MHD Waves 1

Ideal MHD Modes

Single-fluid MHD equations

Ideal MHD collisionless, isotropic, Maxwellian, equilibrium - five moment

Mass conservation $$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation $$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

Energy conservation $$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Faraday's law $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Ampere's law $$\mathbf{J} = \nabla \times \mathbf{B}$$

Ohm's law $$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

Magnetic Gauss $$\nabla \cdot \mathbf{B} = 0$$

Conservative versus Primitive

Conservative (semi) form

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Primitive form

$$\frac{\partial}{\partial t}\rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial}{\partial t} p + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0$$

Maxwell: $\dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$ $\quad \mathbf{J} = \nabla \times \mathbf{B}$ $\quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$

Primary variables: $\rho, \mathbf{u}, p, \mathbf{B}$

Conserved variables: $\rho, \rho\mathbf{u}, \mathscr{E}, \Phi$ $\qquad \mathscr{E} = \dfrac{1}{2}\rho u^2 + \dfrac{p}{\gamma - 1}$ Plasma energy

Secondary/Derived variables: $\mathbf{E}, \mathbf{J}$

Why Waves?

Waves are ubiquitous in magnetized plasmas Just as sound waves are ubiquitous in air

- Waves are the simplest way that a system responds to disturbances and applied forces
- Waves propagate information and energy through a system
- Waves are closely related to shocks, instabilities, and turbulence
- Plasmas display a rich variety of waves within and beyond MHD

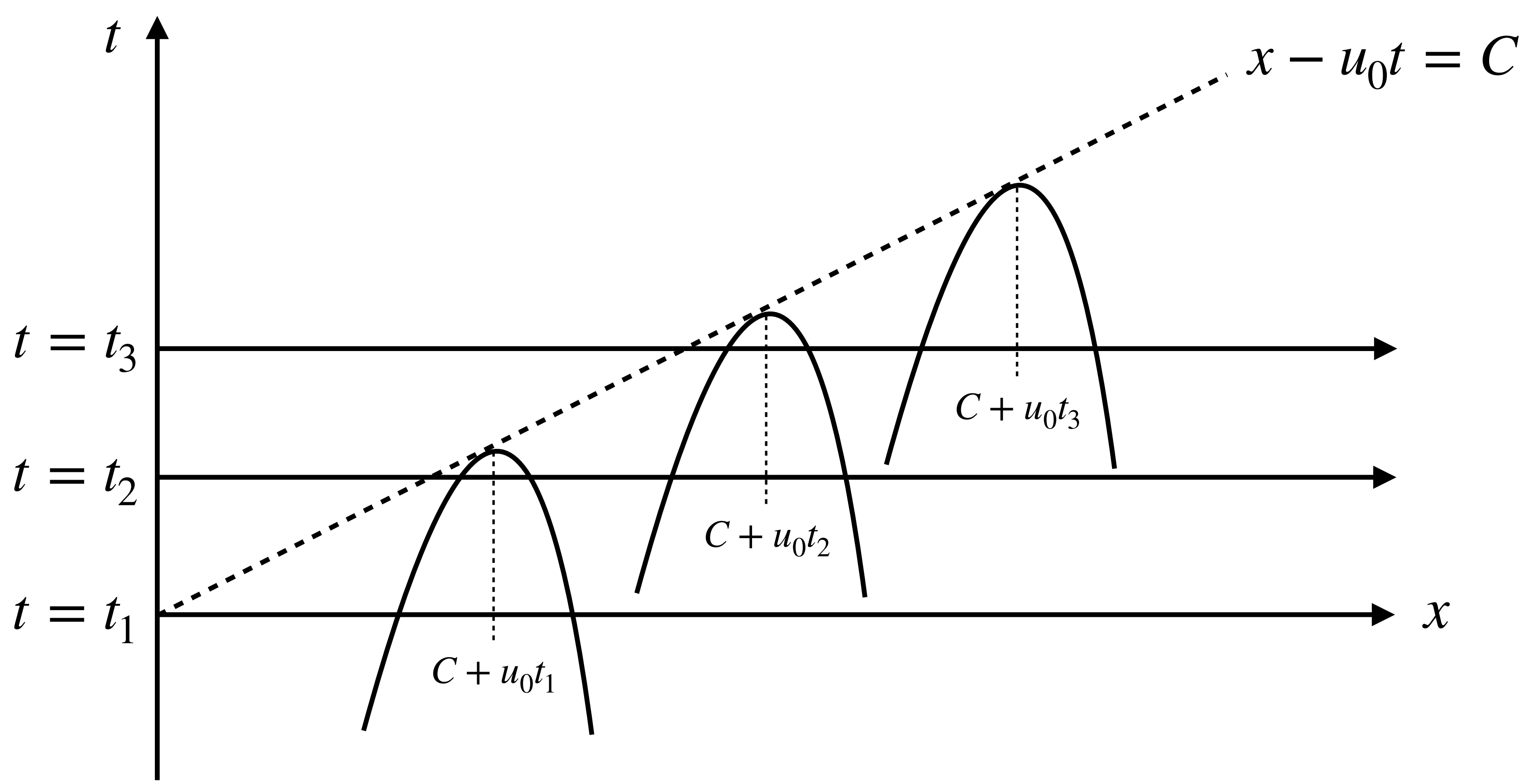
Applications of Wave in Plasma Physics

- Earth's ionosphere, magnetosphere, and solar wind environment
- Solar and stellar physics, Coronal heating
- Acceleration of solar and stellar winds
- Molecular clouds and star formation
- Interstellar medium
- Cosmic ray acceleration and transport
- Accretion disks and jets, Pulsar magnetospheres

First Order Wave Equation

$$\frac{\partial f(x,t)}{\partial t} + u_0 \frac{\partial f(x,t)}{\partial x} = 0$$

The solution goes like $f(x,t) \sim Q(x - u_0 t)$



A simple wave propagation towards +x direction

Second-Order Wave Equation

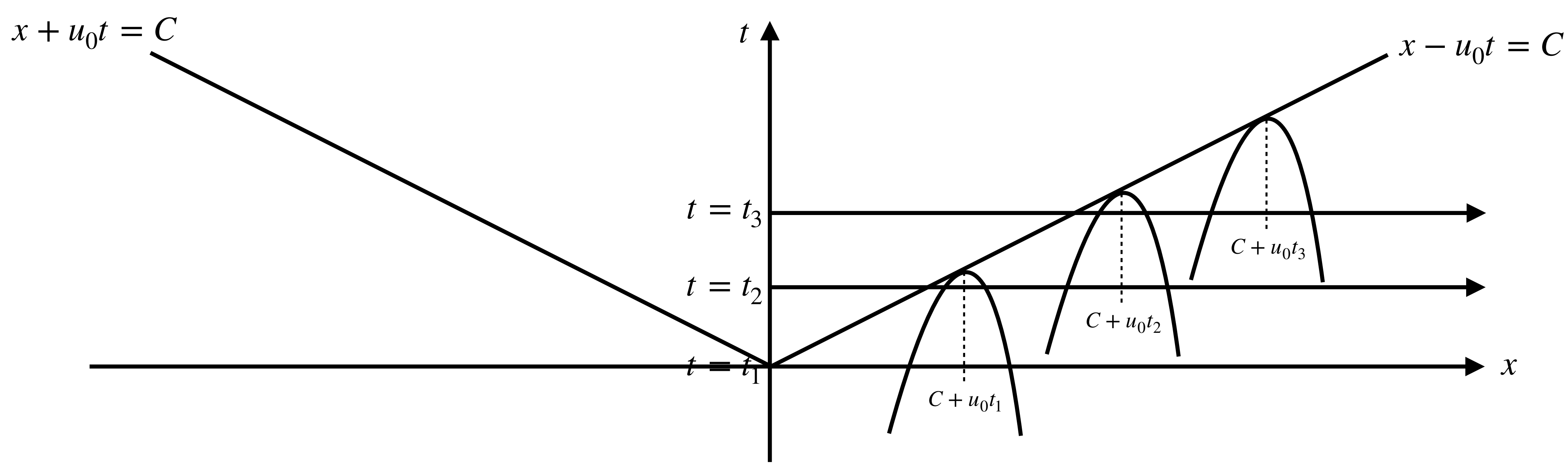
$$\frac{\partial^2 f(x,t)}{\partial t^2} = u_0^2 \frac{\partial^2 f(x,t)}{\partial x^2}$$

The solutions are waves traveling at velocities of $\pm u_0$, The wave equation is a hyperbolic partial differential equation, most of the **Conservation laws** are hyperbolic equations

Define two new variables $\xi(x,t) = x - u_0 t$ $\eta(x,t) = x + u_0 t$ Substitute: $\dfrac{\partial^2 f(\xi,\eta)}{\partial\xi\partial\eta} = 0$

