EASC2410 Special Topic 2

Special Topic: Mathematical Modeling of Epidemics: Python Implementation

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Review of Lecture 16:

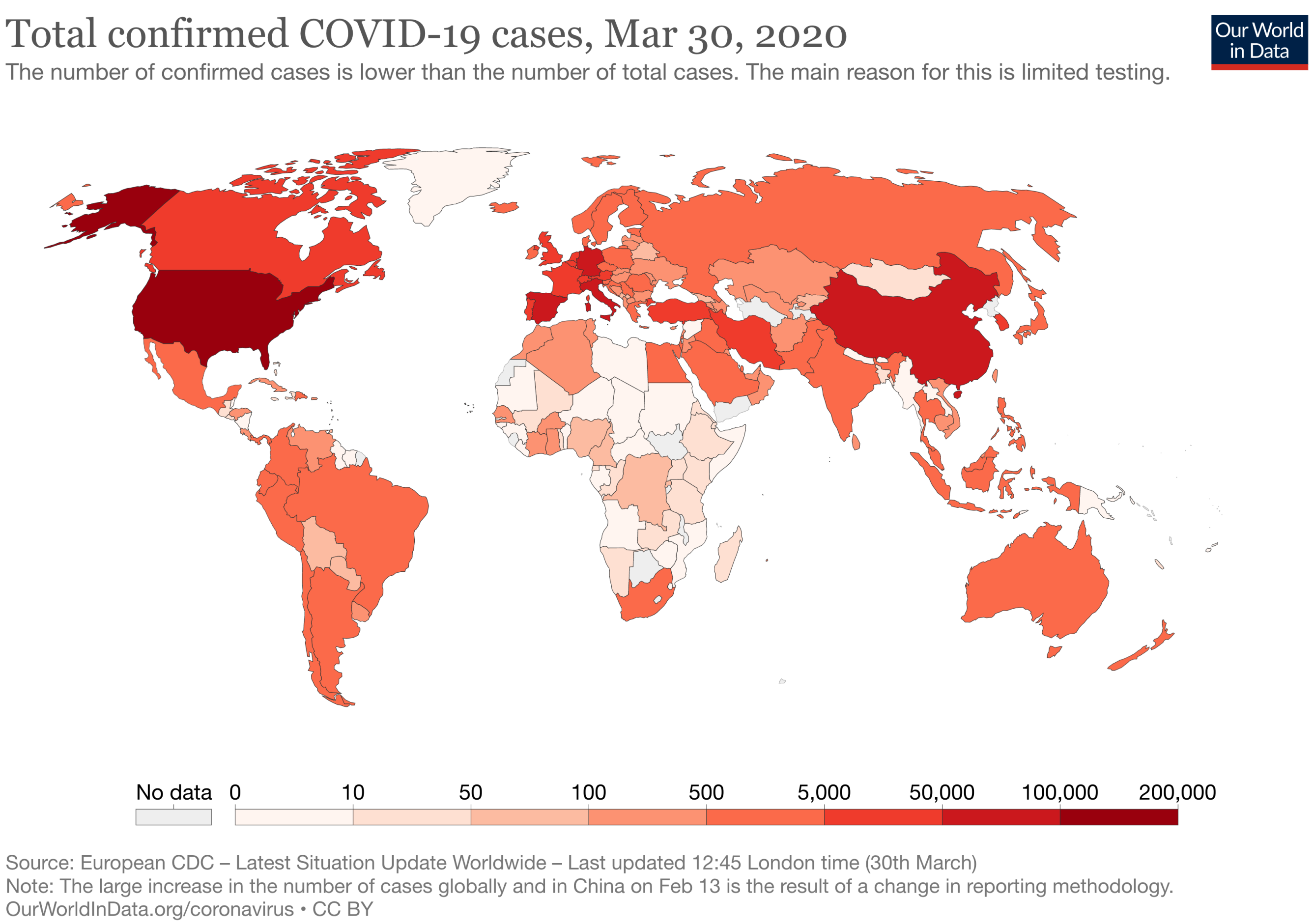
- Program control - if, for loops
- Numpy Arrays
- Pandas data wrangling
- Regression (curve fitting)

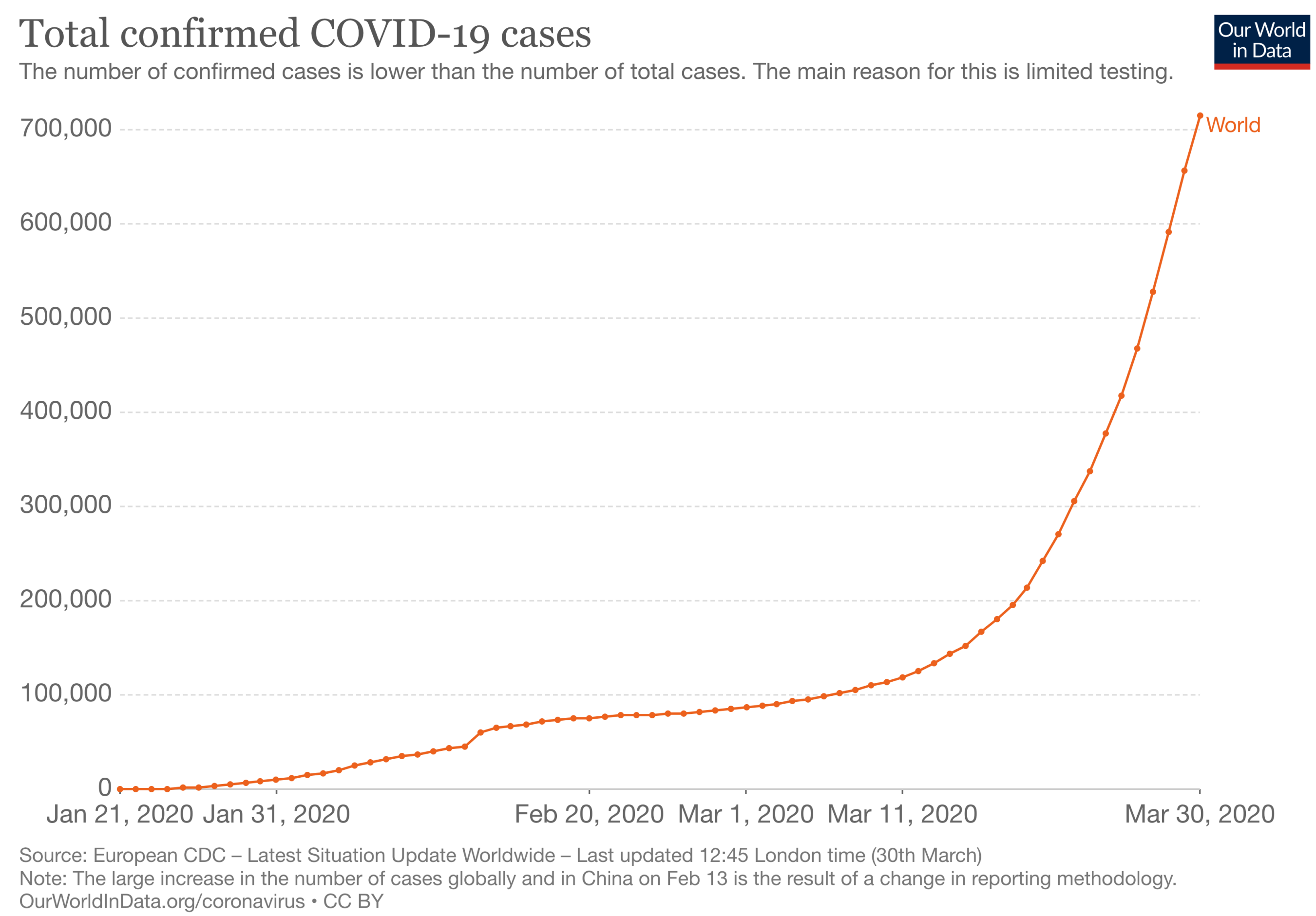
In Lecture 17, we will apply the Python knowledge to model the evolution of epidemics using a set of mathematical equations:

- Mathematical descriptions for infectious disease
- Solving those equations using Python
- Visualize, Analyze and Interpret the results

The COVID-19 Pandemic

The COVID-19 outbreak is an unprecedented global public health challenge. In order for governments, organizations and individuals to respond to it effectively, it will be vital that they have easy access to good, clear data and a good understanding of what can and can not be said based on the available data.

