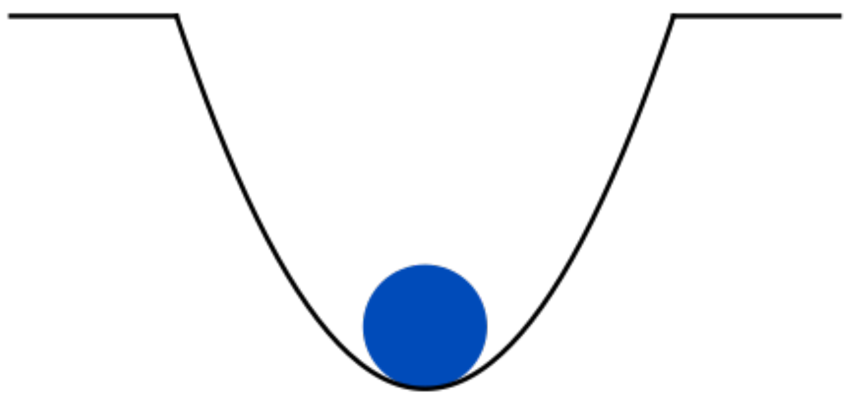
Instability and Discontinuity

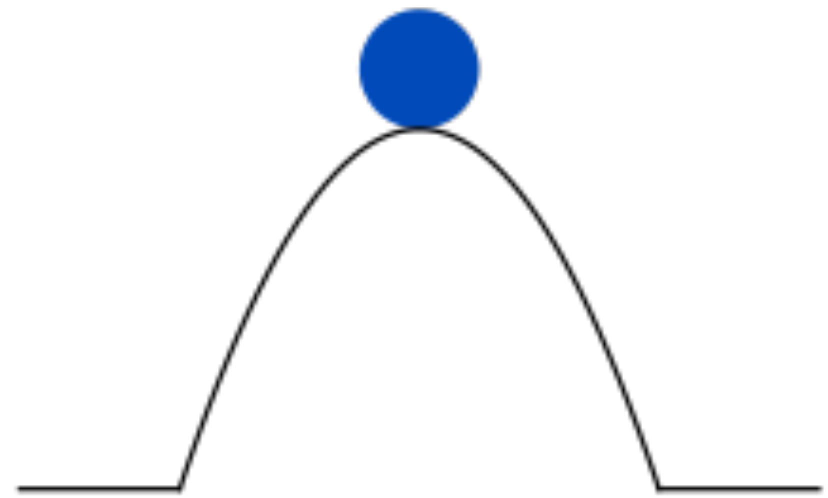
Ideal MHD Solutions and Applications

Basics Equilibrium

Equilibrium: $\frac{d}{dt} \equiv 0$



Stable Equilibrium



Unstable Equilibrium

- If you perturb the stable equilibrium, the ball will oscillate around the equilibrium position
- If you perturb the unstable equilibrium, the ball will roll away from the equilibrium position

If W(x) is the potential energy (e.g., gravitational potential), then the force on the particle is

$$\mathbf{F}_x = -\nabla W = -\frac{\partial W}{\partial x}$$

At the equilibria point: $\frac{\partial W}{\partial x} = 0$

0, equilibria

Use the Taylor expansion:

$$W = W_0 + \cancel{\frac{\partial W}{\partial x}} \Delta x + \underbrace{\frac{1}{2}\frac{\partial^2 W}{\partial x^2}(\Delta x)^2}_{\delta W} = 0$$

$\delta W > 0$ **Stable**

$\delta W < 0$ **unstable**

Why are instabilities important?

- Plasmas are host to numerous instabilities
- An equilibrium will not last or long if it is unstable
- Instabilities play important roles in a variety of space and astrophysical phenomena
  - Solar eruptions
  - Magnetospheric boundary layers
  - Planetary Aurora
  - Plasma transport in giant magnetospheres
  - Particle acceleration
  - Accretion disks
  - Star formation
- Plasma Instabilities are a source turbulence
  - Magnetototational Instability
  - Short wavelength instabilities in fusion devices
- Instabilities = bad confinement!

List of Plasma instabilities

List of plasma instabilities [edit]

- Buneman instability,[3]
  - Farley–Buneman instability,[4][5]
  - Jeans–Buneman instability,[6][7]
  - Relativistic Buneman instability,[8]
- Cherenkov instability,[9]
- Coalescence instability,[10]
  - Non-linear coalescence instability
- Chute instability,
- Collapse instability,
- Cyclotron instabilities, including:
  - Alfven cyclotron instability
  - Cyclotron maser instability,[11]
  - Electron cyclotron instability
  - Electrostatic ion cyclotron Instability
  - Ion cyclotron instability
  - Magnetoacoustic cyclotron instability
  - Proton cyclotron instability
  - Non-resonant beam-type cyclotron instability
  - Relativistic ion cyclotron instability
  - Whistler cyclotron instability
- Diocotron instability,[12] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)[13]
- Double emission instability,
  - Edge-localized modes,[14][15]
  - Explosive instability (or Ballooning instability),[16]
- Double plasma resonance instability,[17]
- Drift instability[18] (a.k.a. drift-wave instability,[19] or universal instability[20])
  - Lower hybrid (drift) instability (in the Critical ionization velocity mechanism)
  - Magnetic drift instability,[21]
  - Slow Drift Instability
- Electrothermal instability
- Fan instability,[22]
- Firehose instability (a.k.a. hose instability)
- Fish instability,
- Free electron maser instability,
- Gyrotron instability,
- Helical (Helix) instability,
- Jeans instability,[23][24]
- Magnetorotational instability (in accretion disks)
- Magnetothermal instability (Laser-plasmas),[25]
- Modulational instability
- Non-Abelian instability,
- Pair instability (in supernovae)
- Parker instability (magnetic buoyancy instability),[26]
- Peratt instability (stacked toroids)
- Pinch instability (a.k.a. Bennett pinch instability),[27][28]
  - Sausage instability (m=0)
  - Kink instability (m=1)
    - Helical kink instability (a.k.a. helical instability)
- Rayleigh-Taylor instability (RTI, a.k.a. gravitational instability)
  - Interchange instability (a.k.a. flute instability),[29]
- Rotating instability,[30]
- Tearing mode instability (or resistive tearing instability[31])
- Two-stream instability (a.k.a. beam-plasma instability, counter-streaming instability
  - Beam acoustic instability
  - Bump-on-tail instability
  - Ion beam instability
  - Weak beam instability
- Weibel instability
  - Chromo–Weibel instability (i.e. non-abelian instability)
  - Filamentation instability (a.k.a. beam-Weibel instability),[32]

Why study MHD instabilities?

- We have learned that ideal MHD is used to describe large-scale, macroscopic plasma behavior (entropy mode, Alfvén mode, magnetosonic mode etc.)

- MHD is often a mediocre approximation of plasma - but it does a good job at describing instabilities (why?)
  - MHD captures most of the essential physics of **force balance**
  - Dynamic pressure, thermal pressure, magnetic pressure and tension
  - MHD instabilities capture the most explosive behavior of plasma

- Instability theory is closely related to the wave analysis

- Ideal MHD does not aways give the right picture
  - There are configurations that are laboratory stable despite being ideal MHD unstable
  - Besides MHD instabilities, there are numerous Kinetic instabilities

General Strategy for Analyzing Instabilities

1. Start from an initial equilibrium: $\rho \frac{D\mathbf{u}}{Dt} + \nabla p = \mathbf{J} \times \mathbf{B} \longrightarrow \mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$

   $\frac{D}{Dt} \equiv 0$

2. Linearize the MHD equations:

$$\frac{\partial}{\partial t}\rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \xrightarrow{\rho = \rho_0 + \rho_1,\ |\rho_1| \ll \rho_0} \frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \rho_0 = 0$$

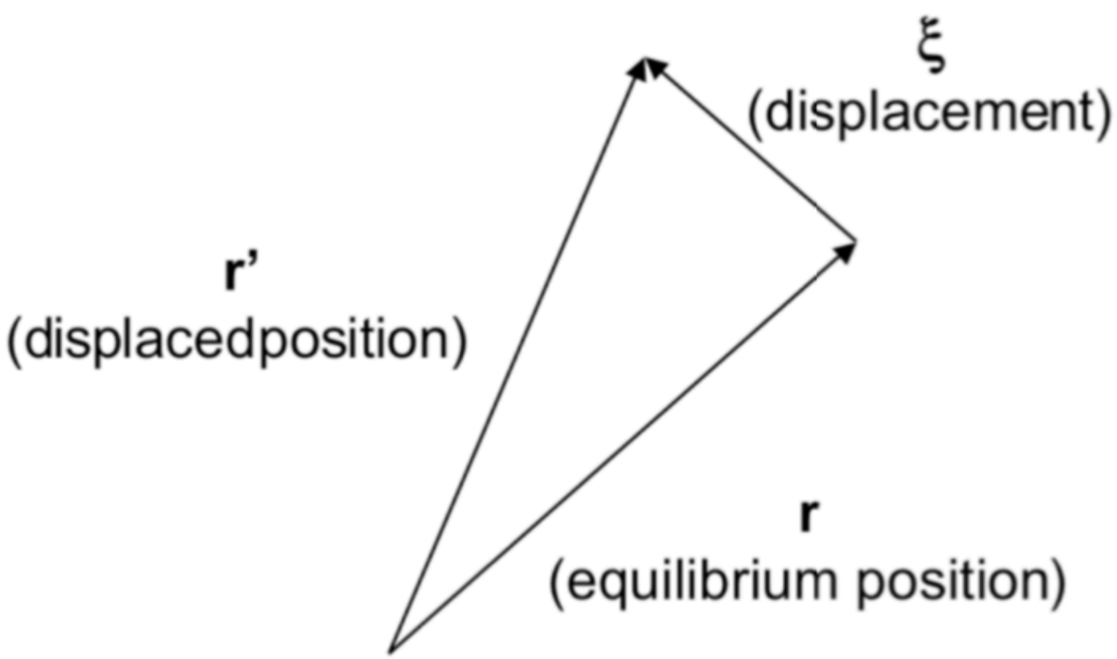
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0 \xrightarrow{\mathbf{u} = \mathbf{u}_1,\ |\mathbf{u}_1| \text{ small}} \rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + V_s^2 \nabla \rho_1 - \mathbf{B}_0 \times \nabla \times \mathbf{B}_1 = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \xrightarrow{\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,\ |\mathbf{B}_1| \text{ small}} \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)$$

3. Slightly perturb the equilibrium: either analytically or numerically
   - If the perturbation grows, the system is unstable
   - If the perturbation does not grow, the system is stable

4. Use a combination of numerical solutions, experiments and observations to study non-linear dynamics of instabilities

The displacement method



if $\xi(\mathbf{r}, t=0) = 0$, then the displacement vector

$$\xi(\mathbf{r}, t) = \int_0^t \mathbf{u}_1(\mathbf{r}, t')dt'$$

then the velocity vector is simply: $\boxed{\dfrac{\partial \xi}{\partial t} = \mathbf{u}_1}$

Now we can re-write the linearized MHD equations in terms of ξ

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0 \xrightarrow{\rho = \rho_0 + \rho_1,\ |\rho_1| \ll \rho_0} \frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 + \mathbf{u}_1\cdot\nabla\rho_0 = 0$$

$$\Big\downarrow\ \mathbf{u}_1 = \frac{\partial \xi}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\rho_1 + \rho_0\nabla\cdot\xi + \xi\cdot\nabla\rho_0\right) = 0 \xleftarrow[\partial_t \text{ and } \nabla \text{ independent}]{\frac{\partial \rho_0}{\partial t} \equiv 0} \frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial t}\cdot\nabla\rho_0 = 0$$

$$\Big\downarrow$$

$$\rho_1 + \rho_0\nabla\cdot\xi + \xi\cdot\nabla\rho_0 = 0 \longrightarrow \boxed{\rho_1 = -\rho_0\nabla\cdot\xi - \xi\cdot\nabla\rho_0}$$

The displacement method

Now look at the linearized induction equation

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \xrightarrow{\mathbf{u}_1 = \frac{\partial \xi}{\partial t}} \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\frac{\partial \xi}{\partial t} \times \mathbf{B}_0) \xrightarrow{\partial_t \text{ and } \nabla \text{ independent}} \frac{\partial \mathbf{B}_1}{\partial t} = \frac{\partial}{\partial t}\left[\nabla \times (\xi \times \mathbf{B}_0)\right]$$

So we get a linear equation for B_1: $\boxed{\mathbf{B}_1 = \nabla \times (\xi \times \mathbf{B}_0)}$

The momentum equation is more complicated: $\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + \nabla p_1 - \mathbf{B}_0 \times \nabla \times \mathbf{B}_1 = 0$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} + \nabla p_1 - \mathbf{B}_0 \times \nabla \times \mathbf{B}_1 = 0 \xrightarrow[\substack{\mathbf{B}_1 = \nabla \times (\xi \times \mathbf{B}_0) \\ p_1 = -\gamma p_0 \nabla \cdot \xi - \xi \cdot \nabla p_0}]{\text{Substitute}}$$

Why?

