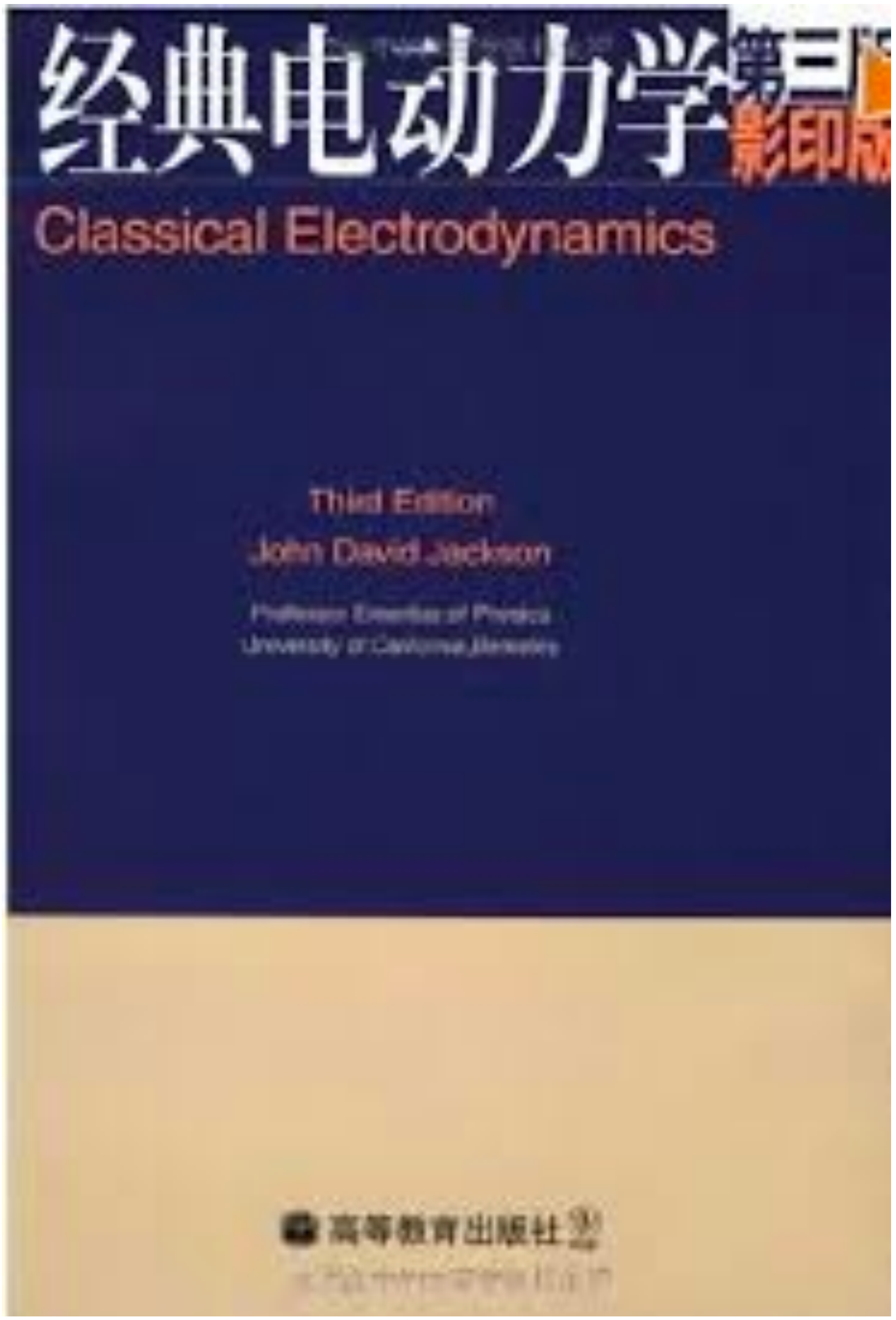
Electrodynamics

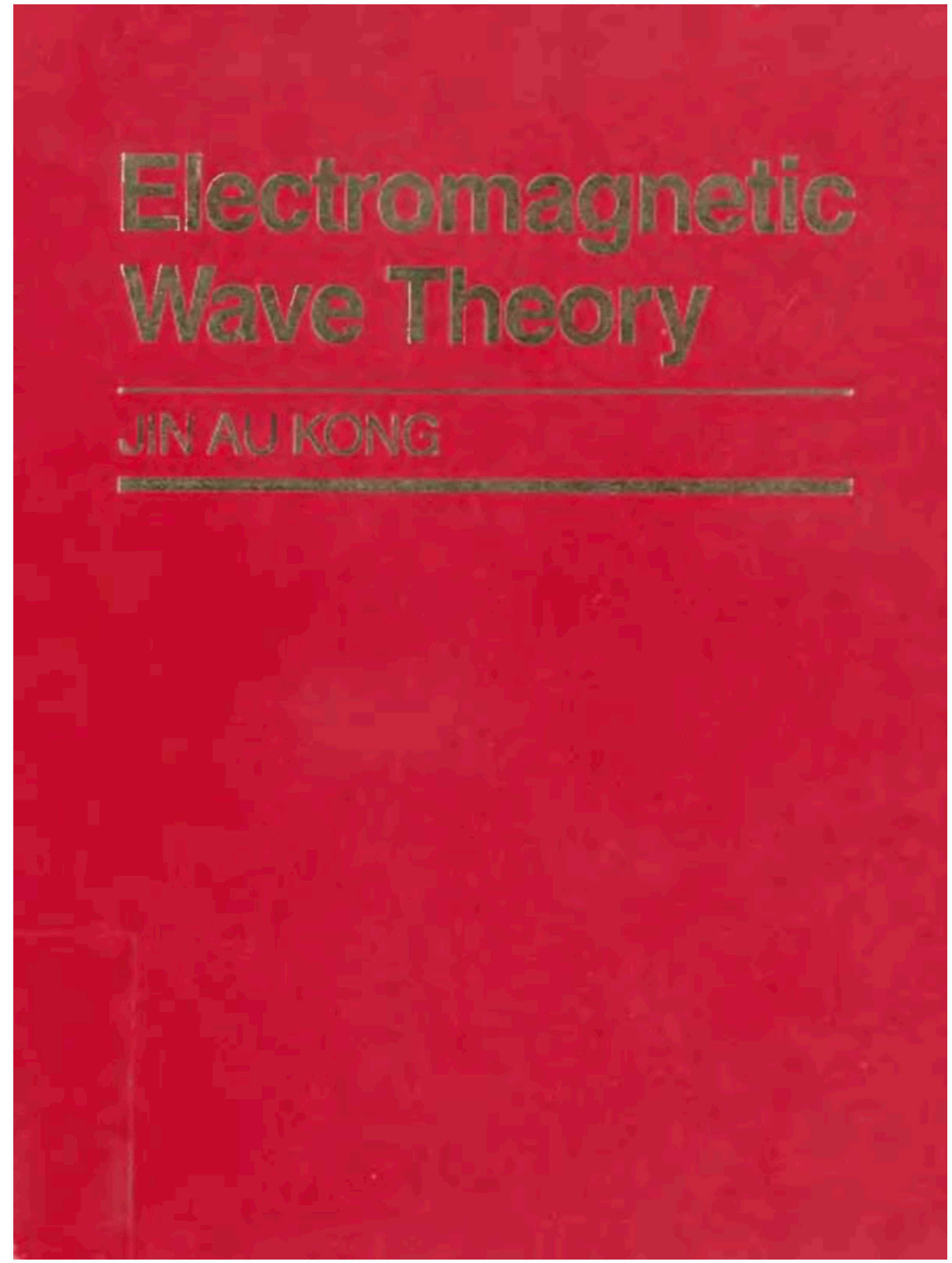
An introduction for MHD

- **The Lorentz Force**
- **The Ohm’s Law**
- **The Ampere’s Law**
- **The Faraday’s Law**
- **Maxwell’s equations for MHD**

Electrodynamics



Jackson



Kong

The Lorentz force

A particle moving with velocity u and carrying a charge q is, in general, subject to electromagnetic forces

$$\mathbf{F} = q\mathbf{E} + q\mathbf{u}\times\mathbf{B} = q(\boxed{\mathbf{E} + \mathbf{u}\times\mathbf{B}})$$

Effective electric field

Electric Force

Magnetic Force

the electric field E is the force per unit charge on a small test charge at rest in the observer's frame of reference.

For the electric field, we have Gauss' law

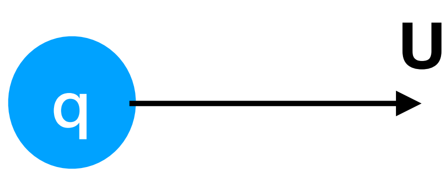
$$\nabla\cdot\mathbf{E} = \frac{\rho_e}{\epsilon_0}$$

no CGS!

The Electric field - revisit

Frames of references

Laboratory Frame



Electric and Magnetic Field

force: $\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

Charge at Rest Frame

Electric Field only

force: $\mathbf{F} = q\mathbf{E}'$

Newtonian relativity (required for MHD) gives

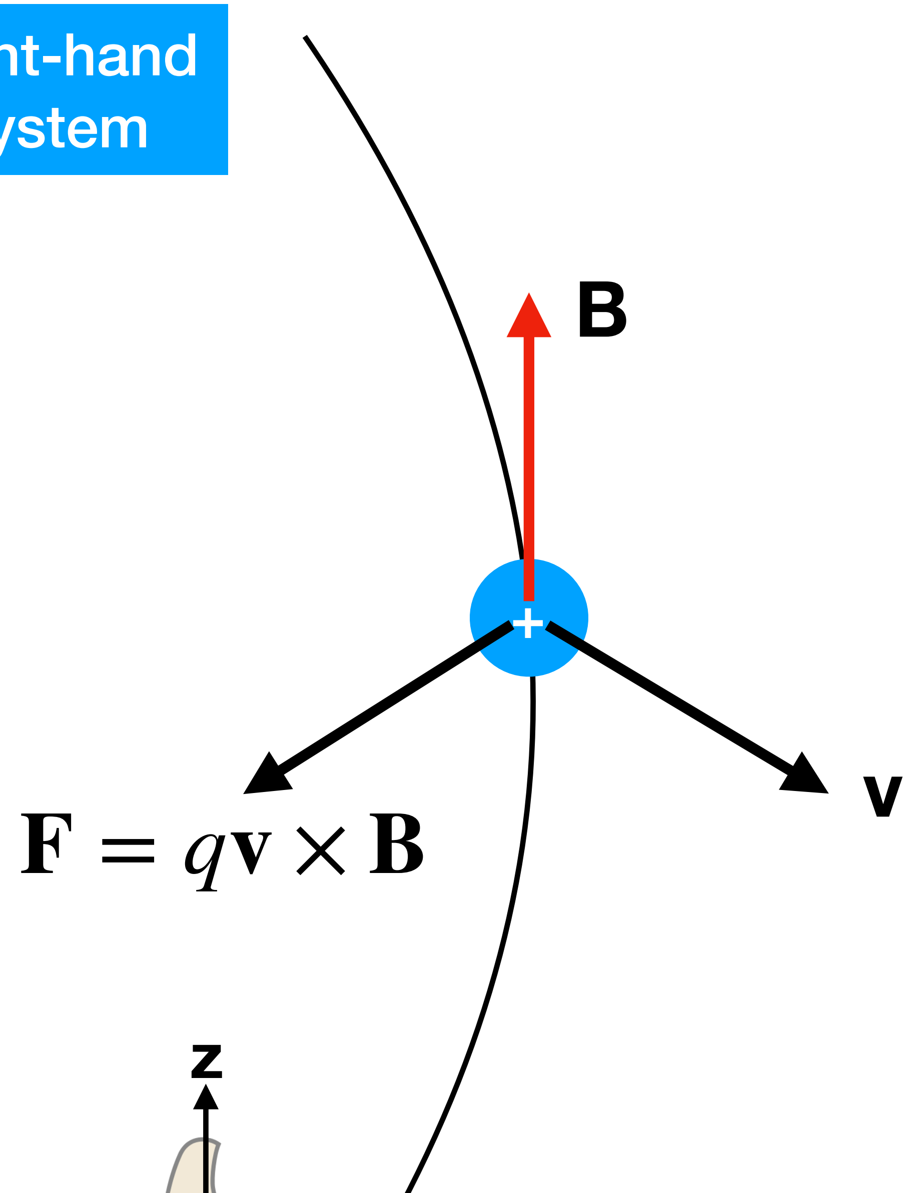
$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

