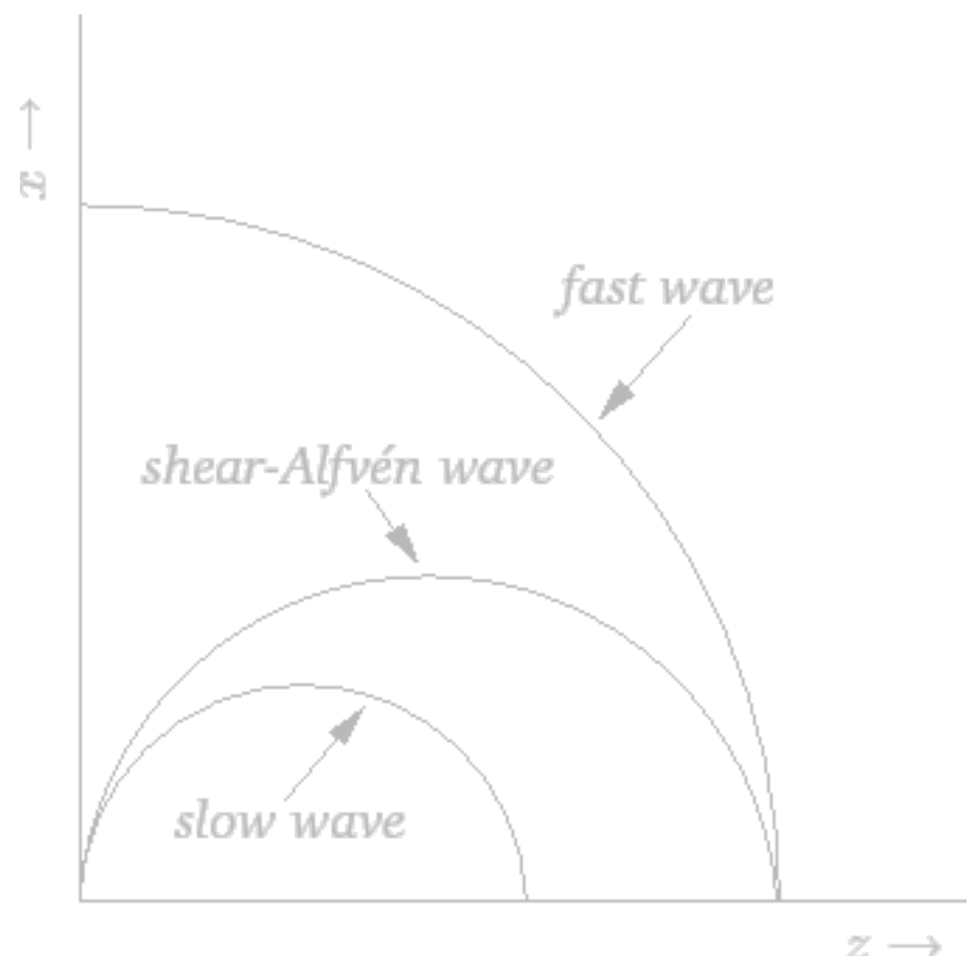
Topic 1

How to Write a 1-D MHD Code using Finite Difference?



Outline

- MHD equations in different forms
- 1-D MHD equations and normalization
- Wave solutions
- Finite difference (revisited)
- Time stepping methods
- Initial and Boundary Conditions
- Put everything together
- Once through the mhd.m code

Differential Equations in Space Plasmas

Fluid Description

$$\frac{\partial}{\partial t}(n_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = S_\alpha$$

Mass conservation

Momentum conservation

$$\frac{\partial}{\partial t}(n_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \mathbf{P}_\alpha) - \frac{n_\alpha q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) = \mathbf{R}_\alpha$$

Thermal Dynamics

Maxwell's equation

$$\frac{\partial p_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla p_\alpha + \gamma p_\alpha \nabla \cdot \mathbf{u}_\alpha = Q_\alpha \qquad \frac{\partial \mathbf{B}}{\partial \mathbf{t}} + \nabla \times \mathbf{E} = 0$$

$$\frac{\partial}{\partial t} : \text{ time derivative}$$

$$\nabla : \text{ spatial derivative}$$

$$= \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

So the first key element of computation space plasma physics is to approximate these derivatives

Kinetic Description

Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = \left(\frac{\delta f_s}{\partial t}\right)_c$$

$$\mathbf{F}_s = q_s/m_s(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lorentz Force

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} + \nabla \times \mathbf{E} = 0$$

Maxwell's equations

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_s q_s \int_{-\infty}^{+\infty} v f_s d^3 v$$

Particle Description

$$m_s n_s \frac{d\mathbf{v}_s}{dt} = q n_s (\mathbf{E} + \mathbf{v}_\mathbf{s} \times \mathbf{B})$$

Equation of motion

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} + \nabla \times \mathbf{E} = 0$$

Maxwell's equations

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_s q_s n_s \mathbf{v}_s$$

Spectrum of Simulating Space Plasmas

Kinetic — **Hybrid** — **Fluid**

high frequencies ← f — *low frequencies*

	ω_{pe}	ω_{pi}	ω_{ci}	ω_a	ω_{ei}
Frequency	electron plasma	ion plasma	ion gyrofrequency	Alfvén	electron-ion collision
Timescale	$\leq 10^{-12}$ s	$\sim 10^{-10} - 10^{-8}$ s		$\sim 10^{-6} - 10^{-3}$ s	10^{1} s
Simulation Methods	Conventional Explicit Particle Codes	Ion-scale Particle Codes, Hybrid Codes		MHD Explicit	MHD Implicit (resistive)

TOPIC 1-2

Guiding Center
Gryokinetic
Implicit particle codes

What is MHD

Magnetohydrodynamics (MHD) is the study of the interaction between magnetic fields and moving, conducting fluids.

MHD = Magnetic field + Hydrodynamics

Maxwell's equations

Fluid Dynamics

Who cares about MHD?

MHD

Engineering

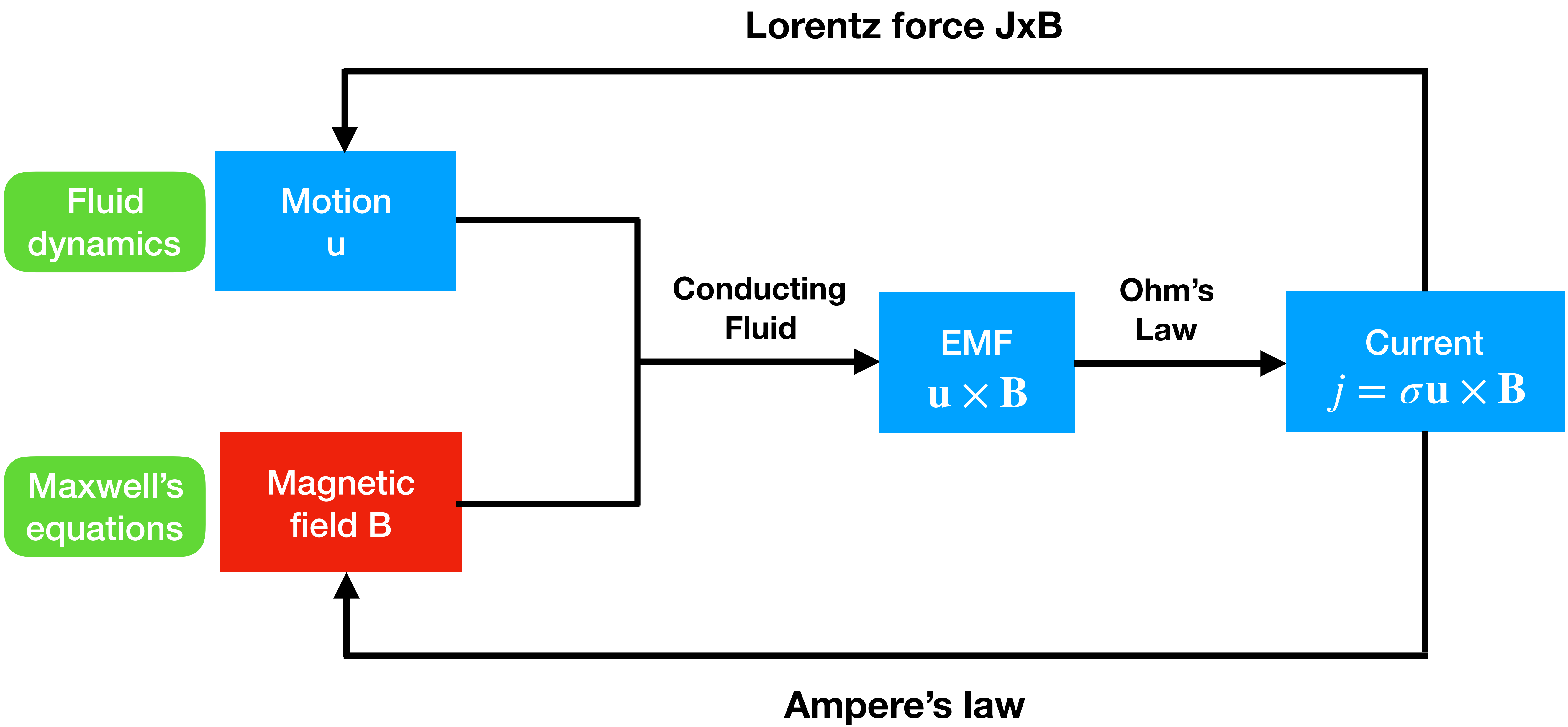
Liquid Matel
engineering pump

Science

Fusion plasma
Space physics
Astrophysics

What is MHD?