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} + \nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) - \mathbf{B}_0 \times \nabla \times (\nabla \times (\xi \times \mathbf{B}_0)) = 0 \longrightarrow \boxed{\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}(\xi)}$$

Where F is known as the ***force operator***:

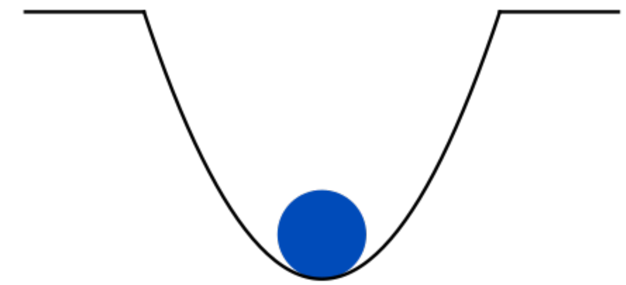
$$\mathbf{F}(\xi) = \nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) + \mathbf{B}_0 \times \nabla \times (\nabla \times (\xi \times \mathbf{B}_0))$$

$$\longrightarrow = \nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\xi \times \mathbf{B}_0) + \nabla \times (\nabla \times (\xi \times \mathbf{B}_0)) \times \mathbf{B}_0$$

Physical Meaning of the force operator

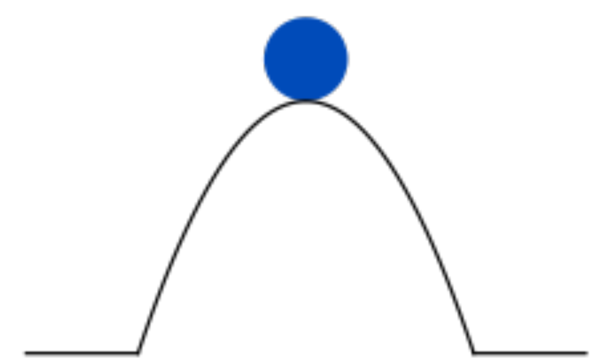
$$\mathbf{F}(\xi) = \nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\xi \times \mathbf{B}_0) + \nabla \times (\nabla \times (\xi \times \mathbf{B}_0)) \times \mathbf{B}_0$$

- If $\xi \cdot \mathbf{F}(\xi) < 0$
    - The displacement and the force are in ***opposite*** directions
    - The force opposes displacement
    - The system is expected to oscillate near the equilibria
    - This case corresponds to a stable perturbation



Stable Equilibrium

- If $\xi \cdot \mathbf{F}(\xi) < 0$
    - The displacement and the force are in ***same*** directions
    - The force enhances displacement
    - The perturbation will grow
    - This case corresponds to an unstable perturbation



Unstable Equilibrium

Note: The ideal MHD equations does not allow ***over-stable*** solutions where the restoring force would be strong enough to over-correct for and amplify the oscillations - requires energy sink or source

Initial Value Formulation for Instabilities

MHD stability can be investigated as an initial value problem by finding numerical or analytical solutions to the displacement equation:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}(\xi)$$

With appropriate initial and boundary conditions (analytical or numerical!)

Recall: in mathematical physics, we separate the space and time dependence of solutions:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \mathbf{F}(\xi) \xrightarrow{\xi(\mathbf{r},t) = \xi(\mathbf{r})T(t)} \rho_0 \frac{\partial^2 \xi T}{\partial t^2} = \mathbf{F}(\xi T) \longrightarrow \rho_0 \xi \frac{\partial^2 T}{\partial t^2} = T\mathbf{F}(\xi)$$

This is because F is linear about xi and T:

$$\begin{aligned}
\mathbf{F}(\xi T) &= \nabla(\gamma p_0 \nabla \cdot \xi T + \xi T \cdot \nabla p_0) + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\xi T \times \mathbf{B}_0) + \nabla \times (\nabla \times (\xi T \times \mathbf{B}_0)) \times \mathbf{B}_0 \\
&= T\left(\nabla(\gamma p_0 \nabla \cdot \xi + \xi \cdot \nabla p_0) + (\nabla \times \mathbf{B}_0) \times (\nabla \times (\xi \times \mathbf{B}_0) + \nabla \times (\nabla \times (\xi \times \mathbf{B}_0)) \times \mathbf{B}_0\right) \\
&= T\mathbf{F}(\xi)
\end{aligned}$$

Normal Mode Formulation for Instabilities

Normal mode solutions

Now we can separate the space and time solutions:

$$\rho_0 \xi \frac{\partial^2 T}{\partial t^2} = T\mathbf{F}(\xi) \longrightarrow \underbrace{\frac{1}{T}\frac{\partial^2 T}{\partial t^2}}_{\text{Function of } t} = \underbrace{\frac{\mathbf{F}(\xi)}{\rho_0 \xi}}_{\text{Function of } r} \longrightarrow \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{\mathbf{F}(\xi)}{\rho_0 \xi} = -\omega^2$$

Then we get to separate equations:

Aka Eigenvalue problem of PDE

$$\frac{d^2 T}{dt^2} = -\omega^2 T \xrightarrow{\text{ODE solution}} T(t) = e^{i\omega t}$$

$$-\omega^2 \rho_0 \xi = \mathbf{F}(\xi) \longrightarrow \xi(\mathbf{r}, t) = \xi(\mathbf{r}) e^{i\omega t}$$

Eigenvalue

*Eigenfunction of **F***

Question: What determines the permitted values of ω^2

- If $\omega^2 > 0$, the solution oscillates and is stable
- If $\omega^2 < 0$, the solution is exponential and is unstable

Why is ω^2 real?

Can also have a discrete set of ω^2 : $\xi(\mathbf{r}, t) = \sum_n \xi_n(\mathbf{r}) e^{i\omega_n t}$

The ideal force operator is self-adjoint

Definition of **self-adjoint**:

For any allowable displacement ξ and η: $$\int \eta \cdot \mathbf{F}(\xi) d\mathbf{r} = \int \xi \cdot \mathbf{F}(\eta) d\mathbf{r}$$

Which is relatively easy to show using the force equation:

$$-\omega^2 \rho_0 \xi = \mathbf{F}(\xi) \xrightarrow{\text{dot } \xi^*} -\omega^2 \rho_0 \xi^* \cdot \xi = \xi^* \cdot \mathbf{F}(\xi) \xrightarrow{\int dV} \omega^2 \int \rho_0 |\xi|^2 dV = -\int \xi^* \cdot \mathbf{F}(\xi) dV$$

Now do the same calculation for the complex conjugate equation:

$$-(\omega^*)^2 \rho_0 \xi^* = \mathbf{F}(\xi^*) \xrightarrow{\text{dot } \xi} -\omega^2 \rho_0 \xi \cdot \xi^* = \xi \cdot \mathbf{F}(\xi^*) \xrightarrow{\int dV}$$

$$(\omega^*)^2 \int \rho_0 |\xi|^2 dV = -\int \xi \cdot \mathbf{F}(\xi^*) dV$$

Subtract the two integral equations:

$$[\omega^2 - (\omega^*)^2] \int \rho_0 |\xi|^2 dV = \int \xi \cdot \mathbf{F}(\xi^*) dV - \int \xi^* \cdot \mathbf{F}(\xi) dV$$

$$0, \quad \int \eta \cdot \mathbf{F}(\xi) d\mathbf{r} = \int \xi \cdot \mathbf{F}(\eta) d\mathbf{r}$$

Only satisfies if $\omega^2 - (\omega^*)^2 = 0$

Solutions are either oscillatory or exponential!

About the Normal Mode Analysis

- This method requires less effort than the initial value formulation
- This method is more amenable to analyze
- This method cannot be used to describe non-linear behavior (after the linearization approximation breaks down)
- Normal model analysis requires that the enginvalues are discrete and distinguishable
- To determine whether the system is stable or unstable, use the energy principle

The Energy Principle

Variational Principle

The kinetic energy of the plasma displacement is calculated as

$$K(\dot{\xi}, \dot{\xi}) = \frac{1}{2}\int \rho_0 \dot{\xi} \cdot \dot{\xi} d\mathbf{r}$$

Using the harmonic analysis analysis:

$$K(\dot{\xi}, \dot{\xi}) = \frac{1}{2}\int \rho_0 \dot{\xi} \cdot \dot{\xi} d\mathbf{r} = -\frac{\omega^2}{2}\int \rho_0 \xi \cdot \xi d\mathbf{r} \xrightarrow[-\omega^2 \rho_0 \xi = \mathbf{F}(\xi)]{\text{Normal Mode}} = \frac{1}{2}\int \xi \cdot \mathbf{F}(\xi) d\mathbf{r}$$

For conservation of energy, the change in potential energy is

$$\delta W(\xi, \xi) = -K(\dot{\xi}, \dot{\xi}) = -\frac{1}{2}\int \xi \cdot \mathbf{F}(\xi) d\mathbf{r} \xrightarrow{\delta W < 0} \text{Unstable}$$

So we get

$$\omega^2 = \frac{\delta W(\xi, \xi)}{K(\xi, \xi)} \longrightarrow \text{Solve for the dispersion relation}$$

The Energy Principle

- Generally we care more about whether or not a configuration is stable than finding the linear growth rate
- The linear growth stage quickly becomes overwhelmed by nonlinear effects
- The energy principle allows us to determine stability but at the cost of losing information about the growth rate

Start from $\delta W(\xi, \xi) = -\frac{1}{2}\int \xi \cdot \mathbf{F}(\xi) d\mathbf{r}$ (work = force x distance)

Express dW as the sum of changes in the potential energy of plasma, the surface and the vacuum:

$$\delta W = \delta W_P + \delta W_S + \delta W_V$$

$$\delta W_P = \frac{1}{2}\int \Big[\underbrace{B_1^2}_{\text{Increase in mag energy}} - \underbrace{\xi \cdot \left(\mathbf{J}_0 \times \mathbf{B}_1\right)}_{\text{Work done by JxB}} - \underbrace{p_1(\nabla \cdot \xi)}_{\text{Adiabatic compression}}\Big] d\mathbf{r}$$

$$\delta W_S = \frac{1}{2}\int (\xi \cdot \hat{\mathbf{n}})^2 \left[\nabla\left(p_0 + \frac{1}{2}B_0^2\right)\right]^2 d\mathbf{S}$$

$$\delta W_V = \int \frac{1}{2} B_0^2 d\mathbf{r}$$

- In case of a deformed boundary, all terms exist - external mode
- In case of a fixed boundary, Ws = Wv = 0 - internal mode

Full Derivation, see Boyd and Sanderson

The Energy Principle

After manipulation the energy principle can be written in a more useful form

$$\delta W_P = \frac{1}{2}\int \Big[\boxed{|B_{1\perp}|^2 + B^2|\nabla\cdot\xi_\perp + 2\xi_\perp\cdot\kappa|^2 + \gamma p|\nabla\cdot\xi|^2} \quad \text{Stabilizing terms}$$

$\kappa = (\mathbf{b}\cdot\nabla)\mathbf{b}$ Curvature of B0

Could be de-stabilizing terms

$$\boxed{-2(\xi_\perp\cdot\nabla p)(\kappa\cdot\xi_\perp^*) - J_\parallel(\xi_\perp^*\times\mathbf{b})\cdot\mathbf{B}_{1\perp}\Big]}\, d\mathbf{r}$$

full derivation, see Freidberg (1987)

Recall unstable criterion: $\delta W < 0$, now we can see why the energy principle is useful

- The first three terms are always stabilizing:
  - The energy required to bend magnetic field lines (shear Alfvén)
  - The energy required to compress B field (magnetosonic mode)
  - The energy required to compress the plasma (slow mode)
- The remaining two terms can be stabilizing or destabilizing
  - Pressure-driven (interchange) instabilities (associated with J_pepr)
  - Current-driven (kink) instability (associated with J_para)

MHD discontinuities

MHD equations are differential equations, why the solutions allow discontinuities?