Newtonian relativity also requires that the magnetic fields are equal in the two frames

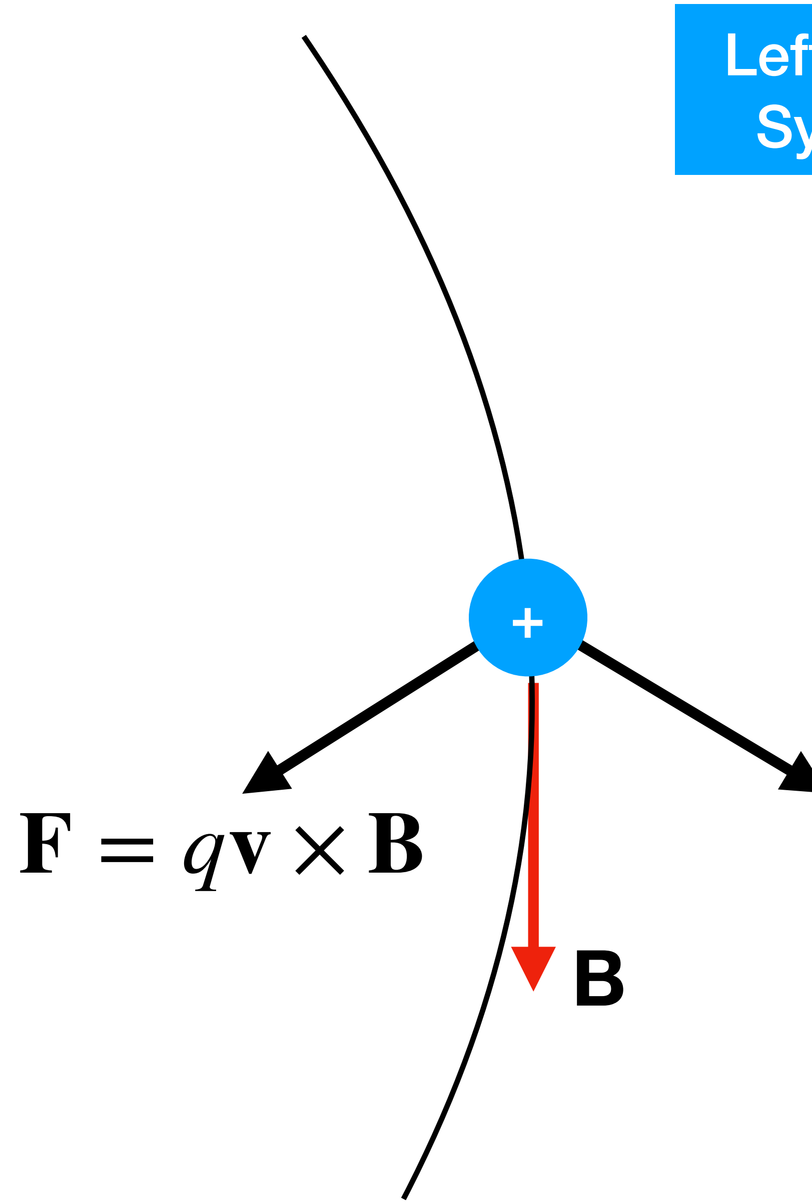
The Magnetic field - revisit

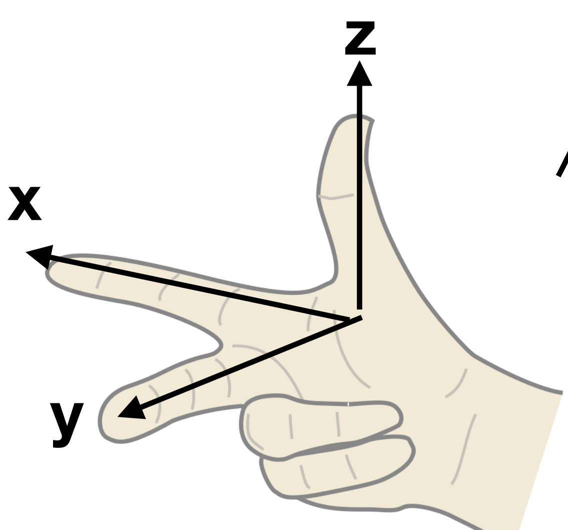
Pseudo vectors

Right-hand System



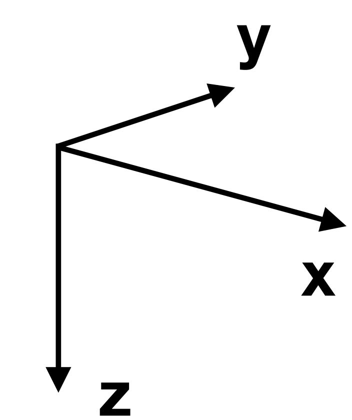
Left-hand System





The magnetic field flips direction when the coordinate system changes form right-handed to left-handed, this is called a **pseudo vector**

Velocity and force are **true vectors**



The Ohm's Law

In a conductor, it is found that the current density J is proportional to the force experienced by the charges. This is known as the **Ohm's** law:

$$\mathbf{J} = \sigma \mathbf{E}$$

In a conducting fluid moving at a velocity of u, the current is calculated as

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Sum over the Lorentz force for individual species:

$$\mathbf{F}_{bulk} = \sum q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \sum q\mathbf{E} + \sum q\mathbf{u} \times \mathbf{B} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Here F is the force per unit volume acting on the conducting fluid

If the conducting fluid travels much slower than the speed of light, the first term vanishes:

$$\mathbf{F}_{bulk} = \cancelto{0}{\rho_e \mathbf{E}} + \mathbf{J} \times \mathbf{B}$$ **why?**

The Ohm's Law

Charge conservation requires $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho_e}{\partial t}$ actually this is simply: $\frac{\partial \rho_e}{\partial t} + \nabla \cdot \rho_e \mathbf{u} = 0$

Taking the divergence of the Ohm's law: $\nabla \cdot \mathbf{J} = \nabla \cdot [\sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})]$

Apply Gauss' law $\longrightarrow$

$$= \sigma \nabla \cdot \mathbf{E} + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B})$$

$$= \frac{\sigma}{\epsilon} \rho_e + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B})$$

Now we get the following PDE:

$$\frac{\partial \rho_e}{\partial t} + \frac{1}{\tau_e} \rho_e + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B}) = 0 \qquad \tau_e = \frac{\epsilon}{\sigma}$$

Charge relaxation time $\approx 10^{-18}$

$$\xrightarrow{u=0} \quad \frac{\partial \rho_e}{\partial t} + \frac{1}{\tau_e} \rho_e = 0 \quad \xrightarrow[\text{Solution}]{\text{relaxation}} \quad \rho_e = \rho_e(0) e^{-\frac{t}{\tau_e}}$$

Since $1/\tau_e$ is a very fast frequency, the d/dt term is small compared to the rest two, we get a quasi-static equation

$$\rho_e = -\epsilon \nabla \cdot (\mathbf{u} \times \mathbf{B})$$

The Ohm's Law

$\rho_e = -\epsilon \nabla \cdot (\mathbf{u} \times \mathbf{B}) \qquad \mathbf{E} \sim \frac{1}{\sigma}\mathbf{J}$

Now we can estimate the magnitude of the $\rho_e \mathbf{E}$ term (compared to the JxB term)

$$\rho_e |\mathbf{E}| \sim \frac{\epsilon_0 u B}{l}\frac{J}{\sigma} = \frac{u\tau_e}{l} JB \sim \frac{u\tau_e}{l} |\mathbf{J} \times \mathbf{B}| \xrightarrow{\frac{u}{l} \sim \tau_{MHD}} \frac{\tau_e}{\tau_{MHD}} |\mathbf{J} \times \mathbf{B}|$$

Since $\tau_e \sim 10^{-18}\ s$, the Lorentz force is simplified as

$$\mathbf{F}_{bulk} = \cancelto{\mathbf{0}}{\rho_e \mathbf{E}} + \mathbf{J} \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$$

Physically, it's because the electron relaxation time is so small compared to the large scale process that MHD is interested.