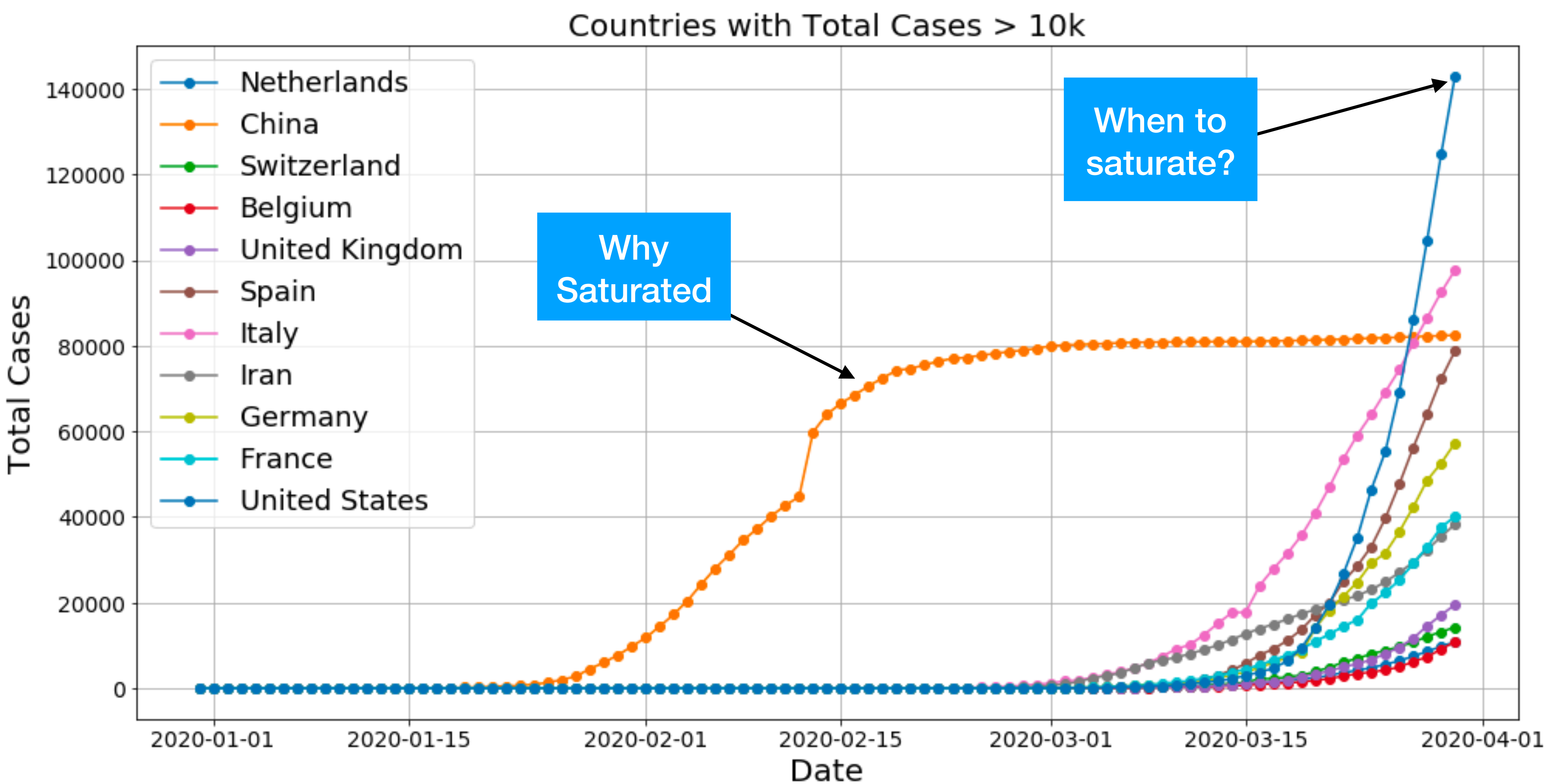




Data is the key - modeling helps interpret the data (to some extent)

time series with selection criteria

Now let's put the time series of total confirmed cases in the same based on total cases > 10000



Here I used a linear scale

The Exponential (R0) Model

Assumptions:

1. Starting from day 0, there was initially I[0] (or I_0) infections;
2. Everyday, one patient infects R_0 persons (the size of the infection grows R_0 times)

$$I[0] = I_0 \qquad I[1] = I[0] + R_0 \cdot I[0] \qquad I[2] = I[1] + R_0 \cdot I[1]$$

$$\text{Infected I[0]} \xrightarrow{\Delta = R_0 \cdot I[0]} \text{Infected I[1]} \xrightarrow{\Delta = R_0 \cdot I[1]} \text{Infected I[2]}$$

Day 0 **Day 1** **Day 2**

So to calculate the number if infections at day n+1:

$$\underbrace{I[n+1]}_{\#\text{ of infected on day } n+1} = \underbrace{I[n]}_{\#\text{ of infected on day } n} + \underbrace{R_0 \cdot I[n]}_{\text{Between day } n+1 \text{ and day } n,\ R_0 * I[n] \text{ new infections!}} \qquad n = 0,1,2,....$$

# of infected on day n+1

# of infected on day n

Between day n+1 and day n, R_0*I[n] new infections!

The mathematical form of the Exponential (R0) Model

$$I(t) : Infected$$

$$\boxed{\dot{I}(t)} = R_0 \cdot I(t)$$

- The time rate of change in I[t] is fixed
- The increase is proportional to I[t], i.e., within one day, the new infections caused by the existing patients is $R_0 I[t]$

Time rate of change in I(t)

$$\dot{I}(t = day\ n) = \frac{I[n+1] - I[n]}{(Day\ n+1) - (Day\ n)} = \frac{I[n+1] - I[n]}{1(day)}$$

$$= I[n+1] - I[n]$$

$$I[n+1] - I[n] = R_0 \cdot I[n] \qquad n = 0,1,2,....$$

A Python implementation of the Exponential (R0) Model

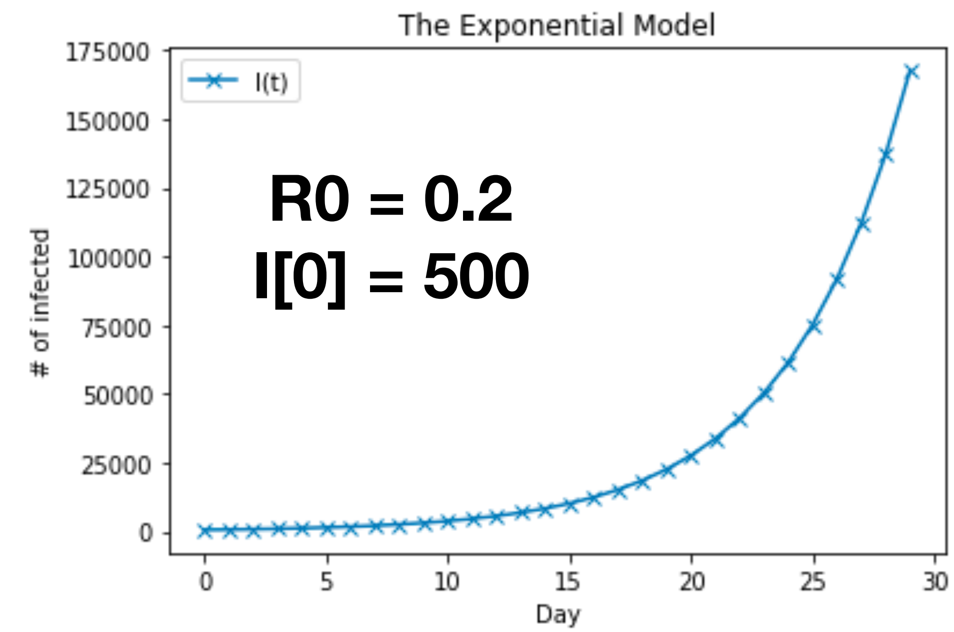
```python
import numpy as np
import matplotlib.pyplot as plt

# simuation Time / Day
T = 11

# Base infection rate
R0 = 2

I = np.zeros(T)

I[0]=1 # on day 0 we had 1 patient

for n in range(T-1):
    I[n+1] = I[n] + I[n]*R0

plt.plot(I,'-x',label='I(t)')
plt.xlabel('Day')
plt.ylabel('# of infected')
plt.title('The Exponential Model')
plt.legend()
```