Dynamics in MHD: coupling between conducting fluid and magnetic field



The nature of MHD: conservation laws

Mathematical Forms of the MHD equations

Primitive form

Fluid Equations

Mass Equation
$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

Velocity Equation
$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0$$

pressure Equation
$$\frac{\partial}{\partial t}p + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} = 0$$

Maxwell's Equations

Faraday
$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

Ampere
$$\mathbf{J} = \nabla\times\mathbf{B}$$
Ignore dB/dt

Ohm
$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0$$
Ignore non-ideal terms

If using the ***convective (total) derivative***

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla\cdot\mathbf{u}$$

the primitive form of the MHD equations is written in a very compact form:

Fluid Equations

Mass Equation
$$\frac{D}{Dt}\rho = -\rho\nabla\cdot\mathbf{u}$$
Compression

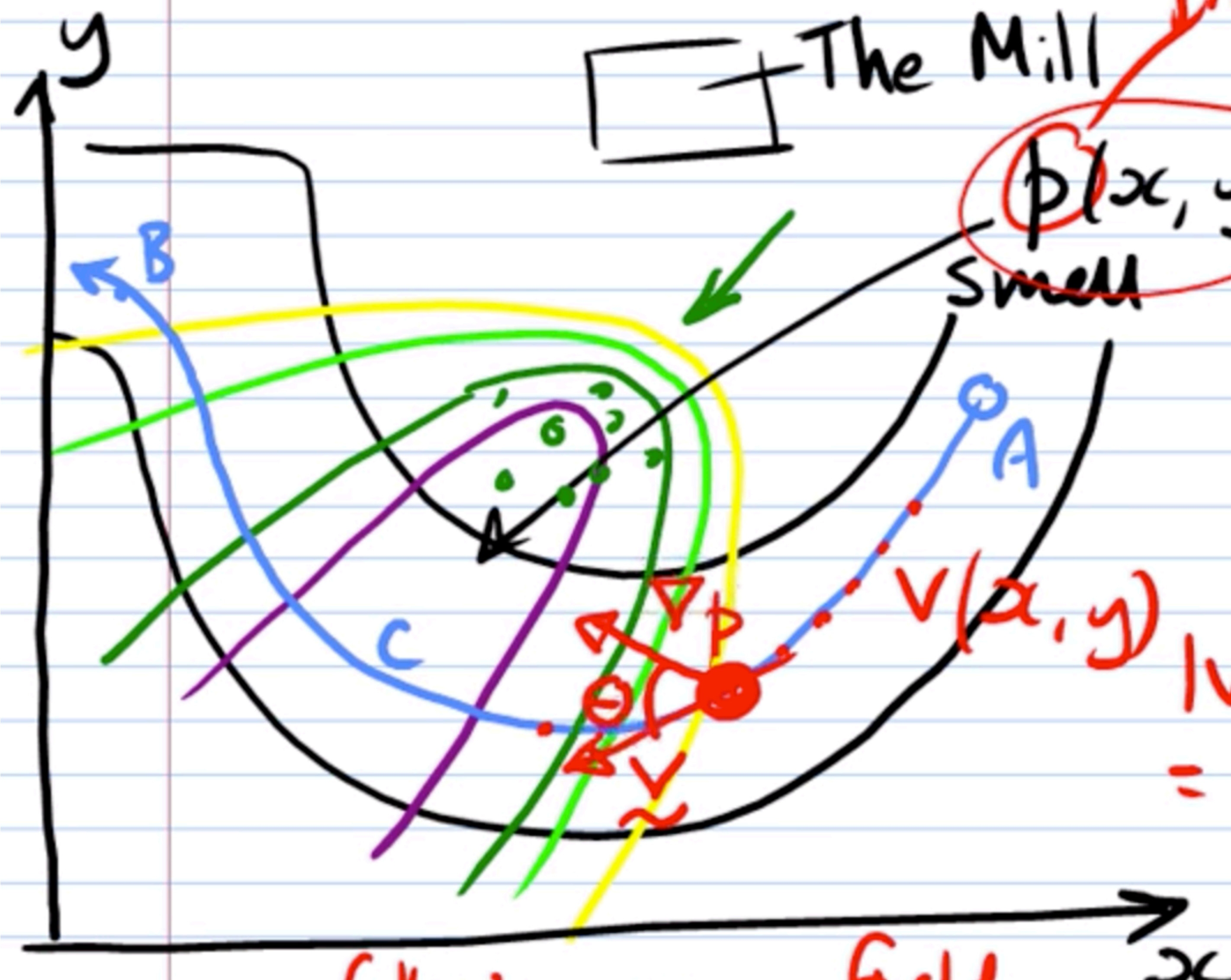
Velocity Equation
$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J}\times\mathbf{B}$$
Acceleration

pressure Equation
$$\frac{D}{Dt}p = -\gamma p\nabla\cdot\mathbf{u}$$
Adiabatic heating

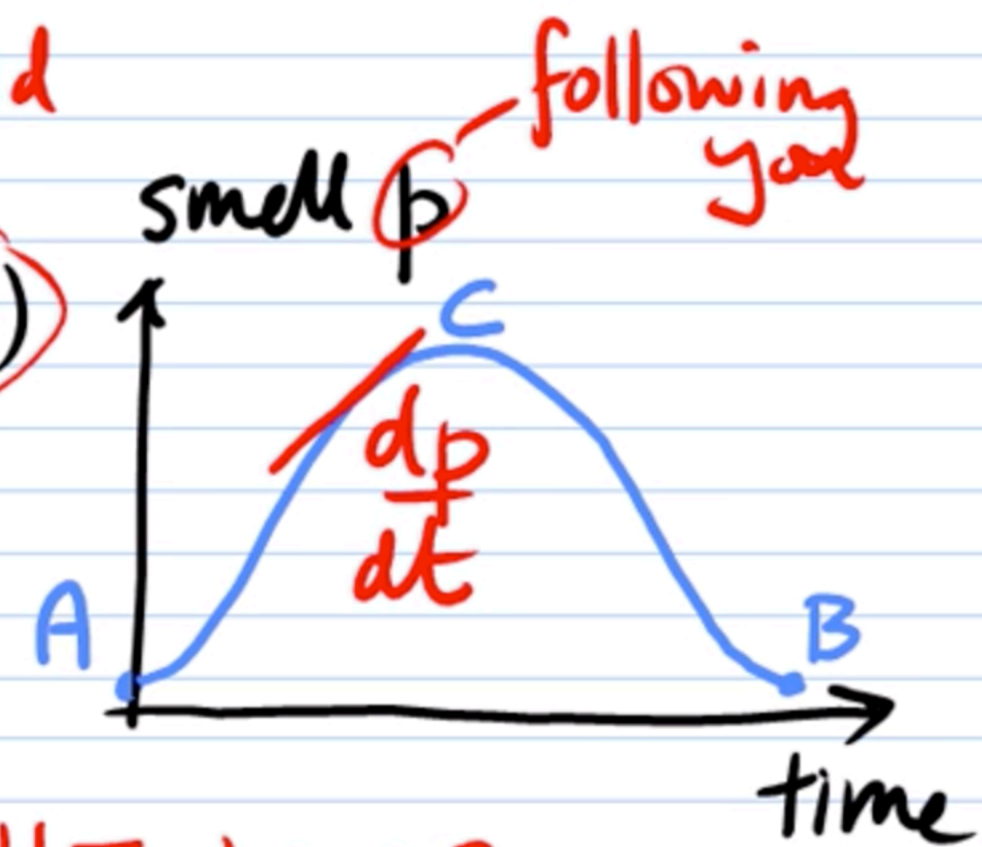
Total derivative form of the single-fluid, ideal MHD equations solve for mass, velocity and pressure (or temperature etc)

Physics

The Convective Derivative $\frac{D}{Dt}$



$$|v||\nabla p| \cos\theta = \underset{\sim}{v} \cdot \nabla p$$



following you — field — field

$$\frac{dp}{dt} = \underset{\sim}{v} \cdot \nabla p + \left.\frac{\partial p}{\partial t}\right|_{x,y}$$

Mathematical Forms of the MHD equations

Conservation form

Fluid Equations

Mass conservation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \mathbf{I}(p + \frac{1}{2}B^2) - \mathbf{B}\mathbf{B}\right] = 0$$

Energy conservation

$$\frac{\partial \mathscr{E}_T}{\partial t} + \nabla \cdot \left[\left(\mathscr{E} + p_{tot}\right)\mathbf{u} - \mathbf{u}\cdot\mathbf{B}\mathbf{B}\right] = 0$$

The conservation form of the single-fluid, ideal MHD equations solve for **mass**, **momentum** and **total energy**

Conservation laws

$$\boxed{\frac{\partial}{\partial t}(\ldots) + \nabla \cdot (\ldots) = 0}$$

Maxwell's Equations

Faraday

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \xrightarrow{\nabla \times (\mathbf{A}\times\mathbf{B}) = \nabla \cdot (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})} \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{B} - \mathbf{B}\mathbf{u}) = 0$$