> In mathematics, a **weak solution** (also called a **generalized solution**) to an ordinary or partial differential equation is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense.

For example the Maxwell's equations can be written in different forms:

Differential Form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Requires E, B differentiable

STRONG solutions

Integral Form

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(\int_S \mathbf{J} \cdot d\mathbf{S} + \epsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S} \right)$$

No requirement on the derivatives of E, B

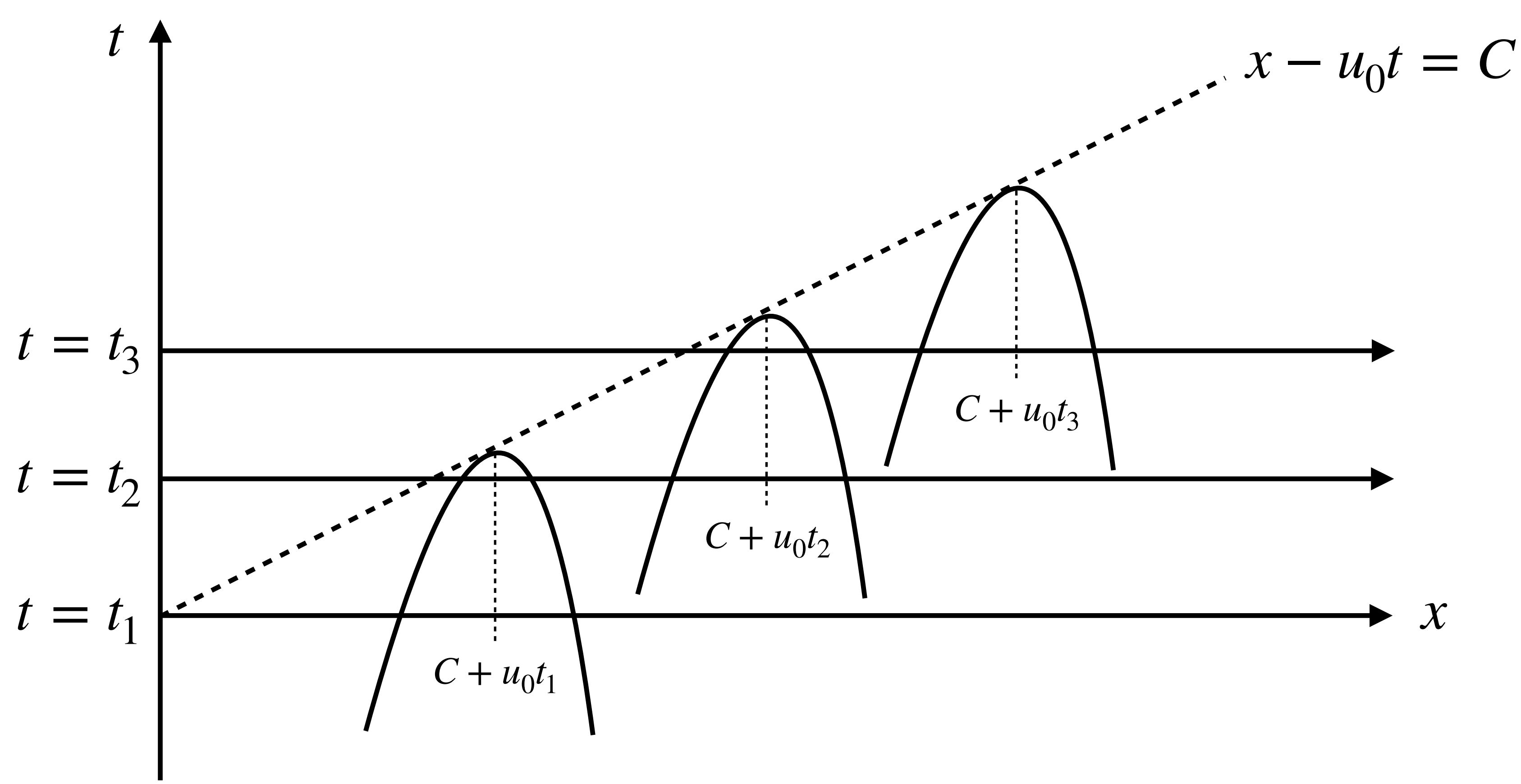
WEAK solutions

Recall Advection Equations

The MHD equations allow the existence of weak solutions, recall in the wave lecture:

$$\frac{\partial f(x,t)}{\partial t} + u_0 \frac{\partial f(x,t)}{\partial x} = 0$$

The solution goes like $f(x,t) \sim \boxed{Q(x - u_0 t)}$ No requirement on the derivatives of E, B



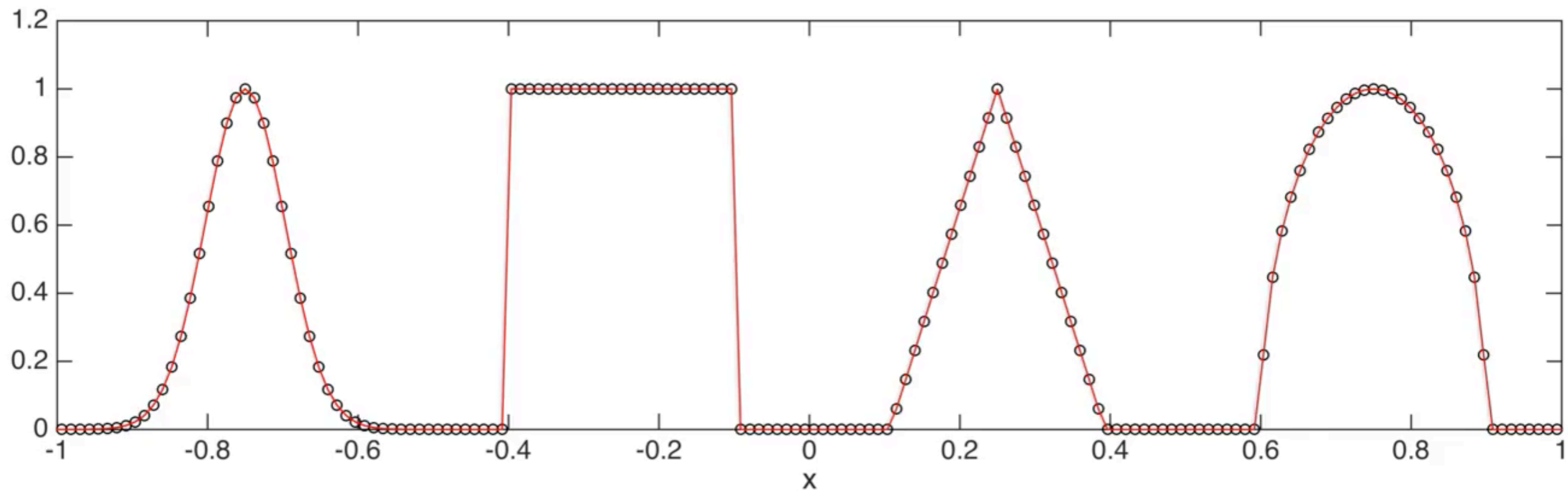
A simple wave propagation towards +x direction

MHD discontinuities

The MHD equations allow the existence of weak solutions, recall in the wave lecture:

$$\frac{\partial f(x,t)}{\partial t} + \frac{\partial u_0 f(x,t)}{\partial x} = 0$$

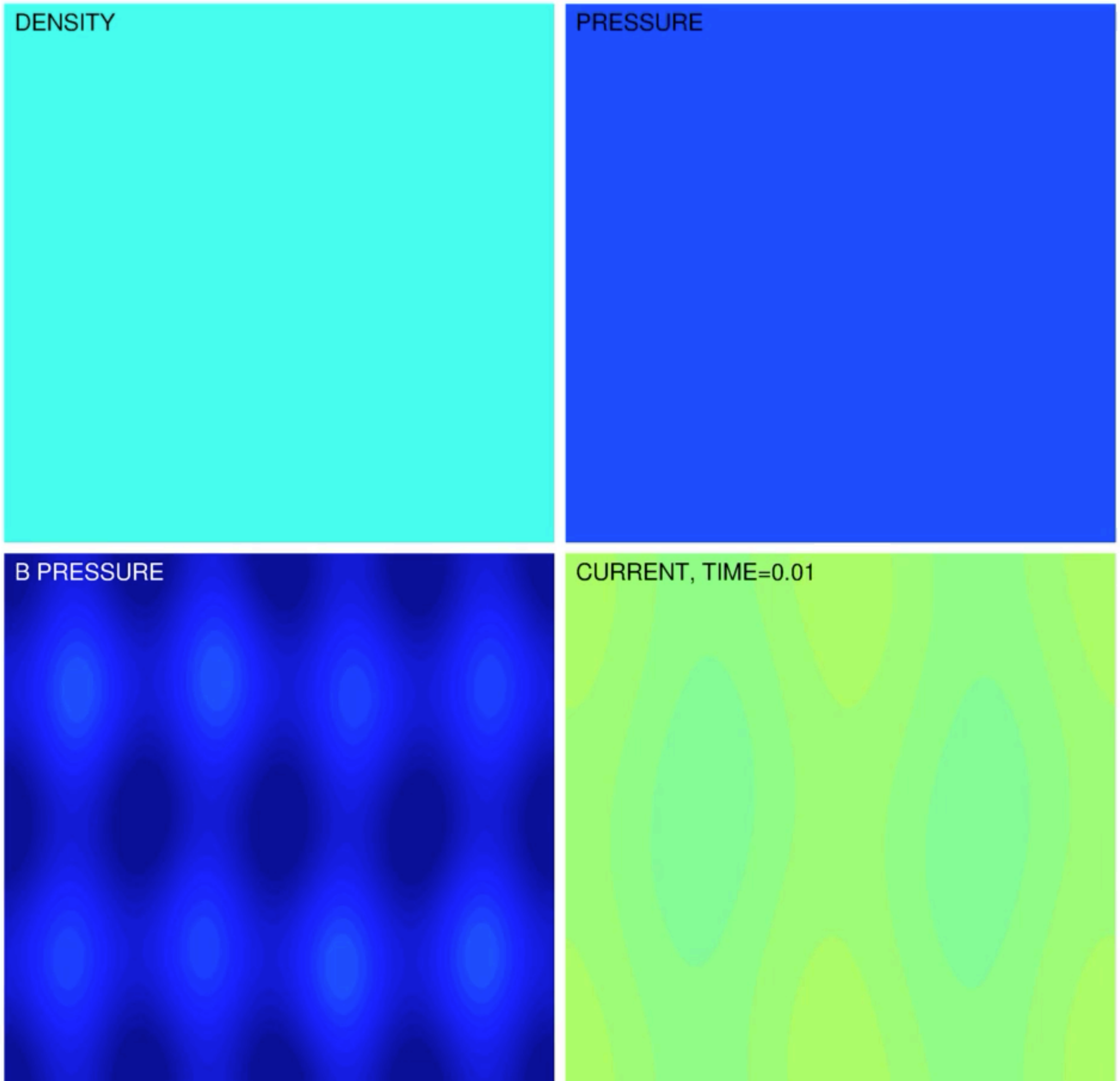
The solution goes like $f(x,t) \sim \boxed{Q(x - u_0 t)}$ No requirement on the derivatives of E, B