Then the solution becomes $f(\xi,\eta) = R(\xi) + L(\eta)$ Or $f(x,t) = R(x - u_0 t) + L(x + u_0 t)$

- where R and L are arbitrary functions traveling at velocities ±u0 (to the right and to the left)



Electromagnetic Waves in Free Space

From Maxwell's equations to Wave Equations

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \xrightarrow{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}\left(\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \longrightarrow \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\xrightarrow{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \nabla \times \nabla \times \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \xrightarrow{\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}} \boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

$$\boxed{\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0}$$

Wave equations

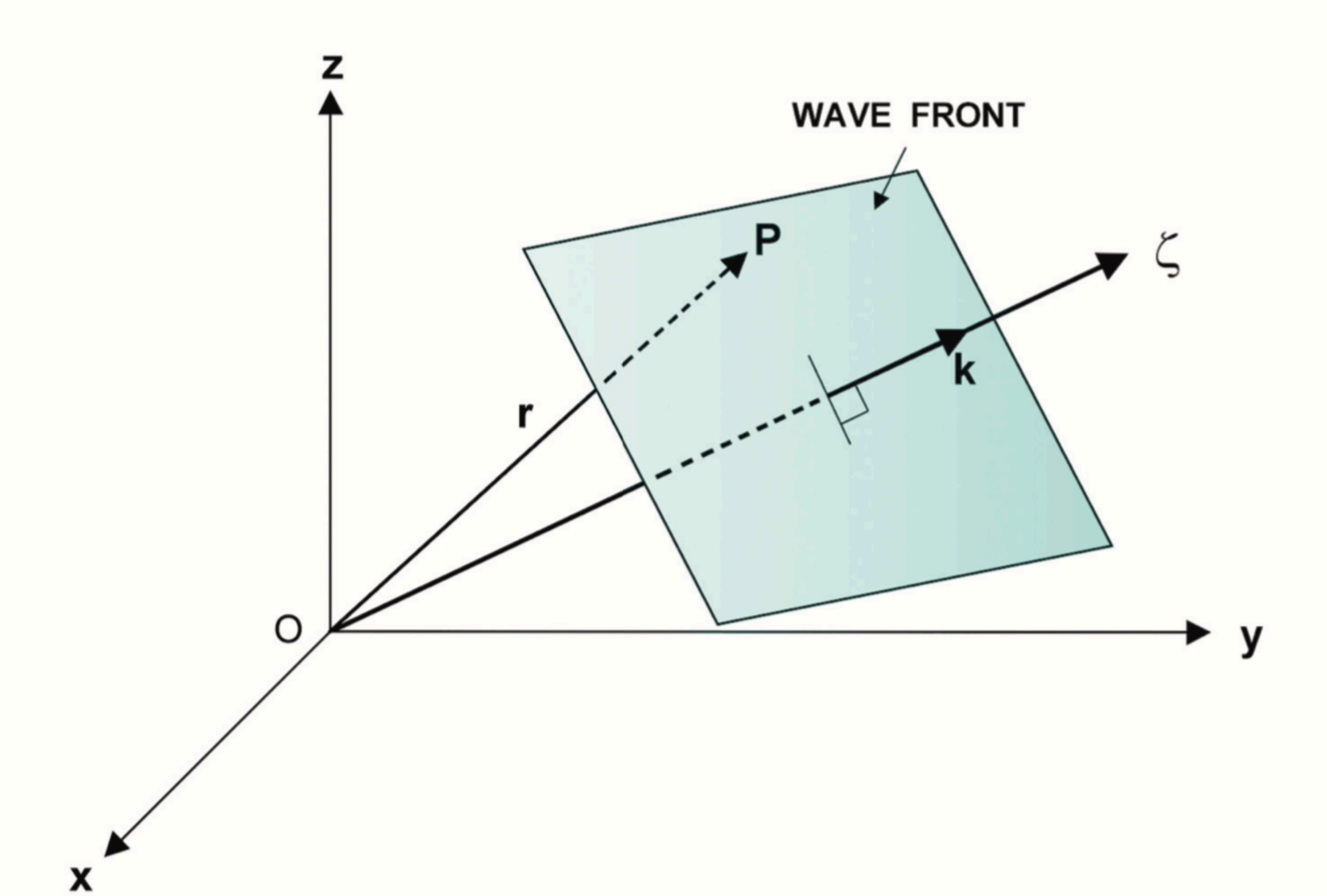
Or we can write the scalar wave equation as

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

$\psi(\mathbf{r}, t)$ ~ any component of E, B

Electromagnetic Waves in Free Space

The Plane wave solution



Wave equation: $$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

Consider transverse plane waves:

- **k** is the direction of propagation
- **E** and **B** lie on a plane *perpendicular* to **k**
- The plane is called the "wave front"

Define $\zeta = \mathbf{k} \cdot \mathbf{r}$

Wave front = constant ζ

Since the field vectors E and B are spatially constant along the wave front (normal to k), but vary only in the ζ direction (and with time), the del operator can be written as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} = \hat{\mathbf{k}}\frac{\partial}{\partial \zeta}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

Speed of light

Hence the wave equation is basically 1-D in terms of the wave front ζ

$$\frac{\partial^2\psi}{\partial\zeta^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \xrightarrow{\text{Plane wave solution}} \psi = f(\zeta - ct) + g(\zeta + ct)$$

Electromagnetic Waves in Free Space

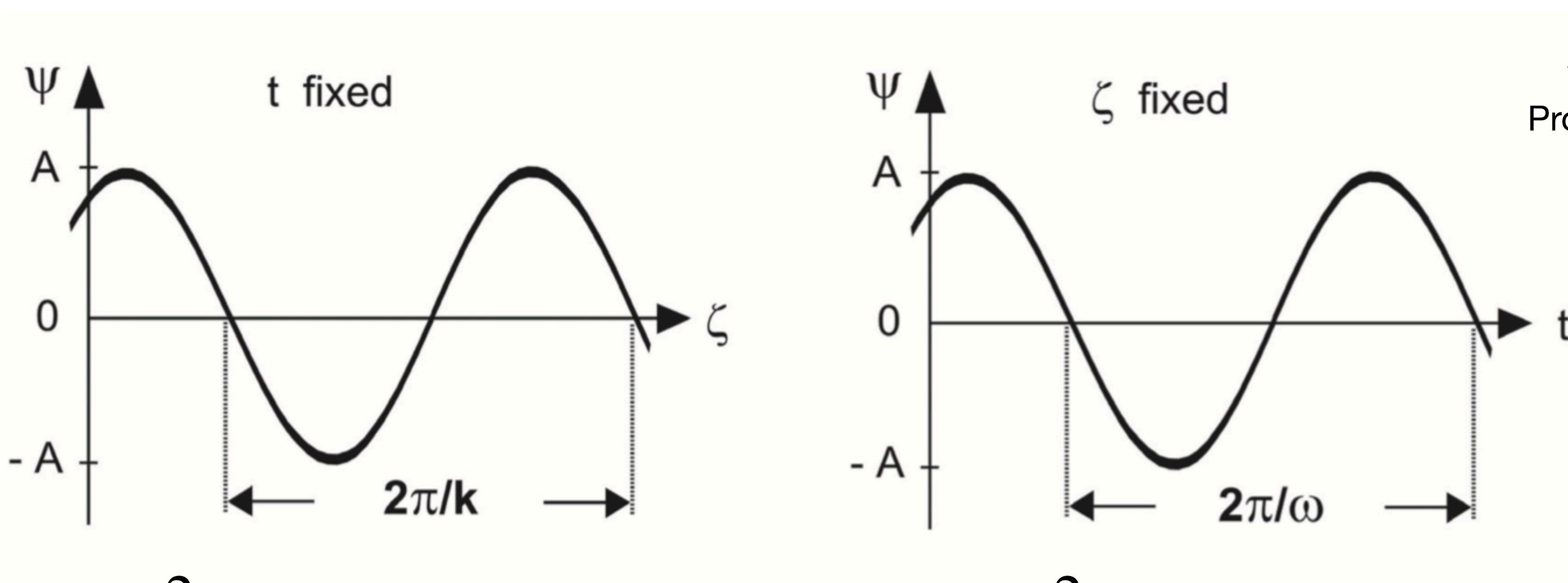
Harmonic waves

A particularly important type of plane waveform is the harmonic wave, which can be written in the form (for propagation in the positive direction)

$$\psi = A\cos[k(\zeta - ct)] = A\cos[k\zeta - kct] = A\cos[k\zeta - \omega t]$$

Angular Frequency $\omega = kc$

Wave Number or Propagation Constant



$\lambda = \dfrac{2\pi}{k}$: Wave Length

$T = \dfrac{2\pi}{\omega}$: Wave Period

Since $\zeta = \mathbf{k}\cdot\mathbf{r}$

A plane wave travels at arbitrary direction is then

$$\psi = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

In view of the argument of the cosine function, the planes of constant phase are defined by the condition

$$\mathbf{k}\cdot\mathbf{r} - \omega t = k\zeta - \omega t = \text{constant}$$

$$\zeta = \frac{\omega}{k}t + C$$

Phase velocity - velocity of the planes ζ : $v_{ph} = \dfrac{\omega}{k}$

Electromagnetic Waves in Free Space

Harmonic waves

The phase velocity is positive for a wave moving in the positive ζ direction, that is, ζ increases as t increases in order to keep (kζ – ωt) constant. If we had taken

$$\psi = A\cos(\mathbf{k}\cdot\mathbf{r} + \omega t)$$ wave moving in the negative ζ direction

A more useful form of the cosine function is to write in a complex form:

$$\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Note: the field quantities are obtained by taking the **real part** of the complex expressions.