Magnetic flux conservation

Ampere

$$\mathbf{J} = \nabla \times \mathbf{B}$$

Ignore dB/dt

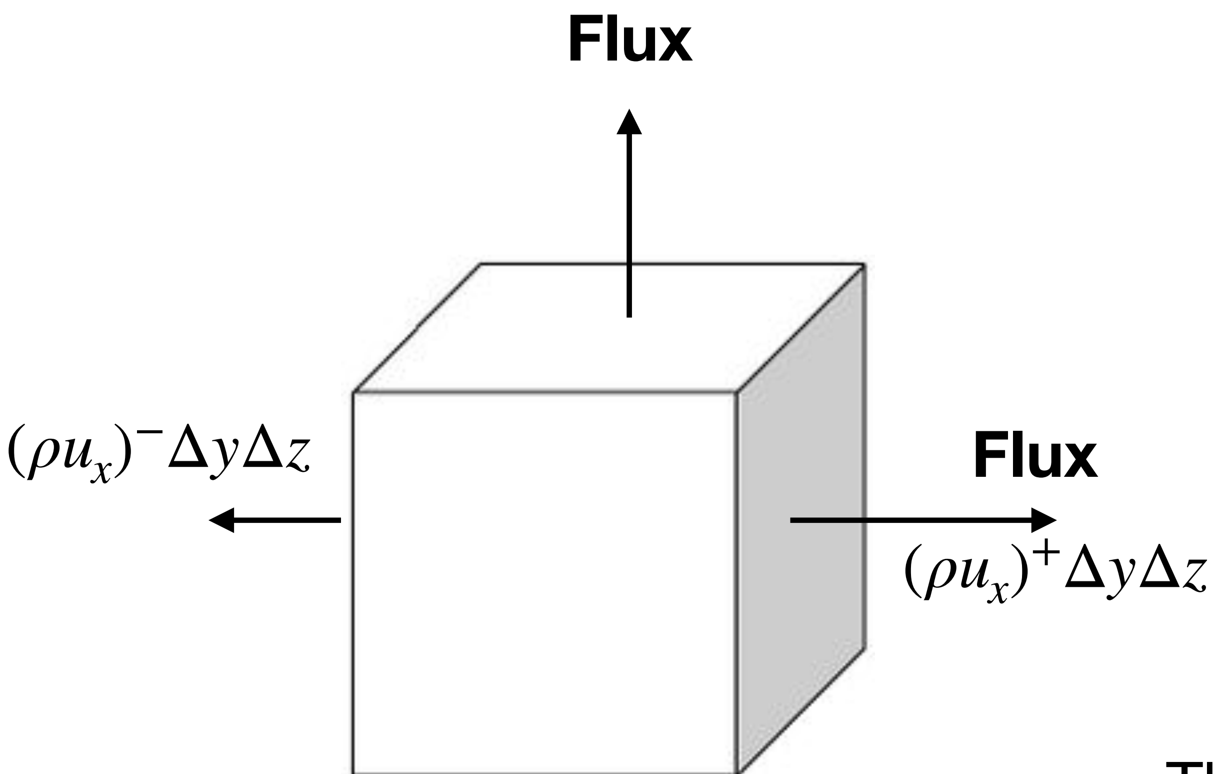
Ohm

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

Ignore non-ideal terms

This is why MHD = conservation laws!

Conservation laws



$$Vol = \Delta x\Delta y\Delta z$$

A general conservation law goes as

$$\text{Changes in Mass} = \Sigma_{Flux}\left(\text{in} - \text{out}\right) + P - L$$

Mass Flux through interface

Local Production and lost

The integral form is written as

$$\frac{\partial}{\partial t}\int_V \rho dV = -\oint \rho\mathbf{u}\cdot d\mathbf{S}$$

$$\downarrow \qquad\qquad \downarrow \tfrac{1}{V}$$

$$V\cdot\frac{\partial\bar{\rho}}{\partial t} \qquad \frac{\partial}{\partial x}\rho u_x + \frac{\partial}{\partial y}\rho u_y + \frac{\partial}{\partial z}\rho u_z$$

$$\frac{1}{V}\oint \rho\mathbf{u}\cdot d\mathbf{S}\,\Big|_{x-dir}$$

$$\downarrow$$

$$\frac{(\rho u_x)^{+}\Delta y\Delta z - (\rho u_x)^{-}\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$$\downarrow$$

$$\frac{(\rho u_x)^{+} - (\rho u_x)^{-}}{\Delta x} \longrightarrow \frac{\partial}{\partial t}\rho u_x$$

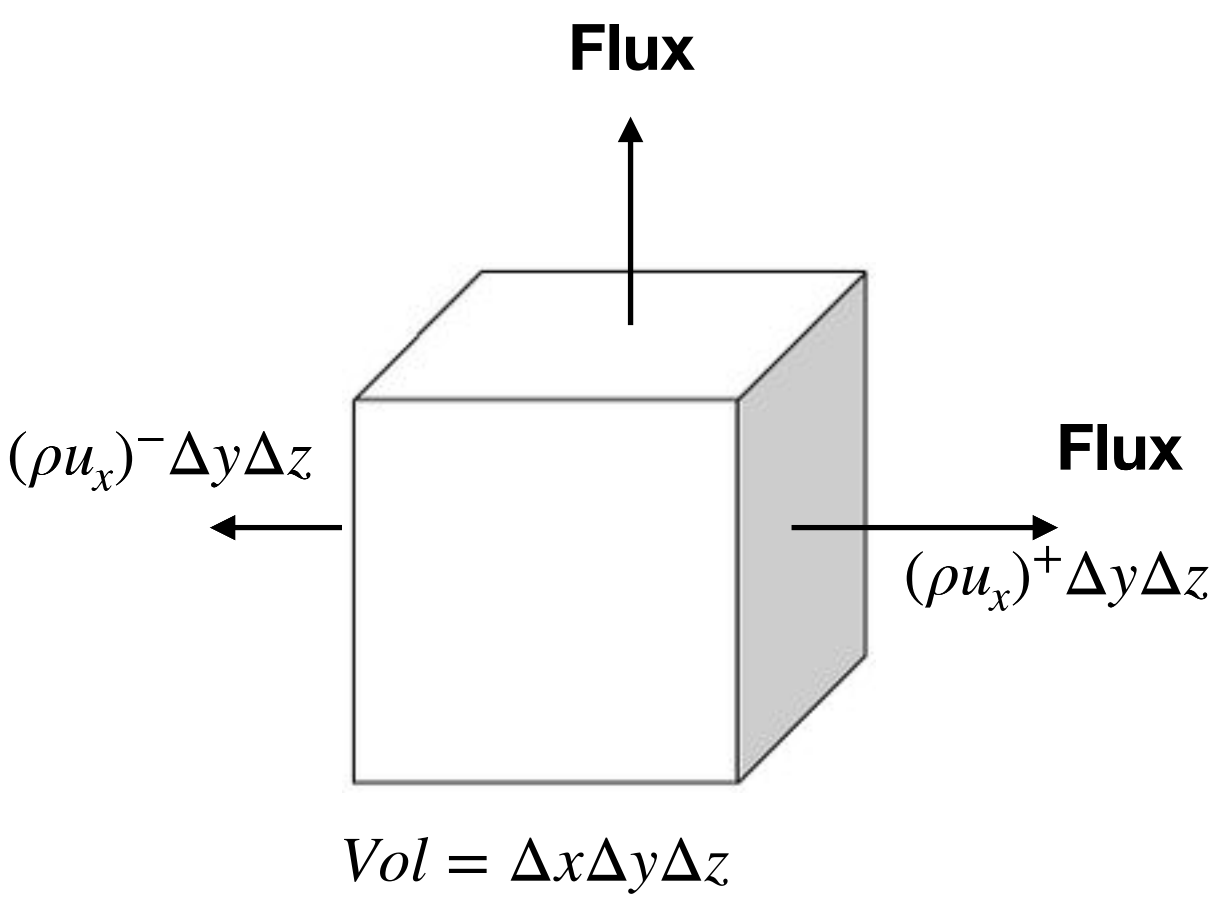
Differential form

$$\boxed{\frac{\partial\bar{\rho}}{\partial t} = -\nabla\cdot\left(\rho\mathbf{u}\right)}$$

Conservation form of differential equations

$$\boxed{\frac{\partial}{\partial t}(\ldots) + \nabla \cdot (\ldots) = 0}$$

Conservation law



The use of such a form of the equations is that one can obtain *local and global conservation laws* and *jump conditions* from them. Moreover, powerful numerical algorithms exist for the solution of such equations.

For example mass conservation:

$$\rho \mathbf{u} \sim \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \frac{\mathrm{m}}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{m}^2 \cdot \mathrm{s}}$$

$$\frac{\partial \bar{\rho}}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

Changes in mass density

Mass fluxes through volume interface

Summary of the ideal MHD equations

Primitive Equations *TOPIC 1-2*

Mass Equation
$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

Velocity Equation
$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0$$

pressure Equation
$$\frac{\partial}{\partial t}p + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} = 0$$

and

Maxwell's Equations

Faraday
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

Ampere
$$\mathbf{J} = \nabla\times\mathbf{B}$$

Ohm
$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0$$

Conservative Equations *TOPIC 3*

Mass conservation
$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{u}) = 0$$

Momentum conservation
$$\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot\left[\rho\mathbf{u}\mathbf{u} + \mathbf{I}(p + \frac{1}{2}B^2) - \mathbf{B}\mathbf{B}\right] = 0$$

Energy conservation
$$\frac{\partial \mathscr{E}_T}{\partial t} + \nabla\cdot\left[\left(\mathscr{E} + p_{tot}\right)\mathbf{u} - \mathbf{u}\cdot\mathbf{B}\mathbf{B}\right] = 0$$

- **Physically**, the primitive and conservative forms of the MHD equations are *equivalent*.
- **Numerically**, the primitive and conservative forms of the MHD equations are ***NOT*** *equivalent*.

The MHD equations to solve

Unit Systems

SI Units

Mass Equation
$$\frac{\partial}{\partial t}\rho = -\nabla \cdot \rho \mathbf{u}$$

Velocity Equation
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

Pressure Equation
$$p = p_g\left(\frac{\rho}{\rho_0}\right)^\gamma$$

Faraday's Law
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Ohm's Law
$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

Ampere's Law
$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$$

CGS Units

Mass Equation
$$\frac{\partial}{\partial t}\rho = -\nabla \cdot \rho \mathbf{u}$$

Velocity Equation
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{c}\mathbf{J} \times \mathbf{B}$$

Pressure Equation
$$p = p_g\left(\frac{\rho}{\rho_0}\right)^\gamma$$

Faraday's Law
$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

Ohm's Law
$$\mathbf{E} = -\frac{1}{c}\mathbf{u} \times \mathbf{B}$$

Ampere's Law
$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B}$$

Depending on the the choice of the unit systems, there are different physical constants floating around - awkward

Normalize the MHD equations

Rule #1: Always normalize your equations

We will use the **normalized** MHD equations in the following lectures and codes for HW

Why normalization? Makes your code broadly applicable - adapt to different systems

How to normalize? Define some characteristic parameters

Let's start with the mass equation:

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\rho\mathbf{u} \xrightarrow[\mathbf{u}=U_0\tilde{\mathbf{u}}]{\rho=\rho_0\tilde{\rho}} \frac{\partial}{\partial t}(\rho_0\tilde{\rho}) = -\nabla\cdot(\rho_0\tilde{\rho}\cdot U_0\tilde{\mathbf{u}})$$

Here ρ_0 and U_0 are some characteristic density and velocity of a system

We also need to normalize the space and time (the derivatives): $t = T_0\tilde{t}$ $\quad \mathbf{x} = L_0\tilde{\mathbf{x}}$

$$\frac{\partial}{\partial t} = \frac{1}{T_0}\frac{\partial}{\partial \tilde{t}} \qquad \nabla = \frac{\partial}{\partial \mathbf{x}} = \frac{1}{L_0}\frac{\partial}{\partial \tilde{\mathbf{x}}} \equiv \frac{1}{L_0}\tilde{\nabla}$$