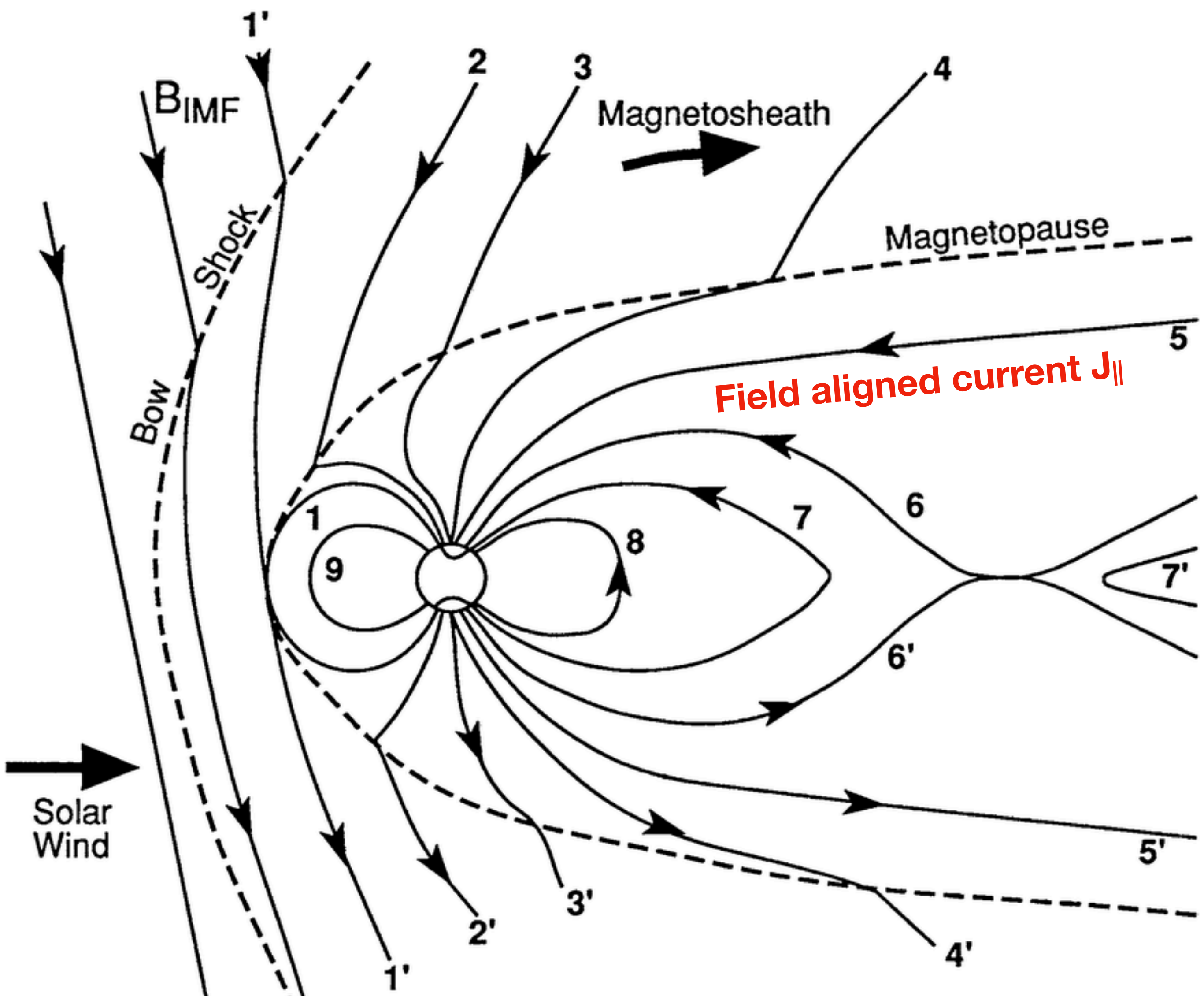
Therefore, we get one of the simplest but also the most important equation in space physics:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t} = 0$$

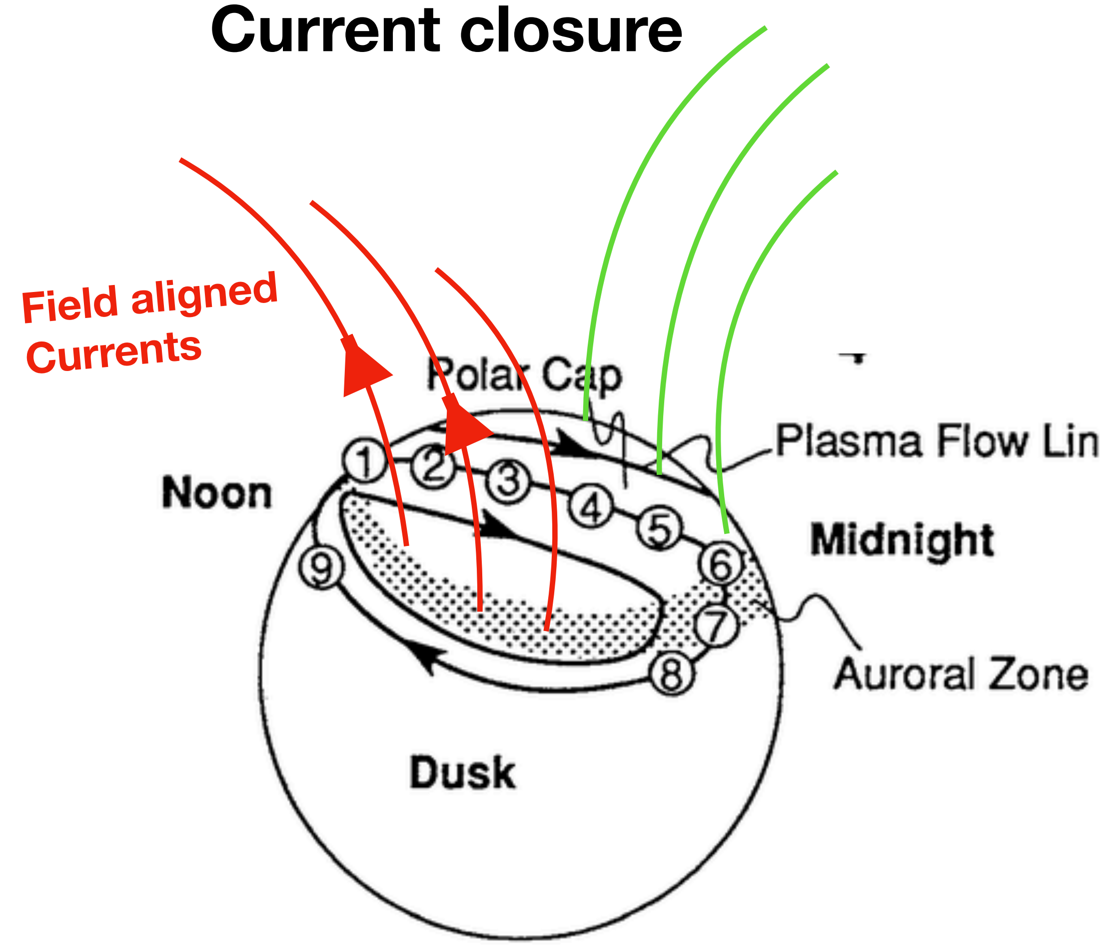
The $\nabla \cdot \mathbf{J} = 0$

Electrostatic Magnetosphere-Ionosphere Coupling

Magnetospheric convection

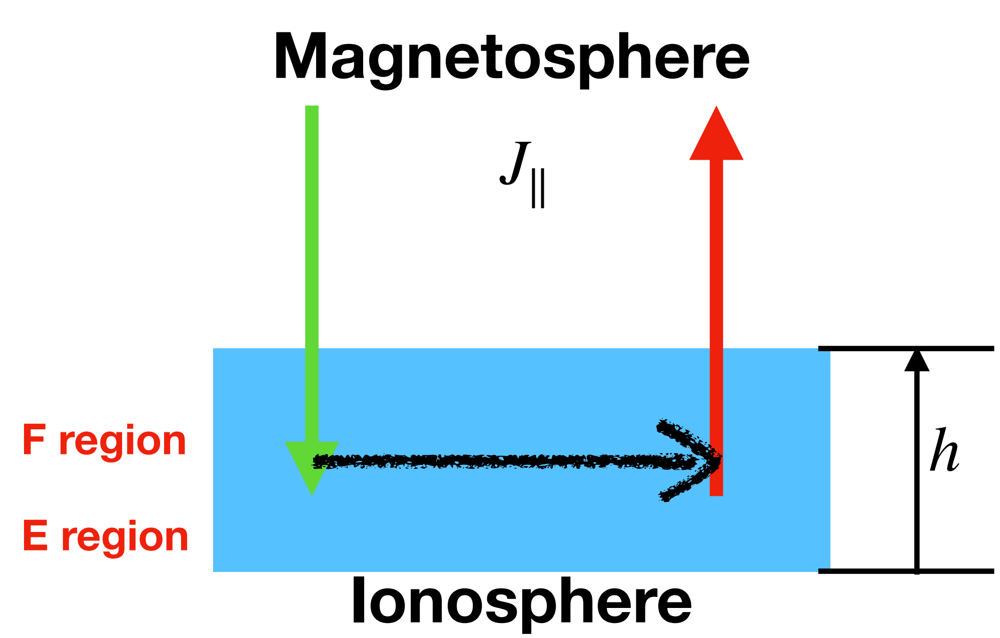


Current closure



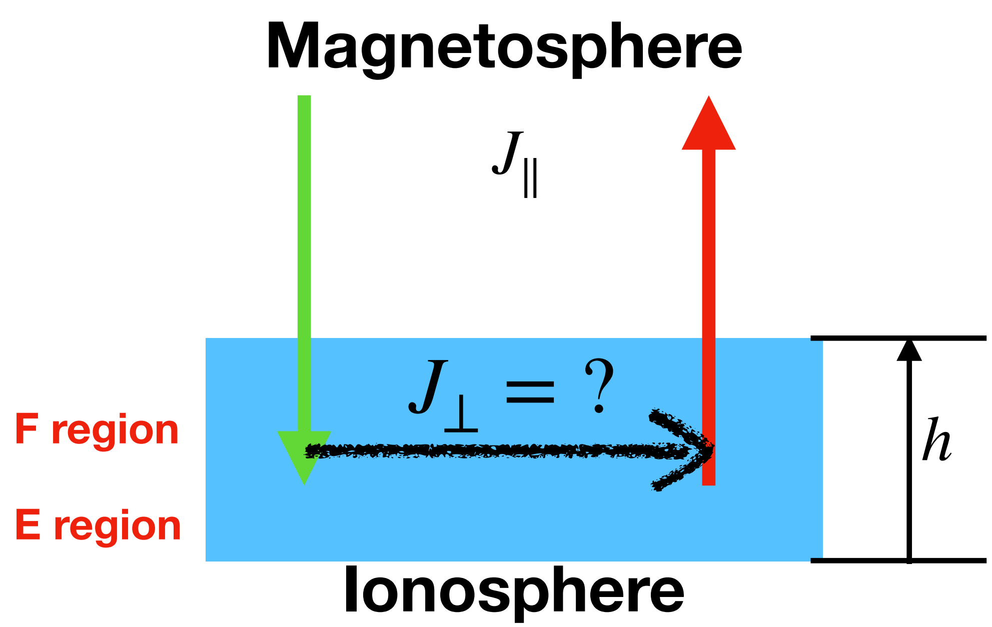
Question: How does $J_{\|}$ close in the ionosphere?

