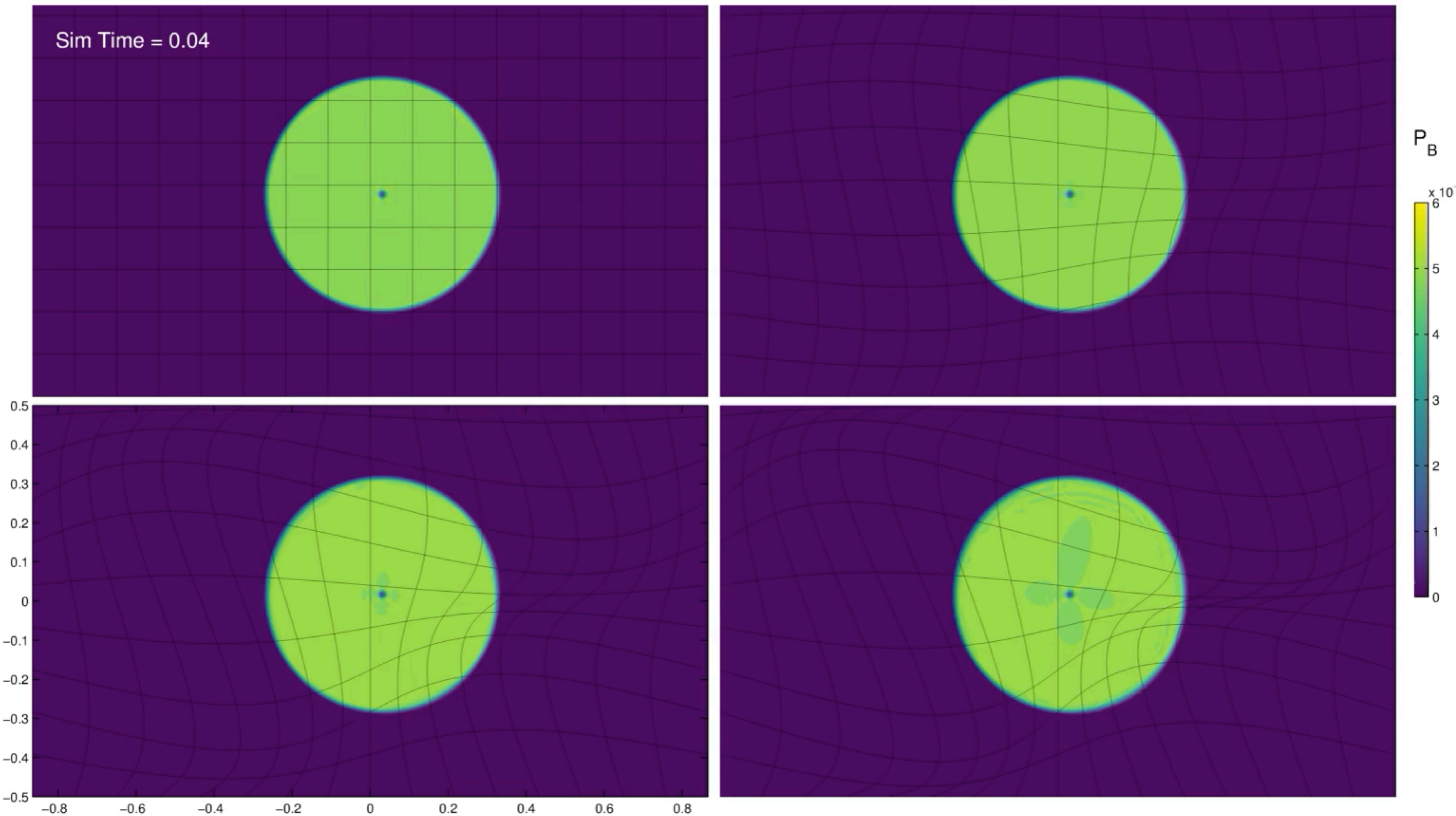
$$\boldsymbol{F}(\boldsymbol{U}^C) = -\begin{bmatrix} \rho \boldsymbol{u} \\ \rho \boldsymbol{u}\boldsymbol{u} + \bar{\boldsymbol{I}} P \\ \boldsymbol{u}(E_P + P) \end{bmatrix},$$