Now the mass equation becomes:

$$\frac{1}{T_0}\frac{\partial}{\partial\tilde{t}}(\rho_0\tilde{\rho}) = -\frac{1}{L_0}\tilde{\nabla}\cdot(\rho_0\tilde{\rho}\cdot u_0\tilde{\mathbf{u}}) \longrightarrow \frac{\partial}{\partial\tilde{t}}(\tilde{\rho}) = -\frac{T_0U_0}{L_0}\tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}}) \xrightarrow{\boxed{\frac{T_0U_0}{L_0}=1}} \boxed{\frac{\partial}{\partial\tilde{t}}(\tilde{\rho}) = -\tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}})}$$

Normalization Relation

Dimensionless Mass Equation

Normalize the MHD equations

Now look at the velocity equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J}\times\mathbf{B} \xrightarrow{\mathbf{J}=\frac{1}{\mu_0}\nabla\times\mathbf{B}} \rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \underbrace{\frac{1}{\mu_0}\nabla\times\mathbf{B}}_{\mathbf{J}}\times\mathbf{B}$$

We can work out the normalization relations for B and p

$$\xrightarrow[p=P_0\tilde{p} \quad \mathbf{B}=B_0\tilde{\mathbf{B}} \quad \nabla=\frac{1}{L_0}\tilde{\nabla}]{\rho=\rho_0\tilde{\rho} \quad \mathbf{u}=U_0\tilde{\mathbf{u}} \quad \frac{\partial}{\partial t}=\frac{1}{T_0}\frac{\partial}{\partial\tilde{t}}} \frac{\rho_0 U_0\tilde{\rho}}{T_0}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = -\frac{P_0}{L_0}\tilde{\nabla}\tilde{p} + \frac{B_0^2}{\mu_0 L_0}\tilde{\nabla}\times\tilde{\mathbf{B}}\times\tilde{\mathbf{B}}$$

Since $\frac{T_0U_0}{L_0}=1$ $\longrightarrow$ $\cancel{\left(\frac{\rho_0 U_0^2}{L_0}\right)}\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = -\cancel{\left(\frac{P_0}{L_0}\right)}\tilde{\nabla}\tilde{p} + \cancel{\left(\frac{B_0^2}{\mu_0 L_0}\right)}\tilde{\nabla}\times\tilde{\mathbf{B}}\times\tilde{\mathbf{B}}$

To make the constants go away, we need: $P_0=\rho_0U_0^2 \qquad B_0=\sqrt{\mu_0\rho_0U_0^2}$

The latter equation means: $U_0=\frac{B_0}{\sqrt{\mu_0\rho_0}}=V_{A0}$ $\longrightarrow$ Characteristic U0 is the Alfvén speed!

Now the velocity equation becomes: $\boxed{\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}\times\tilde{\mathbf{B}}\times\tilde{\mathbf{B}}}$

Dimensionless Velocity Equation

The Convective Derivative term

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{u} \xrightarrow[\mathbf{u} = U_0 \tilde{\mathbf{u}} \qquad \nabla = \frac{1}{L_0}\tilde{\nabla}]{\rho = \rho_0 \tilde{\rho} \qquad \frac{\partial}{\partial t} = \frac{1}{T_0}\frac{\partial}{\partial \tilde{t}}} \frac{1}{T_0}\frac{\partial}{\partial \tilde{t}} + \frac{U_0}{L_0}\hat{\mathbf{u}} \cdot \tilde{\nabla}$$

Since $\frac{T_0 U_0}{L_0} = 1 \longrightarrow \frac{U_0}{L_0} = \frac{1}{T_0}$

The normalization for the convective derivative term goes like

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{u} = \frac{1}{T_0}\left(\frac{\partial}{\partial \tilde{t}} + \cdot \tilde{\nabla}\right) \equiv \boxed{\frac{1}{T_0}\frac{D}{D\tilde{t}}}$$

This is why the convective derivative was treated like partial derivative in the normalization process

Normalize the MHD equations

We still need an equation for the pressure (thermal dynamics)

Assume adiabaticity: $p \propto \rho^\gamma$ (Adiabatic gas law, from statistical mechanics)

We have

$$p = p_g \left(\frac{\rho}{\rho_0}\right)^\gamma \longrightarrow p = p_g \underbrace{\left(\frac{\rho}{\rho_0}\right)}_{\tilde{\rho}}^\gamma \longrightarrow \boxed{p = p_g \tilde{\rho}^\gamma}$$

With normalized rho

p_g Pressure corresponds to ρ_0

The pressure normalization has an interesting physical definition:

$$p = p_g \tilde{\rho}^\gamma \xrightarrow[U_0 = \frac{B_0}{\sqrt{\mu_0 \rho_0}}]{p = P_0 \tilde{p} = \rho_0 U_0^2 = \frac{B_0^2}{\mu_0}} \tilde{p} = \frac{p_g}{B_0^2/\mu_0} \tilde{\rho}^\gamma \longrightarrow \tilde{p} = \frac{p_g}{2P_B} \tilde{\rho}^\gamma \longrightarrow \boxed{\tilde{p} = \frac{\beta}{2} \tilde{\rho}^\gamma}$$

Dimensionless Pressure Equation

Recall the definition of magnetic pressures and plasma beta:

$$P_B = \frac{B^2}{2\mu_0} \qquad \beta = \frac{p_{gas}}{p_B}$$

Normalize the Maxwell's equations

The normalization of the Maxwell's equations are quite straightforward

Let's start with the Ohm's law

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B} \xrightarrow[\mathbf{B}=B_0\tilde{\mathbf{B}} \quad \mathbf{E}=E_0\tilde{\mathbf{E}}]{\mathbf{u}=U_0\tilde{\mathbf{u}}} E_0\tilde{\mathbf{E}} = -U_0B_0\tilde{\mathbf{u}}\times\tilde{\mathbf{B}} \xrightarrow{\boxed{E_0 = U_0B_0}} \boxed{\tilde{\mathbf{E}} = -\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}}$$

Normalization Relation

Dimensionless Ohm's Law

Now combine with the Faraday's law:

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \xrightarrow[\mathbf{E}=U_0B_0\tilde{\mathbf{E}} \quad \nabla = \frac{1}{L_0}\tilde{\nabla}]{\mathbf{B}=B_0\tilde{\mathbf{B}} \quad \frac{\partial}{\partial t} = \frac{1}{T_0}\frac{\partial}{\partial\tilde{t}}} \frac{B_0}{T_0}\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}} = -\frac{U_0B_0}{L_0}\tilde{\nabla}\times\tilde{\mathbf{E}}$$

$$\Big\downarrow \ \frac{T_0U_0}{L_0} = 1$$

$$\boxed{\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}} = -\tilde{\nabla}\times\tilde{\mathbf{E}}}$$

Dimensionless Faraday's Law

Ampere's law is also straightforward:

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B} \longrightarrow J_0\tilde{\mathbf{J}} = \frac{B_0}{\mu_0L_0}\tilde{\nabla}\times\tilde{\mathbf{B}} \xrightarrow{J_0 = \frac{B_0}{\mu_0L_0}} \boxed{\tilde{\mathbf{J}} = \tilde{\nabla}\times\tilde{\mathbf{B}}}$$

This was already included in the velocity equation!

Dimensionless Ampere's Law

The full set of Normalized MHD equations

Dimensionless Equations

Mass Equation

$$\frac{\partial}{\partial \tilde{t}}\tilde{\rho} = -\tilde{\nabla}\cdot(\tilde{\rho}\tilde{\mathbf{u}})$$

Velocity Equation

$$\tilde{\rho}\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} = -\tilde{\nabla}\tilde{p} + \tilde{\mathbf{J}}\times\tilde{\mathbf{B}}$$

Pressure Equation

$$\tilde{p} = \frac{\beta}{2}\tilde{\rho}^{\gamma}$$

Faraday's Law

$$\frac{\partial\tilde{\mathbf{B}}}{\partial\tilde{t}} = -\tilde{\nabla}\times\tilde{\mathbf{E}}$$

Ohm's Law

$$\tilde{\mathbf{E}} = -\tilde{\mathbf{u}}\times\tilde{\mathbf{B}}$$

Ampere's Law

$$\tilde{\mathbf{J}} = \tilde{\nabla}\times\tilde{\mathbf{B}}$$

Normalization Relations

$$L_0 = T_0 U_0 \qquad U_0 = B_0/\sqrt{\mu_0\rho_0}$$

$$E_0 = U_0 B_0 \qquad J_0 = \frac{B_0}{\mu_0 L_0}$$

Interpretations:

- this set of normalized equation is called "dimensionless", e.g., u = 1 means the magnitude of the velocity is one Alfvén speed - very convenient in understanding the solutions
- We ignore the ***tilde*** in the following analysis and use the following set of MHD equations:

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\rho\mathbf{u}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J}\times\mathbf{B}$$

$$p = \frac{\beta_0}{2}\rho^{\gamma}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B}$$

$$\mathbf{J} = \nabla\times\mathbf{B}$$

Dimensionless Ideal MHD equations

This is the set of equations solved in mhd.m

Physics

The $\mathbf{J} \times \mathbf{B}$ force term

The plasma velocity is influenced by the magnetic field through the Lorentz force:

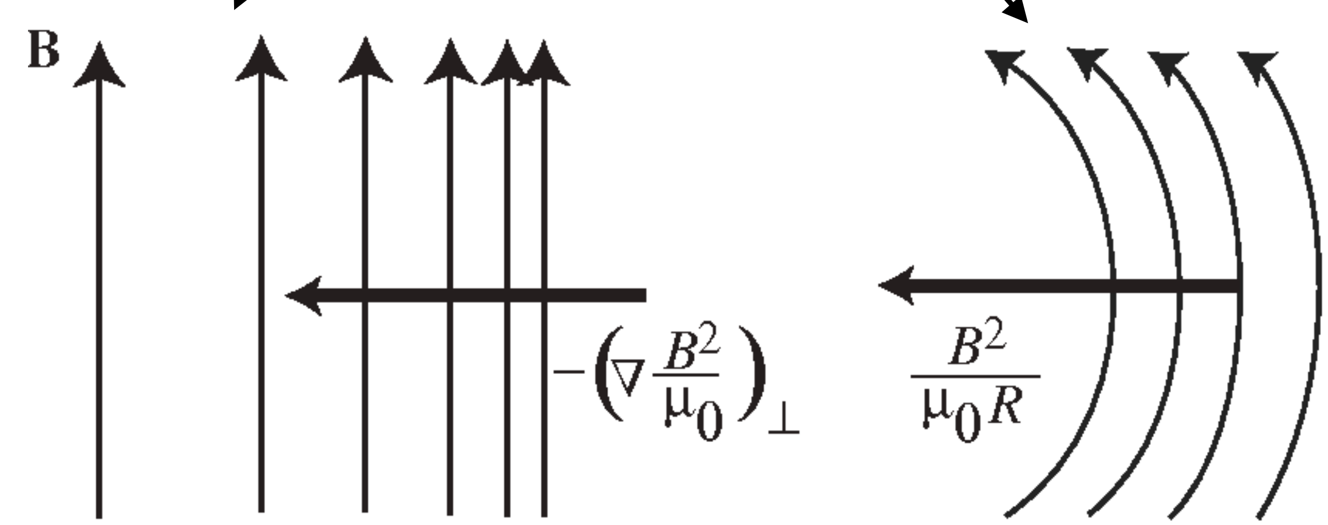
$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \boxed{\mathbf{J} \times \mathbf{B}}$$

$$\underbrace{-\mathbf{J} \times \mathbf{B}}_{\text{Lorentz Force}} = \mathbf{B} \times \nabla \times \mathbf{B} \xrightarrow[\substack{\nabla \cdot (\mathbf{AB}) = \nabla \mathbf{A} \cdot \mathbf{B} + \nabla \mathbf{B} \cdot \mathbf{A} \\ \nabla \cdot \mathbf{B} \equiv 0}]{\mathbf{A} \times \nabla \times \mathbf{B} = \nabla \mathbf{B} \cdot \mathbf{A} - \nabla \cdot (\mathbf{AB}) - \mathbf{B} \nabla \cdot \mathbf{A}} \nabla \underbrace{\left(\frac{1}{2} B^2\right)}_{\text{Magnetic pressure}} - \underbrace{(\mathbf{B} \cdot \nabla)\mathbf{B}}_{\text{Magnetic Tension}} = \nabla \cdot \underbrace{\left(\mathbf{I}\frac{1}{2}B^2 - \mathbf{BB}\right)}_{\boxed{\text{Maxwell Stress Tensor}}}$$

If div B isn't zero, the Lorentz force becomes:

$$= \nabla \cdot \left(\mathbf{I}\frac{1}{2}B^2 - \mathbf{BB}\right) - \underbrace{\mathbf{B}(\nabla \cdot \mathbf{B})}_{\text{Unphysical Parallel Acceleration}}$$

This is why div B = 0 is crucial in MHD!



Magnetic pressure

Magnetic Tension

1-D MHD equations

If only consider **variations** in the x-direction

The mass equation becomes:

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\rho\mathbf{u} \xrightarrow[\mathbf{u}=u_x\hat{\mathbf{x}}+u_y\hat{\mathbf{y}}+u_z\hat{\mathbf{z}}]{\nabla=\frac{\partial}{\partial x}\hat{\mathbf{x}}} \frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}\hat{\mathbf{x}}\cdot\rho(u_x\hat{\mathbf{x}}+u_y\hat{\mathbf{y}}+u_z\hat{\mathbf{z}})$$

$$\longrightarrow \boxed{\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u_x)}$$

1-D Mass equation

The velocity equation is slightly more complicated:

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J}\times\mathbf{B} \xrightarrow{\frac{D}{Dt}=\frac{\partial}{\partial t}+\nabla\cdot\mathbf{u}} \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{u} = \frac{1}{\rho}\left(-\nabla p + \mathbf{J}\times\mathbf{B}\right)$$

$$\xrightarrow[\mathbf{u}=u_x\hat{\mathbf{x}}+u_y\hat{\mathbf{y}}+u_z\hat{\mathbf{z}}]{\nabla=\frac{\partial}{\partial x}\hat{\mathbf{x}}} \frac{\partial\mathbf{u}}{\partial t}+u_x\frac{\partial\mathbf{u}}{\partial x} = \frac{\partial p}{\partial x}\hat{\mathbf{x}}+\mathbf{J}\times\mathbf{B}$$

We get three equations for the velocity $\longrightarrow$

$$\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x} = \frac{\partial p}{\partial x}+(J_yB_z-J_zB_y)$$

$$\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x} = +(J_zB_x-J_xB_z)$$

$$\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x} = +(J_xB_y-J_yB_x)$$

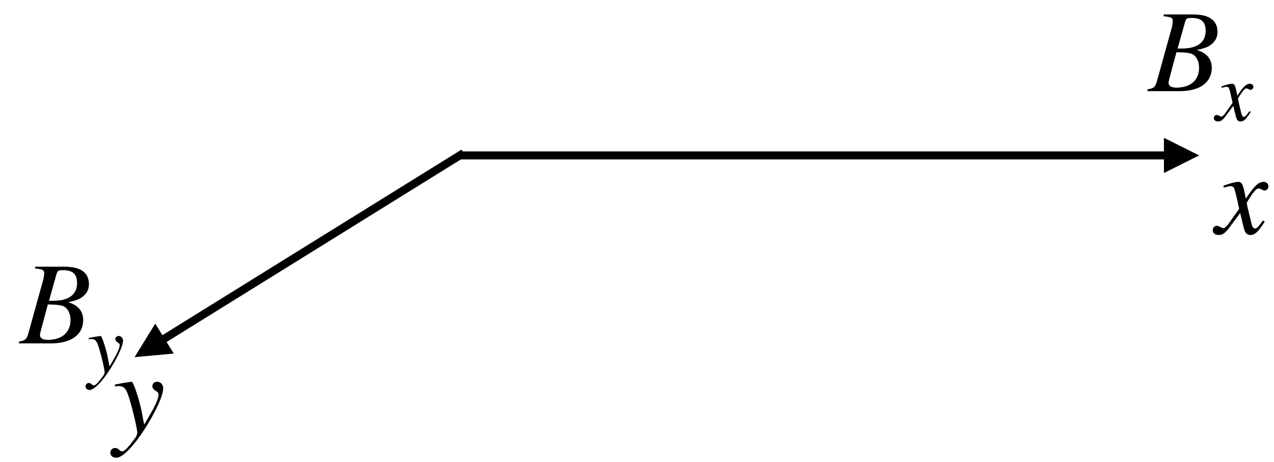
1-D MHD equations

Ampere's law in 1-D becomes:

$$\mathbf{J} = \nabla \times \mathbf{B} \xrightarrow[\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}]{\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}}} \mathbf{J} = \frac{\partial}{\partial x}\hat{\mathbf{x}} \times \left(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}\right) \longrightarrow \mathbf{J} = \frac{\partial B_y}{\partial x}\hat{\mathbf{z}} - \frac{\partial B_z}{\partial x}\hat{\mathbf{y}}$$

Since we're only looking at 1-D variations

So without losing generality, we are limiting the components of B field to be only Bx and By



Thus the Ampere's law gives only one component of the current Jz:

$$\mathbf{J} = \frac{\partial B_y}{\partial x}\hat{\mathbf{z}} - \cancelto{0}{\frac{\partial B_z}{\partial x}\hat{\mathbf{y}}} \xrightarrow{\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}} \mathbf{J} = \frac{\partial B_y}{\partial x}\hat{\mathbf{z}} \equiv J_z\hat{\mathbf{z}} \longrightarrow \boxed{J_z = \frac{\partial B_y}{\partial x}}$$

1-D Ampere's Law

Now the JxB term in the momentum equation only has two components

$$\mathbf{J} \times \mathbf{B} = J_z\hat{\mathbf{z}} \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}) = J_zB_x\hat{\mathbf{y}} - J_zB_y\hat{\mathbf{x}}$$

1-D MHD equations

With the definition of J and JxB:

$$\mathbf{J} = \frac{\partial B_y}{\partial x}\hat{\mathbf{z}} \equiv J_z\hat{\mathbf{z}} \qquad \mathbf{J}\times\mathbf{B} = J_z\hat{\mathbf{z}}\times(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}) = J_zB_x\hat{\mathbf{y}} - J_zB_y\hat{\mathbf{x}}$$

Revisit the velocity equations:

1-D Velocity equation (x-dir)

x-dir $$\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} = \frac{\partial p}{\partial x} + (\cancel{J_yB_z} - J_zB_y)/\rho \longrightarrow \boxed{\frac{\partial u_x}{\partial t} = -u_x\frac{\partial u_x}{\partial x} + \frac{1}{\rho}\left(-\frac{\partial p}{\partial x} - J_zB_y\right)}$$

$J_y \equiv 0$

y-dir $$\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} = +(J_zB_x - \cancel{J_xB_z})/\rho \longrightarrow \boxed{\frac{\partial u_y}{\partial t} = -u_x\frac{\partial u_y}{\partial x} + \frac{1}{\rho}J_zB_{x0}}$$

$J_x \equiv 0$

Why?

1-D Velocity equation (y-dir)

$J_x \equiv 0$ $J_y \equiv 0$

z-dir $$\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} = +(\cancel{J_xB_y} - \cancel{J_yB_x})/\rho \longrightarrow \frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} = 0 \longrightarrow u_z = \text{const}$$

z-dir is not needed!

Magnetic divergence must vanish:

$$\nabla\cdot\mathbf{B} = \frac{\partial}{\partial x}\hat{\mathbf{x}}\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}) = \frac{\partial B_x}{\partial x} \equiv 0 \longrightarrow \boxed{B_x = \text{const} \equiv \mathrm{B}_{x0}}$$

Constraint of Bx

1-D MHD equations

Now the ones left are the 1-D Ohm's law (trivial):

E-field only has z-component!