Now the spatial derivative and time derivative can be written in a very convenient form:

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}\left[Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right] = -i\omega \cdot Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = -i\omega\psi$$

$$\nabla\psi = \nabla\left[Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right] = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \cdot \nabla(i\mathbf{k}\cdot\mathbf{r}) = i\mathbf{k}Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = i\mathbf{k}\psi$$

Or simply introduce the operators:

$$\frac{\partial}{\partial t} = -i\omega \qquad \nabla = i\mathbf{k}$$

Which means under harmonic wave assumption, the differential equations become algebra equations

Electromagnetic Waves in Free Space

Transverse Waves

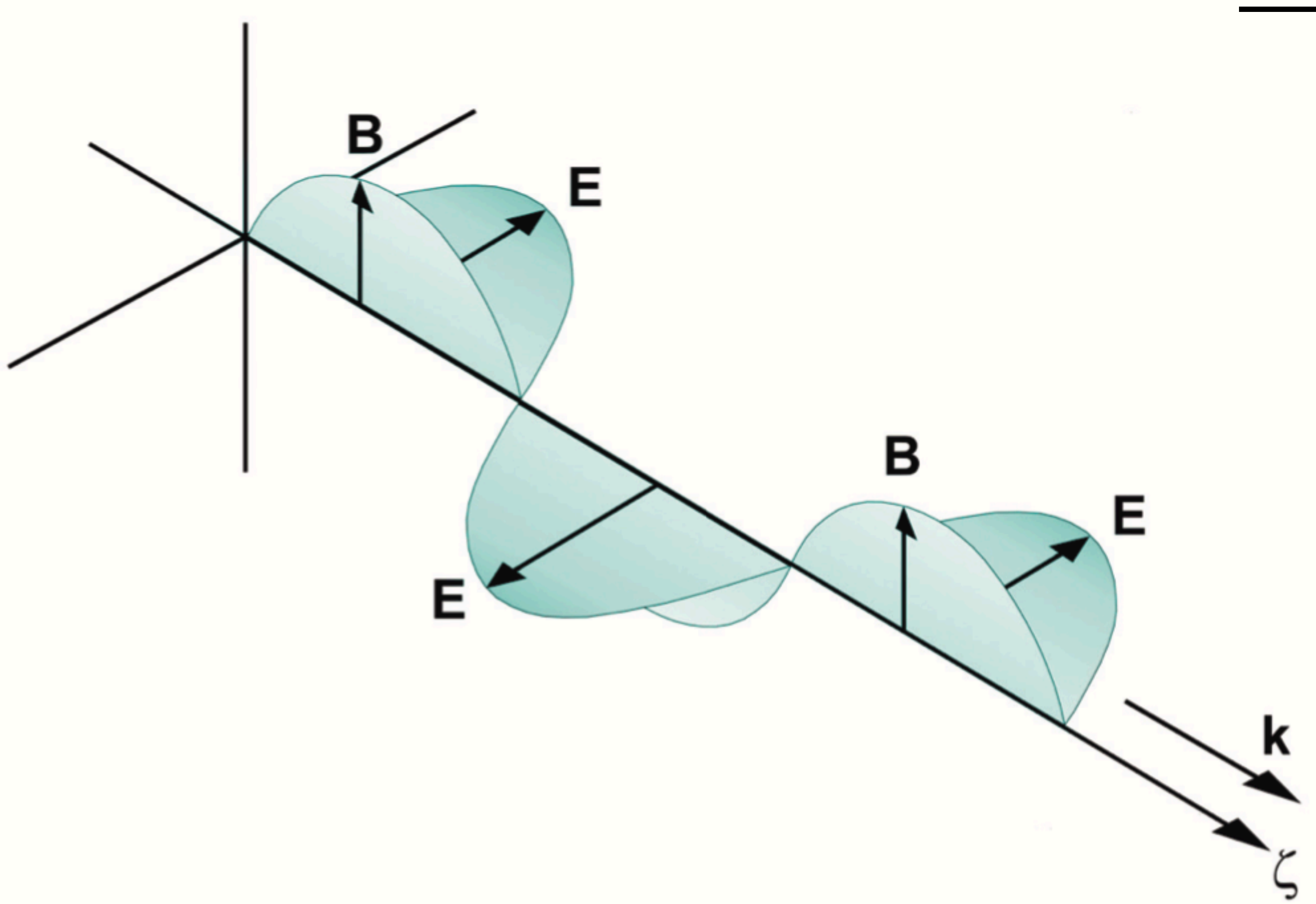
$$\nabla \cdot \mathbf{E} = 0 \qquad \mathbf{k} \cdot \mathbf{E} = 0 \longrightarrow \mathbf{E} \perp \mathbf{k}$$

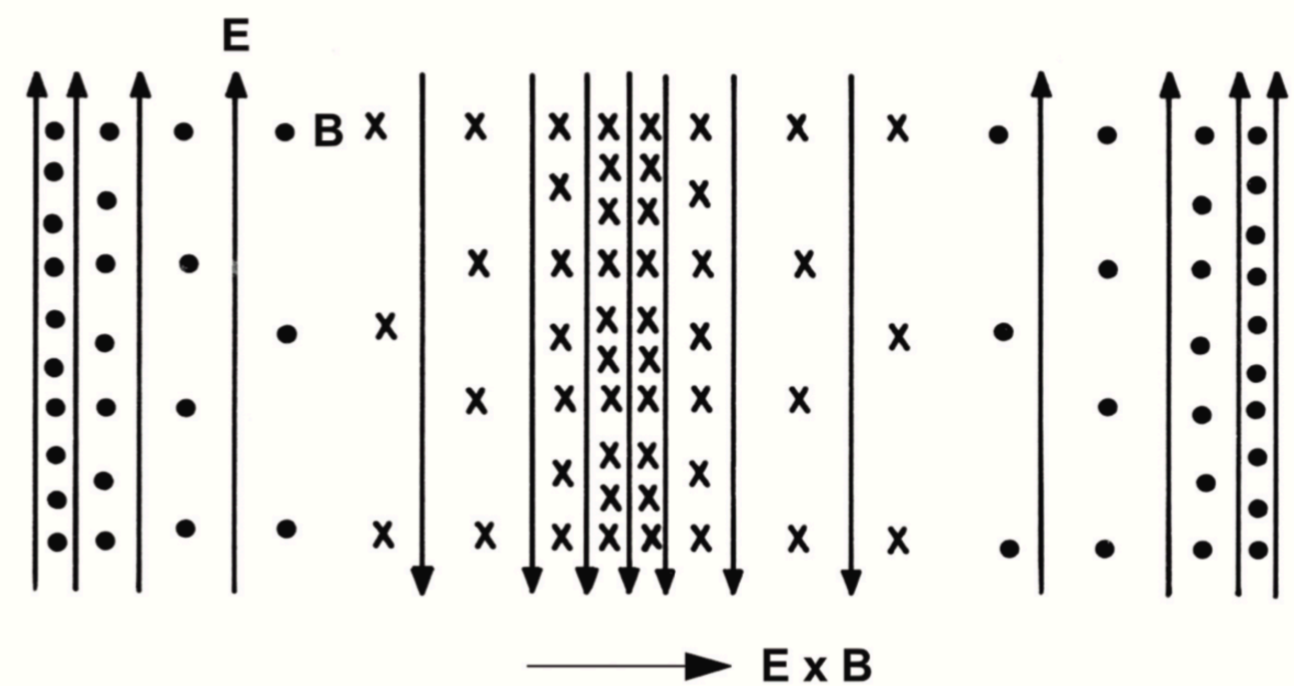
$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{k} \cdot \mathbf{B} = 0 \longrightarrow \mathbf{B} \perp \mathbf{k}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \longrightarrow \mathbf{E} \perp \mathbf{B} \qquad \frac{E}{B} = \frac{\omega}{k} = c$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2}\mathbf{E} \longrightarrow \mathbf{k} \times \mathbf{k} \times \mathbf{B} = -\frac{\omega^2}{c^2}\mathbf{B} \longrightarrow -k^2\mathbf{B} = -\frac{\omega^2}{c^2}\mathbf{B}$$