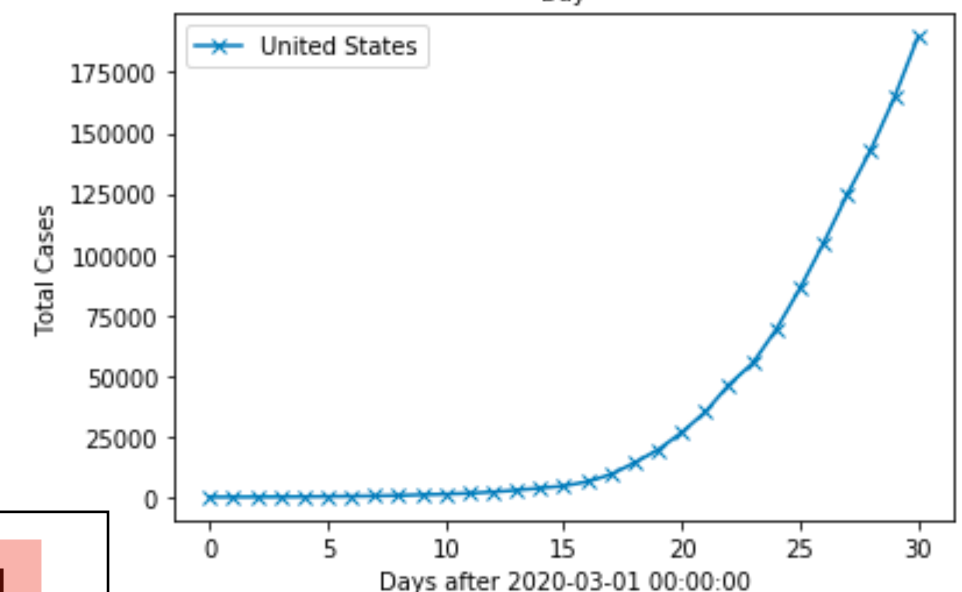
$$I[0] = I_0$$

Infected I[0]

Day 0

Plotting codes





So to calculate the number if infections at day n+1:

$$I[n+1] = I[n] + R_0 \cdot I[n]$$

$$n = 0,1,2,....$$

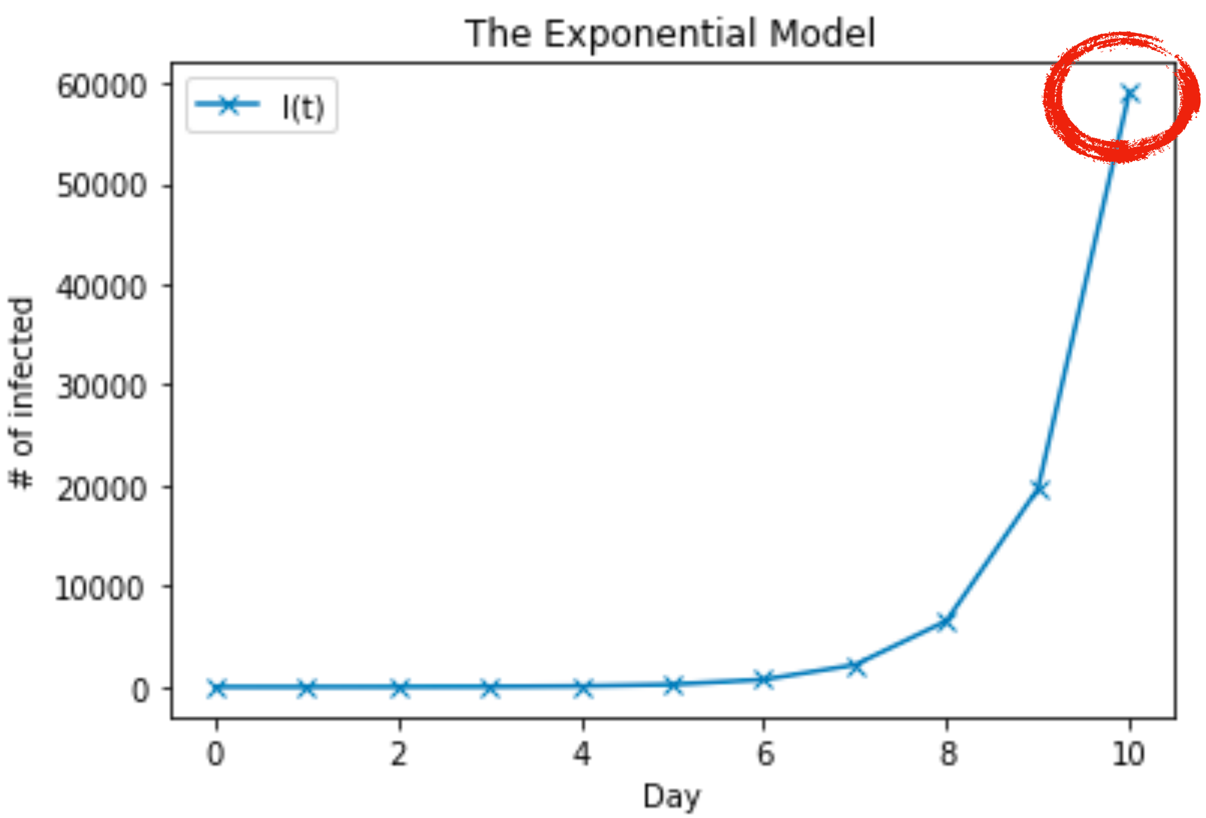
# of infected on day n+1

# of infected on day n

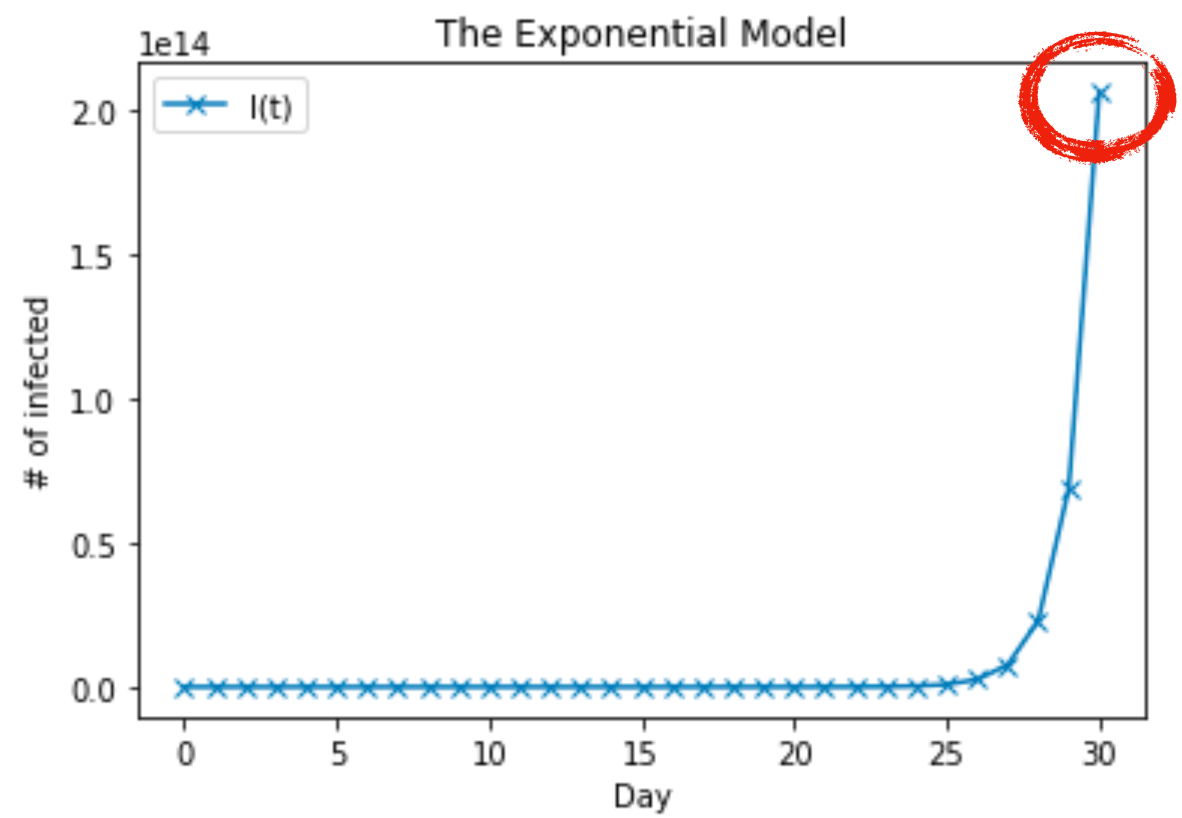
Between day n+1 and day n, $R_0 * I[n]$ new infections!