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B} \xrightarrow[\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}]{\mathbf{u} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}} \mathbf{E} = -(u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}) \longrightarrow \boxed{E_z = -u_xB_y + u_yB_{x0}}$$

1-D Ohm's Law (z-dir)

And the Faraday's law:

No equation needed for Bx!

$B_x \equiv B_{x0} = \text{const}$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \xrightarrow[\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}},\ \mathbf{E} = E_z\hat{\mathbf{z}}]{\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}}} \frac{\partial}{\partial t}(\cancel{B_x\hat{\mathbf{x}}} + B_y\hat{\mathbf{y}}) = -\frac{\partial}{\partial x}\hat{\mathbf{x}}\times E_z\hat{\mathbf{z}}$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}$$

only the y-component is needed:

$$\boxed{\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x}}$$

1-D Faraday's Law

And the ideal gas law (no derivatives):

$$\boxed{p = \frac{\beta_0}{2}\rho^\gamma}$$

Ideal Gas Law

1-D MHD equations

Plasma variables

$$\rho, u_x, u_y, p$$

Field variables

$$B_y, E_z, J_z$$

Put the equations together:

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot \rho \mathbf{u}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

$$p = \frac{\beta_0}{2}\rho^{\gamma}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u_x)$$

$$\frac{\partial u_x}{\partial t} = -u_x \frac{\partial u_x}{\partial x} + \frac{1}{\rho}\left(-\frac{\partial p}{\partial x} - J_z B_y\right)$$

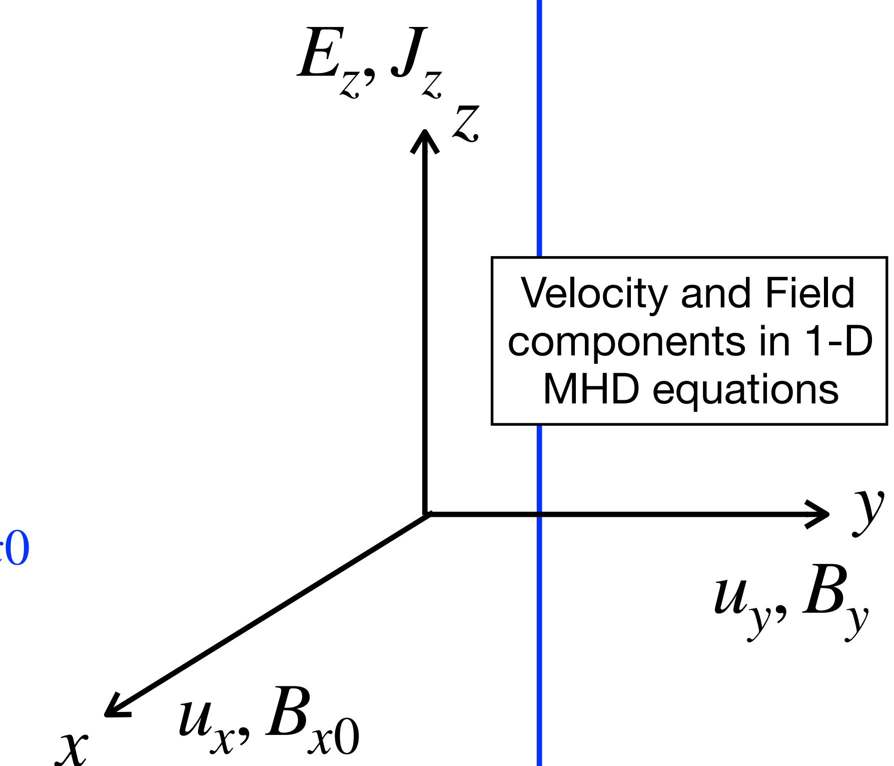
$$\frac{\partial u_y}{\partial t} = -u_x \frac{\partial u_y}{\partial x} + \frac{1}{\rho} J_z B_{x0}$$

$$p = \frac{\beta_0}{2}\rho^{\gamma}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$E_z = -u_x B_y + u_y B_{x0}$$

$$J_z = \frac{\partial B_y}{\partial x}$$



Velocity and Field components in 1-D MHD equations

Physics

Waves in the 1-D MHD equations

First, for a linear polarized traveling wave, assume all perturbed terms vary with x and t like

$$\sim e^{i(kx-\omega t)}$$

The spatial and time derivatives become simply algebra calculations: $\dfrac{\partial}{\partial x} = ik \quad \dfrac{\partial}{\partial t} = -i\omega$

Now let's linearize the MHD equations and see what's the relationship between omega and k

Assuming $\rho = \rho_0,\ u_x = 0,\ u_y = \delta u_y\ B_y = \delta B_y$

$$\mathbf{J} = \nabla \times \mathbf{B} \longrightarrow J_z = \frac{\partial B_y}{\partial x} \longrightarrow \delta J_z = ik\delta B_y$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} \longrightarrow E_z = -u_x B_y + u_y B_{x0} \longrightarrow \delta E_z = \delta u_y B_0$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{J} \times \mathbf{B} \longrightarrow -i\omega\rho_0\delta u_y = \delta J_z B_0$$

$\dfrac{D\mathbf{u}}{Dt} \approx \dfrac{\partial \mathbf{u}}{\partial t}$

Linearize

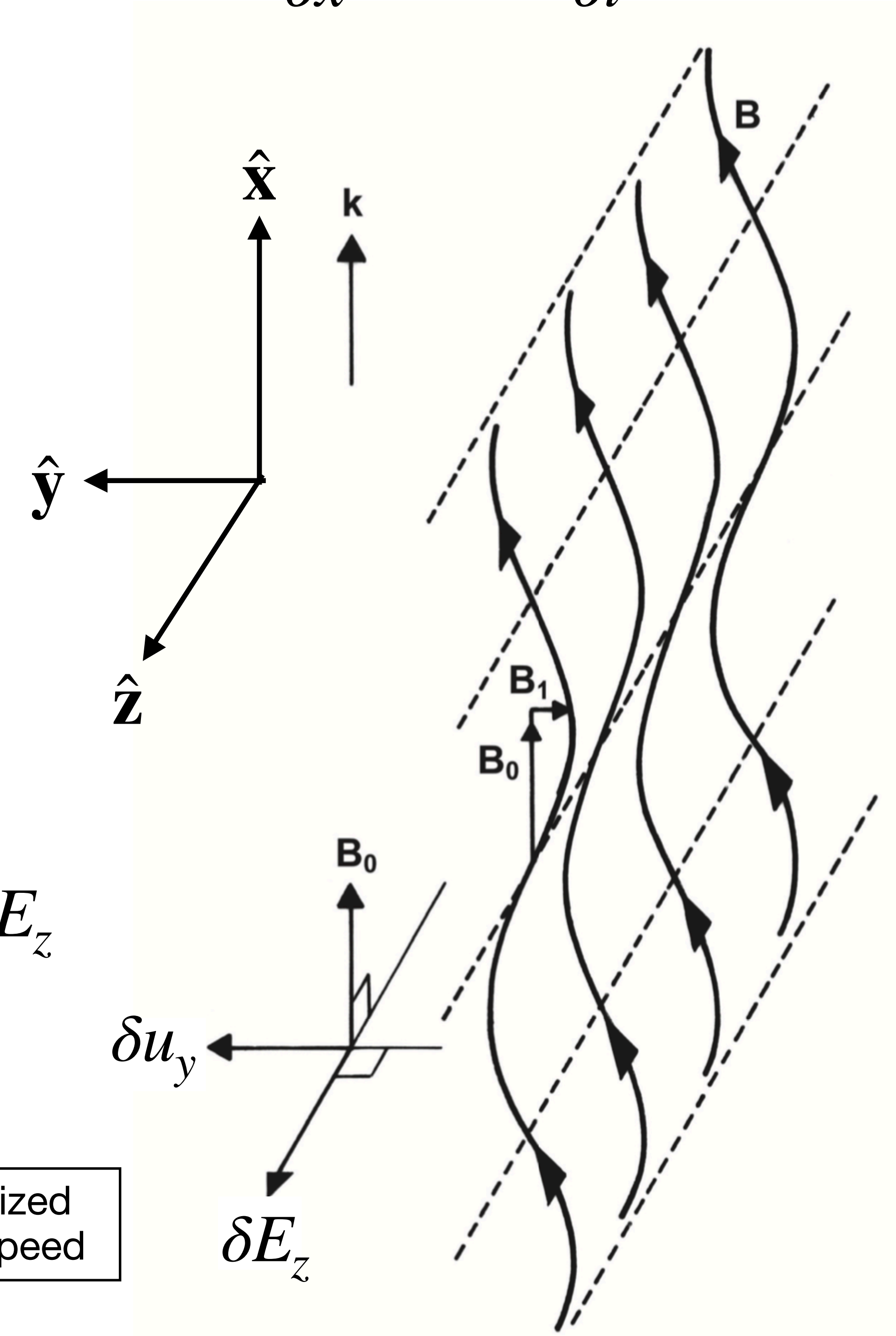
$p \approx 0$

Cold plasma Approximation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \longrightarrow -i\omega\delta B_y = ik\delta E_z$$

$$\Longrightarrow \boxed{\left(\frac{\omega}{k}\right)^2 = \frac{B_0^2}{\rho_0} = 1}$$