Slab geometry



The $\nabla \cdot \mathbf{J} = 0$

Slab geometry



Electrostatic M-I Coupling is simplified as a simple current closure problem starting from

$$\nabla \cdot \mathbf{J} = 0$$

$$\nabla \times \mathbf{E} = 0$$

Separate the perp and parallel direction: $\nabla \cdot \mathbf{J} = \nabla_{\perp} \cdot \mathbf{J} + \frac{\partial}{\partial z}\mathbf{J} = 0 \longrightarrow \nabla_{\perp} \cdot \mathbf{J} = -\frac{\partial}{\partial z}\mathbf{J}$

Now integrate over altitude and use the classic Ohm's law $\mathbf{J} = \sigma(z)\mathbf{E}$

$$\int_{100km}^{500km} \left(\nabla_{\perp} \cdot \mathbf{J} = -\frac{\partial}{\partial z}\mathbf{J} \right) \longrightarrow \nabla_{\perp} \cdot \boxed{\int_{100km}^{500km} \sigma(z)dz} \mathbf{E} = -\int_{100km}^{500km} \frac{\partial}{\partial z}\mathbf{J}dz$$

Height-integrated conductivity - conductance $\sum$

$$\nabla_{\perp} \cdot \Sigma\mathbf{E} = J_{\|} - \cancelto{0}{J(100km)} \longrightarrow \Sigma \nabla_{\perp}^{2} \Phi = J_{\|} \longrightarrow \mathbf{u} = \frac{-\nabla \Phi \times \mathbf{B}}{B^2}$$

Remix

Gamera

Inner Boundary Condition for Gamera

Ampere's Law

The Ampere's law describes the magnetic field generated by a given distribution of total current:

$$\nabla \times \mathbf{B} = \mu \left[\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right]$$

The dE/dt term is called the *displacement current*. To see why it's needed in the Maxwell's equation, let's take the divergence of the Ampere's law:

$$\nabla \cdot \left(\nabla \times \mathbf{B} = \mu \left[\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right] \right) \longrightarrow \nabla \cdot \cancelto{0}{\nabla \times \mathbf{B}} = \nabla \cdot \left[\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right] = 0$$

We get the charge conservation equation:

$$\nabla \cdot \mathbf{J} = -\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = -\frac{\partial \rho_e}{\partial t}$$

However, the displacement current is **not** needed in (most) MHD, because the time variation of the charge rho_e is the electron relaxation time:

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \sim \frac{\epsilon_0}{\sigma} \frac{\partial \mathbf{J}}{\partial t} \sim \frac{\tau_e}{\tau_{MHD}} \mathbf{J} \ll \mathbf{J} \longrightarrow \nabla \times \mathbf{B} = \mu \left[\mathbf{J} + \cancelto{0}{\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \right]$$

Ampere's Law - Biot-Savart Law $\frac{\partial}{\partial t} \approx 0$

For infinite domains, the Ampere's low is inverted as:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{r^3} d\mathbf{x}'$$

This is basically the Green's function of the 1st order differential equation, which is equivalent to the Ampere's law used in MHD

$$\nabla \times \mathbf{B} = \mu \mathbf{J}$$

In fact, the Biot-Savart form is the true character of the Ampere's law:

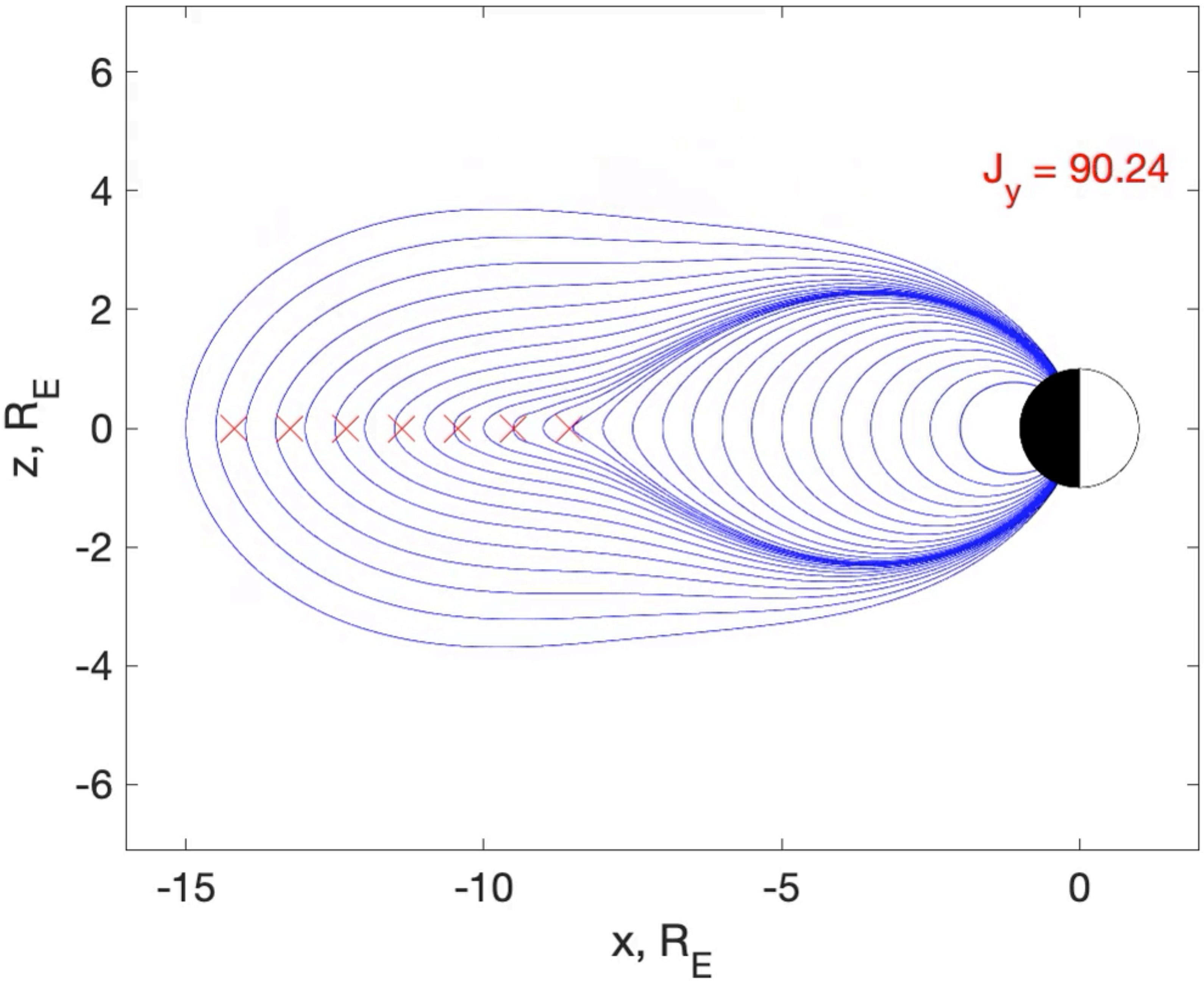
In MHD, the Ampere's law tells us the magnetic field associated with a given current distribution

Force-Free Field: $\nabla \times \mathbf{B} = \alpha \mathbf{B}$

Understanding the Ampere's Law

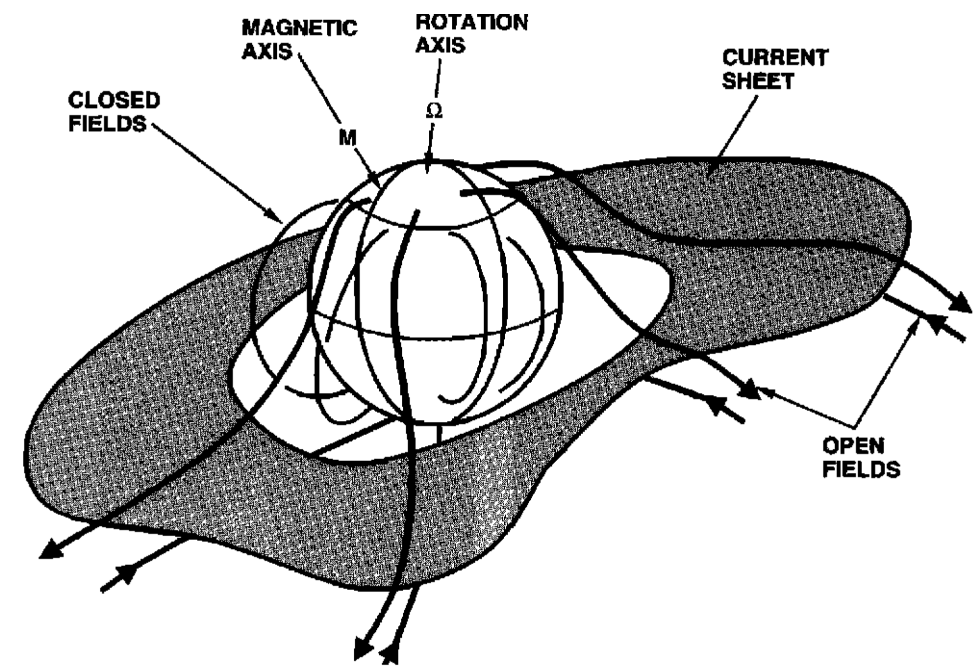
Current sheets

Magnetospheres

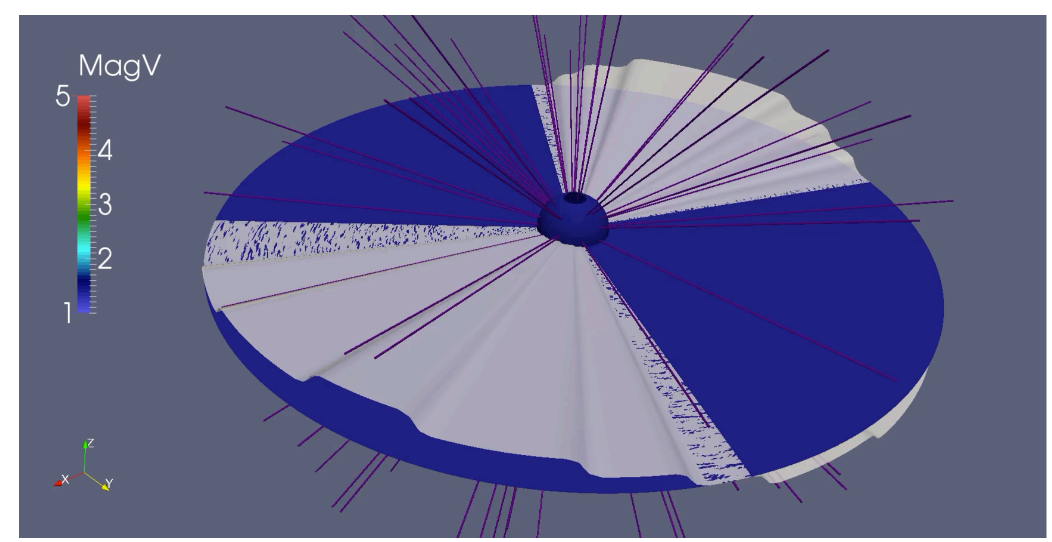


J in the y-direction, causing the tail to "stretch"