The problem of the Exponential (R0) Model

```
# simuation Time / Day
T = 11

# Base infection rate
R0 = 2
```




```
# simuation Time / Day
T = 31

# Base infection rate
R0 = 2
```



- **Starting from day 0, 1 infection**
- **At day 11, 60k infections**
- **At day 31, 10^{14} infections (more than the whole population of the world!)**

What's the problem here?

The S-I model

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Starting from day 0, there was initially I[0] infections;
3. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
4. New infections are removed from S

λ : contact rate

	Suspectible	Infected	Newly Infected
Day n	$S[n]$	$I[n]$	$\lambda \cdot I[n] \cdot S[n]$
Day n+1	$S[n] - \lambda \cdot I[n]S[n]$	$I[n] + \lambda \cdot I[n]S[n]$	

# of susceptible on day n+1, S[n+1]

# of infected on day n+1, I[n+1]

The S-I model

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Starting from day 0, there was initially I[0] infections;
3. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
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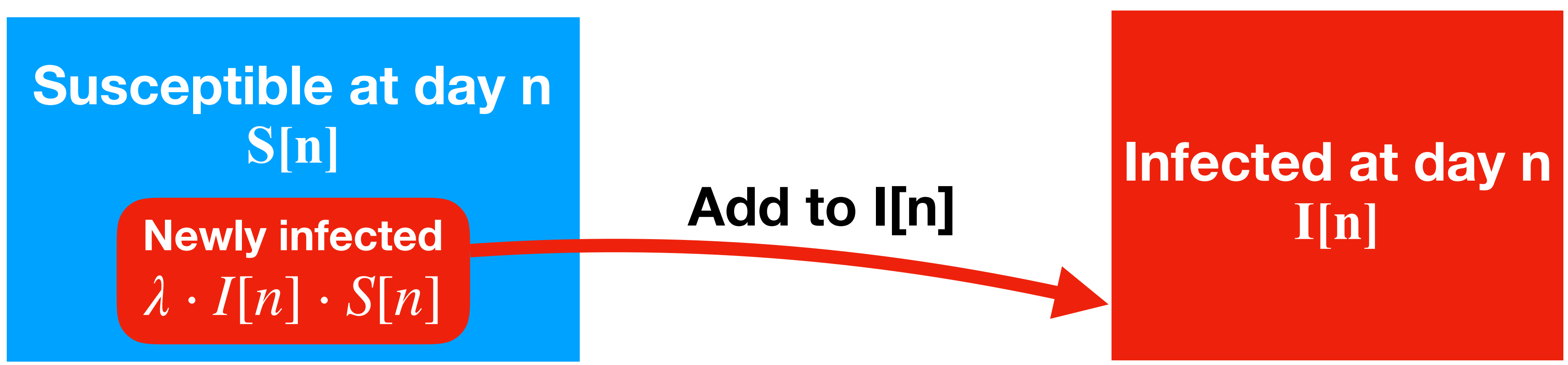
λ : contact rate



So to calculate the number if infections at day n+1:

$$S[n+1] = S[n] - \boxed{\lambda \cdot I[n] \cdot S[n]}$$

Removed from S[n]

$$I[n+1] = I[n] + \boxed{\lambda \cdot I[n] \cdot S[n]} \qquad n = 0,1,2,....$$

# of infected on day n+1

# of infected on day n

Between day n+1 and day n, lamda*I[n] new infections!

Added to I[n]

A Python implementation of the S-I Model

```python
# population
N = 1e7
# simuation Time / Day
T = 70
# susceptiable ratio
s = np.zeros([T])
# infective ratio
i = np.zeros([T])
# contact rate
lamda = 0.5

# initial infective people
i[0] = 5 / N
s[0] = 1
for t in range(T-1):
    s[t + 1] = s[t] - i[t] * lamda * s[t]
    i[t + 1] = i[t] + i[t] * lamda * s[t]

plt.plot(i*N,label='infected')
plt.plot(s*N,label='susceptible')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I Model')
plt.legend()
```

Plotting codes

Initialization

- **Setting N, T and lambda**
- **Setting I and S at day 0**

N: total # of population (susceptible)
T: total days of model simulation
lamda = contact rate

Susceptible at day n

New infections removed at day n

$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n]$$

$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n]$$

Infected at day n

New infections Added at day n

The mathematical form of the S-I Model

$$S(t) : Susceptible$$

$$I(t) : Infected$$

Time rate of change in S(t)

$$\boxed{\dot{S}(t)} = -\lambda \cdot S(t) \cdot I(t)$$

$$\boxed{\dot{I}(t)} = \lambda \cdot S(t) \cdot I(t)$$

Time rate of change in I(t)