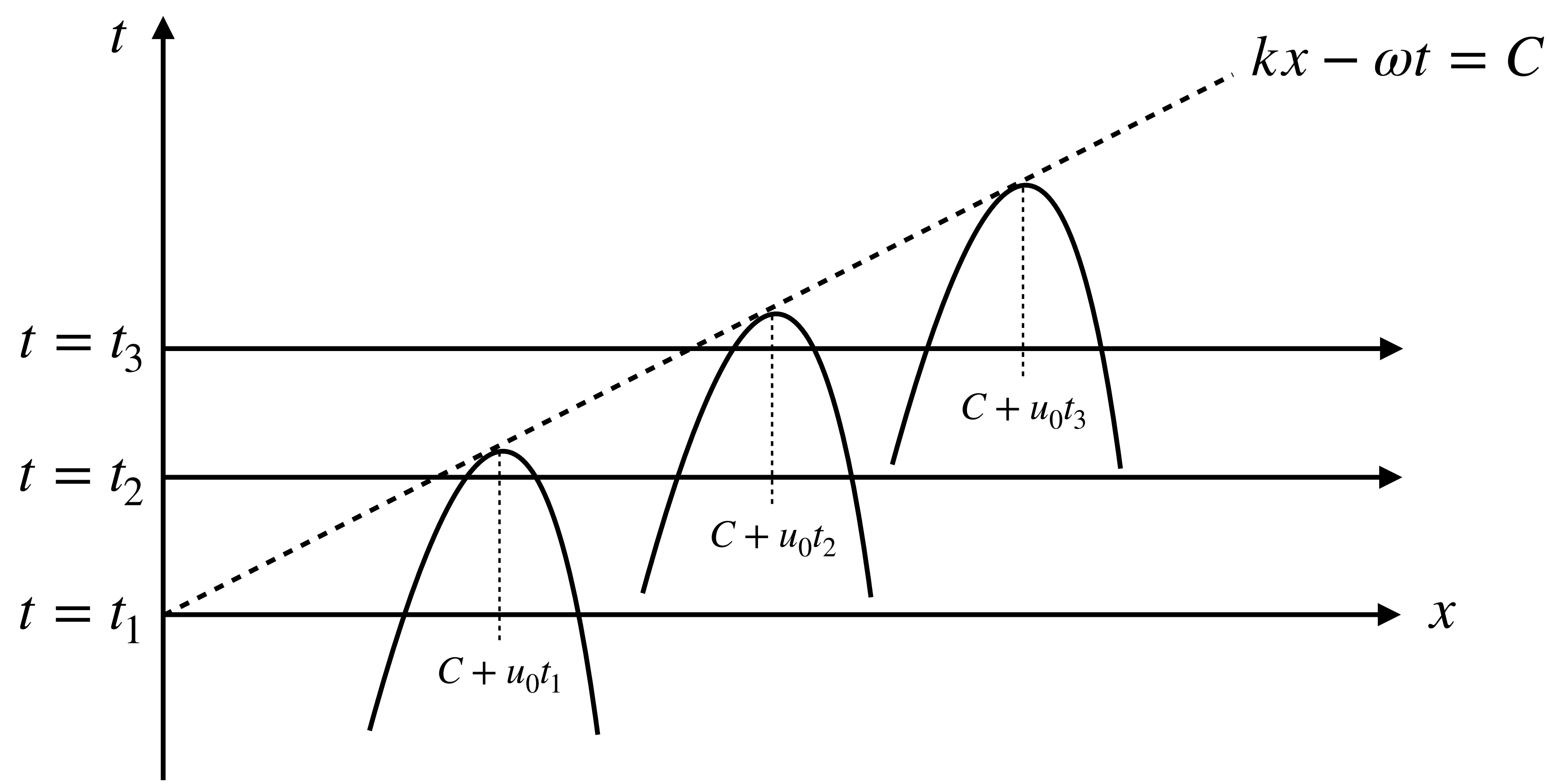
Normalized Alfvén speed



Traveling versus Standing Waves

In the Alfvén wave solution, note that $\dfrac{\omega}{k} = \pm 1$

- If k > 0, we get a wave traveling the +x direction - easy to see that from the phase of the wave: $(kx - \omega t)$ needs to keep constant
- If k < 0, we get a wave traveling the -x direction



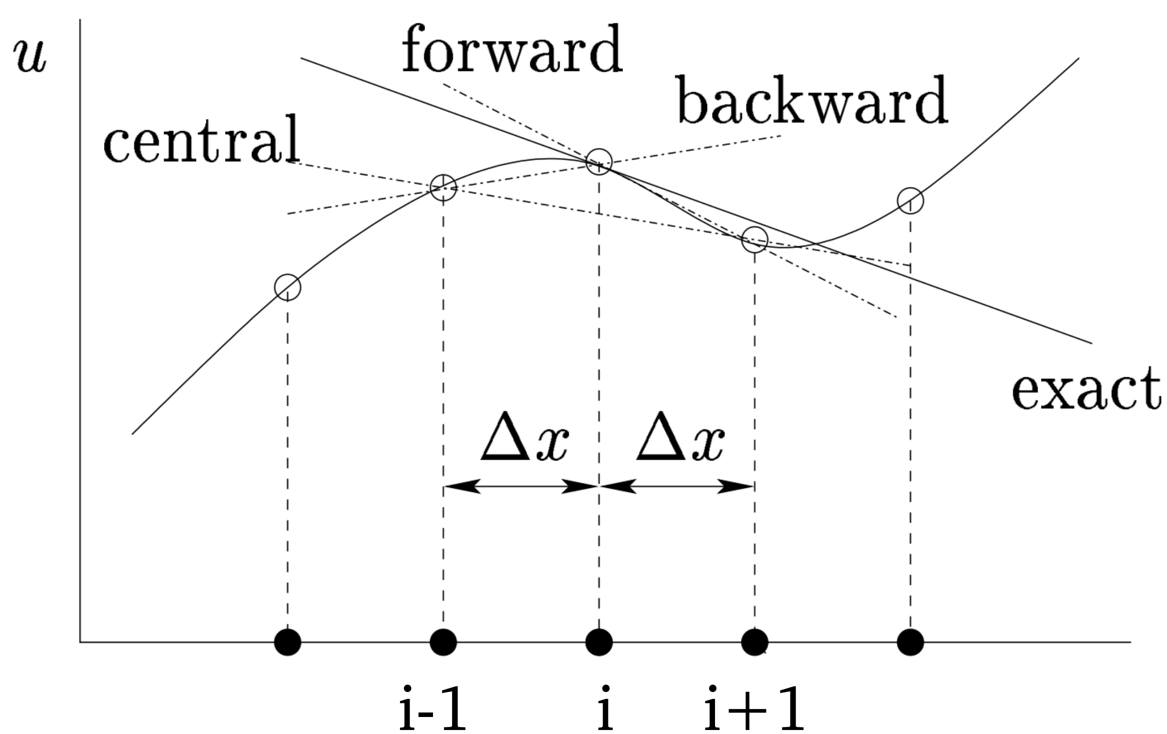
To keep $kx - \omega t$ constant: $kx - \omega t = \text{const} \longrightarrow \dfrac{dx}{dt} = \dfrac{\omega}{k} = \pm 1$

Wave traveling in both directions!

How to approximate partial derivatives

We've already learned the finite difference method

Geometric interpretation



$$\left(\frac{\partial u}{\partial t}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x}$$ Forward difference

$$\left(\frac{\partial u}{\partial t}\right)_i \approx \frac{u_i - u_{i-1}}{\Delta x}$$ Backward difference

$$\left(\frac{\partial u}{\partial t}\right)_i \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$ Central difference

This is based on the Taylor series expansion $u(x) = \sum_{n=0}^{\infty} \frac{(x - x_i)^n}{n!} \left(\frac{\partial^n u}{\partial x^n}\right)_i$

Let's try two expansions around x_i

T1: $$u_{i+1} = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{1}{2}\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{1}{6}\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

T2: $$u_{i-1} = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{1}{2}\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{1}{6}\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

Numerical differentiation

Truncation errors

Using the Taylor series expansion, we can evaluate the accuracy of the finite difference

T1: $$u_{i+1} = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{1}{2}\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{1}{6}\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \underbrace{\frac{u_{i+1} - u_i}{\Delta x}}_{\text{Forward difference approximation}} \underbrace{- \frac{1}{2}\Delta x \left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{1}{6}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots}_{\text{Truncation Error, leading term } \mathcal{O}(\Delta x)}$$

Using T2, we can get the backward difference approximation of the first-derivative

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \underbrace{\frac{u_i - u_{i-1}}{\Delta x}}_{\text{Backward difference approximation}} \underbrace{+ \frac{1}{2}\Delta x \left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{1}{6}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots}_{\text{Truncation Error, leading term } \mathcal{O}(\Delta x)}$$

Numerical differentiation

Truncation errors - central difference

Using the Taylor series expansion, we can evaluate the accuracy of the finite difference

T1: $$u_{i+1} = u_i + \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{1}{2}\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{1}{6}\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

T2: $$u_{i-1} = u_i - \Delta x \left(\frac{\partial u}{\partial x}\right)_i + \frac{1}{2}\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{1}{6}\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

T1 - T2:

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_i = \underbrace{\frac{u_{i+1} - u_{i-1}}{2\Delta x}}_{\text{Central difference approximation}} \underbrace{- \frac{1}{6}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots}_{\text{Truncation Error, leading term } \mathcal{O}(\Delta x^2)}$$

Definition of leading truncation error term:

$$\epsilon_\tau = \alpha_m (\Delta x)^m + \alpha_{m+1} (\Delta x)^{m+1} + \ldots \approx \alpha_m (\Delta x)^m \rightarrow \mathcal{O}(\Delta x^m)$$

Recall the Linear Advection Equation

$$\left(\frac{\partial u}{\partial t}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x}$$

Forward difference

$$\frac{\partial Q}{\partial t} + u_0 \frac{\partial Q}{\partial x} = 0$$

Forward difference

Forward difference

Euler Time-Stepping

$$\left.\frac{\partial Q}{\partial t}\right|_{t=t_n} = \frac{Q_i^{n+1} - Q_i^n}{\Delta t} + \mathcal{O}(\Delta t)$$

$$\left.\frac{\partial Q}{\partial t}\right|_{x=x_i} = \frac{Q_{i+1}^n - Q_i^n}{\Delta x} + \mathcal{O}(\Delta x)$$

Combine the two numerical derivatives

$$\frac{Q_i^{n+1} - Q_i^n}{\Delta t} = u_0 \frac{Q_{i+1}^n - Q_i^n}{\Delta x} + \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t) \longrightarrow$$

Forward Euler method

$$Q_i^{n+1} \approx \underbrace{Q_i^n + \frac{u_0 \Delta t}{\Delta x}(Q_{i+1}^n - Q_i^n)}_{\text{Values on step n (known)}}$$

If we know all the Q values at time t = t_n, then we can calculate the Q values at t = t_n+1. This is known as an explicit scheme of first-order accuracy

Can we do this to our MHD equations?