A simple wave propagation towards +x direction

MHD discontinuities

Non-linear Evolution



Orszag-Tang MHD Vortex Simulation

$$\rho(x, y, t=0) = \gamma^2$$

$$p(x, y, t=0) = \gamma$$

$$v_x(x, y, t=0) = -\sin y$$

$$v_y(x, y, t=0) = \sin x$$

$$B_x(x, y, t=0) = -\sin y$$

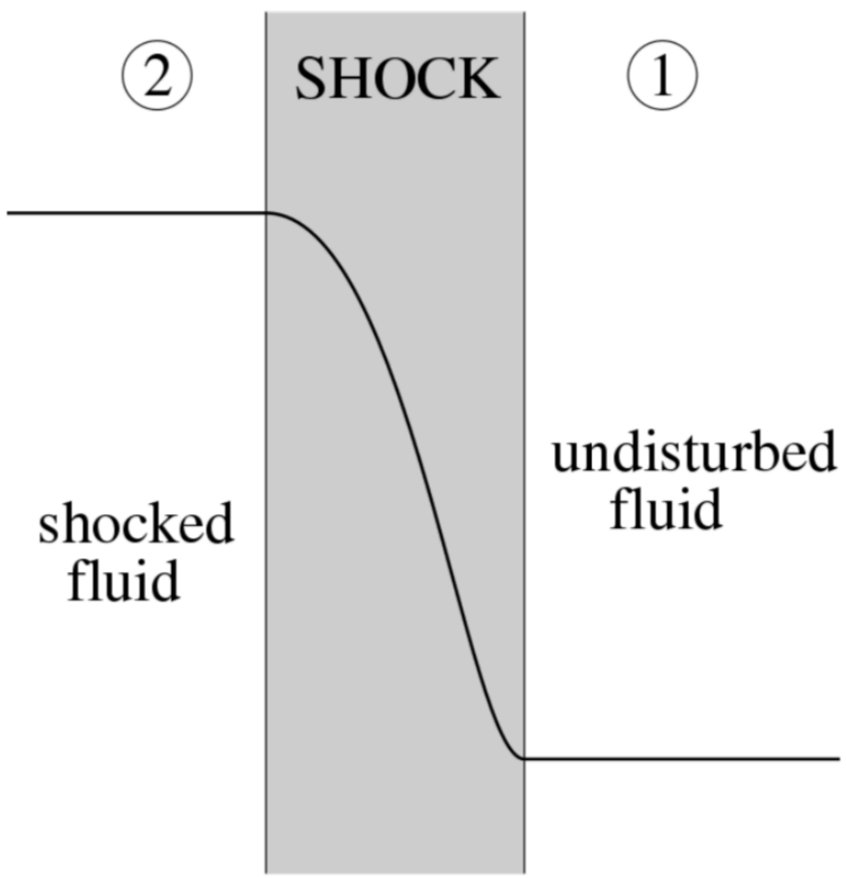
$$B_y(x, y, t=0) = \sin 2x$$

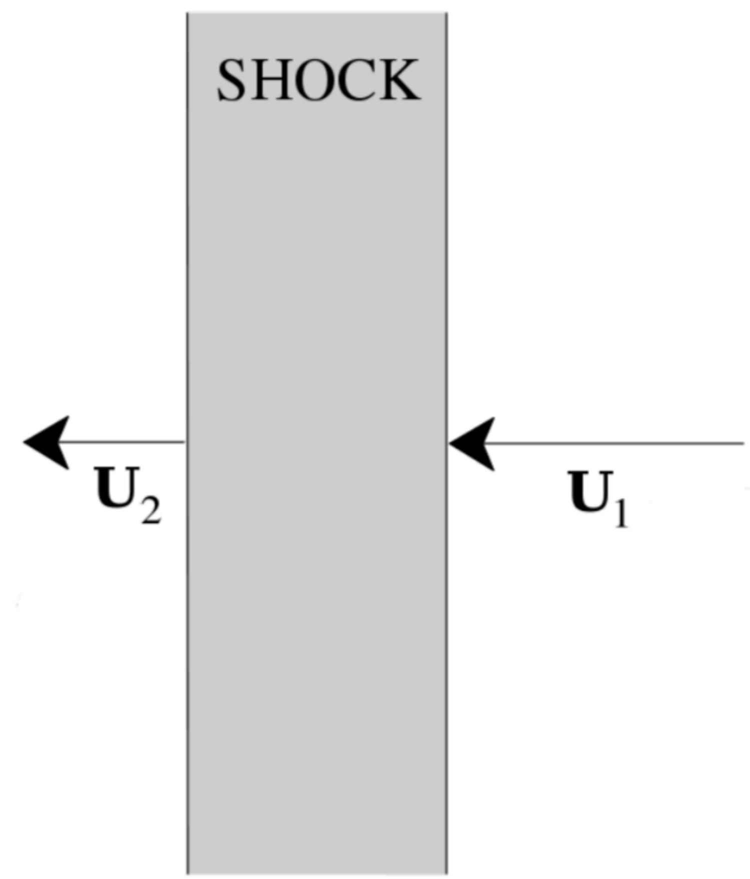
Smooth initial conditions quickly evolves to discontinuities (turbulence)

MHD discontinuities

So the MHD equations allow the existence a special type of flow solutions, named discontinuities (including shocks)

Let's start from hydrodynamic discontinuities:



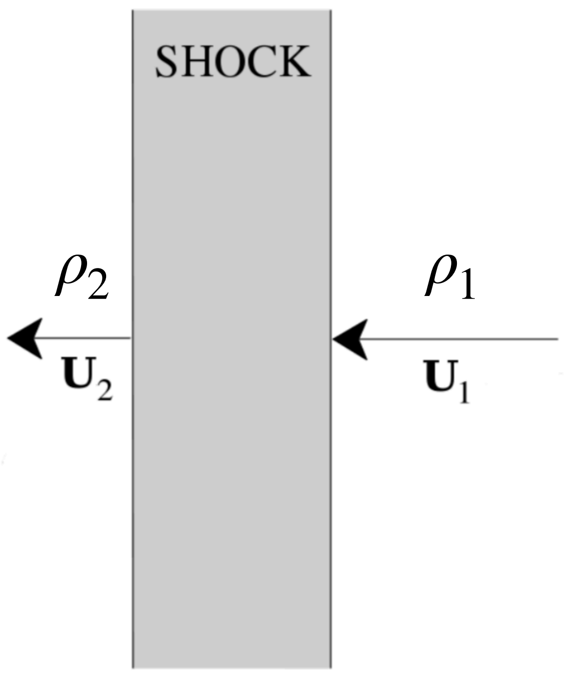


1-D flow in x $\quad \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} = 0$

In the frame with the moving shock $\quad \dfrac{\partial}{\partial t} = 0$

Hydrodynamic discontinuities

Mass conservation

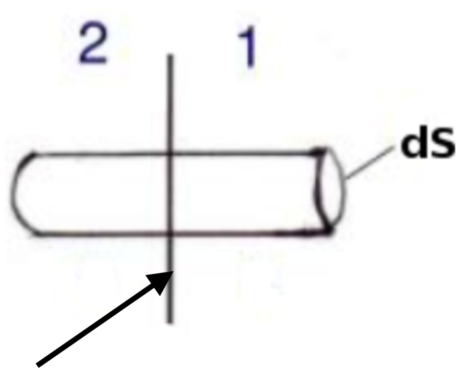


- The mass flux approaching the shock is rho1U1
- The mass flux leaving the shock is rho2U2
- In a steady state, the mass flux must be balanced:

$$\rho_1 U_1 = \rho_2 U_2$$

Now put the continuity equation in an integral form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \xrightarrow[\int_V (\,) dV]{\frac{\partial}{\partial t} = 0} \quad \int_V \nabla \cdot (\rho \mathbf{u}) dV = \oint_S (\rho \mathbf{u}) \cdot d\mathbf{S} = 0$$



Shock front

$$\xrightarrow{\text{Surface Integral}} \quad \oint_S (\rho \mathbf{u}) \cdot d\mathbf{S} = -\int_1 (\rho_1 U_1) dS + \int_2 (\rho_2 U_2) dS \quad \xrightarrow{\text{No net flux}} \quad = 0$$

$$\Rightarrow \; \rho_1 U_1 = \rho_2 U_2$$

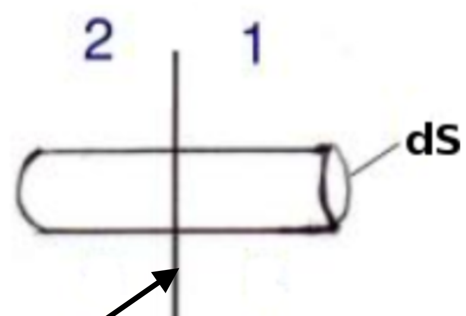
Hydrodynamic discontinuities

Momentum conservation

Now put the steady-state momentum equation in an integral form:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) = 0 \xrightarrow[\int_V (\,)dV]{\frac{\partial}{\partial t}=0} \int_V \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p)dV \xrightarrow{\text{Gauss}} \oint_S (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) \cdot d\mathbf{S}$$

Evaluate the integral near the shock front:



$$\Rightarrow \rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2$$

similarly, conservation of energy yields: $\epsilon_P = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}$

$$\frac{\partial \epsilon_P}{\partial t} + \nabla \cdot \mathbf{u}(\epsilon_P + p) = 0 \xrightarrow[\int_V (\,)dV]{\frac{\partial}{\partial t}=0} \int_V \nabla \cdot \mathbf{u}(\epsilon_P + p)dV = 0 \xrightarrow{\text{Gauss}} \oint_S \mathbf{u}(\epsilon_P + p) \cdot d\mathbf{S} = 0$$

Evaluate the integral near the shock front: $\Rightarrow \frac{1}{2}\rho_1 U_1^3 + \frac{\gamma p_1 U_1}{\gamma - 1} = \frac{1}{2}\rho_2 U_2^3 + \frac{\gamma p_2 U_2}{\gamma - 1}$

Hydrodynamic discontinuities

Notations

There are several standard ways to write ***jump conditions*** across discontinuities :

$$
\begin{aligned}
[\![\phi]\!] &= 0 \\
[\phi]_1^2 &= 0 \\
[\phi] &= 0 \\
\phi_2 - \phi_1 &= 0
\end{aligned}
$$

These are all **equivalent**

To summarize, the jump conditions across a hydrodynamic discontinuities are written as

Conservation of Mass $\quad [\rho U_x] = 0 \;\Rightarrow\; \rho_1 U_1 = \rho_2 U_2$

Conservation of Momentum $\quad [\rho U_x^2 + p] = 0 \;\Rightarrow\; \rho_1 U_1^2 + p_1 = \rho_2 U_2^2 + p_2$

Conservation of Energy $\quad [U_x \epsilon + p U_x] = 0 \;\Rightarrow\Rightarrow\; \dfrac{1}{2}\rho_1 U_1^3 + \dfrac{\gamma p_1 U_1}{\gamma - 1} = \dfrac{1}{2}\rho_2 U_2^3 + \dfrac{\gamma p_2 U_2}{\gamma - 1}$

Now we can derive the properties of the flow near a hydrodynamic discontinuity

Hydrodynamic discontinuities

Properties

Now we have three equations with six unknown - infinite set of solutions, but

The density ratio cross the shock is $\frac{\rho_1}{\rho_2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \equiv \chi$

$M = \frac{U_1}{V_{s1}}$

The sonic Mach number

The velocity ratio cross the shock is $\frac{U_2}{U_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$

The pressure ratio is then calculated $\frac{p_2}{p_1} = \frac{2\gamma M^2 - (\gamma-1)}{\gamma+1}$

The temperature ratio is then $\frac{T_2}{T_1} = \frac{[(\gamma-1)M^2+2][2\gamma M^2-(\gamma-1)]}{(\gamma+1)^2 M^2}$