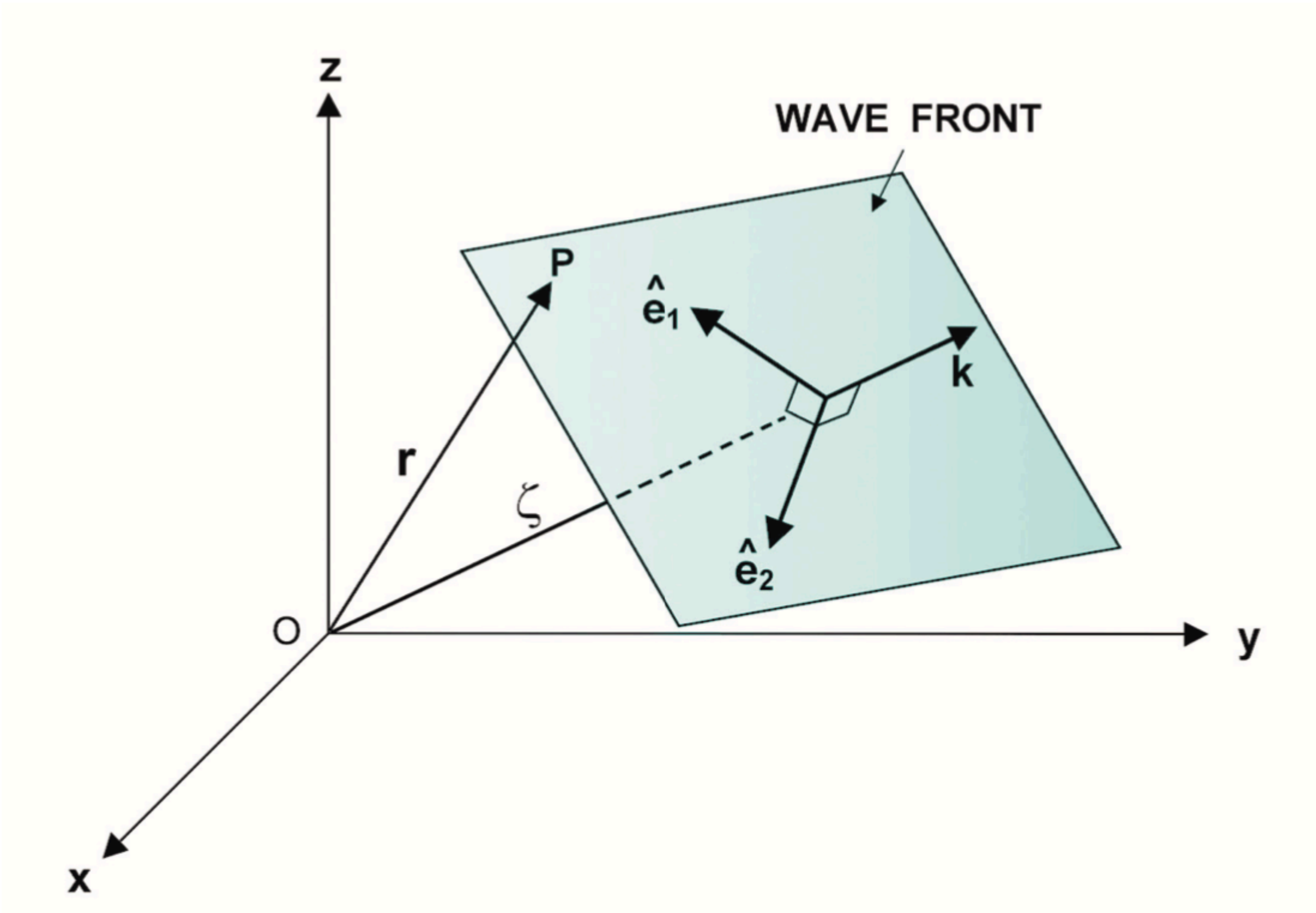
$$\longrightarrow \omega^2 - k^2c^2 = 0$$ **Dispersion Relation**





Electromagnetic Waves in Free Space

Polarization



For plane waves, the electric field is written as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \mathbf{E}_0 e^{i(k\zeta-\omega t)}$$

If $\mathbf{E}_0$ is a constant vector, then the direction of the electric field is not changed - ***linearly polarized***

For a general plane wave solution:

$$\mathbf{E}(\mathbf{r}, t) = (\hat{\mathbf{e}}_1 E_1 + \hat{\mathbf{e}}_2 E_2) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$= (\hat{\mathbf{e}}_1 E_1^0 e^{i\alpha_1} + \hat{\mathbf{e}}_2 E_2^0 e^{i\alpha_2}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Case $\alpha_1 = \alpha_2$:

$$\mathbf{E}_0 = \underbrace{(\hat{\mathbf{e}}_1 E_1^0 + \hat{\mathbf{e}}_2 E_2^0)}_{\text{Fixed direction}} e^{i\alpha_1} \cdot e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

Linearly polarized

Case $\alpha_1/\alpha_2 = \pm\, i$:

$E_1^0 = E_2^0$ $\quad \mathbf{E}_0 = E_1^0(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2) e^{i\alpha_1} \cdot e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ **Circularly polarized**

$E_1^0 \neq E_2^0$ $\quad \mathbf{E}_0 = (\hat{\mathbf{e}}_1 E_1^0 \pm i\hat{\mathbf{e}}_2 E_2^0) e^{i\alpha_1} \cdot e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ **Elliptically polarized**

Electromagnetic Waves in Free Space

Wave Packets and Group Velocity

A single harmonic wave: $\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

A superposition of plane harmonic waves $\psi(\zeta, t) = \int_{-\infty}^{+\infty} A(k)e^{i(k\zeta-\omega t)}dk$

Wave Packet: a small spread in wave numbers about a central wave number k_0.

Consider now that we have a wave packet, in which the range of values of k is small and is centered about some specific wave number k_0:

$$k_0 - \delta k \leq k \leq k_0 + \delta k$$

Try Taylor expansion near k0:

$$\omega(k) = \omega_0 + (k - k_0)\left(\frac{\partial\omega}{\partial k}\right)_{k_0}$$

The phase factor \zeta is now written as

$$[k\zeta - \omega(k)t] = k_0\zeta - \omega_0 t + \omega_0 + (k - k_0)\left[\zeta - \left(\frac{\partial\omega}{\partial k}\right)_{k_0} t\right]$$

Thus the group velocity $v_g = \left(\frac{\partial\omega}{\partial k}\right)_{k_0} \longrightarrow \nabla_k\omega \text{ or } \frac{\partial\omega}{\partial\mathbf{k}}$

Sound Waves

The most fundamental type of wave motion that propagates in a compressible, nonconducting fluid is that of longitudinal sound waves.

Based on the adiabatic energy equation:

Sound Speed $V_s = \sqrt{\dfrac{\gamma p}{\rho}}$

$$p\rho^{-\gamma} = const \xrightarrow{\nabla} \nabla(p\rho^{-\gamma}) = 0 \longrightarrow \nabla p = \frac{\gamma p}{\rho}\nabla\rho = V_s^2 \nabla\rho$$

Adiabatic sound speed

Now let's take a look at the origin of sound waves from the MHD equations, first ignore the JxB force

Primitive form

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0$$

$$\frac{\partial}{\partial t}p + \mathbf{u}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{u} = 0$$

$$\rho = \rho_0 + \rho_1,\ |\rho_1| \ll \rho_0$$
$$p = p_0 + p_1,\ p_1 \ll p_0$$
$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1,\ |\mathbf{u}_1|\ \text{small}$$
$$\longrightarrow$$

Linearlized form

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0\cdot\nabla\right)\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0\cdot\nabla\right)\mathbf{u}_1 + \frac{1}{\rho_0}\nabla p_1 = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0\cdot\nabla\right)p_1 + \gamma p_0\nabla\cdot\mathbf{u}_1 = 0$$

$\partial_t\rho_0 = 0$ $\quad$ $\mathbf{u}_1\cdot\nabla\rho_1 \sim 0$ $\quad$ $\nabla\rho_0 = 0$ $\quad$ $\rho_1\nabla\cdot\mathbf{u}_1 \sim 0$ $\quad$ $\nabla\cdot\mathbf{u}_0 = 0$