Heliosphere

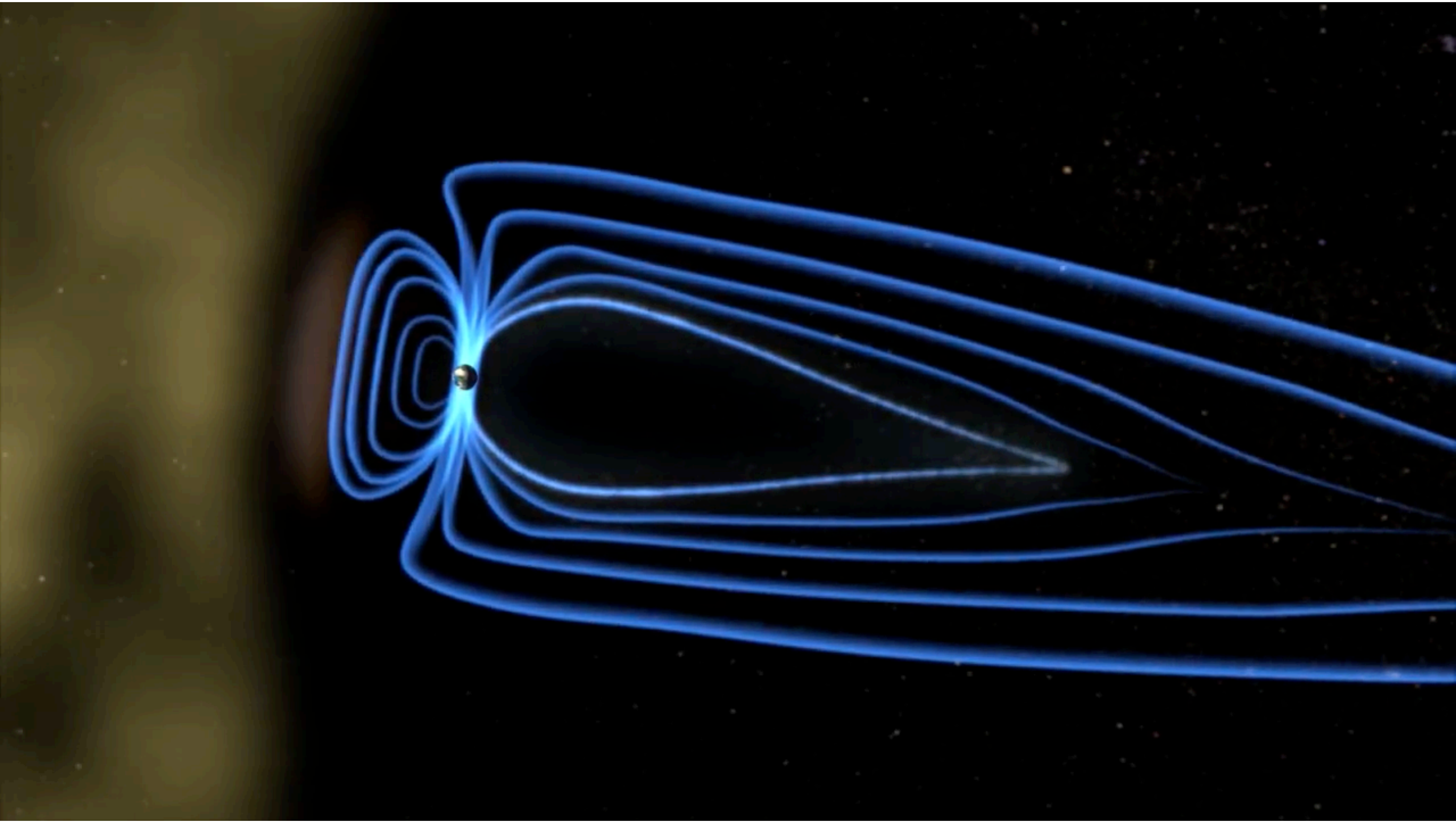


e 1. Schematic of the heliospheric current sheet. The shaded current sheet separates fields fror



Understanding the Ampere's Law

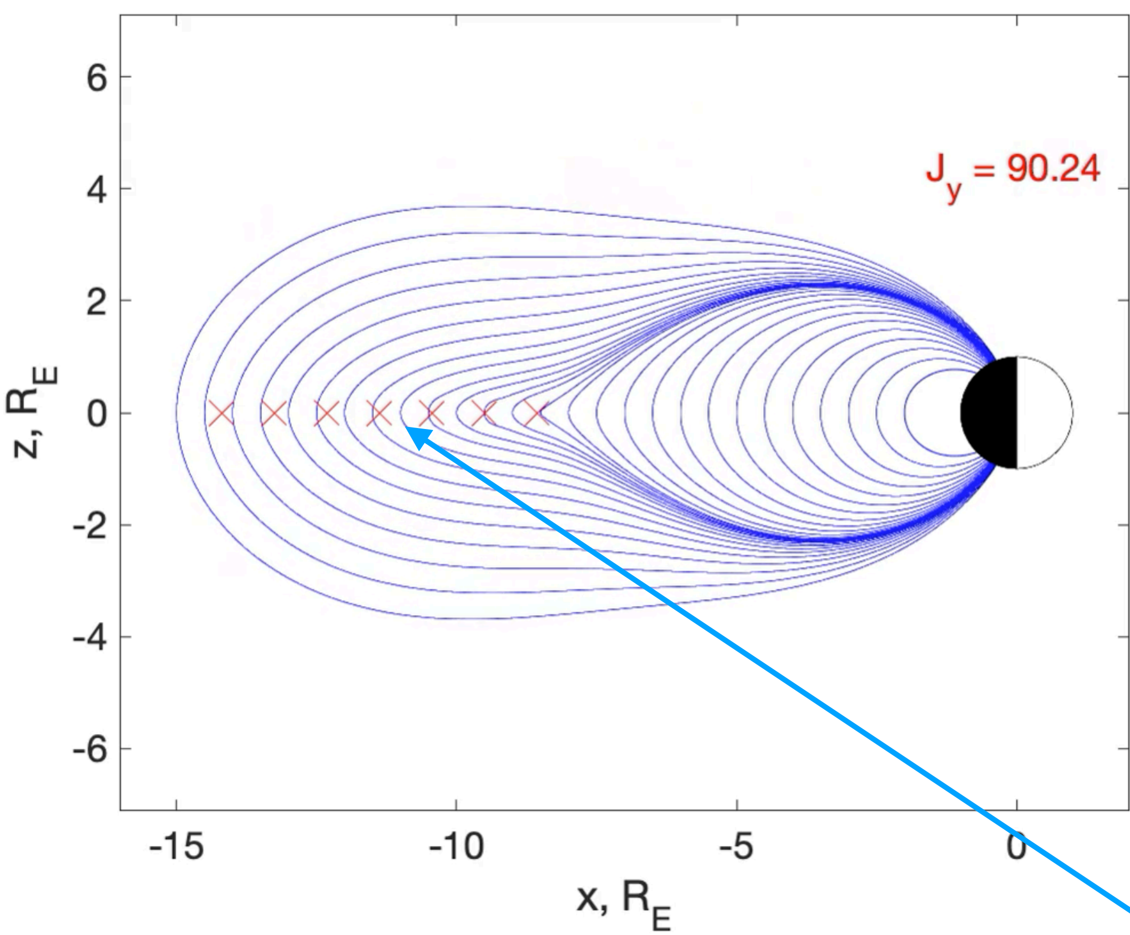
Substorm Current Wedge



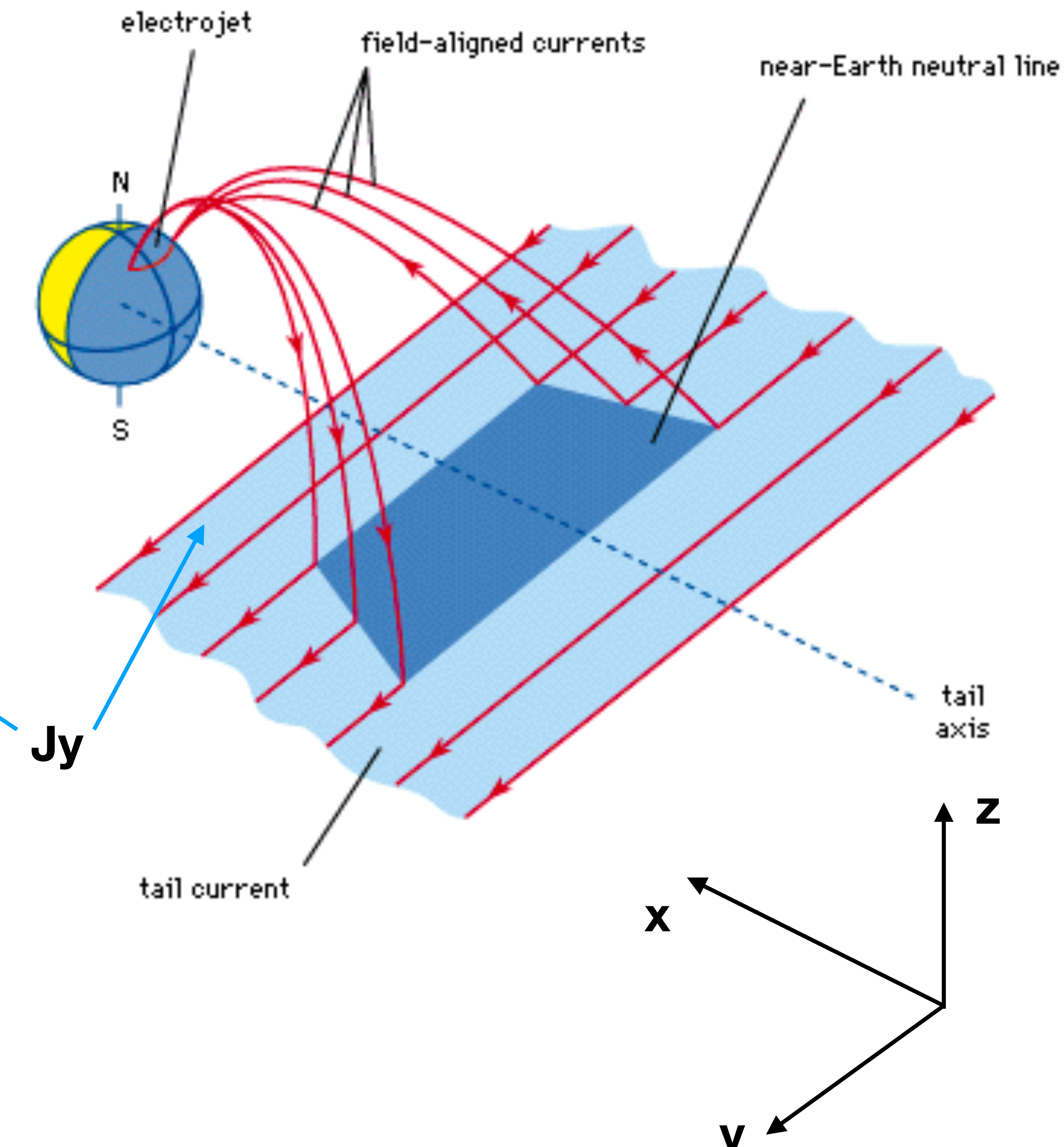
Understanding the Ampere’s Law

Substorm Current Wedge

2-D view



3-D view



$$\nabla \times \mathbf{B} = \mu \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = 0$$

Faraday's Law

The differential form of the Faraday's law is a special case of the integral form:

Integral form

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\oint_C \mathbf{E}\cdot d\mathbf{l}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} \longrightarrow \frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

Differential form

The differential form of the Faraday's low only tells us about the magnetic field induced by a time-varying magnetic field

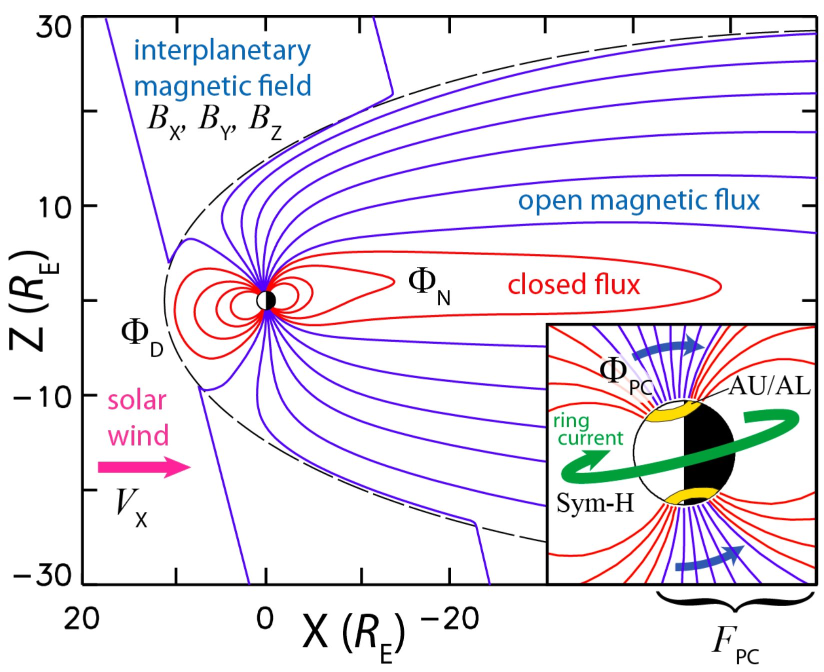
The differential form of the Faraday's law ensures solenoidal B field by taking the divergence of the equations:

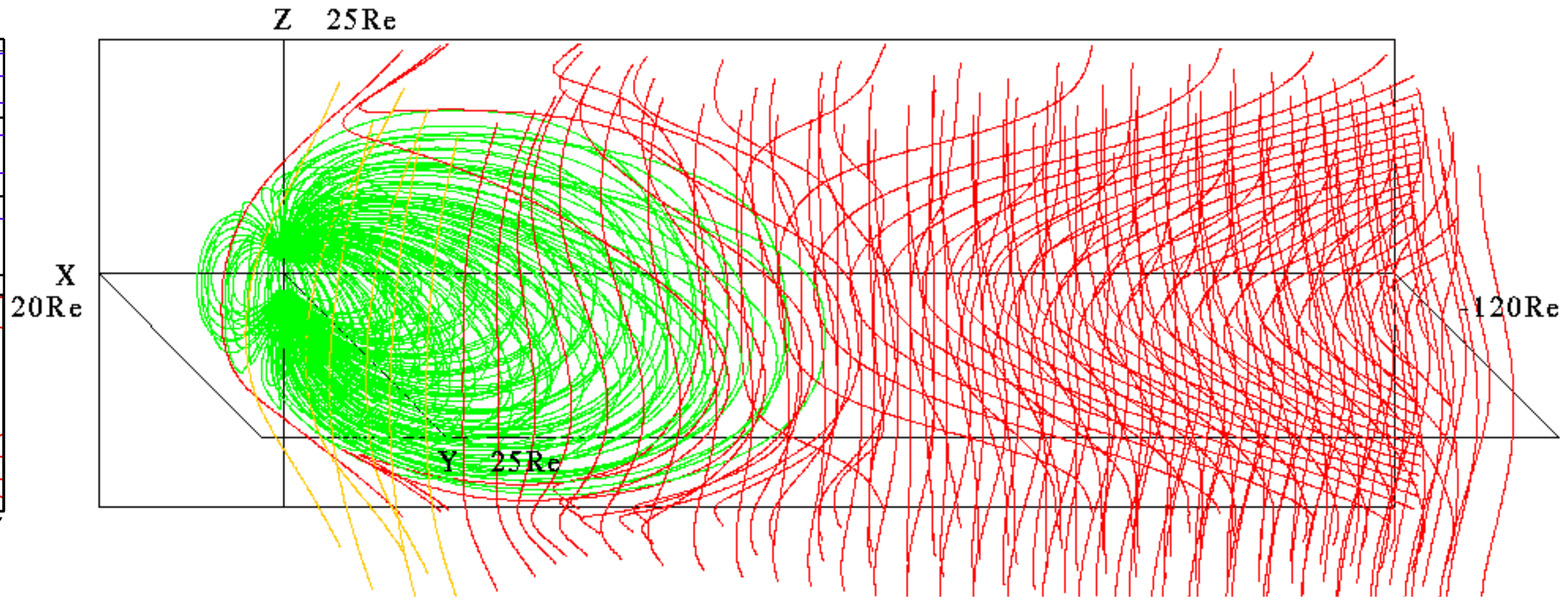
$$\nabla\cdot\left(\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\right) \xrightarrow{\nabla\cdot\nabla\times = 0} \frac{\partial}{\partial t}\nabla\cdot\mathbf{B} = -\nabla\cdot\nabla\times\mathbf{E} = 0$$

Div B = 0 is an initial condition of the Maxwell's eqn

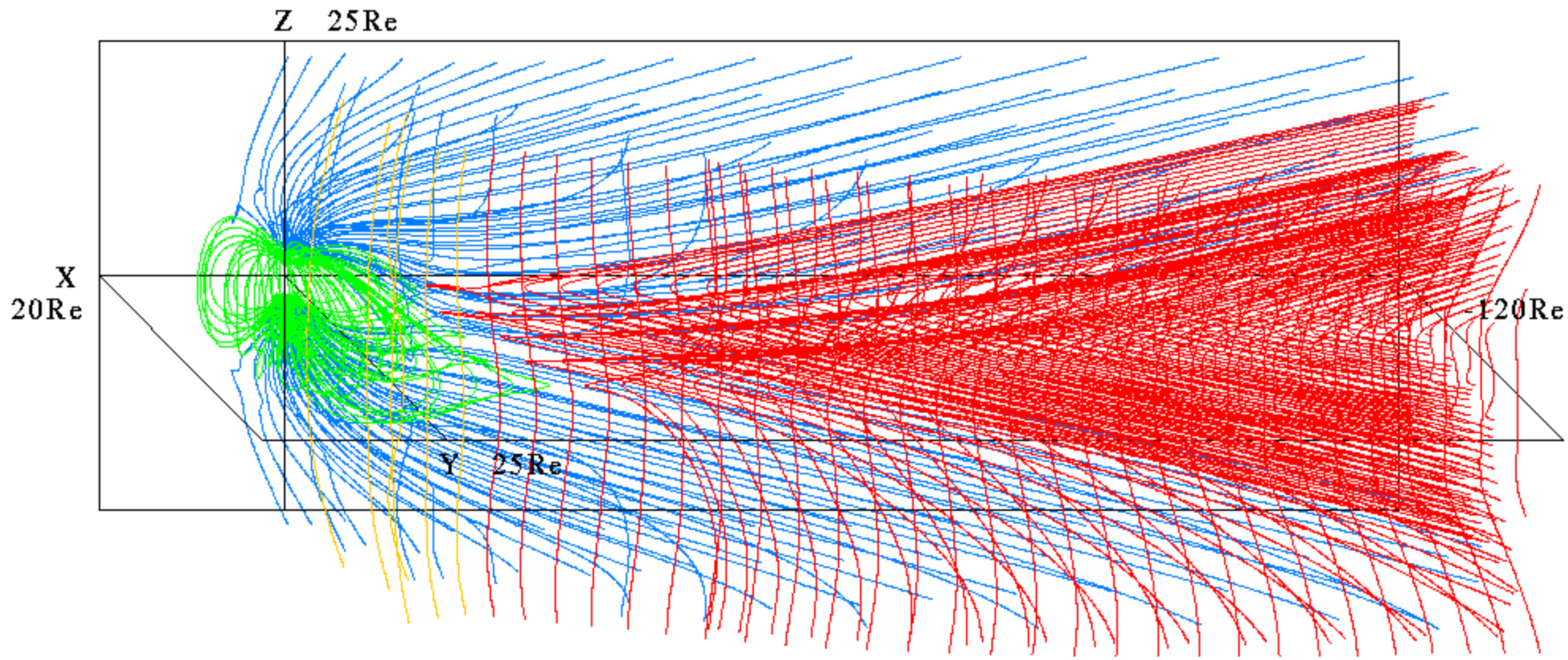
Faraday's Law

Generation of Open Flux





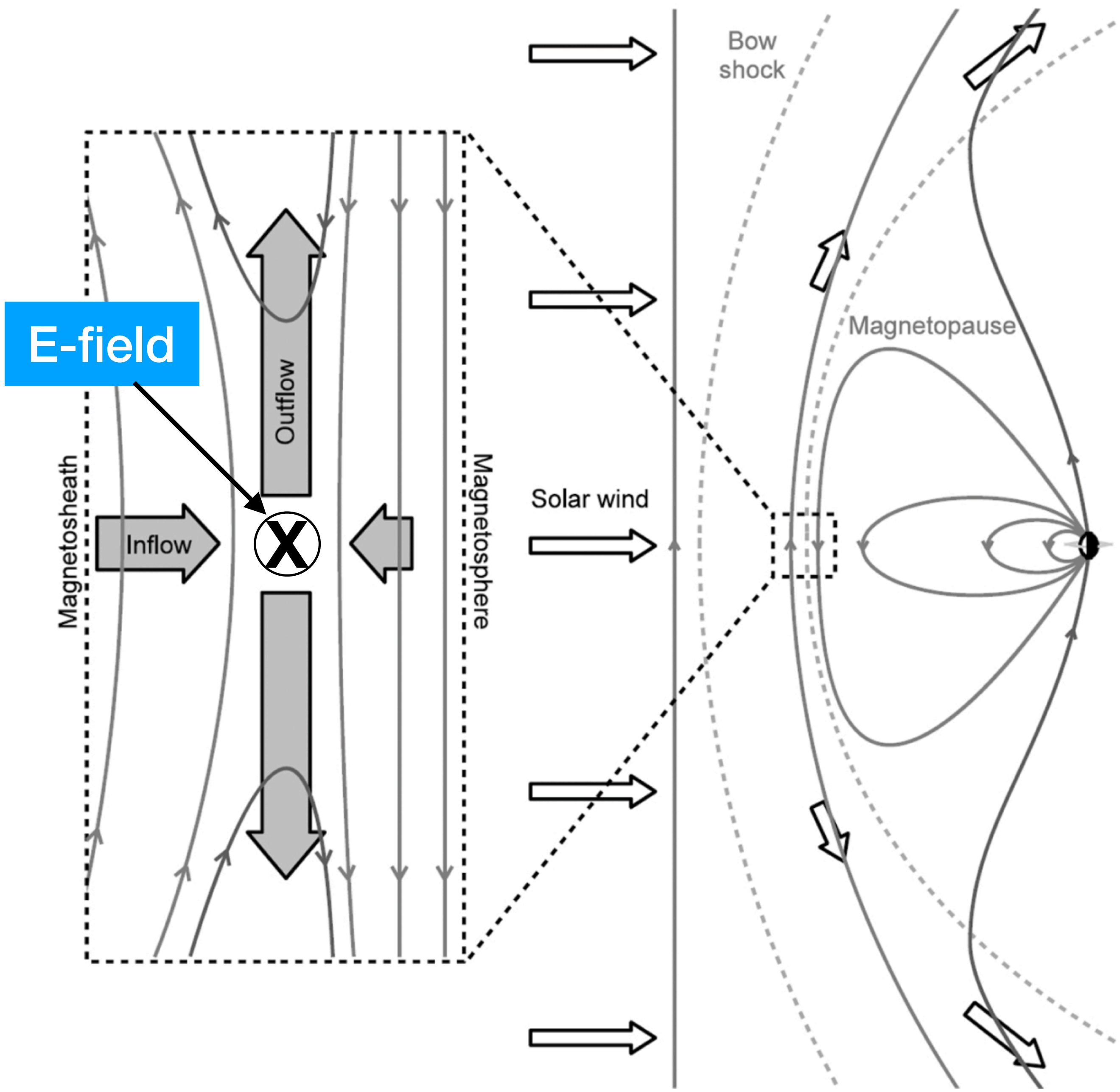
Question:How much open flux is generated by the SW interaction?