$$\boldsymbol{M}(\boldsymbol{U}^C) = -\begin{bmatrix} 0 \\ \nabla \cdot \left(\bar{\boldsymbol{I}} \frac{B^2}{2} - \boldsymbol{B}\boldsymbol{B} \right) \\ \boldsymbol{u} \cdot \nabla \cdot \left(\bar{\boldsymbol{I}} \frac{B^2}{2} - \boldsymbol{B}\boldsymbol{B} \right) \end{bmatrix}.$$



Extending to Multi-Dimensional Problems

2-D Advection



Extending to Multi-Dimensional Problems

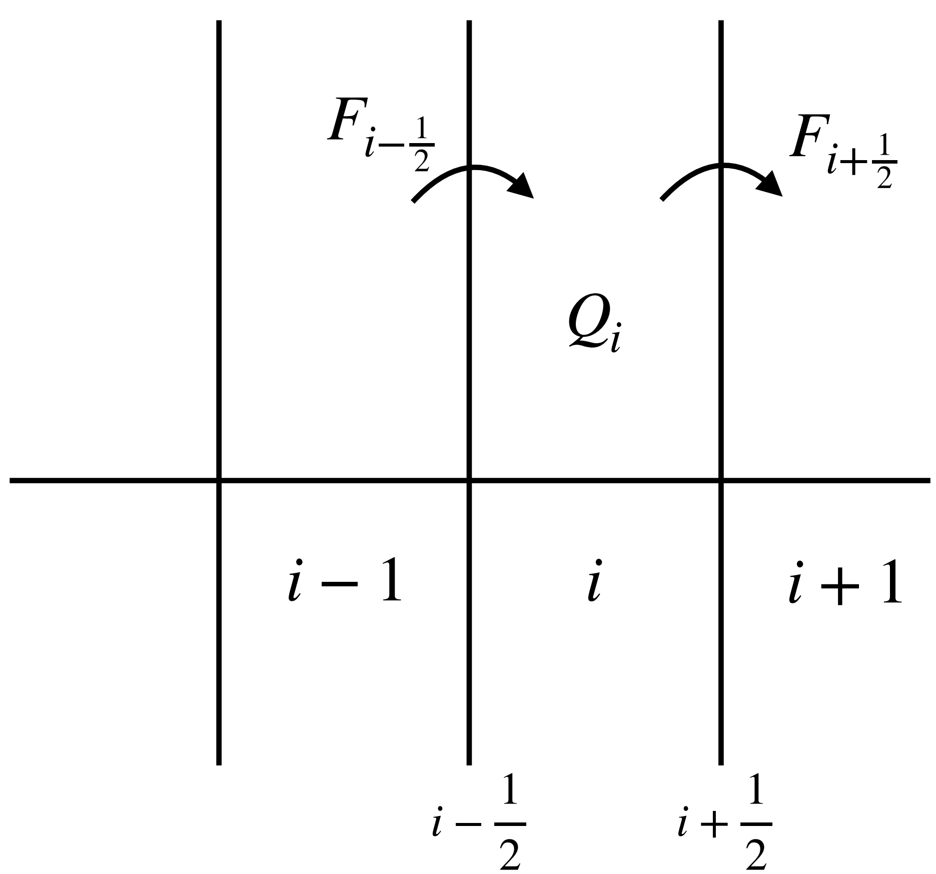
2-D Linear Advection $\frac{\partial}{\partial t}\rho + \mathbf{u}_0 \cdot \nabla \rho = 0$