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \rho_1) + (\mathbf{u}_0 + \cancel{\mathbf{u}_1})\cdot\nabla(\cancel{\rho_0} + \rho_1) + (\rho_0 + \cancel{\rho_1})\nabla\cdot(\cancel{\mathbf{u}_0} + \mathbf{u}_1) = 0$$

Sound Waves

$$\left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)\mathbf{u}_1+\frac{1}{\rho_0}\nabla p_1=0 \xrightarrow{\left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)} \left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)^2\mathbf{u}_1+\frac{1}{\rho_0}\left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)\nabla p_1=0$$

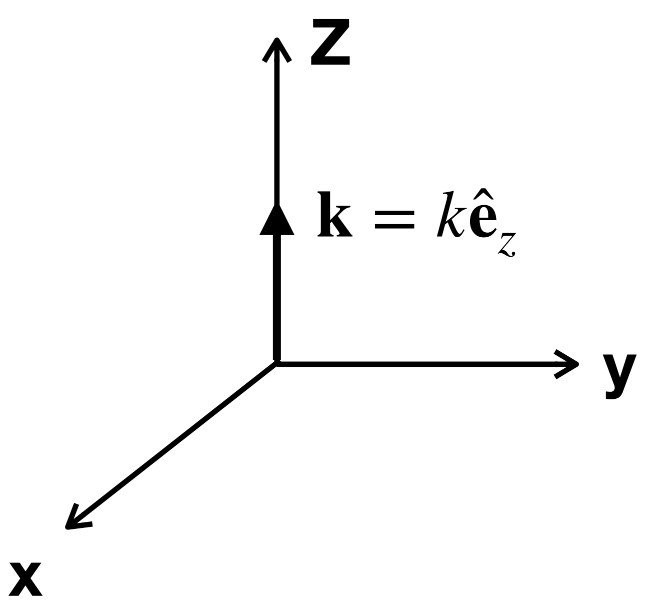
$$\Big\downarrow \left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)p_1=-\gamma p_0\nabla\cdot\mathbf{u}_1$$

Eigenvalue equation

$$\boxed{\left[\left(\omega-\mathbf{k}\cdot\mathbf{u_0}\right)^2\bar{\mathbf{I}}-V_s^2\mathbf{k}\mathbf{k}\right]\cdot\mathbf{u}_1=0} \xleftarrow{\mathbf{u}_1\sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}} \left(\frac{\partial}{\partial t}+\mathbf{u}_0\cdot\nabla\right)^2\mathbf{u}_1-V_s^2\nabla\nabla\cdot\mathbf{u}_1=0$$

$\omega'=\left(\omega-\mathbf{k}\cdot\mathbf{u_0}\right)$ Doppler shifted frequency

Consider $\mathbf{u_0}=0$ Static media



Restrict wave propagation in z-direction only

$$\left[\left(\omega-\mathbf{k}\cdot\mathbf{u_0}\right)^2\bar{\mathbf{I}}-V_s^2\mathbf{k}\mathbf{k}\right]\cdot\mathbf{u}_1=0$$

$$\longrightarrow \begin{bmatrix}\omega^2 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega^2-V_s^2k^2\end{bmatrix}\cdot\begin{bmatrix}u_{1x}\\ u_{1y}\\ u_{1z}\end{bmatrix}=0$$

Eigenvalue problem

Sound Waves

Solution 1: compressional, longitudinal sound waves

$$\omega^2 u_{1x} = 0$$
$$\omega^2 u_{1y} = 0$$
$$(\omega^2 - V_s^2 k^2) u_{1z} = 0$$

$$\longrightarrow$$

$$u_{1x} = 0$$
$$u_{1y} = 0$$
$$\omega = \pm kV_s,\ u_{1z} \text{ arbitrary}$$

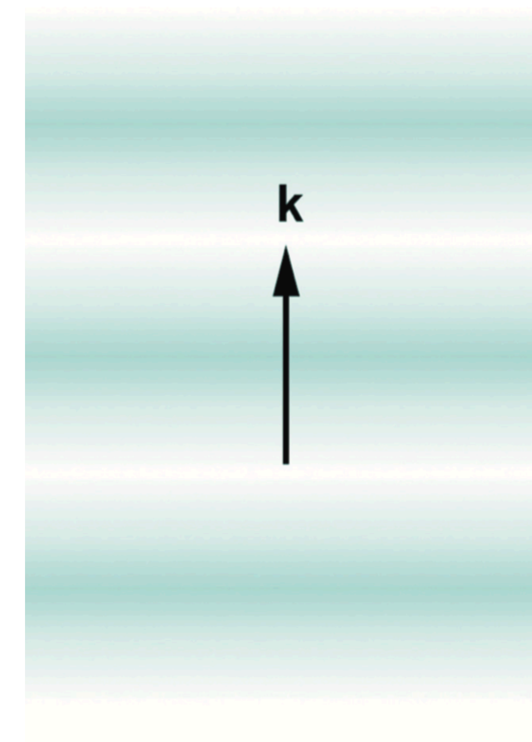
$$\longrightarrow$$

Sound wave

Longitudinal

$$\mathbf{u_1} \| \mathbf{k}$$
$$\nabla \cdot \mathbf{u_1} \neq 0$$

Compressible



Solution 2: trivial, entropy perturbation

$$\omega^2 u_{1x} = 0$$
$$\omega^2 u_{1y} = 0$$
$$(\omega^2 - V_s^2 k^2) u_{1z} = 0$$

$$\longrightarrow$$

$$u_{1x}, u_{1y} \text{ arbitrary}$$
$$u_{1z} = 0$$
$$\omega^2 = 0$$

$$\longrightarrow$$

$$\nabla \cdot \mathbf{u_1} = 0$$

Entropy wave

it represents a perturbation of the density and, hence, of the entropy function $S \equiv p\rho^{-\gamma}$, since the pressure is not perturbed.

If u0 is non-zero, then the entropy mode is written as

$$(\omega - ku_0)^2 = 0$$

Linearized MHD equations

Primitive form

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0 \longrightarrow \boxed{\frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 = 0} \quad (1)$$

$$\rho = \rho_0 + \rho_1,\ \ |\rho_1| \ll \rho_0$$

$$\mathbf{u} = \mathbf{u}_1,\ \ |\mathbf{u}_1|\ \text{small}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0 \xrightarrow[\text{Adiabatic}]{\nabla p = V_s^2\nabla\rho} \rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + V_s^2\nabla\rho - \mathbf{J}\times\mathbf{B} = 0$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,\ \ |\mathbf{B}_1|\ \text{small}$$

$$\rho = \rho_0 + \rho_1,\ \ |\rho_1| \ll \rho_0$$

$$\mathbf{u} = \mathbf{u}_1,\ \ |\mathbf{u}_1|\ \text{small}$$

$$\boxed{\rho_0\frac{\partial \mathbf{u}_1}{\partial t} + V_s^2\nabla\rho_1 - \mathbf{B}_0\times\nabla\times\mathbf{B}_1 = 0} \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) \longrightarrow \boxed{\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{u}_1\times\mathbf{B}_0)} \quad (3)$$

$$\frac{\partial}{\partial t}\ (2) \longrightarrow \rho_0\frac{\partial^2 \mathbf{u}_1}{\partial t^2} + V_s^2\nabla\frac{\partial\rho_1}{\partial t} + \mathbf{B}_0\times\nabla\times\frac{\partial \mathbf{B}_1}{\partial t} = 0 \longrightarrow \text{Use (1) (3)}$$

$$\longrightarrow \frac{\partial^2 \mathbf{u}_1}{\partial t^2} - V_s^2\nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A\times\{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\} = 0$$

Linearized Momentum Equation

$$\mathbf{V}_A = \frac{\mathbf{B}_0}{\sqrt{\rho_0}}$$

Vector Alfvén velocity

Linearized MHD equations

Magnetosonic Mode $\mathbf{k} \cdot \mathbf{V}_A = 0$

Again, consider plane wave solution: $\mathbf{u}_1 \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \longrightarrow \frac{\partial}{\partial t} = -i\omega \quad \nabla = i\mathbf{k}$

the linearized momentum equation $$\frac{\partial^2 \mathbf{u}_1}{\partial t^2} - V_s^2 \nabla(\nabla \cdot \mathbf{u}_1) + \mathbf{V}_A \times \{\nabla \times [\nabla \times (\mathbf{u}_1 \times \mathbf{V}_A)]\} = 0$$