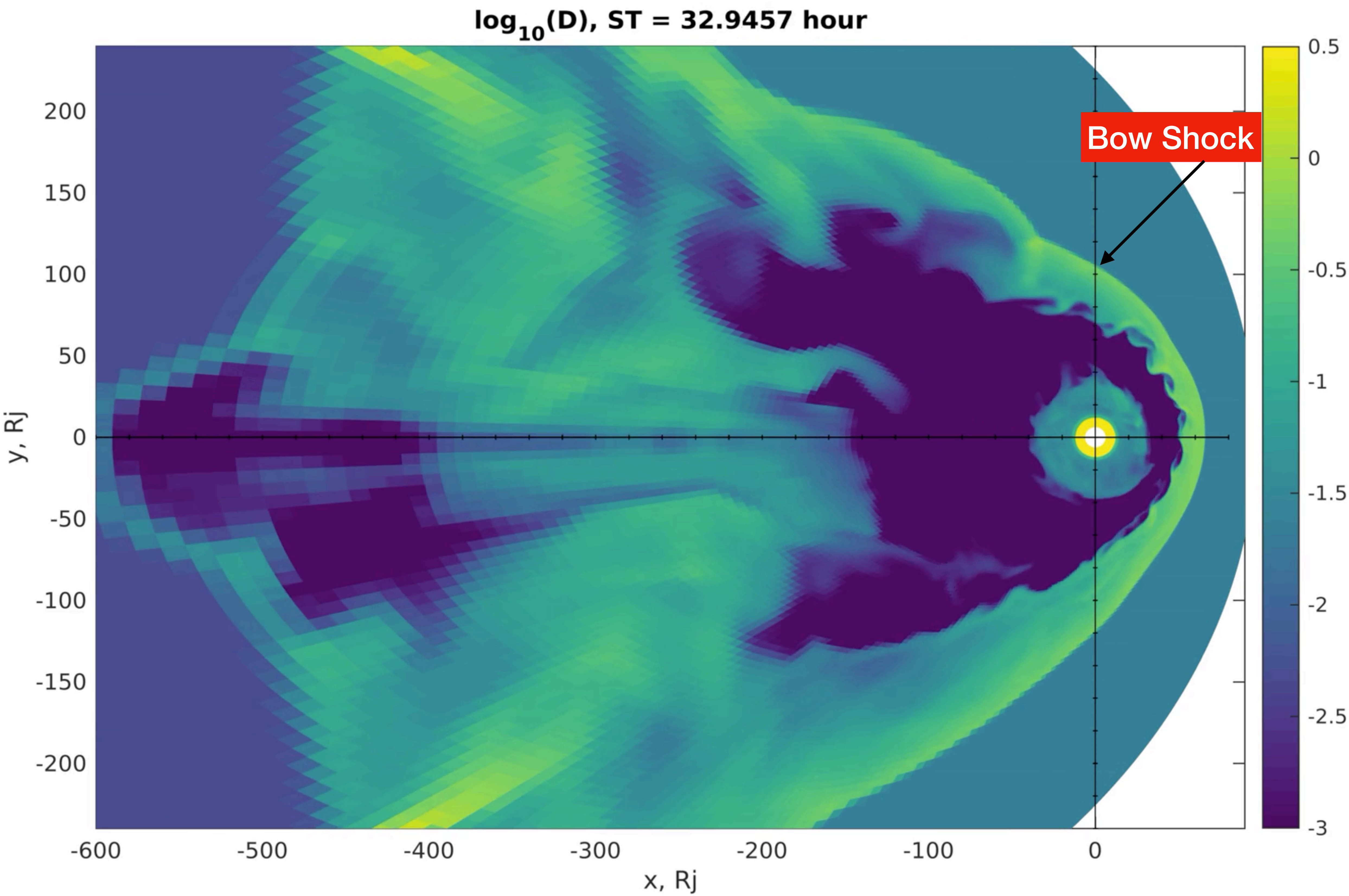
So,
1. When M>1 discontinuities occur, which is known as “shocks”
2. Shocks are compressible: $\rho_2 > \rho_1;\ p_2 > p_1$
3. Shocks must be dissipative - entropy increases (why?)

The flow is called hypersonic when $M \to \infty$

The density ratio becomes $\frac{\rho_1}{\rho_2} \to \frac{\gamma+1}{\gamma-1} \approx 4$

Hydrodynamic discontinuities

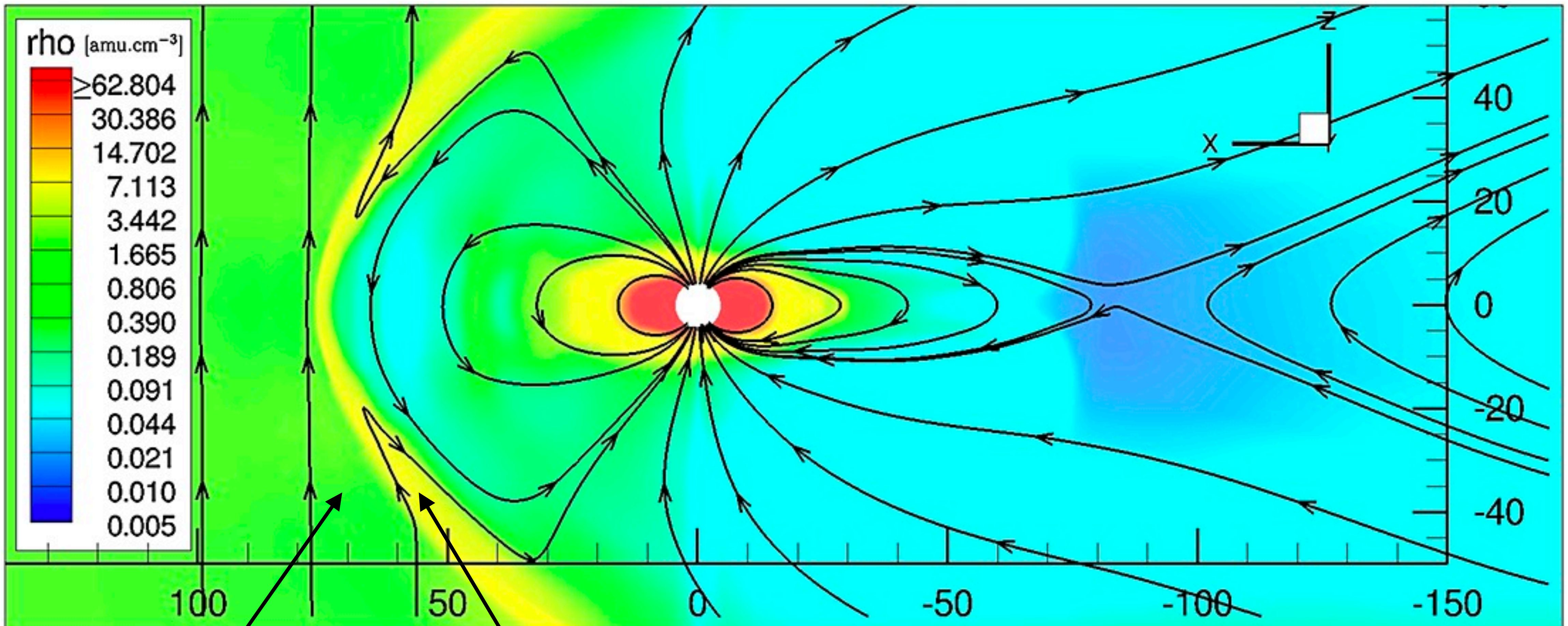
A Jovian Magnetosphere simulation from GAMERA



Hydrodynamic discontinuities

Examples

A Global Jovian Magnetosphere Model



Cháne et al 2013

$\rho_1 \approx 0.4$ $\rho_1 \approx 8$

So $\dfrac{\rho_2}{\rho_1} \approx 20$

But

The flow is called hypersonic when $M \to \infty$

The density ratio becomes $\dfrac{\rho_1}{\rho_2} \to \dfrac{\gamma + 1}{\gamma - 1} \approx 4$

Hydrodynamic discontinuities

Strong shock limit and dissipation

The flow is called hypersonic when $M \to \infty$

The density ratio becomes $\dfrac{\rho_1}{\rho_2} \to \dfrac{\gamma+1}{\gamma-1} \approx 4$

The post-shock pressure is $p_2 \approx \dfrac{2\gamma}{\gamma+1} M^2 p_1 = \dfrac{2}{\gamma+1}\rho_1 U_1^2 = \dfrac{3}{4}\rho_1 U_1^2$

The post-shock temperature $T_2 \approx \dfrac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2 T_1 = \dfrac{3}{16}\dfrac{M}{k_B} U_1^2$

In the rest frame of a strong shock, the post-shock kinetic and thermal energy are

$$\frac{1}{2}U_2^2 \approx \frac{1}{32}U_1^2$$

$$\frac{3}{2}\frac{kT_2}{M} \approx \frac{9}{32}U_1^2$$

So roughly half of the pre-shock kinetic energy is converted to thermal energy. That's why people say the temperature of the magnetosheath is determined by the solar wind velocity

Dissipation near discontinuities

Why the primitive form won't work

A shock converts supersonic gas into denser, slower moving, higher pressure, subsonic gas. Thus it increases the specific entropy of the gas by an amount

$$\Delta s = s_2 - s_1 = c_V \ln\left(\frac{p_2}{\rho_2^\gamma}\right) - c_V \ln\left(\frac{p_1}{\rho_1^\gamma}\right) = c_V \ln\left(\frac{p_2}{p1}\right) - c_V \gamma \ln\left(\frac{\rho_2}{\rho_1}\right) > 0$$

In another terminology, a shock shifts gas to a higher adiabat. An adiabat is a locus of constant entropy ($T \sim \rho^{\gamma-1}$) in the density-temperature plane. Gas can move adiabatically along an adiabat, while changes in entropy move it from one adiabat to another

This means dissipation is needed in describing the correct behavior of shocks

$$\frac{\partial}{\partial t}\rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial}{\partial t} p + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

- **No explicit dissipation mechanism describing the shock**
- **Does not guarantee conservation of energy numerically**
- **Thus end up with wrong jump conditions**

Applications of the momentum equation

Shock location

Now here’s another problem （那么问题又来了）

JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 103, NO. A1, PAGES 225–235, JANUARY 1, 1998

A global magnetohydrodynamic simulation of the Jovian magnetosphere

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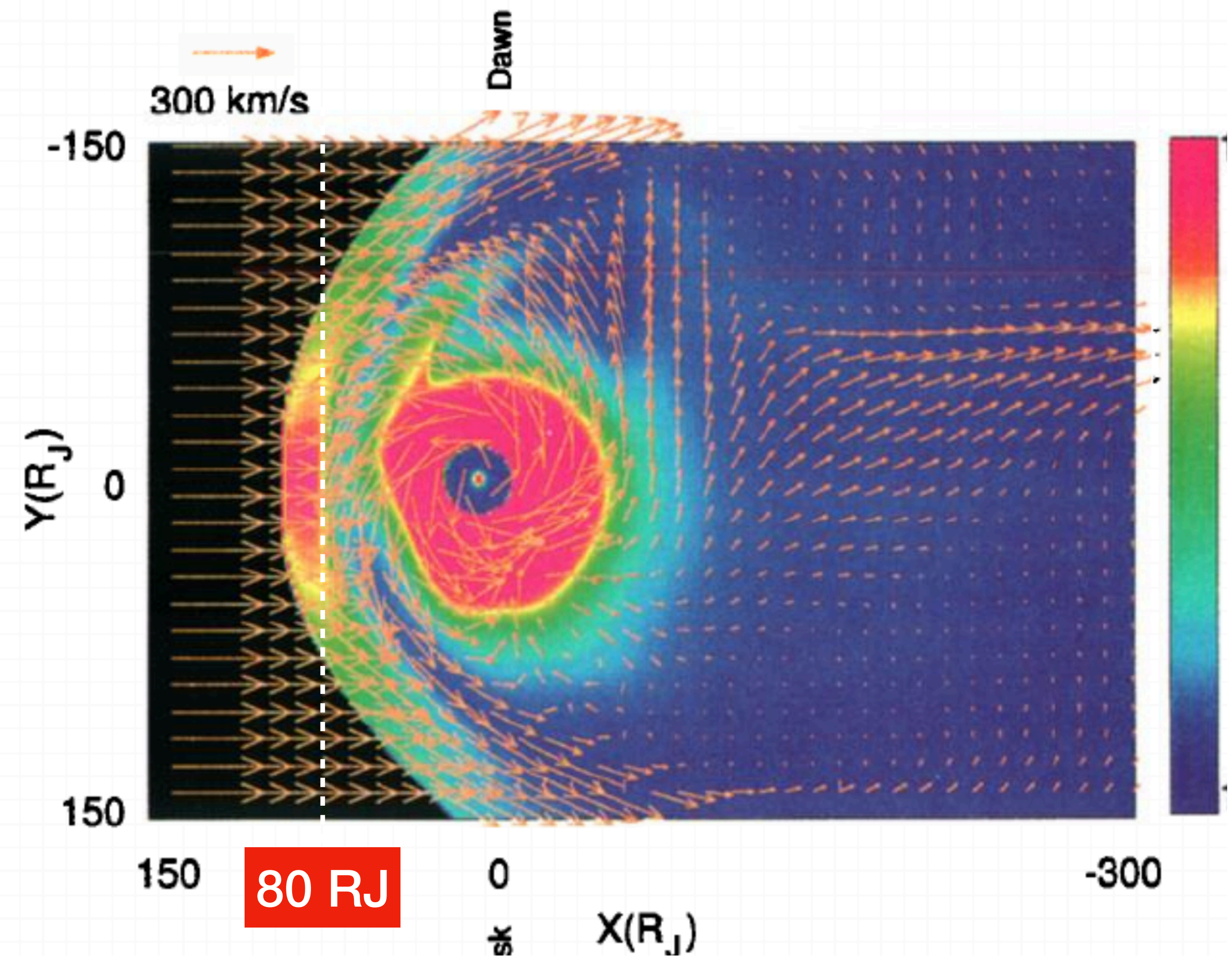
Raymond J. Walker, Margaret G. Kivelson[1]
Institute of Geophysics and Planetary Physics, University of California, Los Angeles

$$\partial \rho / \partial t = -\nabla \cdot (\mathbf{v}\rho) + D\nabla^2 \rho$$
$$\partial \mathbf{v} / \partial t = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla P/\rho + (\mathbf{J}\times\mathbf{B})/\rho + \mathbf{g} + \mathbf{\Phi}/\rho$$
$$\partial P / \partial t = -(\mathbf{v}\cdot\nabla)P - \gamma P\nabla\cdot\mathbf{v} + D_p\nabla^2 P$$
$$\partial \mathbf{B} / \partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2 B$$
$$\mathbf{J} = \nabla\times(\mathbf{B}-\mathbf{B}_d)$$



The Ogino MHD model for the jovian magnetosphere gives 80 RJ as the stand-off distance - it’s “consistent” with observations!