A First-Order Scheme

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(\rho u_x)$$

$$\frac{\partial u_x}{\partial t} = -u_x\frac{\partial u_x}{\partial x} + \frac{1}{\rho}\left(-\frac{\partial p}{\partial x} - J_z B_y\right)$$

$$\frac{\partial u_y}{\partial t} = -u_x\frac{\partial u_y}{\partial x} + \frac{1}{\rho}J_z B_{x0}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial E_z}{\partial x}$$

$$E_z = -u_x B_y + u_y B_{x0}$$

$$J_z = \frac{\partial B_y}{\partial x}$$

$$p = \frac{\beta_0}{2}\rho^\gamma$$

$$\rho_i^{n+1} \approx \rho_i^n + \frac{\Delta t}{\Delta x}(u^n_{x,i+1}\rho^n_{i+1} - u^n_{x,i}\rho^n_i)$$

$$u^{n+1}_{x,i} \approx u^n_{x,i} - u_{x,i}\frac{\Delta t}{\Delta x}(u^n_{x,i+1} - u^n_{x,i}) + \frac{\Delta t}{\rho^n_i}\left[\frac{1}{\Delta x}\left(p^n_{i+1} - p_i\right) - J^n_{z,i}B^n_{y,i}\right]$$

$$u^{n+1}_{y,i} \approx u^n_{y,i} - u_{x,i}\frac{\Delta t}{\Delta x}(u^n_{y,i+1} - u^n_{y,i}) + \frac{\Delta t}{\rho^n_i}J^n_{z,i}B_{x0}$$

$$B^{n+1}_y \approx B^n_y + \frac{\Delta t}{\Delta x}(E^n_{z,i+1} - E^n_{z,i})$$

$$E^n_{z,i} = -u^n_{x,i}B^n_{y,i} + u^n_{y,i}B_{x0}$$

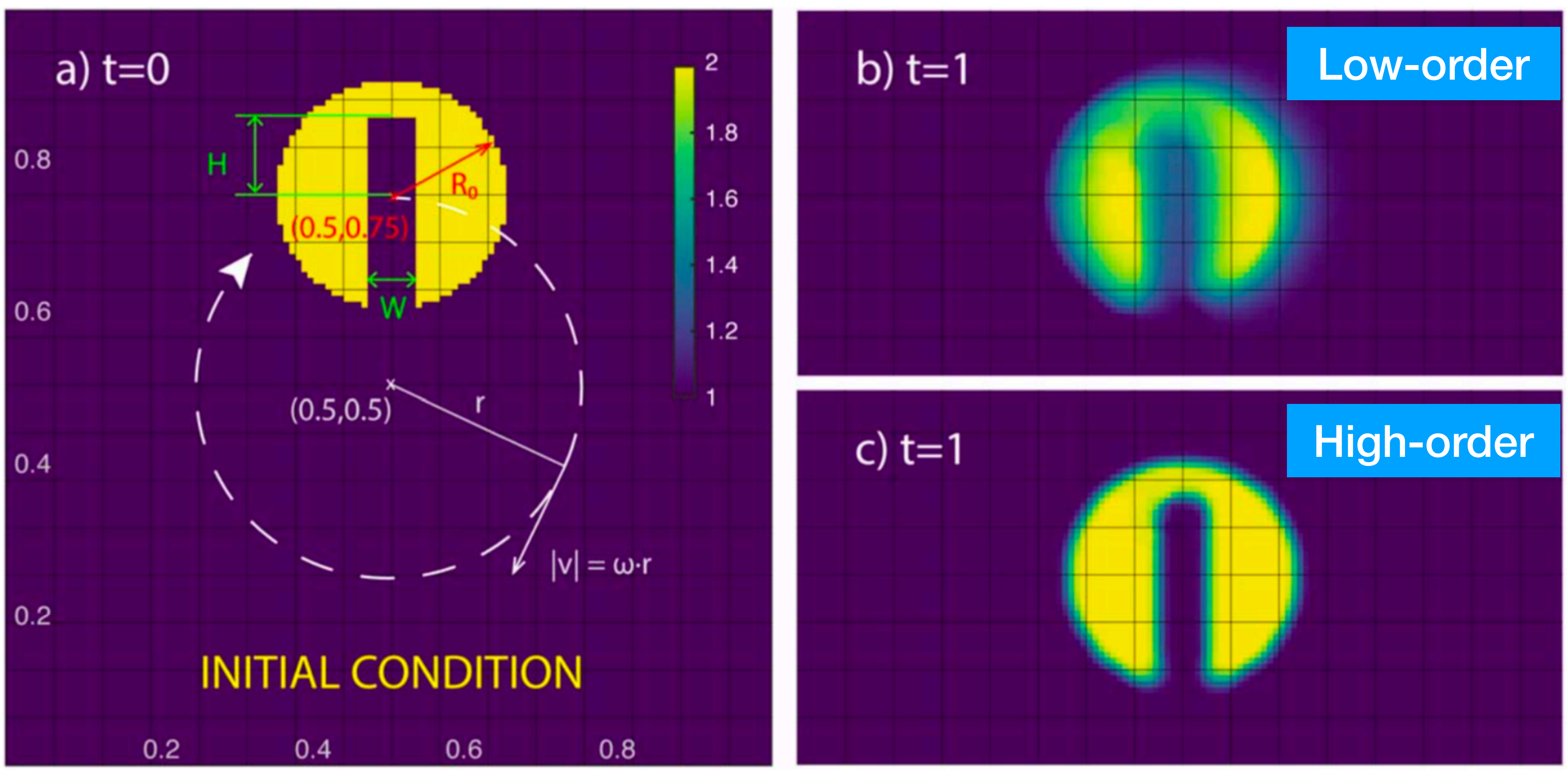
$$J^n_{z,i} = \frac{1}{\Delta x}\left(B^n_{y,i+1} - B^n_{y,i}\right)$$

$$p^n_{z,i} = \frac{\beta_0}{2}(\rho^n_i)^\gamma$$

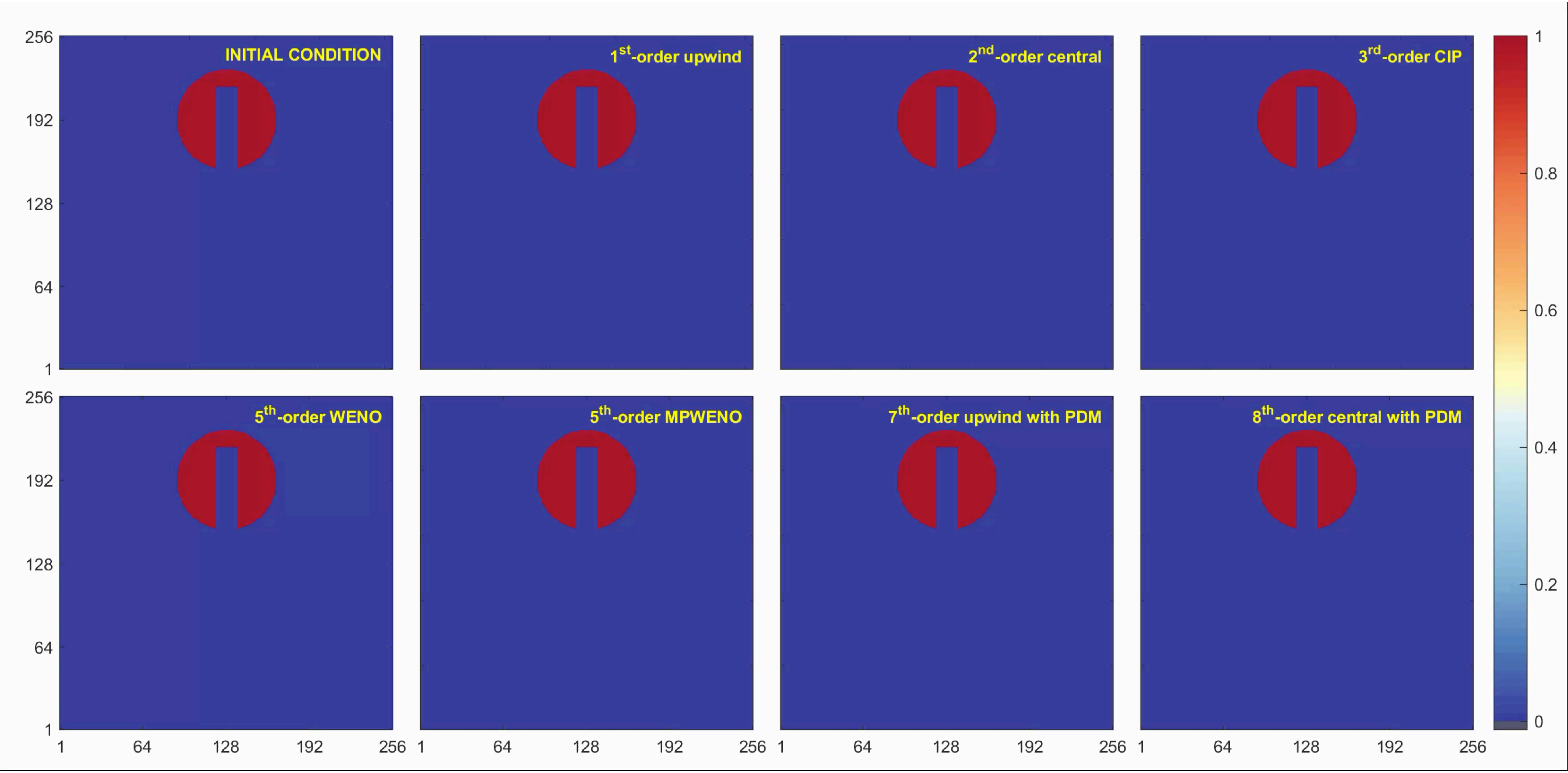
Why not using First-order schemes

Pros:
- Simple, straightforward to implement
- Usually quite stable

Cons:
- Not accurate - only first order approximations
- Converges at a very slow rate - linearly with grid size



Advection w/ different schemes



Movie credit: ZZ Li, Earth and Space Sciences@USTC