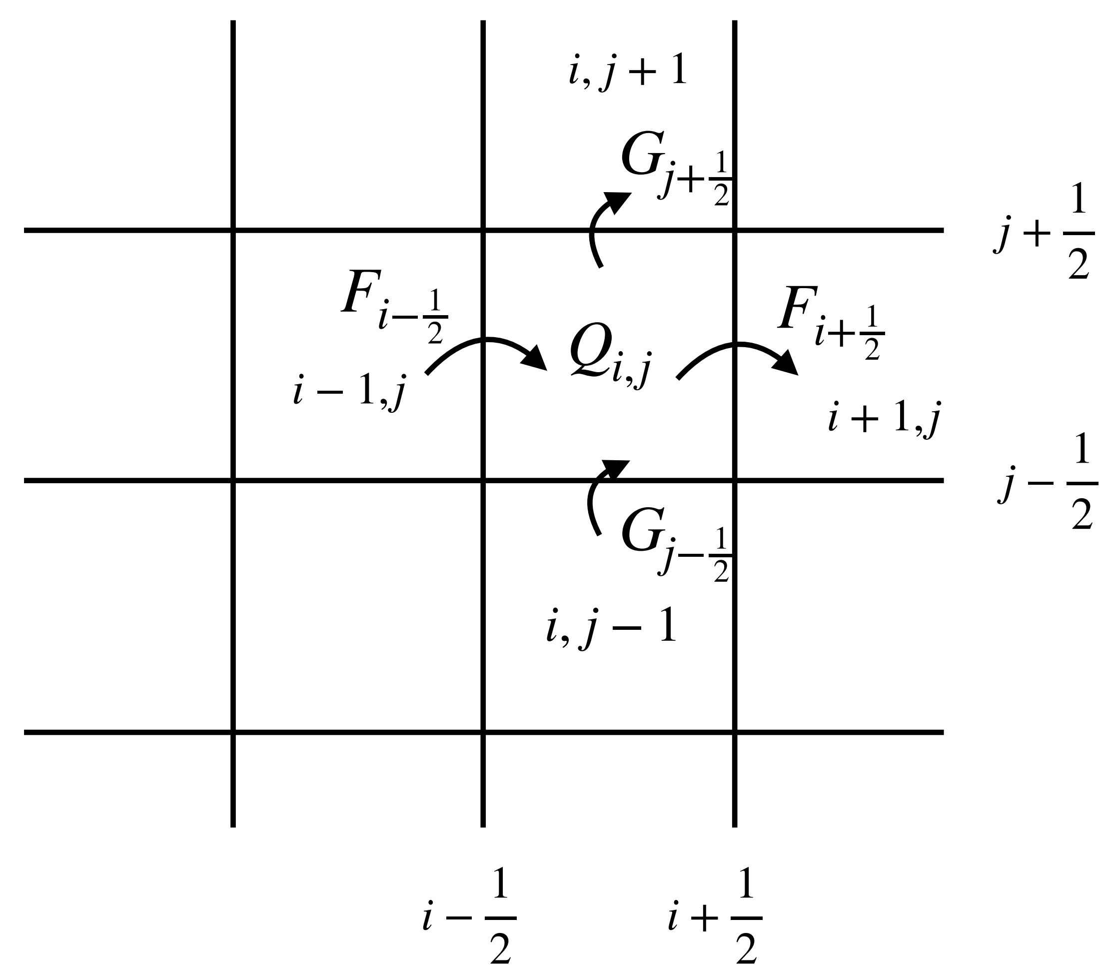
Now both rho and u are functions of x and y: $\rho(x, y), \mathbf{u}_0 = (u_x, u_y)$

The equation is still linear and is a simple extension of the 1-D equation:

1-D $$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} = 0$$

2-D $$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} + \frac{\partial}{\partial y}\underbrace{(u_y\rho)}_{G(\rho)} = 0$$