Becomes $$-\omega^2 \mathbf{u}_1 - V_s^2 (\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k} + \mathbf{V}_A \times \{\mathbf{k} \times [\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{V}_A)]\} = 0$$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

bac-cab law

$$-\omega^2 \mathbf{u}_1 + (V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k} + (\mathbf{k} \cdot \mathbf{V})_A[(\mathbf{k} \cdot \mathbf{V}_A)\mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_1)\mathbf{k} - (\mathbf{k} \cdot \mathbf{u}_1)\mathbf{V}_A] = 0$$

Propagation perpendicular to B0: $\xrightarrow{\mathbf{k} \cdot \mathbf{V}_A = 0} \quad -\omega^2 \mathbf{u}_1 + (V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k} = 0$

$$\longrightarrow \mathbf{u}_1 = \frac{1}{\omega^2}(V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k} \longrightarrow \mathbf{u}_1 \parallel \mathbf{k}$$

longitudinal wave

$$\xrightarrow{\mathbf{u}_1 \cdot \mathbf{k} = u_1 k} 1 = \frac{k^2}{\omega^2}(V_s^2 + V_A^2) \longrightarrow \frac{\omega}{k} = \sqrt{V_s^2 + V_A^2}$$

Magnetosonic wave

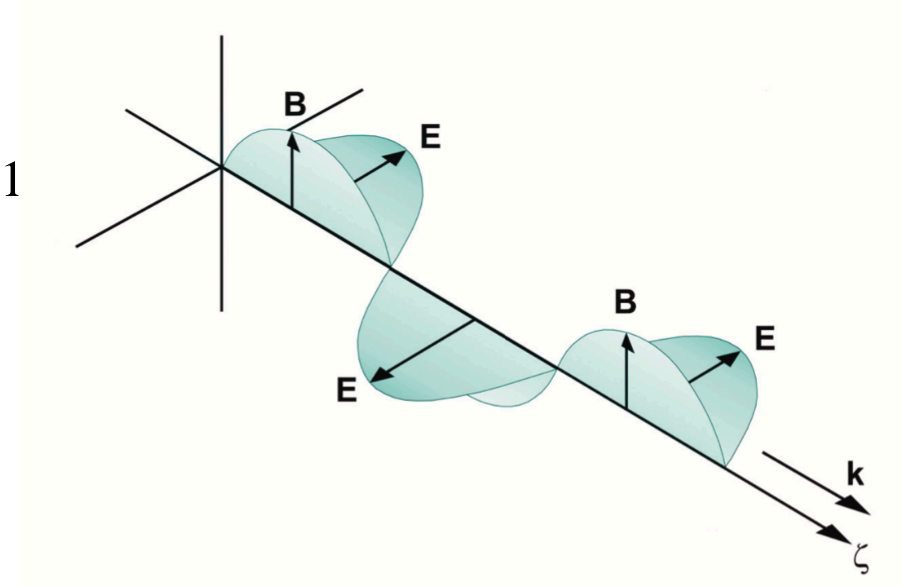
Linearized MHD equations

Magnetosonic Mode $\mathbf{k}\cdot\mathbf{V}_A = 0$

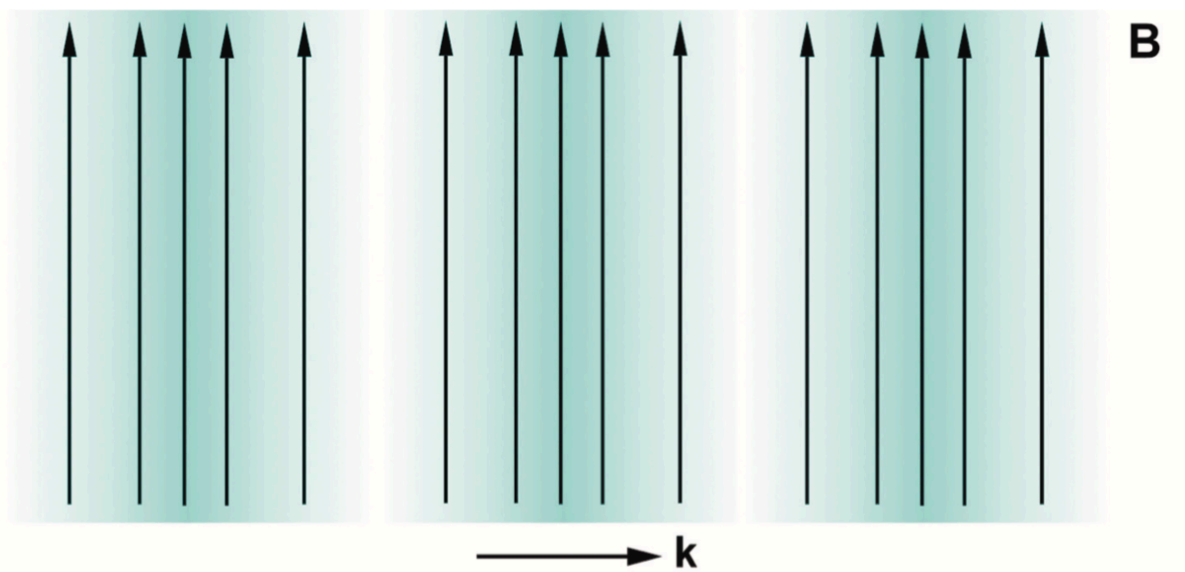
Use the linearized form of the Faraday's law

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) \longrightarrow -i\omega \mathbf{B}_1 = i\mathbf{k} \times (\mathbf{u}_1 \times \mathbf{B}_0) \xrightarrow[\mathbf{k}\cdot\mathbf{B}_0 = 0]{\text{bac-cab law}} \boxed{\mathbf{B}_1 = \frac{u_1}{(\omega/k)}\mathbf{B}_0}$$

Use the linearized form of the Ohm's law $\boxed{\mathbf{E}_1 = -\,\mathbf{u}_1 \times \mathbf{B}_0}$



- This wave is similar to an electromagnetic wave $\mathbf{E}_1 \perp \mathbf{k},\quad \mathbf{E}_1 \perp \mathbf{B}_1$
- This wave is a longitudinal wave since $\mathbf{u}_1 \parallel \mathbf{k}$
- It's called the magnetosonic wave, and non-dispersive, aka compressional Alfvén wave or fast Alfvén wave



the magnetosonic wave produces compressions and rarefactions in the magnetic field lines without changing their direction.

Linearized MHD equations

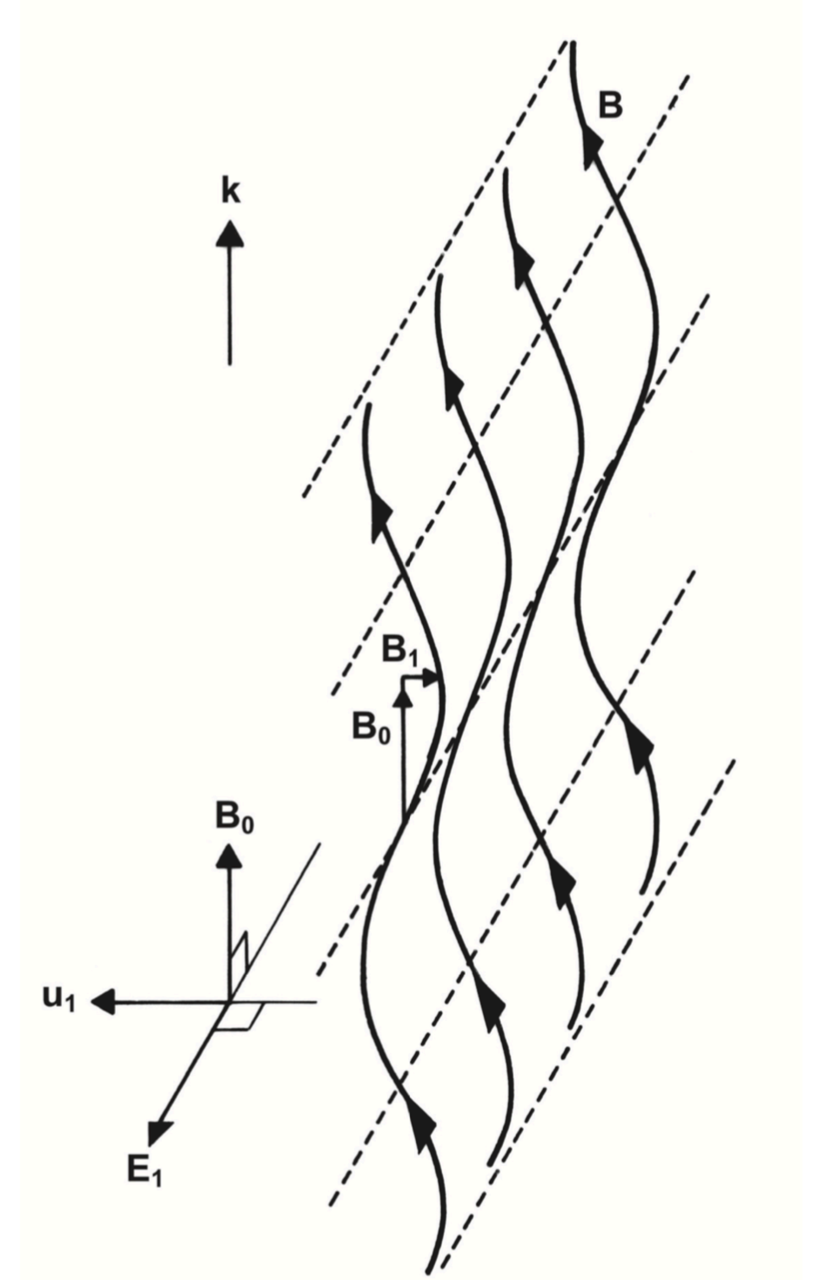
Shear and Slow Mode $\mathbf{k} \parallel \mathbf{V}_A$