What’s the problem here?

Application to the Earth's bow shock

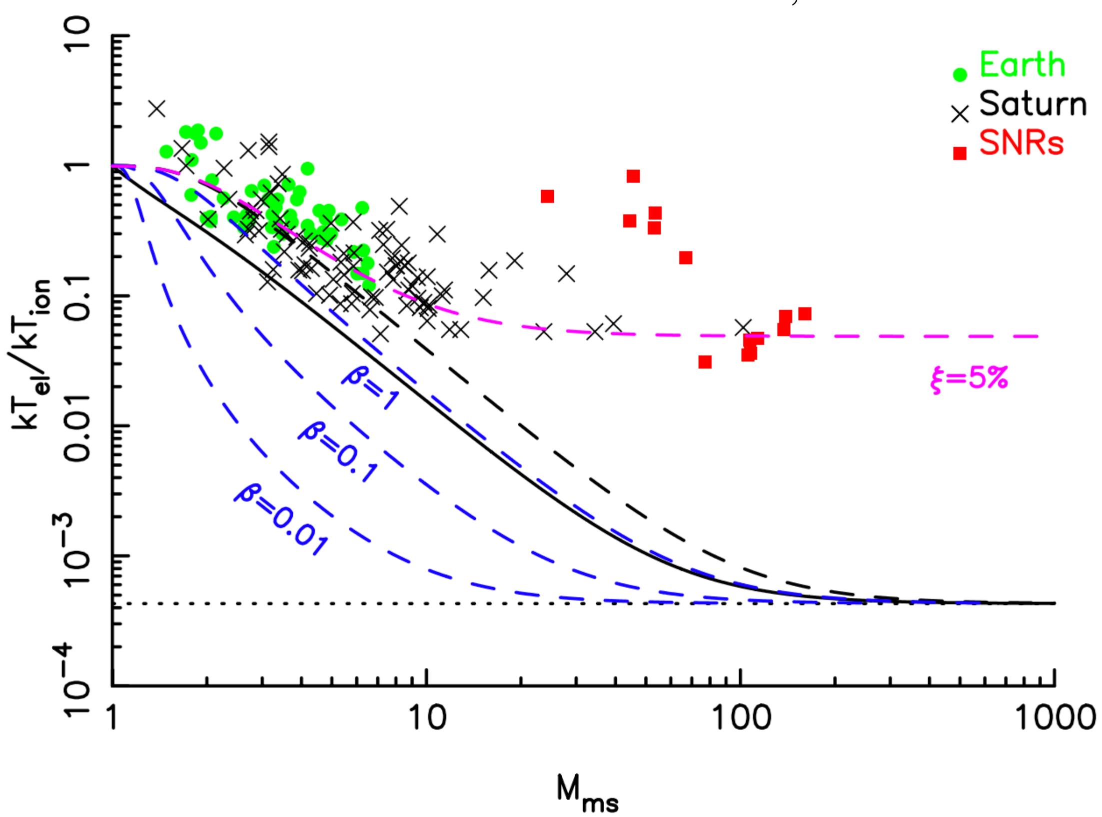
In the 1-D hydro model, the post-shock temperature for each species goes

$$T_2 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2 T_1 = \frac{2(\gamma-1)}{(\gamma+1)^2} m_s V_S^2 = \frac{3}{16} m_s V_S^2$$

Which depends on particle mass, i.e., the electron temperature << ion temperature across a strong shock

If we apply the HD jump conditions to both species (electrons and ions) separately, it is relatively straightforward to show that after the shock, the temperature ratio becomes

$$\frac{T_{e,2}}{T_{i,2}} = \frac{m_e}{m_i} \frac{2\left(\frac{m_i}{m_e}\right)\chi^2 + M^2(\gamma-1)(\chi^2-1)}{2\chi^2 + M^2(\gamma-1)(\chi^2-1)} \qquad \chi = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$$



When the flow is hypersonic $M \to \infty$

$$\frac{T_{e,2}}{T_{i,2}} \approx \frac{m_e}{m_i}$$

Which means ions are much more heated across the bow shock. This is why Ti/Te -> 10 in the Earth's magnetosphere

Application to the Earth's bow shock

Parameter	Interplanetary (Pre-shock)	Magnetosheath (Post-shock)
Bulk velocity, km s^{-1}	280 ± 30	180 ± 20
ϕ ('spacecraft' co-ordinates), deg	$9{\cdot}9 \pm 1$	$30{\cdot}4 \pm 4$
θ ('spacecraft' co-ordinates), deg	$2{\cdot}2 \pm 1$	$21{\cdot}1 \pm 4$
Ion density, cm^{-3}	11 ± 3	31 ± 8
Ion temperature, °K	3×10^4 °K (mean)	$5{\cdot}0$–$7{\cdot}5 \times 10^5$ °K (thermal part)
Electron temperature, °K	2×10^5 °K	5×10^5 °K
Magnetic field strength, γ	$3{\cdot}5$	$13{\cdot}1$
R-component	$+2{\cdot}3$	$+8{\cdot}2$
T-component	$-2{\cdot}0$	$-8{\cdot}3$
N-component	$+1{\cdot}8$	$+6{\cdot}1$

Table 1. Measured plasma and magnetic field parameters adjacent to bow shock—Pioneer VI

Mihalov et al. [1969]

From Hydrodynamics to MHD

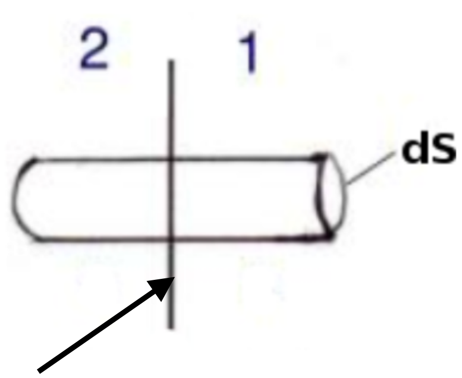
Additional constraints

- Conservation of mass is unchanged $\boxed{[\rho(\mathbf{u}\cdot\hat{\mathbf{n}})] = 0}$ Mass jump condition for MHD
- Conservation of momentum must include the Lorentz force terms
- Conservation of energy must include magnetic energy
- Need two additional constraints
  - Divergence-free field constraint
  - Conservation of Magnetic flux

Start from the conservation form of the momentum equation:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla\cdot\left[\rho\mathbf{u}\mathbf{u} + \mathbf{I}(p + \frac{1}{2}B^2) - \mathbf{B}\mathbf{B}\right] = 0 \xrightarrow[\int_V (\,)dV]{\frac{\partial}{\partial t}=0} \int_V \nabla\cdot\left[\rho\mathbf{u}\mathbf{u} + \mathbf{I}(p + \frac{1}{2}B^2) - \mathbf{B}\mathbf{B}\right] dV$$

$$\xrightarrow{\text{Gauss}} \oint_S \left[\rho\mathbf{u}\mathbf{u} + \mathbf{I}(p + \frac{1}{2}B^2) - \mathbf{B}\mathbf{B}\right]\cdot d\mathbf{S} = 0$$



Shock front

$$\xrightarrow{\text{Surface Integral}} \boxed{\left[\rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}}) + (p + \frac{1}{2}B^2)\hat{\mathbf{n}} - (\mathbf{B}\cdot\hat{\mathbf{n}})\mathbf{B}\right] = 0}$$

Momentum jump condition for MHD

From Hydrodynamics to MHD

Additional constraints

Conservation form of the total energy equation:

$$\frac{\partial \mathscr{E}_T}{\partial t} + \nabla \cdot \left[\left(\mathscr{E} + p_{tot} \right) \mathbf{u} - \mathbf{u} \cdot \mathbf{BB} \right] = 0 \xrightarrow[\int_V (\,) dV]{\frac{\partial}{\partial t} = 0} \int_V \nabla \cdot \left[\left(\mathscr{E} + p_{tot} \right) \mathbf{u} - \mathbf{u} \cdot \mathbf{BB} \right] dV = 0$$

$$\xrightarrow{\text{Gauss}} \oint_S \left[\left(\mathscr{E} + p_{tot} \right) \mathbf{u} - \mathbf{u} \cdot \mathbf{BB} \right] \cdot \hat{\mathbf{n}} dS = 0 \longrightarrow \boxed{\left[\left(\mathscr{E} + p_{tot} \right) (\mathbf{u} \cdot \hat{\mathbf{n}}) - \mathbf{u} \cdot \mathbf{B}(\mathbf{B} \cdot \hat{\mathbf{n}}) \right] = 0}$$

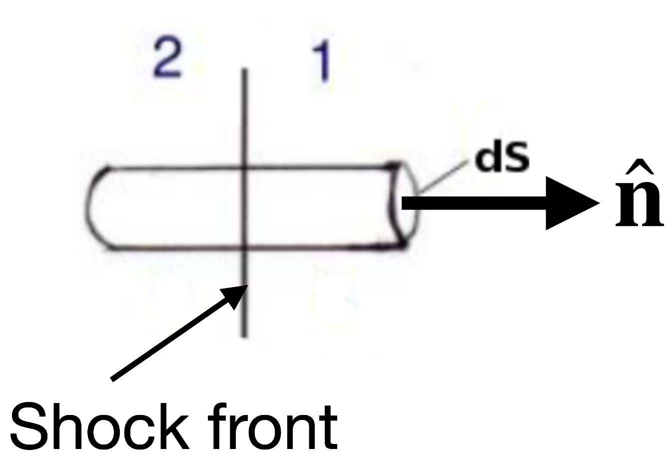
Energy jump condition for MHD

$$p_{tot} = p + \frac{1}{2} B^2 \qquad \mathscr{E}_T = \frac{1}{2}\rho u^2 + \rho e + \frac{1}{2} B^2$$

The divergence constraint $\nabla \cdot \mathbf{B} = 0$ gives:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \longrightarrow \boxed{[\mathbf{B} \cdot \hat{\mathbf{n}}] = 0}$$

Normal component of B continuous



The steady state Faraday's law $\nabla \times \mathbf{E} = 0$ gives:

$$\boxed{[\hat{\mathbf{n}} \times \mathbf{E}] = 0}$$

Tangential component of E continuous

The Rankine-Hugoniot jump conditions

Summary for MHD

$[\rho(\mathbf{u}\cdot\hat{\mathbf{n}})] = 0$	Conservation of Mass
$\left[\rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}}) + (p + \frac{1}{2}B^2)\hat{\mathbf{n}} - (\mathbf{B}\cdot\hat{\mathbf{n}})\mathbf{B}\right] = 0$	Conservation of Momentum
$\left[\left(\mathscr{E} + p_{tot}\right)(\mathbf{u}\cdot\hat{\mathbf{n}}) - \mathbf{u}\cdot\mathbf{B}(\mathbf{B}\cdot\hat{\mathbf{n}})\right] = 0$	Conservation of Energy
$[\mathbf{B}\cdot\hat{\mathbf{n}}] = 0$	Divergence Free condition
$[\hat{\mathbf{n}}\times\mathbf{E}] = 0$	Conservation of Magnetic Flux

- The jump conditions contain rich physics related to different wave modes in MHD
- Define $\theta = \hat{\mathbf{b}}\cdot\mathbf{n}$, the shock can be categorized as

  1. $\theta = 0$ Corresponds to a **parallel shock**
  2. $\theta = \pi/2$ Corresponds to a **perpendicular shock**
  3. $0 < \theta < \pi/2$ Corresponds to an **oblique shock**

- The jump conditions also allow three types of discontinuities that a **NOT** shocks