$$I[n+1] - I[n] = \lambda \cdot S[n] \cdot I[n] \quad n = 0,1,2,....$$

A Python implementation of the S-I Model

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# population
N = 1e7
# simuation Time / Day
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s = np.zeros([T])
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# contact rate
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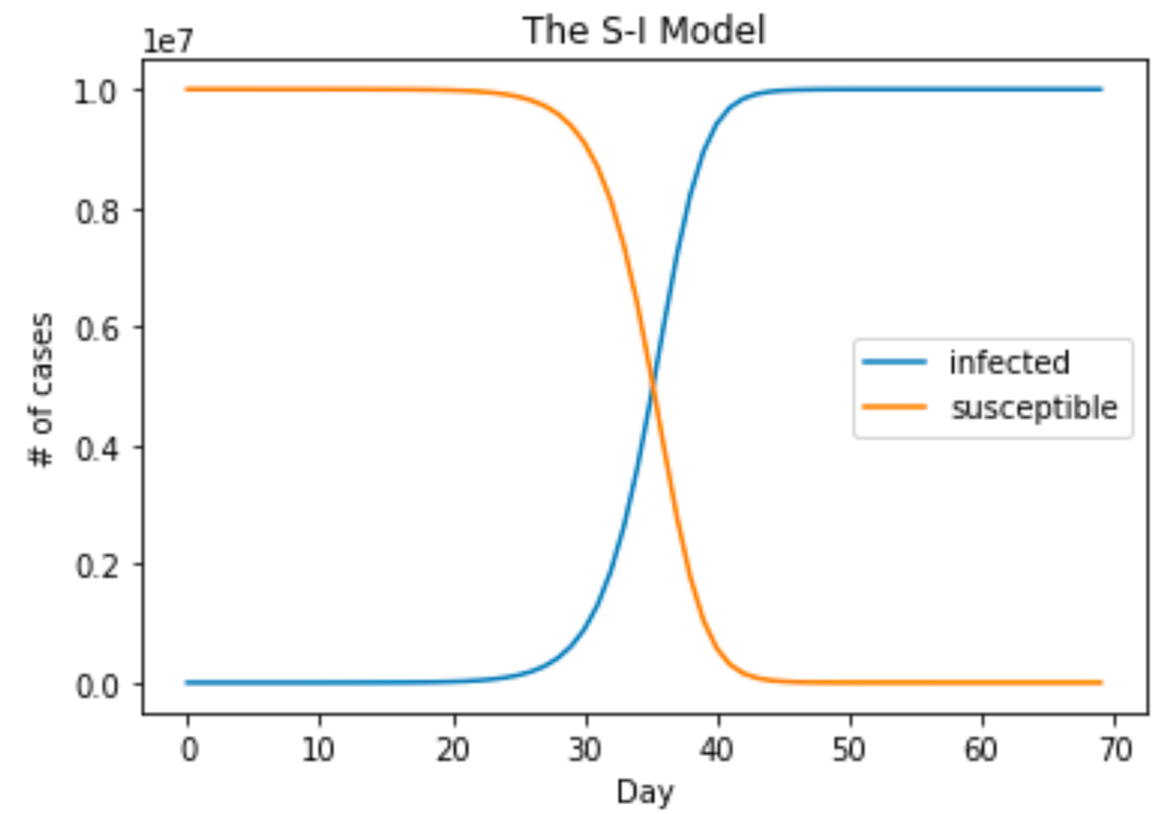
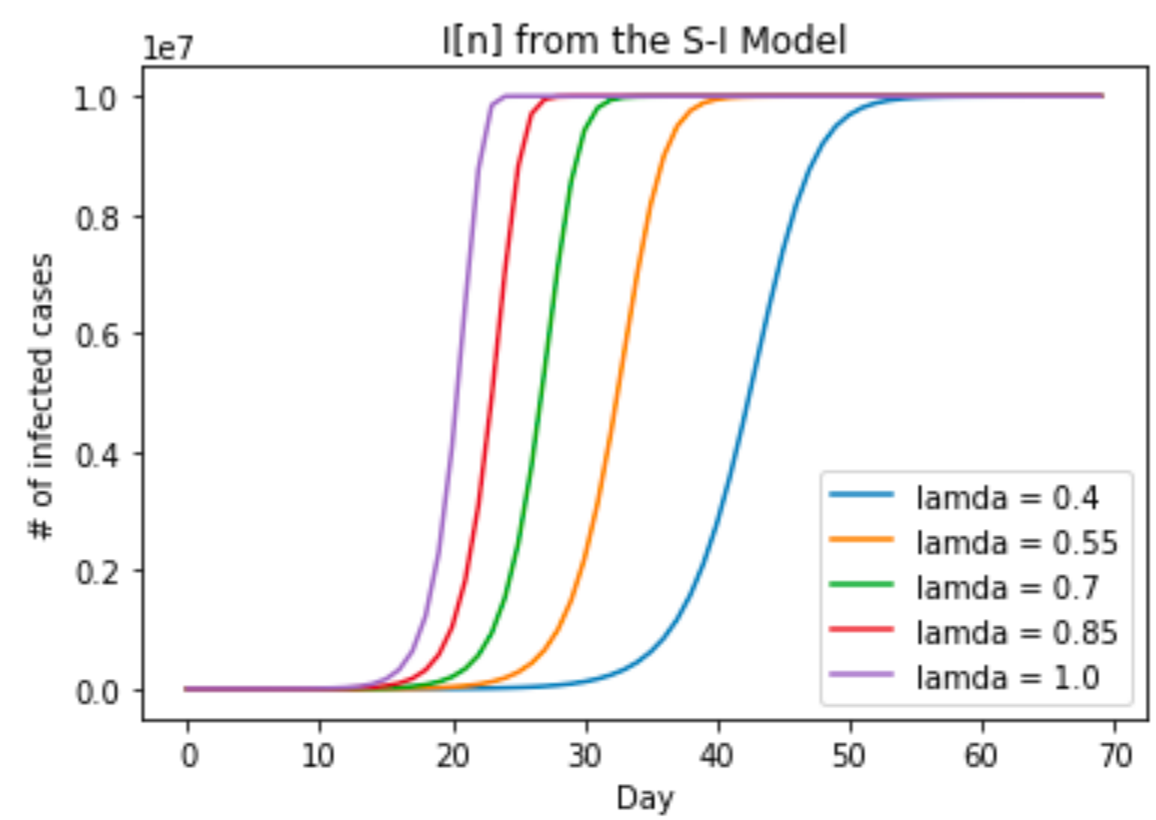
# initial infective people
i[0] = 5 / N
s[0] = 1
for t in range(T-1):
    s[t + 1] = s[t] - i[t] * lamda * s[t]
    i[t + 1] = i[t] + i[t] * lamda * s[t]

plt.plot(i*N,label='infected')
plt.plot(s*N,label='susceptible')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I Model')
plt.legend()
```



Plotting codes

- **Starting from day 0, 1 infection**
- **Lamda determines the rate of infection**



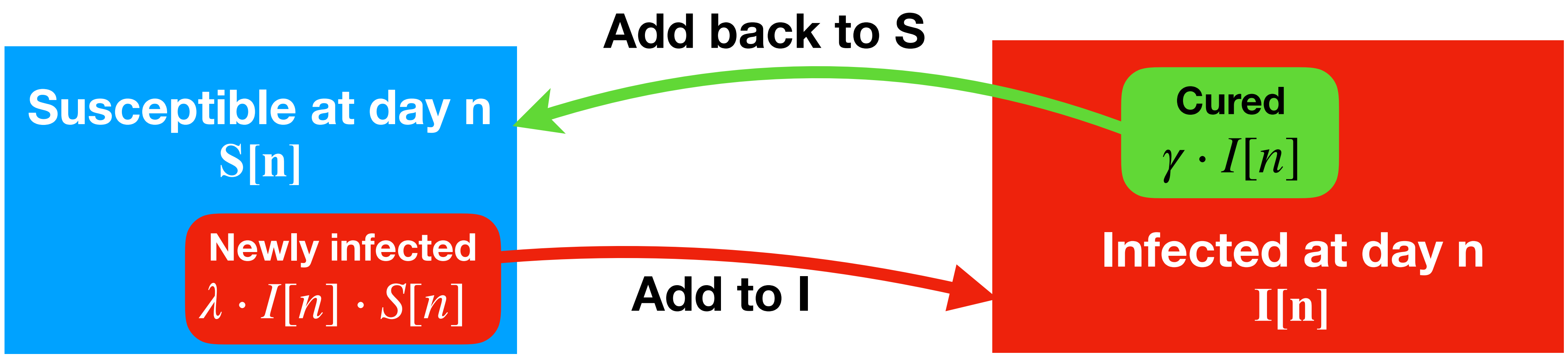
What's the problem here?

The S-I-S model (no immunity)

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
3. New infections are removed from S (added to I)
4. New cured is $\gamma \cdot I$, they are added back to S (no immunity)

λ : contact rate
γ : cure rate



	Suspectible (S)	Infected (I)	Newly Infected	Cured
Day n	$S[n]$	$I[n]$	$\lambda \cdot I[n] \cdot S[n]$	$\gamma \cdot I[n]$

Day n+1

$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n] + \gamma \cdot I[n]$$

$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n] - \gamma \cdot I[n]$$

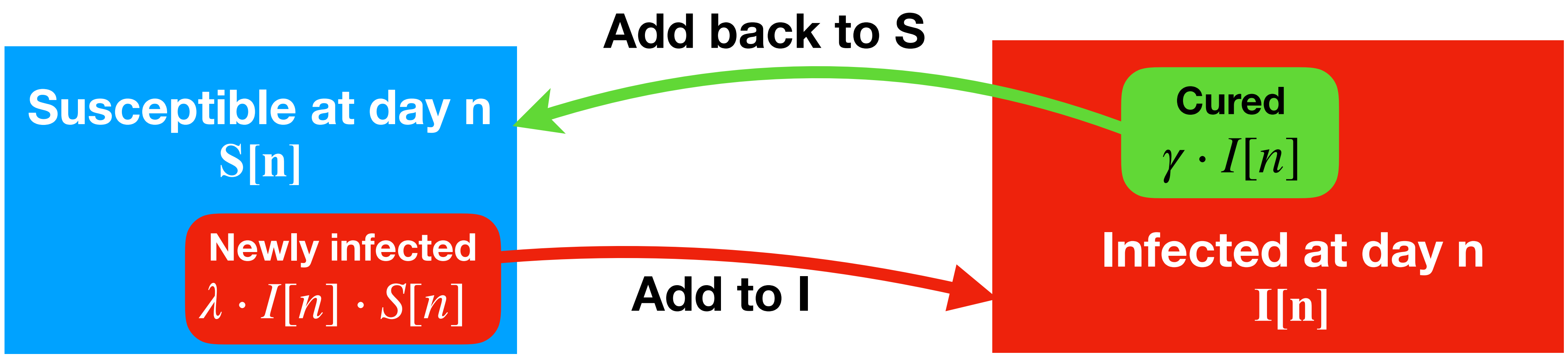
The S-I-S model (no immunity)

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
3. New infections are removed from S (added to I)
4. New cured is $\gamma \cdot I$, they are added back to S (no immunity)

λ : contact rate

γ : cure rate



$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n] + \underbrace{\gamma I[n]}_{\text{Add back to } S[n]}$$

$$n = 0,1,2,....$$

$$\underbrace{I[n+1]}_{\text{\# of infected on day n+1}} = \underbrace{I[n]}_{\text{\# of infected on day n}} + \underbrace{\lambda \cdot I[n] \cdot S[n]}_{\text{Between day n+1 and day n, new infections through contact}} - \underbrace{\gamma I[n]}_{\text{\# of cured on day n}}$$

Add back to S[n]

# of cured on day n

# of infected on day n

# of infected on day n+1

Between day n+1 and day n, new infections through contact

A Python implementation of the S-I-S Model