$$-\omega^2\mathbf{u}_1 + (V_s^2 + V_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} + (\mathbf{k}\cdot\mathbf{V})_A[(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{u}_1 - (\mathbf{V}_A\cdot\mathbf{u}_1)\mathbf{k} - (\mathbf{k}\cdot\mathbf{u}_1)\mathbf{V}_A] = 0$$

Propagation perpendicular to B0: $\xrightarrow[\mathbf{k}\cdot\mathbf{V}_A = kV_A]{\mathbf{k} \parallel \mathbf{V}_A}$ $(k^2V_A^2 - \omega^2)\mathbf{u}_1 + (V_s^2/V_A^2 - 1)k^2(\mathbf{u}_1\cdot\mathbf{V}_A)\mathbf{V}_A = 0$

Type 1: u1 perpendicular to B0 and k $\xrightarrow{\mathbf{u}_1\cdot\mathbf{V}_A = 0}$ $\dfrac{\omega}{k} = V_A$

shear Alfvén wave



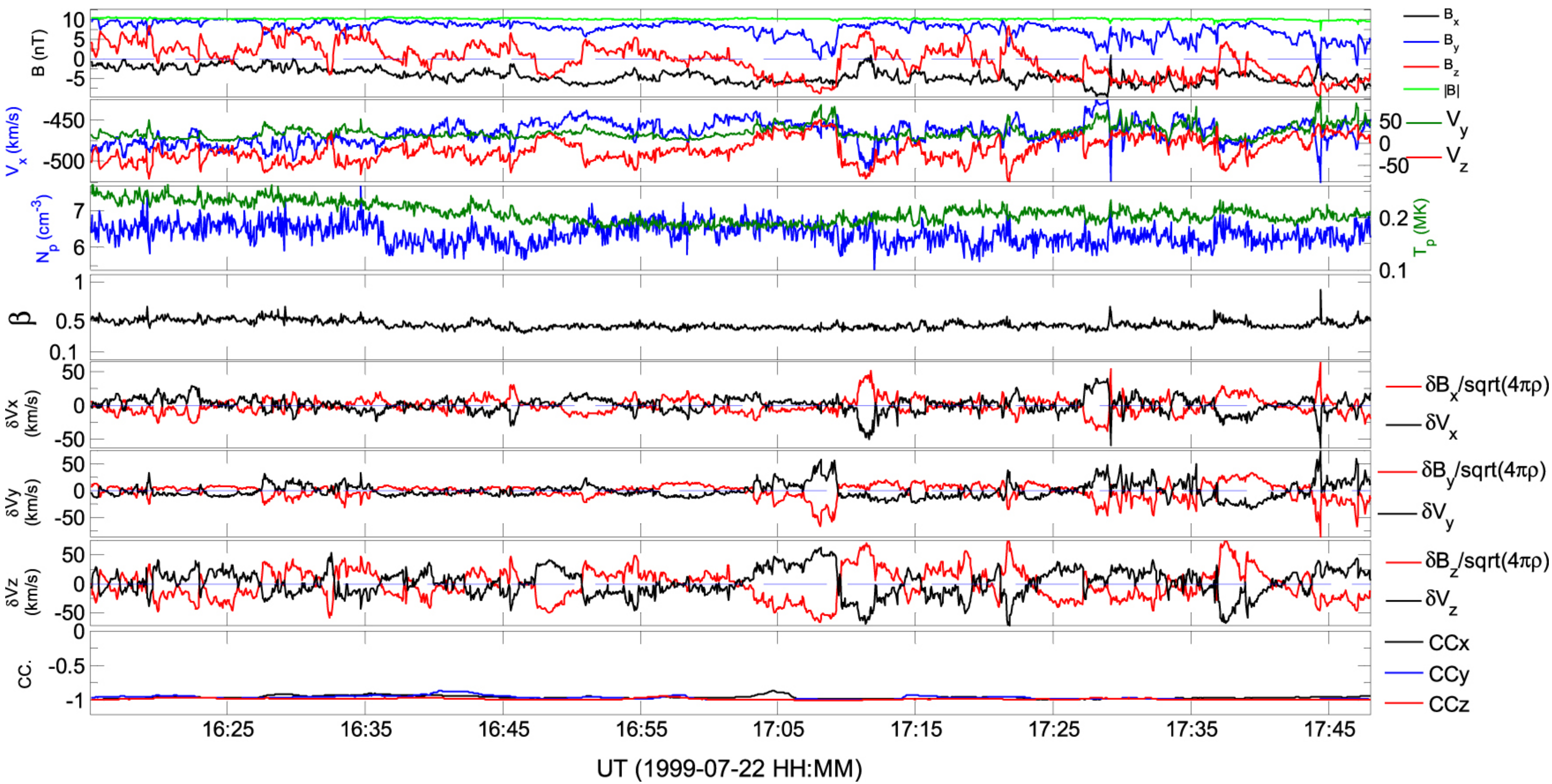
The magnetic perturbation in shear Alfvén wave is

$$\mathbf{B}_1 = -\frac{B_0}{(\omega/k)}\mathbf{u}_1 \longrightarrow \mathbf{B}_1\cdot\mathbf{B}_0 = 0$$

- The small component B_1, when added to B_0, gives the magnetic field lines a sinusoidal ripple
- The Alfvén wave involves no fluctuations in the fluid density or pressure
- Magnetic energy density of the wave motion equals to the kinetic energy density of the fluid oscillation

$$\frac{1}{2}B_1^2 = \frac{1}{2}\rho_0 u_1^2$$

In-situ measurements of Shear mode



Spacecraft observations provide highly detailed localized information

Anticorrelations between δB and δV in Wind data are due to Alfvén waves in the solar wind near 1 AU (Shi et al. 2015)