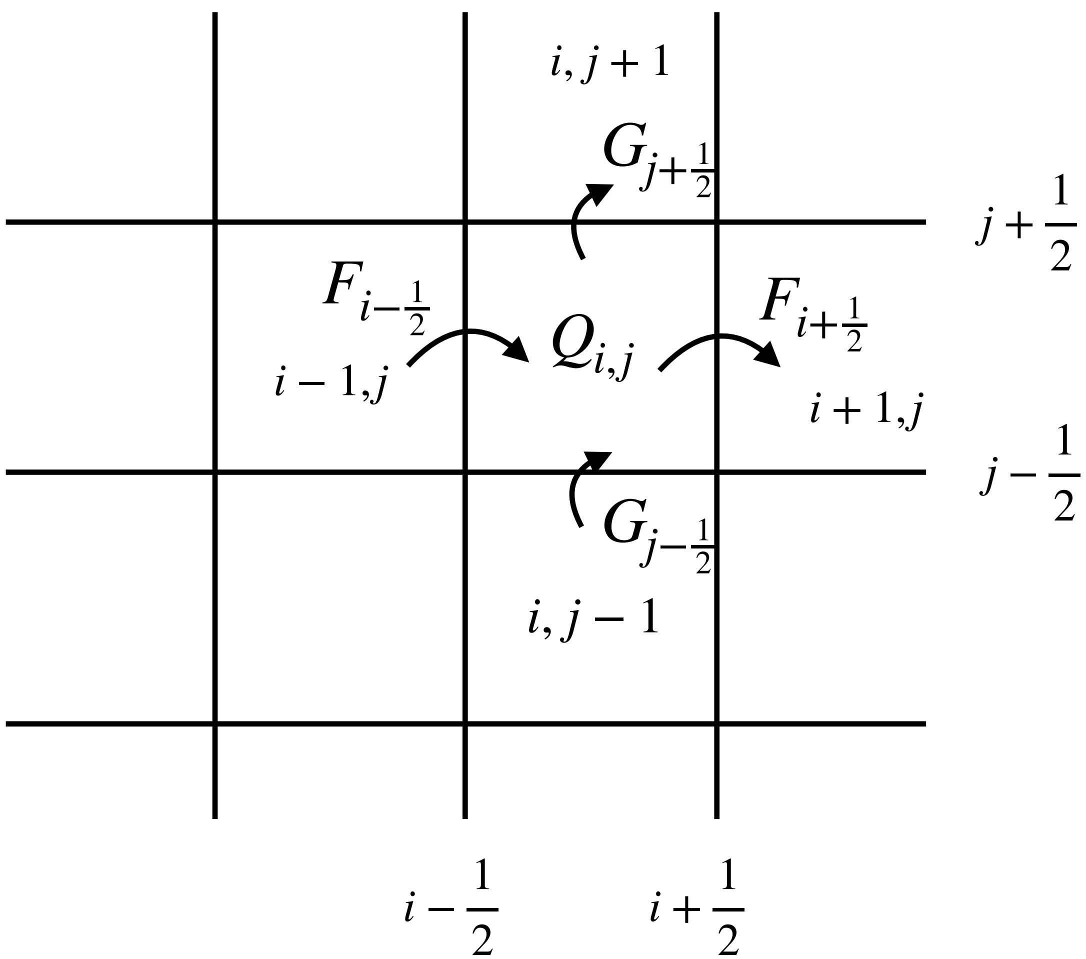




Extending to Multi-Dimensional Problems

Two-Dimensional Algorithm for Advection

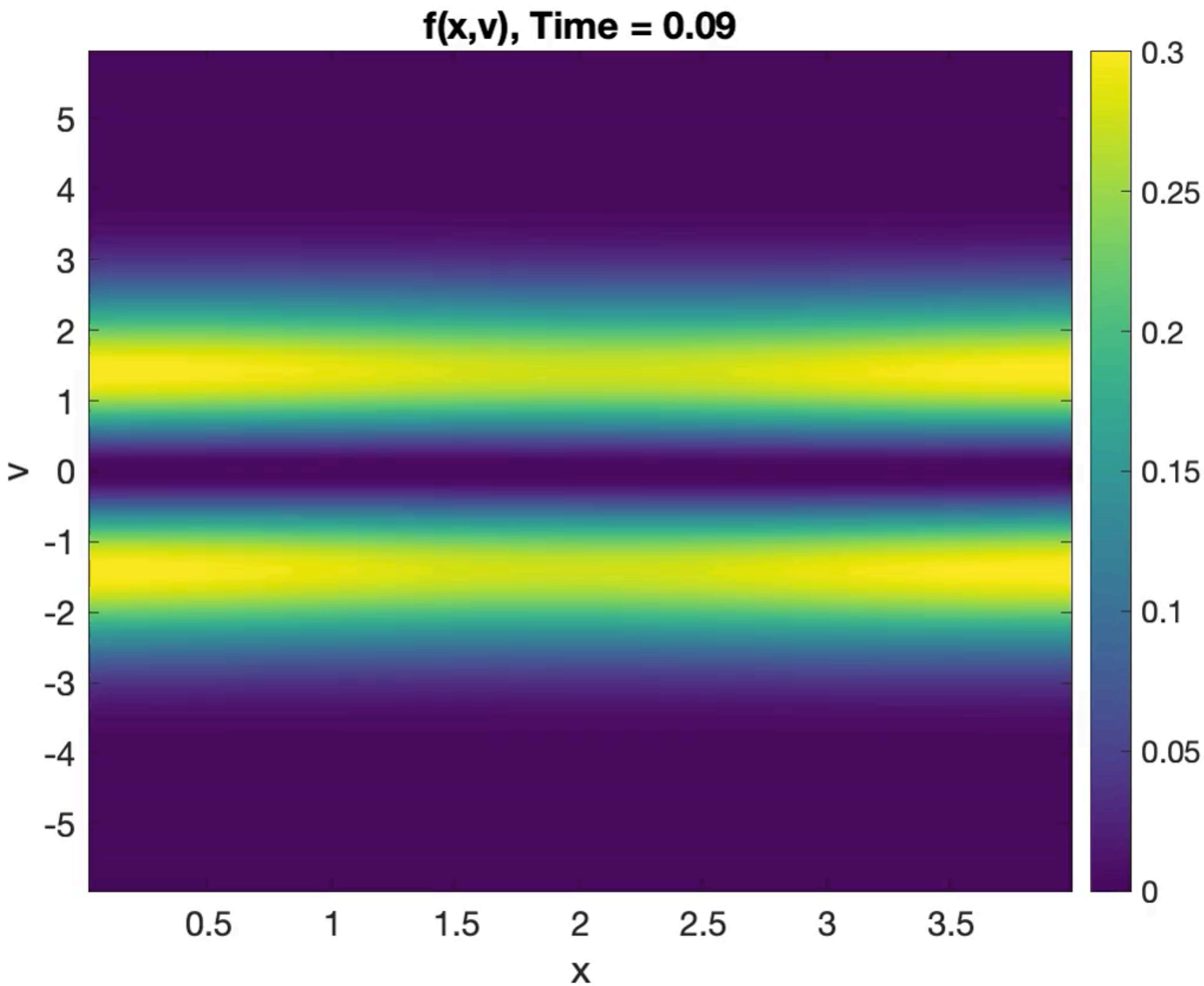
2-D $$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\underbrace{(u_x\rho)}_{F(\rho)} + \frac{\partial}{\partial y}\underbrace{(u_y\rho)}_{G(\rho)} = 0$$



$$\frac{\partial}{\partial t}\bar{Q}_i = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) - \frac{1}{\Delta y}(G_{j+\frac{1}{2}} - G_{j-\frac{1}{2}})$$

Vlasov-Poisson Simulations

Two-stream Instability



1D-1V Vlasov Equation

$$\frac{\partial f_s}{\partial t} + v\frac{\partial f_s}{\partial x} + \frac{q_s E}{m_s}\frac{\partial f_s}{\partial v} = 0$$