```python
# susceptiable ratio
s = np.zeros([T])
# infective ratio
i = np.zeros([T])

# contact rate
lamda = 1.0
# recover rate
gamma = 0.3

# initial infective people
i[0] = 45.0 / N
s[0] = 1.0

for t in range(T-1): # step through day 0 to day T-1

    s[t + 1] = s[t] - i[t] * lamda * s[t] + gamma*i[t]
    i[t + 1] = i[t] + i[t] * lamda * s[t] - gamma*i[t]

plt.plot(i*N,label='infected')
plt.plot(s*N,label='susceptible')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I-S Model')
plt.legend()
```

Plotting codes

Initialization

- Setting N, T lambda, gamma
- Setting I and S at day 0

N: total # of population (susceptible)
T: total days of model simulation
lamda = contact rate
gamma = recover rate

$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n] + \gamma I[n]$$
$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n] - \gamma I[n]$$

Infected at day n

Newly infected at day n

Recovered at day n

The mathematical form of the S-I-S Model

$$S(t) : Susceptible$$
$$I(t) : Infected$$

Time rate of change in S(t)

$$\boxed{\dot{S}(t)} = -\lambda \cdot S(t) \cdot I(t) + \gamma I(t)$$
$$\boxed{\dot{I}(t)} = \lambda \cdot S(t) \cdot I(t) - \gamma I(t)$$

Time rate of change in I(t)

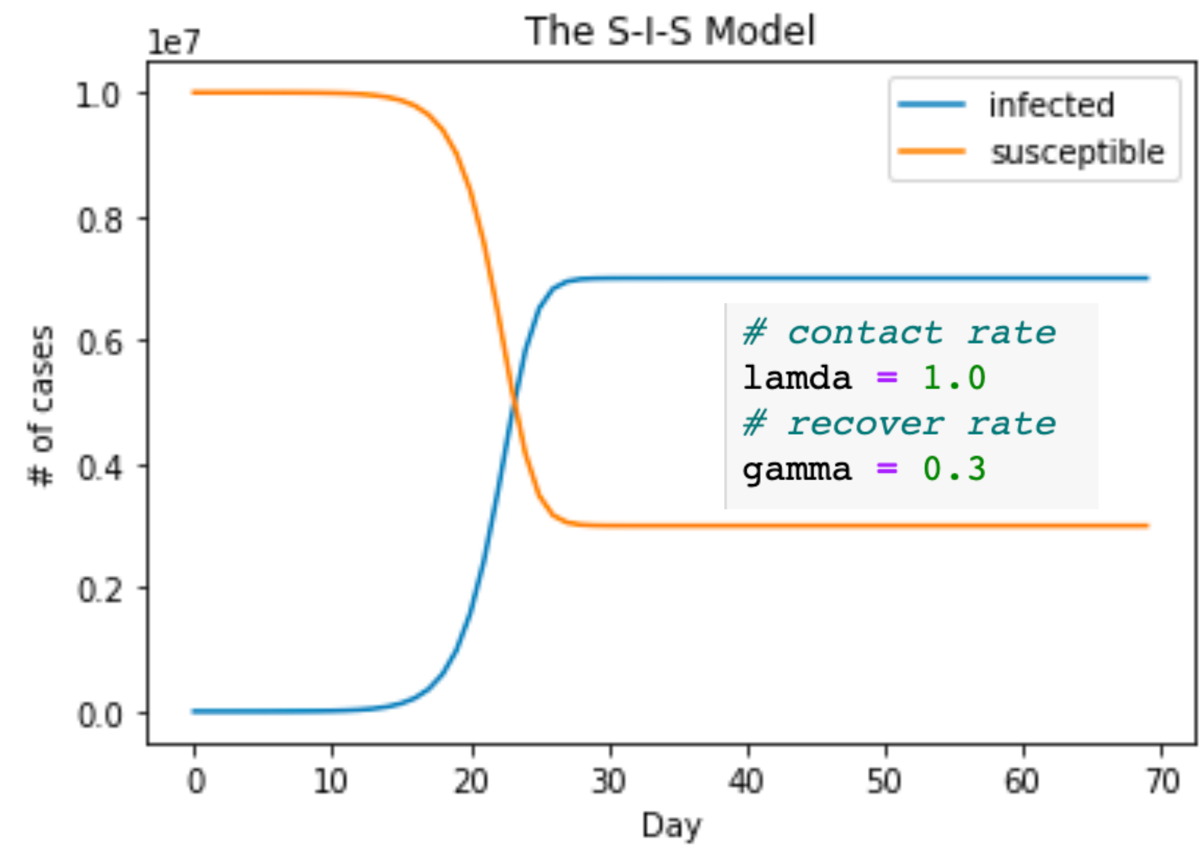
$$I[n+1] - I[n] = \lambda \cdot S[n] \cdot I[n] - \gamma \cdot I[n]$$
$$n = 0,1,2,....$$

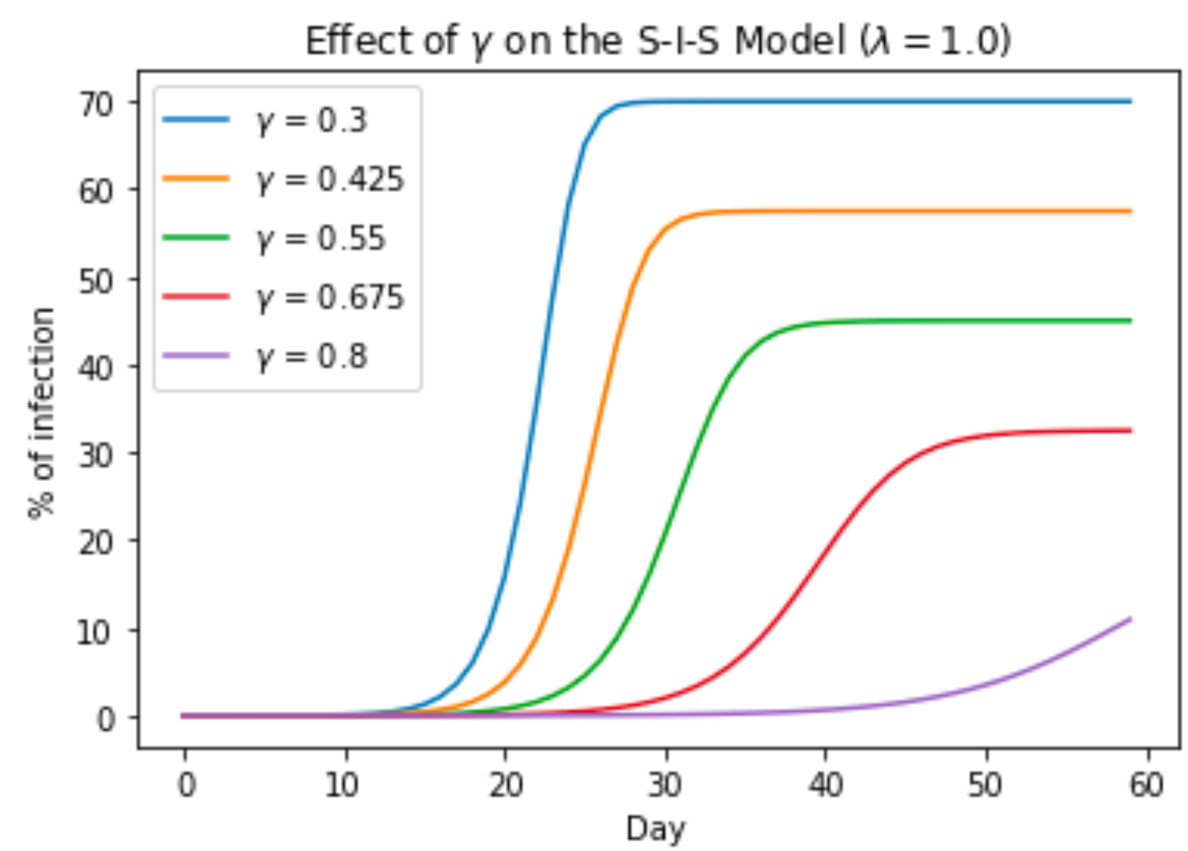
A Python implementation of the S-I-S Model