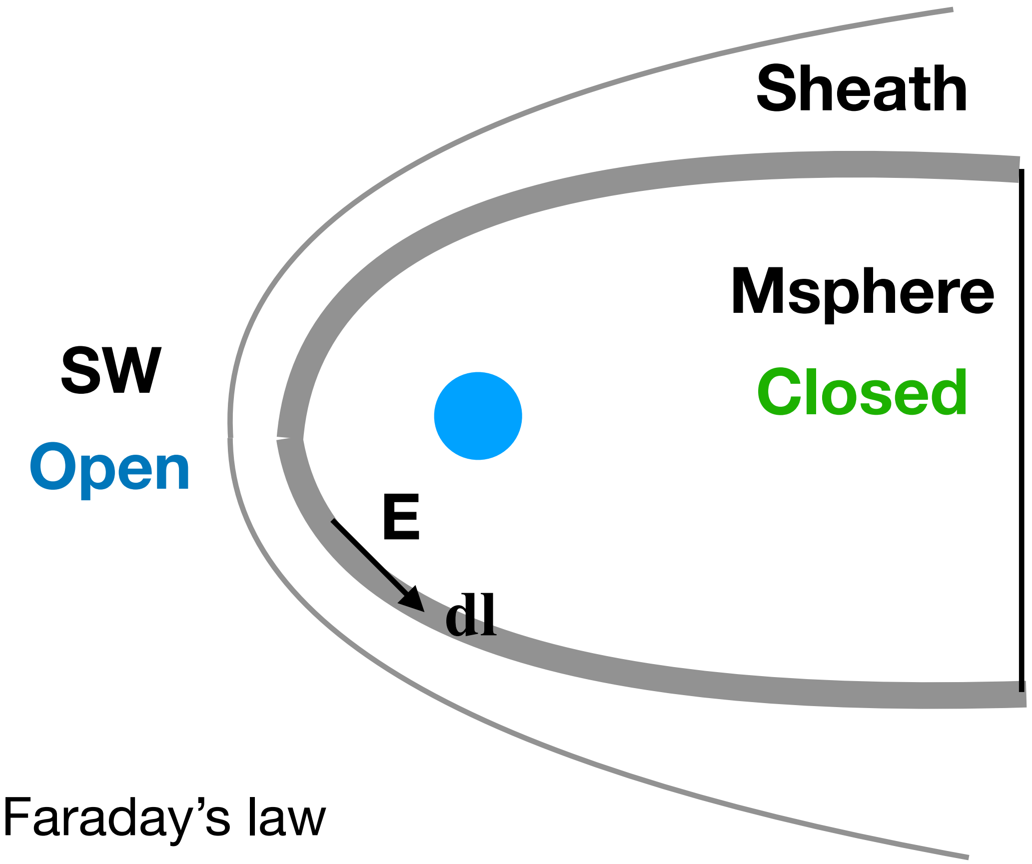
Faraday’s Law

Generation of Open Flux

Magnetic Reconnection



Open Flux Generation



Faraday’s law

$$\frac{\partial \Phi}{\partial t} = \mathbf{E} \cdot \mathbf{dl}$$

To zeroth-order, in a quasi-steady/balanced state:

$$\Delta\Phi = \Delta t(\mathbf{E} \cdot \mathbf{dl}) = \Delta t \cdot \Phi_{recon}$$

$$\Phi_{recon} \approx 200\ \mathrm{kV} \qquad \Delta t \approx 1\ \mathrm{hr}$$

$$\Delta\Phi \approx 0.72 \times 10^{9}\ \mathrm{Wb} = 0.72\ \mathrm{GWb}$$

So the reconnection potential represents the rate of the creating of open magnetic flux.

Faraday's Law

Vector Potential

Given that the magnetic field must be solenoidal $\nabla \cdot \mathbf{B} = 0$

Which means mathematically the magnetic field can be expressed using a vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

This definition automatically ensures the B field is solenoidal. Substitute A for B in the Faraday's law

$$\nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \longrightarrow \nabla \times \left(\mathbf{E} - \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \longrightarrow \mathbf{E} = \boxed{-\frac{\partial \mathbf{A}}{\partial t}} \boxed{-\nabla V}$$

Where V is an arbitrary scalar function.

In applications the electric field is usually separated in to divergence-free and curl-free parts:

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s, \quad \nabla \cdot \mathbf{E}_i = 0; \quad \nabla \times \mathbf{E}_s = 0;$$

The Full set of Maxwell's equations

The full set of Maxwell's equations is (differential form)

Gauss' Law
$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}$$

Solenoidal B
$$\nabla \cdot \mathbf{B} = 0$$

Faraday's Law
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Displacement Current

Ampere-Maxwell eqn
$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \boxed{\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \right)$$

In addition, we have

Current continuity
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_e}{\partial t}$$

Lorentz force
$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

The Displacement Current and EM Waves

The displacement current not only fixes the inconsistencies in the Ampere's law, it introduces electromagnetic waves:

Let's take the curl of the Ampere-Maxwell equation:

$$\nabla \times \left[\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \right] \xrightarrow{\nabla \cdot \mathbf{B} = 0} \nabla^2 \mathbf{B} = -\mu_0 \nabla \times \mathbf{J} - \frac{1}{c^2} \frac{\partial \nabla \times \mathbf{E}}{\partial t}$$

Now combine with Faraday's law, we get

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t} = -\mu_0 \nabla \times \mathbf{J}$$

If we ignored the displacement current, the equation becomes a magnetostatic one:

$$\nabla^2 \mathbf{B} = -\mu_0 \nabla \times \mathbf{J}$$

Mathematically, the displacement current term dE/dt transforms a Poisson equation to a wave equation - electromagnetic waves traveling at the speed of light c

The Reduced Maxwell's equations for MHD

The Lorentz force is written as

$$\mathbf{F} = \underbrace{\cancelto{0}{q\mathbf{E}}}_{\text{tiny}} + \underbrace{q\mathbf{u}}_{\text{J}} \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$$

Current continuity

$$\nabla \cdot \mathbf{J} = -\cancelto{0}{\frac{\partial \rho_e}{\partial t}} \approx 0$$

Displacement current

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \approx 0$$

Now we get

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad (\nabla \cdot \mathbf{B} = 0) \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \quad (\nabla \cdot \mathbf{J} = 0) \end{aligned}}$$

Lorentz force and current

$$\mathbf{J} = \sigma\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}$$

The Poynting Theorem

Use the Faraday's law and Ampere's law

$$\frac{d}{dt}\int_V \frac{\mathbf{B}^2}{\mu} dV = -\int_V \mathbf{J}\cdot\mathbf{E}dV - \oint_S \frac{1}{\mu}\mathbf{E}\times\mathbf{B}\cdot d\mathbf{S}$$

Then applying the Ohm's law to get

$$\int_V \mathbf{J}\cdot\mathbf{E}dV = \frac{1}{\sigma}\int_V \mathbf{J}^2 dV + \int_V (\mathbf{J}\times\mathbf{B})\cdot\mathbf{u}dV$$

Combining the two we obtain the following energy equation

$$\frac{d}{dt}\int_V \frac{\mathbf{B}^2}{2\mu} dV = -\frac{1}{\sigma}\int_V \mathbf{J}^2 dV - \int_V (\mathbf{J}\times\mathbf{B})\cdot\mathbf{u}dV - \oint_S \frac{1}{\mu}\mathbf{E}\times\mathbf{B}\cdot d\mathbf{S}$$

Changes in magnetic energy

Ohmic/ Joule Dissipation

To internal energy

Work done by the Lorentz Force

To/from kinetic energy

Poynting Vector

Electromagnetic energy flow in/out a volume

Question: where is the electric field energy term?

The Poynting Theorem

JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 115, A10311, doi:10.1029/2010JA015768, 2010

Recommended Reading

On the ionospheric application of Poynting's theorem

A. D. Richmond[1]

Received 2 June 2010; revised 11 July 2010; accepted 13 July 2010; published 14 October 2010.

[1] It has been proposed that the geomagnetic field-aligned component of the perturbation Poynting vector above the ionosphere, as obtained from the cross product of the electric and magnetic perturbation fields observed on a spacecraft, may be used to estimate the field line-integrated electromagnetic energy dissipation in the ionosphere below. This paper clarifies conditions under which this approximation may be either valid or invalid. It is shown that the downward field-aligned component of the perturbation Poynting vector can underestimate the electromagnetic energy dissipation in regions of high ionospheric Pedersen conductance, and it can significantly overestimate the dissipation in regions of low conductance. Local values of upward perturbation Poynting vector do not necessarily correspond to net ionospheric generation of electromagnetic energy along that geomagnetic field line. An Equipotential Boundary Poynting Flux (EBPF) theorem is presented for quasi-static electromagnetic fields as follows: when a volume of the ionosphere is bounded on the sides by an equipotential surface and on the bottom by the base of the conducting ionosphere, then the area integral of the downward normal component of the perturbation Poynting vector over the top of that volume equals the energy dissipation within the volume. This equality does not apply to volumes with arbitrary side boundaries. However, the EBPF theorem can be applied separately to different components of the electric potential, such as the large- and small-scale components. Since contours of the small-scale component of potential tend to close over relatively localized regions, the associated small-scale structures of downward perturbation Poynting vector tend to be dissipated locally.