  1. Contact discontinuity
  2. Tangential discontinuity
  3. Rotational discontinuity

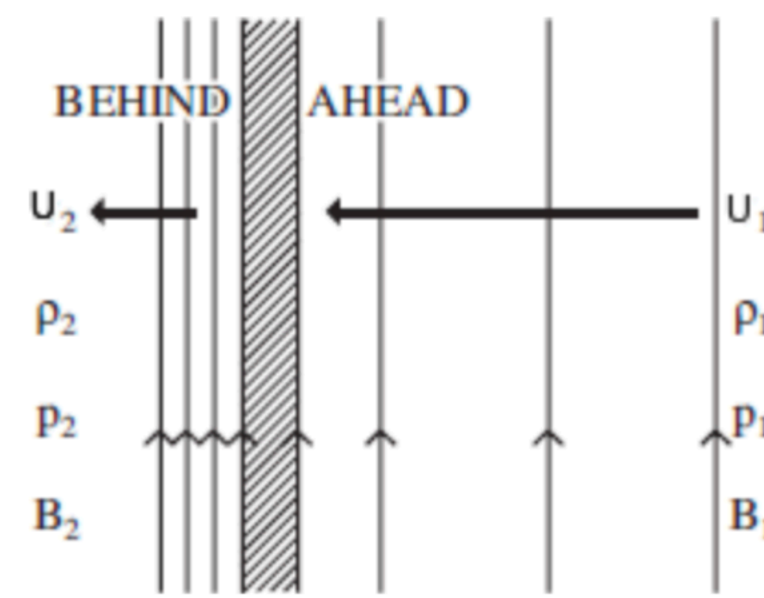
Parallel and Perpendicular Shocks

$$\theta = \hat{\mathbf{b}} \cdot \mathbf{n} \equiv 0$$

- Both U1 and B1 are parallel to n
- Often favorable to particle acceleration (kinetic physics) - why?
- Simplest case: the magnetic field is parallel to the shock velocity and constant in front of and behind the shock (B1 = B2)
  - B drops out of the Rankine-Hugoniot jump conditions
  - The solution reduced to a hydrodynamic shock
- Other possibility: a switch-on shock where |B2|>|B1|
  - The normal component is conserved
  - B2 has a tangential component
  - Some of the flow energy is converted into magnetic energy

$$\theta = \hat{\mathbf{b}} \cdot \mathbf{n} \equiv \pi/2$$

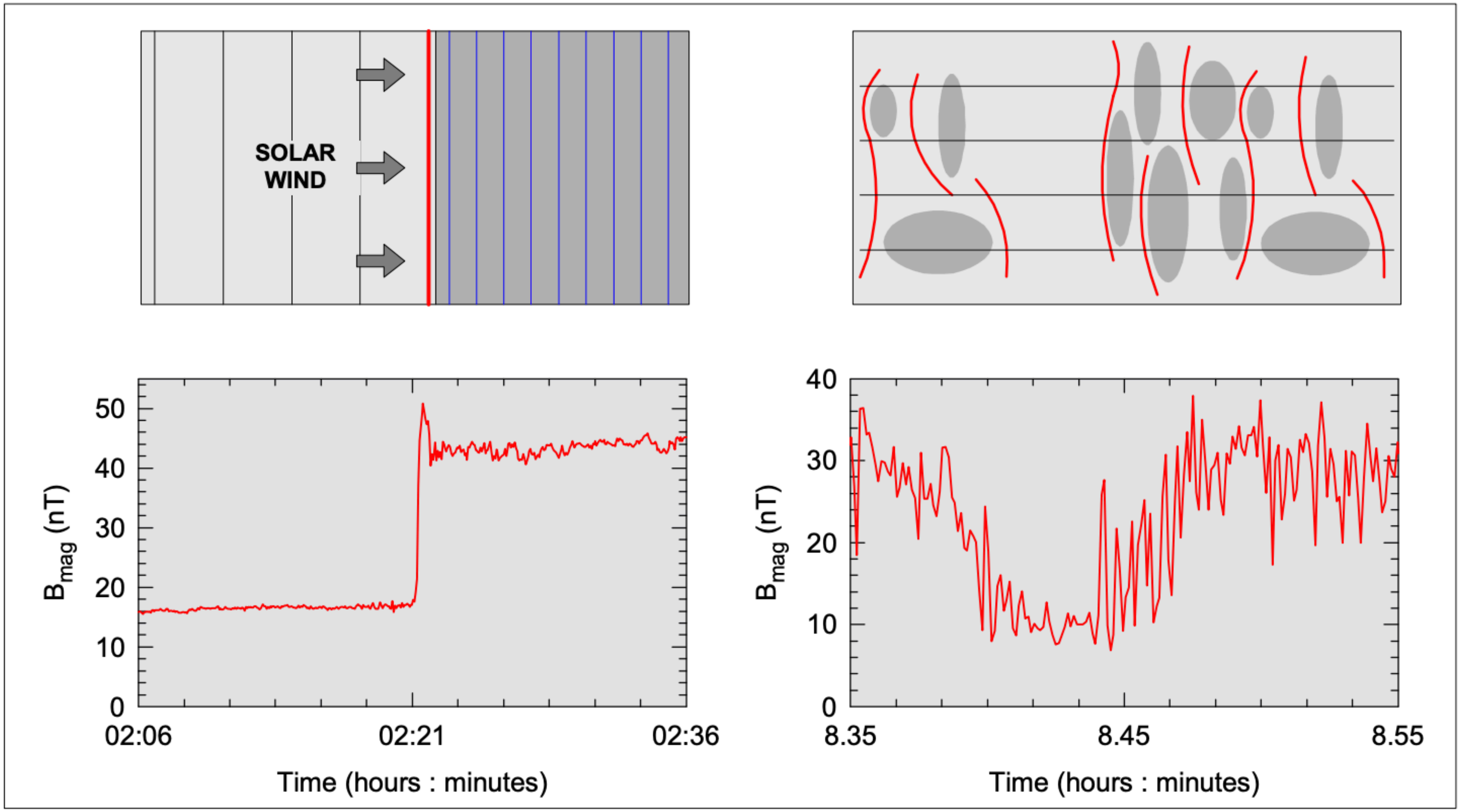
- The shock speed must exceed the fast model speed in the upstream
- The upstream and down stream magnetic fields are both tangential to the shock front
- Flow energy is converted to magnetic energy and heat



The Earth's bow shock in observations

Perpendicular shock: short, sharp jump, simple structure

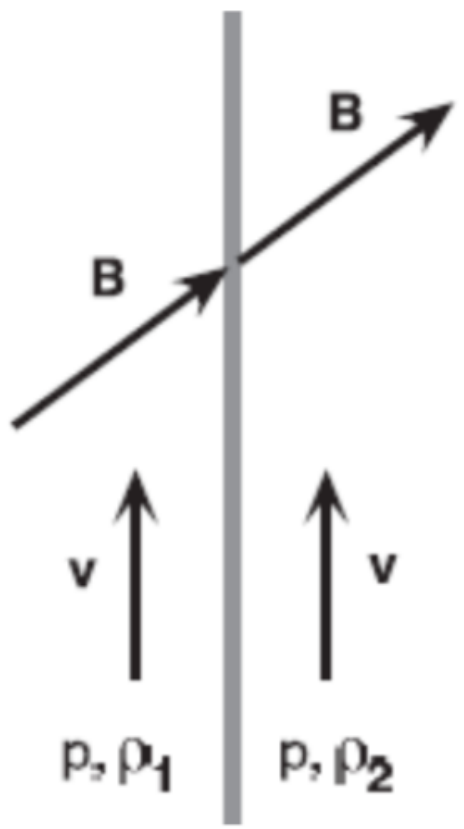
Parallel shock: the shock front forms, breaks up, then re-forms; complex, fast varying structures



Cluster observations

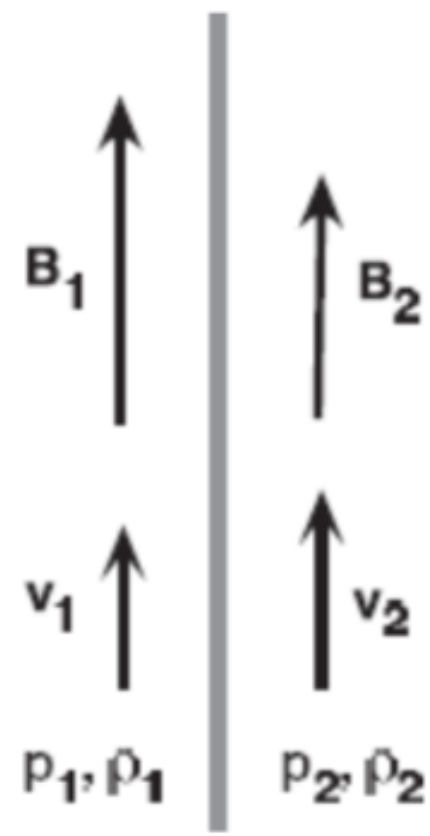
Discontinuities

Contact



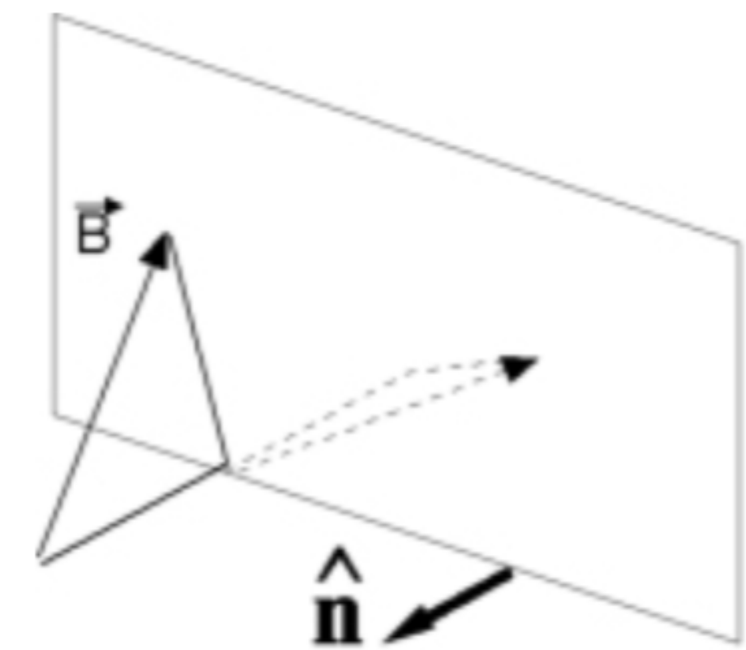
- Only density is discontinuous
- No flow across the discontinuity
- Because temperature gradient is non-zero, thermal conduction will not let a contact discontinuity last long

Tangential



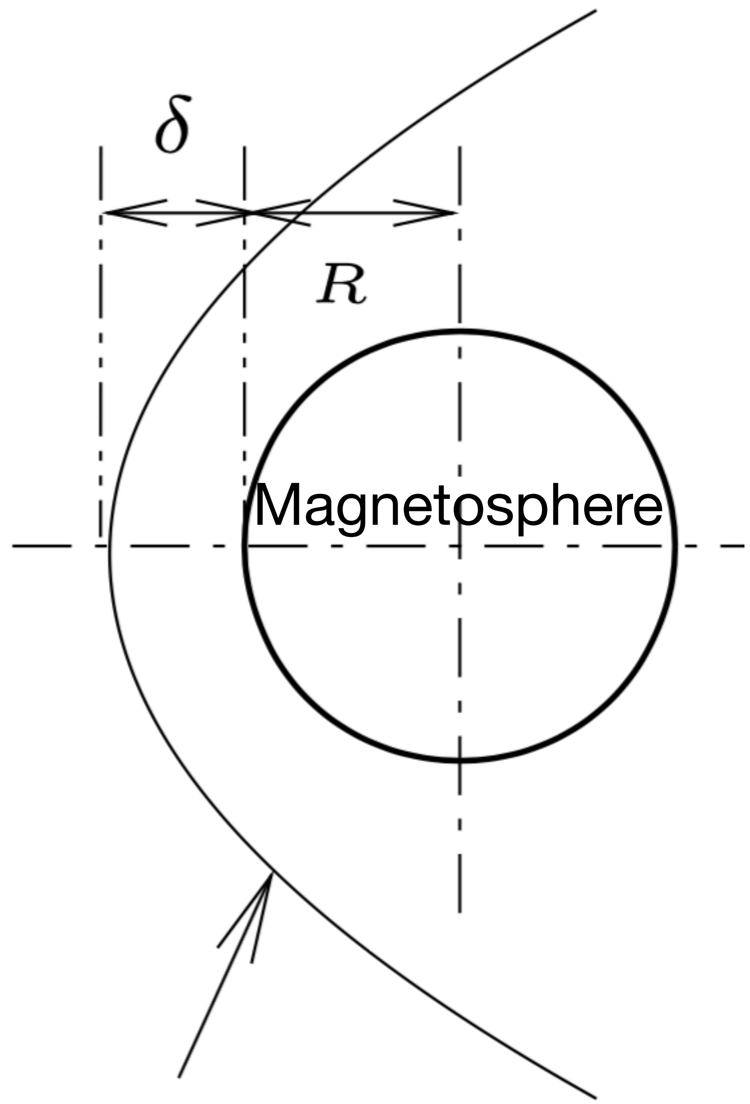
- Both B and U are tangential to the discontinuity
- No flow across the discontinuity
- Total pressure (thermal+mag) must be continuous across the discontinuity