```python
# susceptiable ratio
s = np.zeros([T])
# infective ratio
i = np.zeros([T])

# contact rate
lamda = 1.0
# recover rate
gamma = 0.3

# initial infective people
i[0] = 45.0 / N
s[0] = 1.0

for t in range(T-1): # step through day 0 to day T-1

    s[t + 1] = s[t] - i[t] * lamda * s[t] + gamma*i[t]
    i[t + 1] = i[t] + i[t] * lamda * s[t] - gamma*i[t]

plt.plot(i*N,label='infected')
plt.plot(s*N,label='susceptible')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I-S Model')
plt.legend()
```

Initialization

- Setting N, T lambda, gamma
- Setting I and S at day 0





- **Starting from day 0, 1 infection**
- **Lamda determines the rate of infection**
- **Gamma determines the rate of cure**

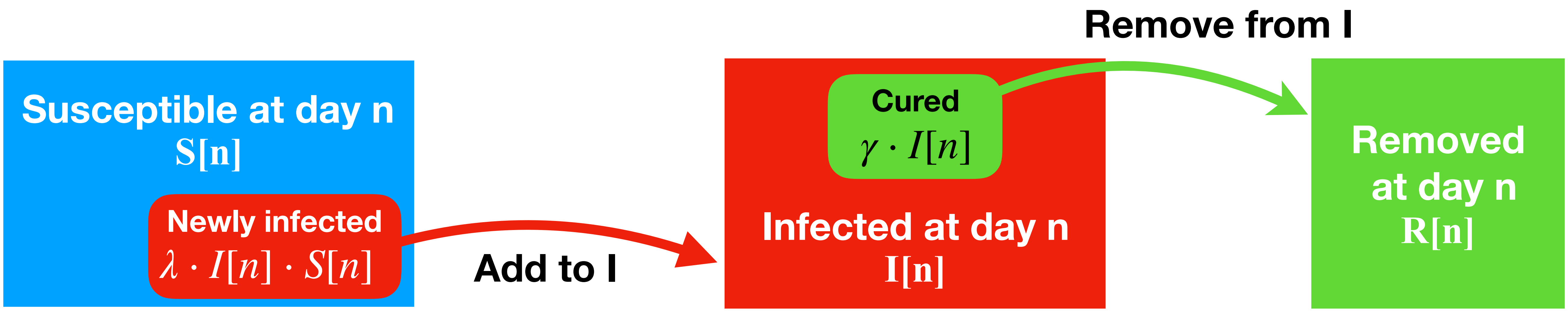
What's the problem here?

The S-I-R model (immunity)

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
3. New infections are removed from S (added to I)
4. New cured is $\gamma \cdot I$, they are removed from I (immunity)

λ : contact rate
γ : cure rate



	Suspectible (S)	Infected (I)	Removed (R)	Newly Infected	Cured (immuned)
Day n	$S[n]$	$I[n]$	$R[n]$	$\lambda \cdot I[n] \cdot S[n]$	$\gamma \cdot I[n]$

Day n+1

$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n]$$

$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n] - \gamma \cdot I[n]$$

$$R[n+1] = R[n] + \gamma \cdot I[n]$$

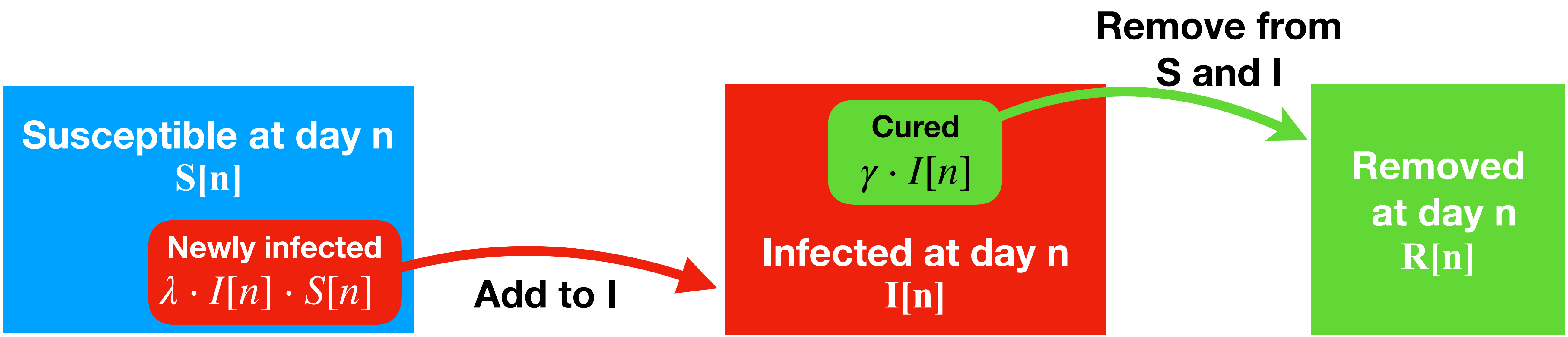
The S-I-R model (immunity)

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Everyday, new infection is through contact rate $\lambda \cdot I \cdot S$
3. New infections are removed from S (added to I)
4. New cured is $\gamma \cdot I$, they are removed from I (immunity)

λ **: contact rate**

γ **: cure rate**



$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n]$$

$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n] - \gamma I[n]$$

$$R[n+1] = R[n] + \gamma I[n]$$

# of recovered on day n

Between day n+1 and day n, new infections through contact

The mathematical form of the S-I-R Model

$$S(t) : Susceptible$$

$$I(t) : Infected$$

$$R(t) : Removed$$

Time rate of change in S(t)

$$\dot{S}(t) = -\overset{\text{Contact}}{\lambda \cdot S(t) \cdot I(t)}$$

Time rate of change in I(t)

$$\dot{I}(t) = \underset{\text{Contact}}{\lambda \cdot S(t) \cdot I(t)} - \underset{\text{Removed}}{\gamma I(t)}$$

Time rate of change in R(t)

$$\dot{R}(t) = \underset{\text{Removed}}{\gamma I(t)}$$

A Python implementation of the S-I-R Model

```python
N = 1e7 + 10 + 5
# simuation Time / Day
T = 170
# susceptiable ratio
s = np.zeros([T])
# infective ratio
i = np.zeros([T])
# remove ratio
r = np.zeros([T])

# contact rate
lamda = 0.2586
# recover rate
gamma = 0.0821

# initial infective people
i[0] = 10.0 / N
s[0] = 1e7 / N
for t in range(T-1):
    i[t + 1] = i[t] + i[t] * lamda * s[t] - gamma*i[t]
    s[t + 1] = s[t] - lamda * s[t] * i[t]
    r[t + 1] = r[t] + gamma*i[t]

plt.plot(s*N,label='susceptible')
plt.plot(i*N,label='infected')
plt.plot(r*N,label='removed')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I-R Model')
plt.legend()
```

Plotting codes

Initialization

- Setting N, T lambda, gamma
- Setting I and S at day 0

N: total # of population (susceptible)
T: total days of model simulation
lamda = contact rate
gamma = recover rate

$$S[n+1] = S[n] - \lambda \cdot I[n] \cdot S[n]$$
$$I[n+1] = I[n] + \lambda \cdot I[n] \cdot S[n] - \gamma I[n]$$
$$R[n+1] = R[n] + \gamma I[n]$$

Infected at day n

Newly infected at day n

Recovered at day n

A Python implementation of the S-I-R Model