Citation: Richmond, A. D. (2010), On the ionospheric application of Poynting's theorem, *J. Geophys. Res.*, *115*, A10311, doi:10.1029/2010JA015768.

The Transport equation for B Field

Combine the Faraday's law and the Ohm's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

+

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\longrightarrow \quad \frac{\partial \mathbf{B}}{\partial t} - \nabla \times \mathbf{E} = -\nabla \times \left[\frac{J}{\sigma} - \mathbf{u} \times \mathbf{B}\right]$$

use Ampere's law, we get

$$\xrightarrow{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}} \quad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu\sigma}\nabla^2 \mathbf{B} \qquad \text{(induction equation)}$$

$\frac{1}{\mu\sigma}$ Is in m²/s, known as the "***magnetic diffusivity***"

This is also one of the key equations in MHD. Two special cases:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \times \mathbf{B})$$

Convection dominant

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu\sigma}\nabla^2 \mathbf{B}$$

Diffusion dominant

ionosphere?

What's "Perfect Conducting" in MHD?

The Alfvén Theorem

The differential form of the Faraday's law says the magnetic field moves together with the flow, but it's less intuitive in terms of the derivative form

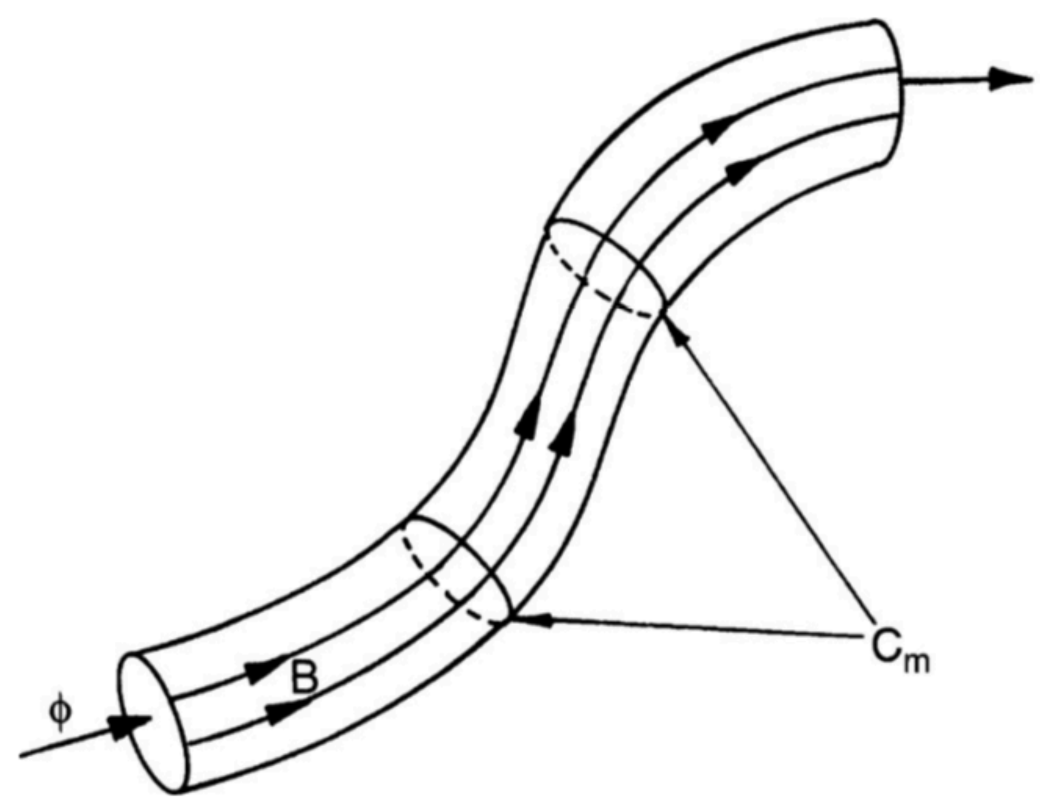
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{u} \times \mathbf{B})$$

If we use the integral form of the Faraday's law, things get much easier to understand.

$$\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = -\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{\sigma}\oint_C \mathbf{J} \cdot d\mathbf{l} \xrightarrow{\sigma = \infty} 0$$

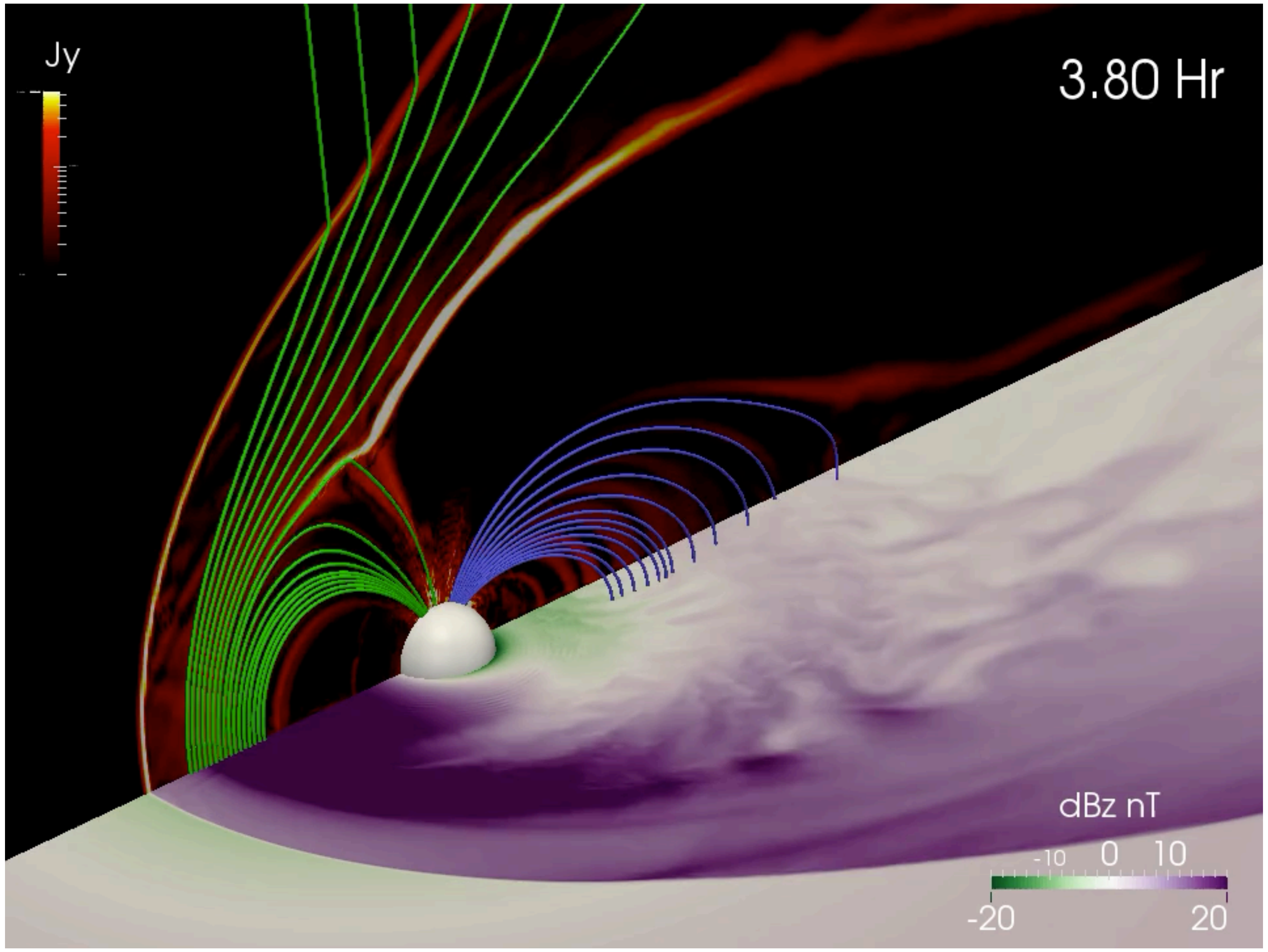
Then the Faraday's low is written as

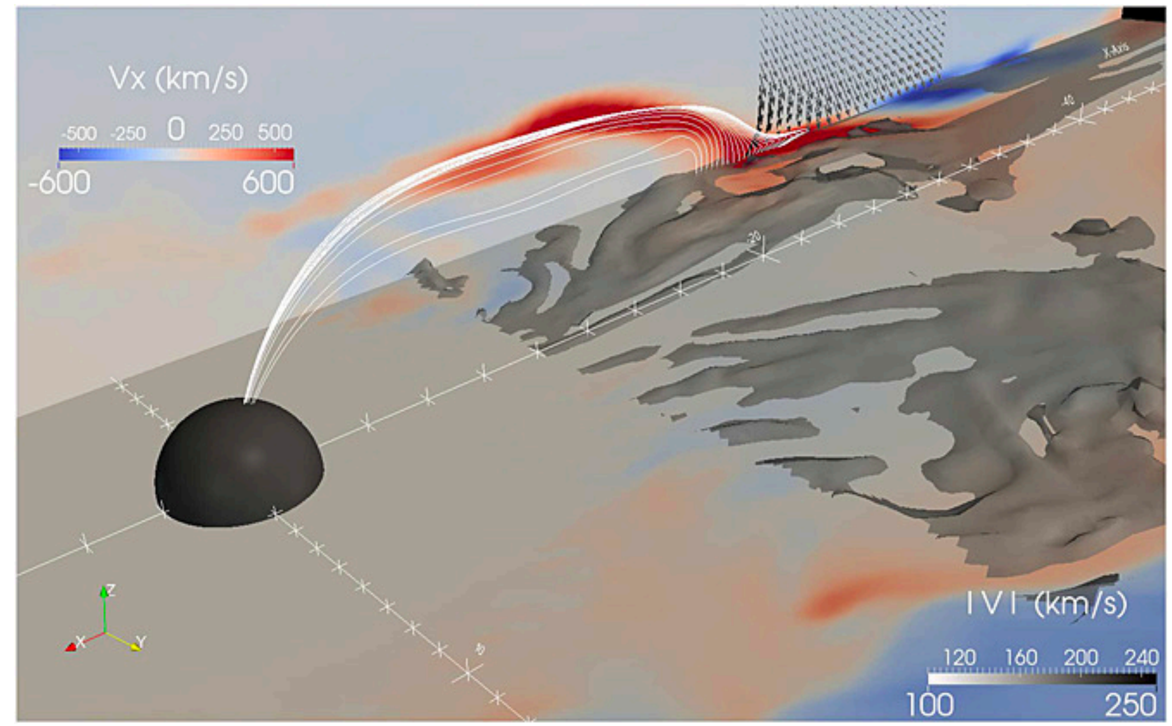
$$\boxed{\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = \frac{d\Phi}{dt} = 0}$$



What's "Perfect Conducting" in MHD?

BBF and Dipolarization front





$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = \frac{d\Phi}{dt} = 0$$