Rotational



- The magnetic and plasma flow changes direction but not the magnitude
- There are mass flow across the discontinuity
- Incompressible - a shear mode
- Common in the SW

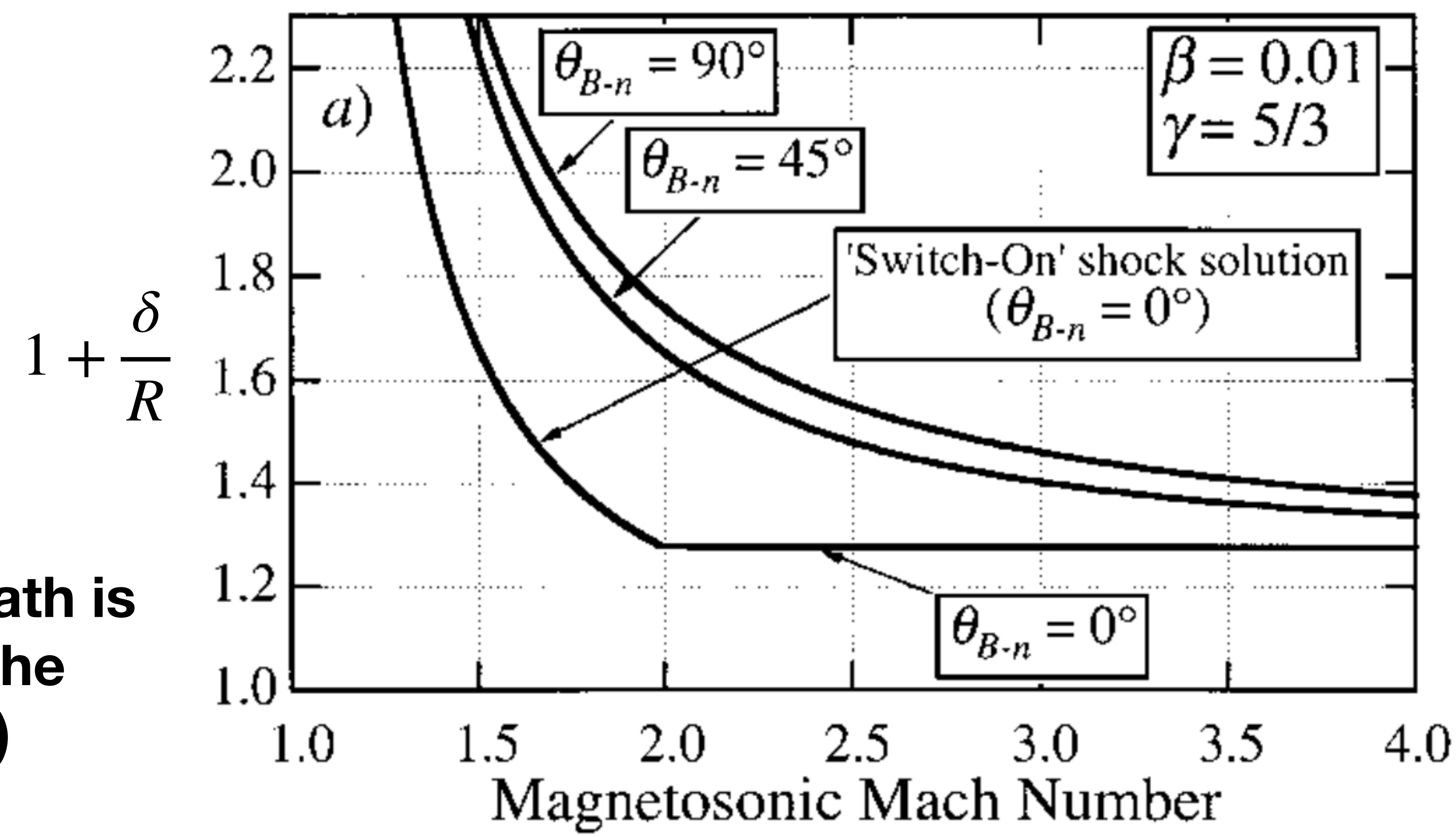
Example - apply to the magnetosheath



Question: what is delta? (the width of the magnetosheath)

Using the ***Rankine-Hugoniot*** conditions, Petrinec and Russell derived the following theoretical estimation of delta:

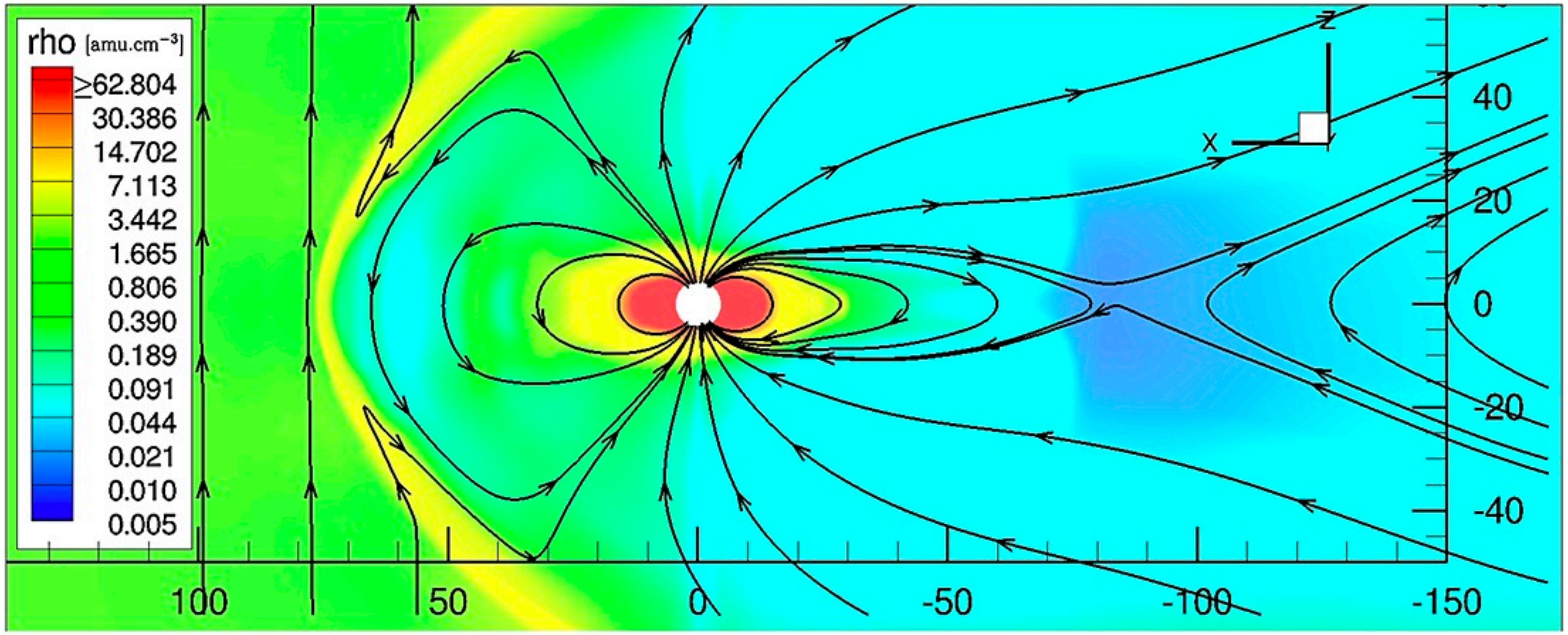
$$\frac{\delta}{R} = 1.1\frac{(\gamma-1)M^2+2}{(\gamma+1)(M^2-1)} \xrightarrow{M\to\infty} \approx \frac{1}{4}$$

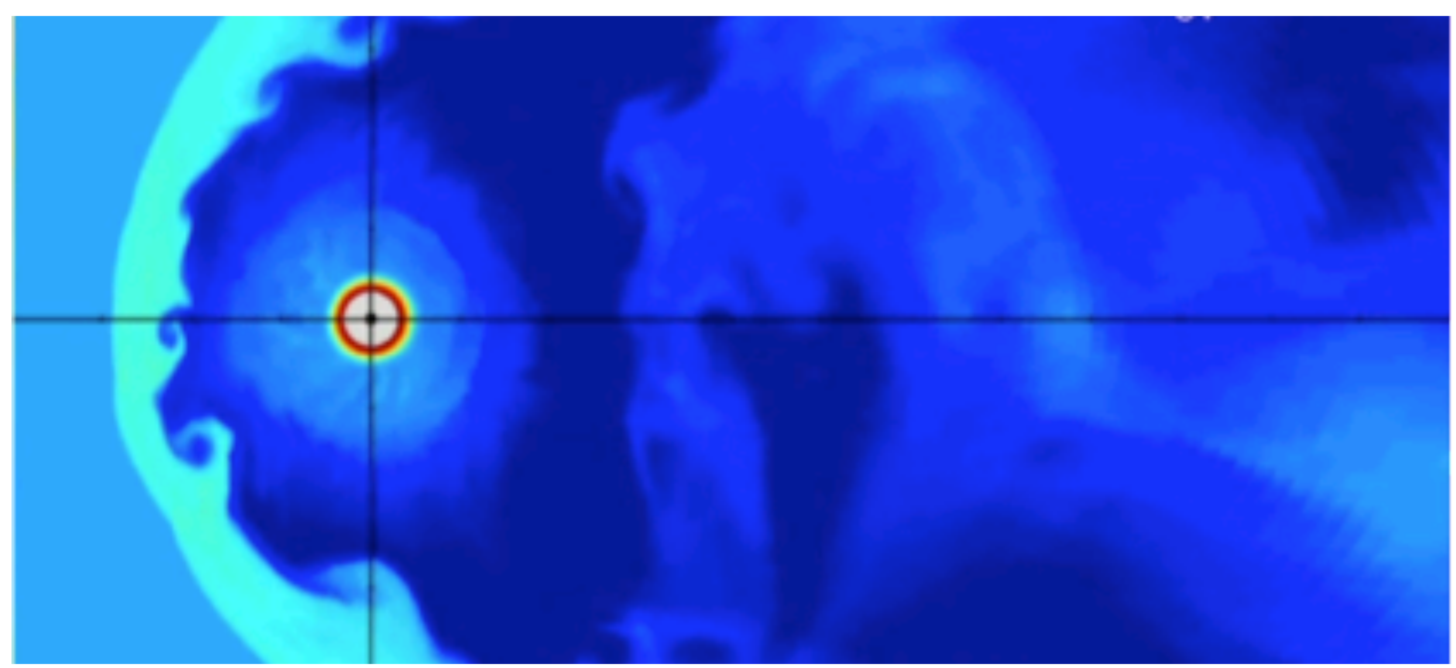


So the width of a magnetosheath is usually between 1/2-1/4 of the obstacle (magnetosphere)

Example - apply to the magnetosheath

Jupiter simulations





Which solution is more likely correct?

Limitation of the MHD analysis of shocks

- The ideal MHD approximation breaks down near the discontinuities
  - Dissipation mechanisms are not ideal MHD
  - Often need kinetic theory to describe the full behavior
- Other effects may also play important roles
  - Radiative cooling
  - Partial ionization
  - Thermal conduction
- Partical acceleration and plasma energization not considered
- Shock may give rise to various instabilities
  - Weibel instability
  - Streaming instability
  - Rayleigh-Taylor instability
  - Richmyer-Mashkov instability
  - Bell's instability

Collisionless shocks are bizarre - and the **key of numerical MHD** is basically solving for discontinuities!