```python
N = 1e7 + 10 + 5
# simuation Time / Day
T = 170
# susceptiable ratio
s = np.zeros([T])
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i = np.zeros([T])
# remove ratio
r = np.zeros([T])

# contact rate
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# recover rate
gamma = 0.0821

# initial infective people
i[0] = 10.0 / N
s[0] = 1e7 / N
for t in range(T-1):
    i[t + 1] = i[t] + i[t] * lamda * s[t] - gamma*i[t]
    s[t + 1] = s[t] - lamda * s[t] * i[t]
    r[t + 1] = r[t] + gamma*i[t]

plt.plot(s*N,label='susceptible')
plt.plot(i*N,label='infected')
plt.plot(r*N,label='removed')
plt.xlabel('Day')
plt.ylabel('# of cases')
plt.title('The S-I-R Model')
plt.legend()
```
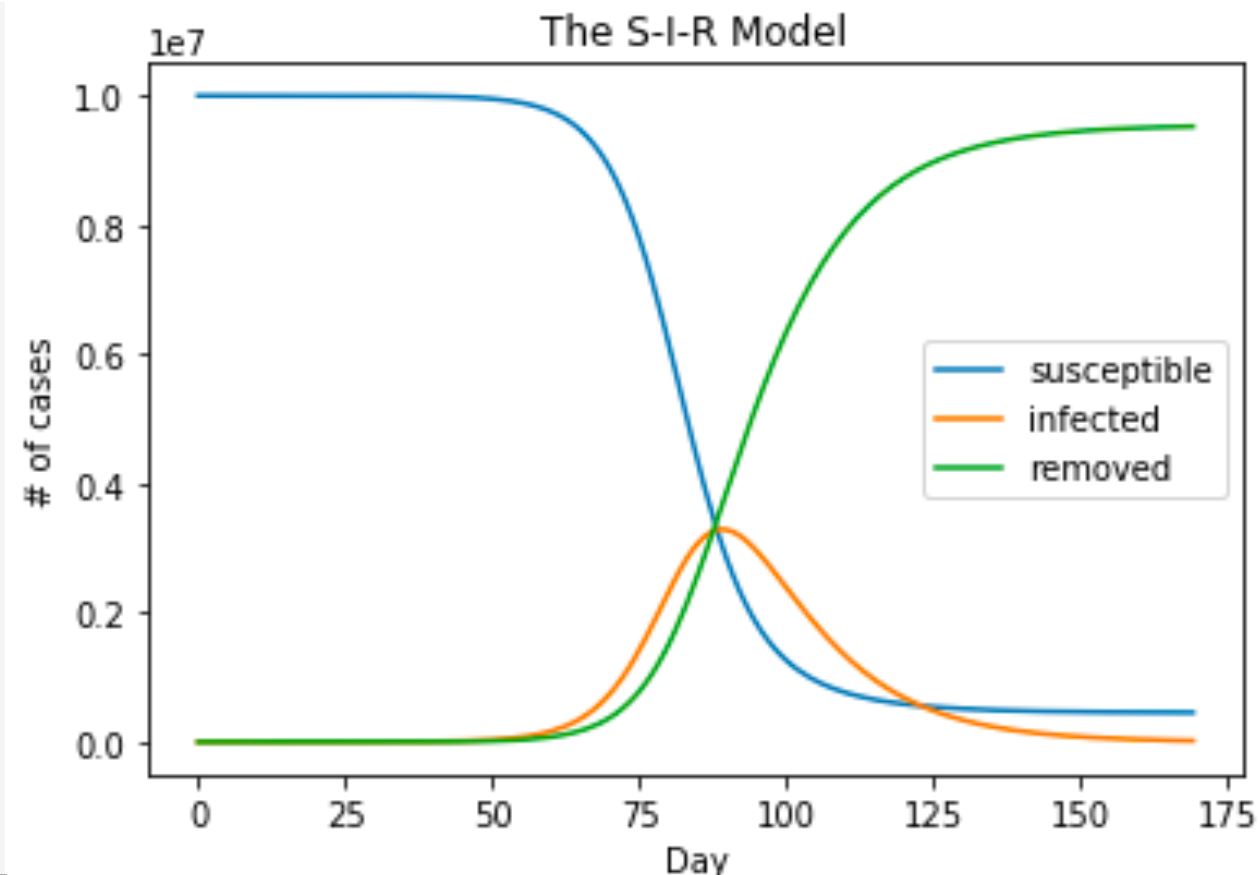

From SARS data

Plotting codes



- **Starting from day 0, 10 infections within 1M population**
- **Lamda determines the rate of infection**
- **Gamma determines the rate of cure**
- **Here the parameters are from data collected during the SARS**
- **The pandemic peaks around day 90 and decays after day 150 (~ 5 months)**

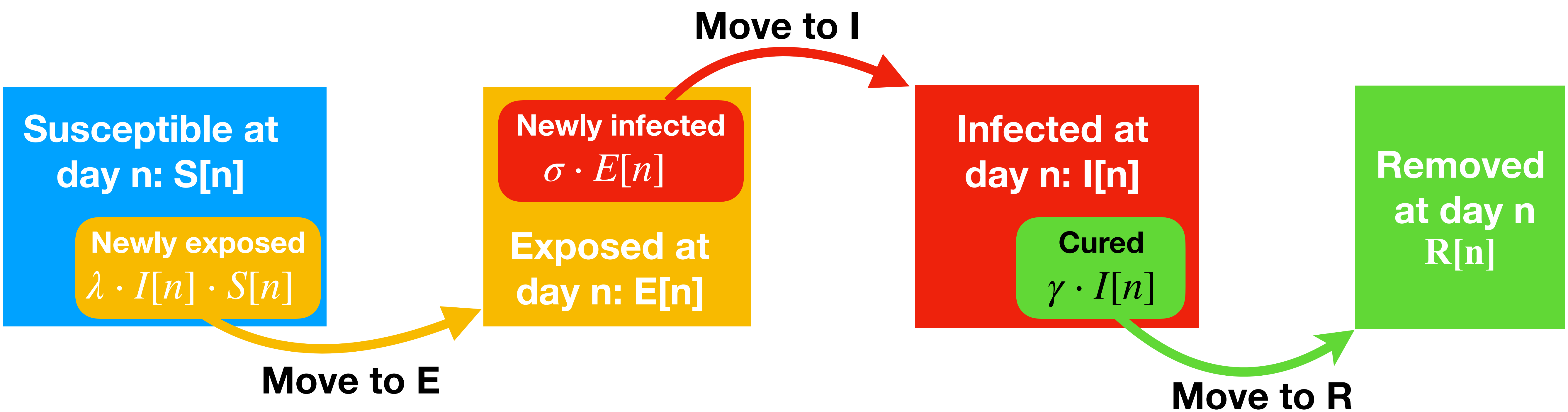
Now, what's still missing here?

The S-E-I-R model

λ : **contact rate**
σ : **incubation**
γ : **cure rate**

Assumptions:

1. The total population (susceptible) is N_0, i.e., $S[0] = N_0$
2. Everyday, newly **exposed** is through contact rate $\lambda \cdot I \cdot S$
3. New **infections** are from the exposed population due to incubation $\sigma \cdot E$
4. New **cured** is $\gamma \cdot I$, they are removed from S and I (immunity)



	Susceptible (S)	Exposed (E)	Infected (I)	Removed (R)	Newly exposed	Newly infected	Cured (immuned)
Day n	$S[n]$	$E[n]$	$I[n]$	$R[n]$	$\lambda \cdot I[n] \cdot S[n]$	$\sigma \cdot E[n]$	$\gamma \cdot I[n]$

Day n+1

$$S[n+1] =$$
$$E[n+1] =$$
$$I[n+1] =$$
$$R[n+1] =$$

?