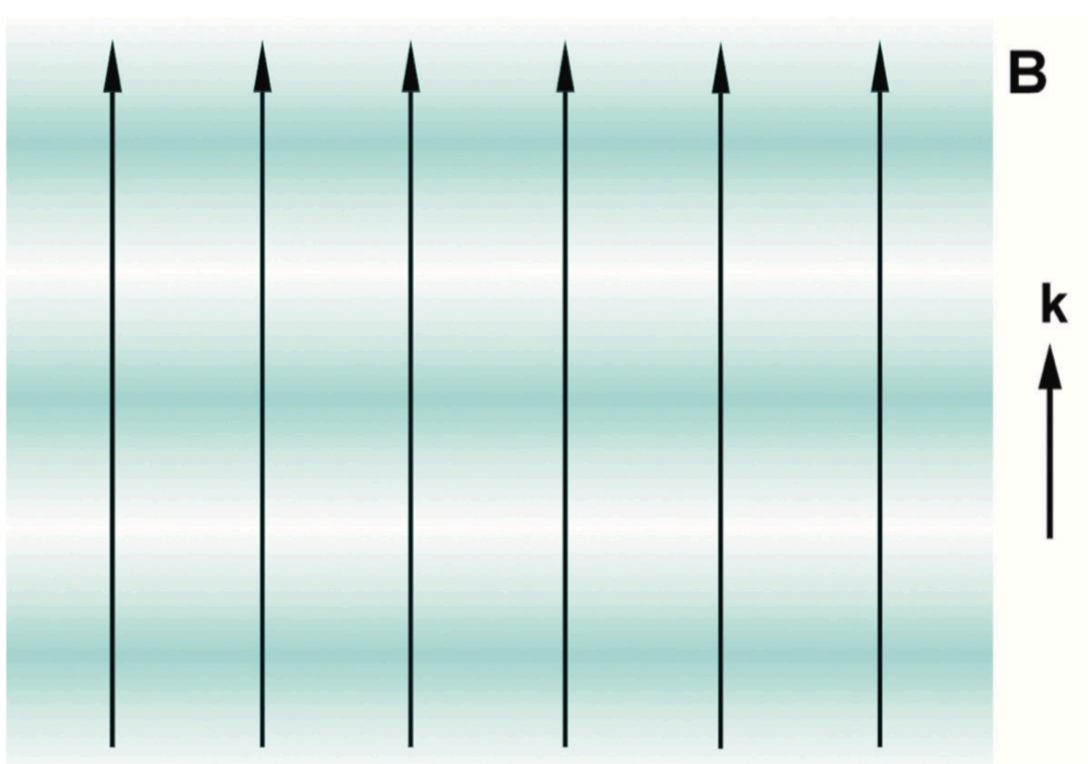
Linearized MHD equations

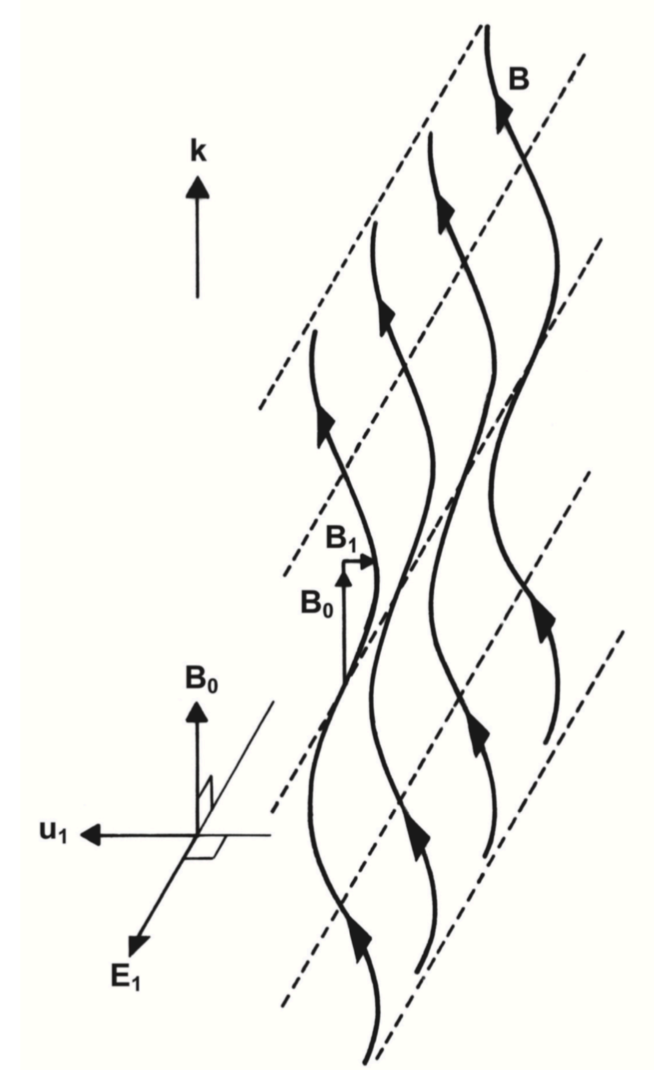
Shear and Slow Mode $\mathbf{k} \parallel \mathbf{V}_A$

$$-\omega^2\mathbf{u}_1 + (V_s^2 + V_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} + (\mathbf{k}\cdot\mathbf{V})_A[(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{u}_1 - (\mathbf{V}_A\cdot\mathbf{u}_1)\mathbf{k} - (\mathbf{k}\cdot\mathbf{u}_1)\mathbf{V}_A] = 0$$

Propagation perpendicular to B0: $\xrightarrow[\mathbf{k}\cdot\mathbf{V}_A = kV_A]{\mathbf{k} \parallel \mathbf{V}_A}$ $(k^2V_A^2 - \omega^2)\mathbf{u}_1 + (V_s^2/V_A^2 - 1)k^2(\mathbf{u}_1\cdot\mathbf{V}_A)\mathbf{V}_A = 0$

Type 2: u1 parallel to B0 and k $\longrightarrow$ $\dfrac{\omega}{k} = V_s$ Longitudinal (ordinary) sound wave

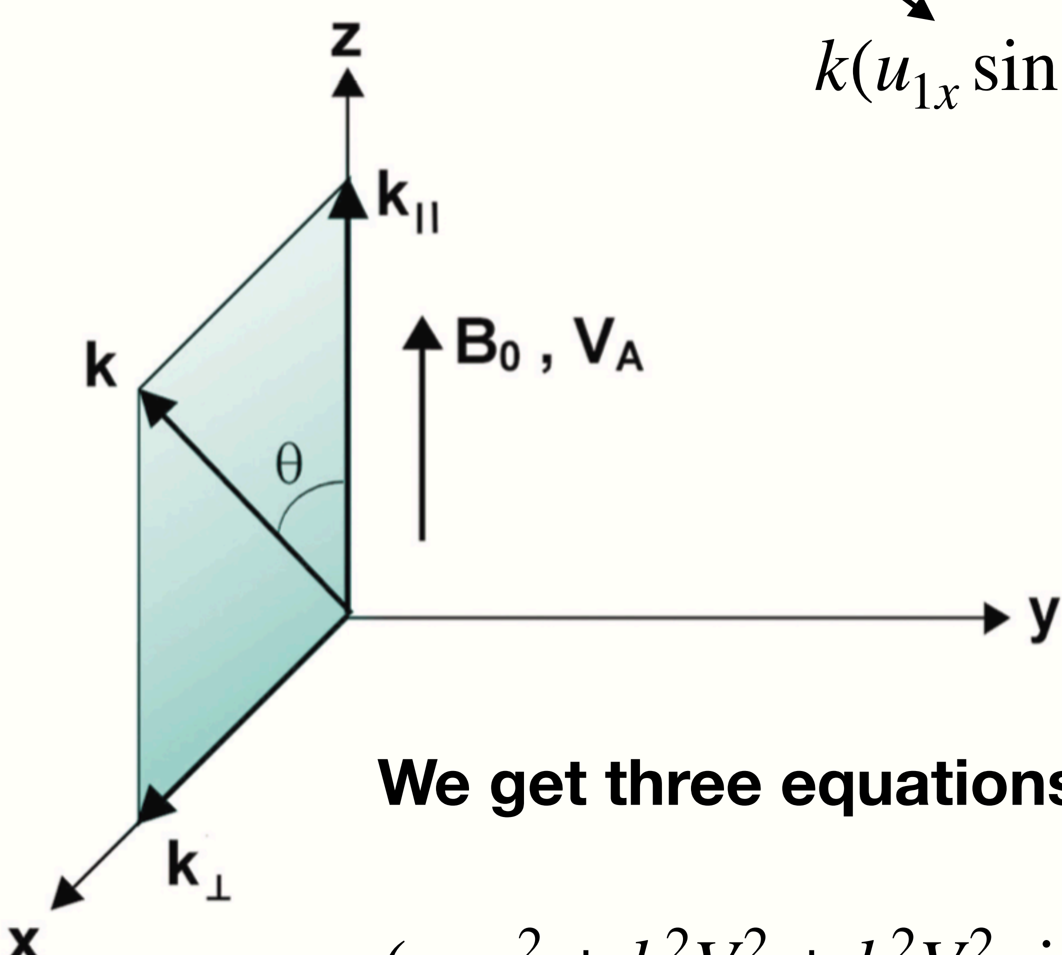




Linearized MHD equations

Propagation at arbitrary directions

$$-\omega^2\mathbf{u}_1 + (V_s^2 + V_A^2)\underbrace{(\mathbf{k}\cdot\mathbf{u}_1)}_{k(u_{1x}\sin\theta + u_{1z}\cos\theta)}\mathbf{k} + \underbrace{(\mathbf{k}\cdot\mathbf{V}_A)}_{kV_A\cos\theta}[(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{u}_1 - \underbrace{(\mathbf{V}_A\cdot\mathbf{u}_1)}_{V_A u_{1z}}\mathbf{k} - (\mathbf{k}\cdot\mathbf{u}_1)\mathbf{V}_A] = 0$$



Define

$$\mathbf{k} = k(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)$$

$$\mathbf{V}_A = V_A\hat{\mathbf{z}}$$

$$\mathbf{u}_1 = u_{1x}\hat{\mathbf{x}} + u_{1y}\hat{\mathbf{y}} + u_{1z}\hat{\mathbf{z}}$$

We get three equations

$$(-\omega^2 + k^2V_A^2 + k^2V_s^2\sin^2\theta)\, u_{1x} + (k^2V_s^2\sin\theta\cos\theta)\, u_{1z} = 0$$

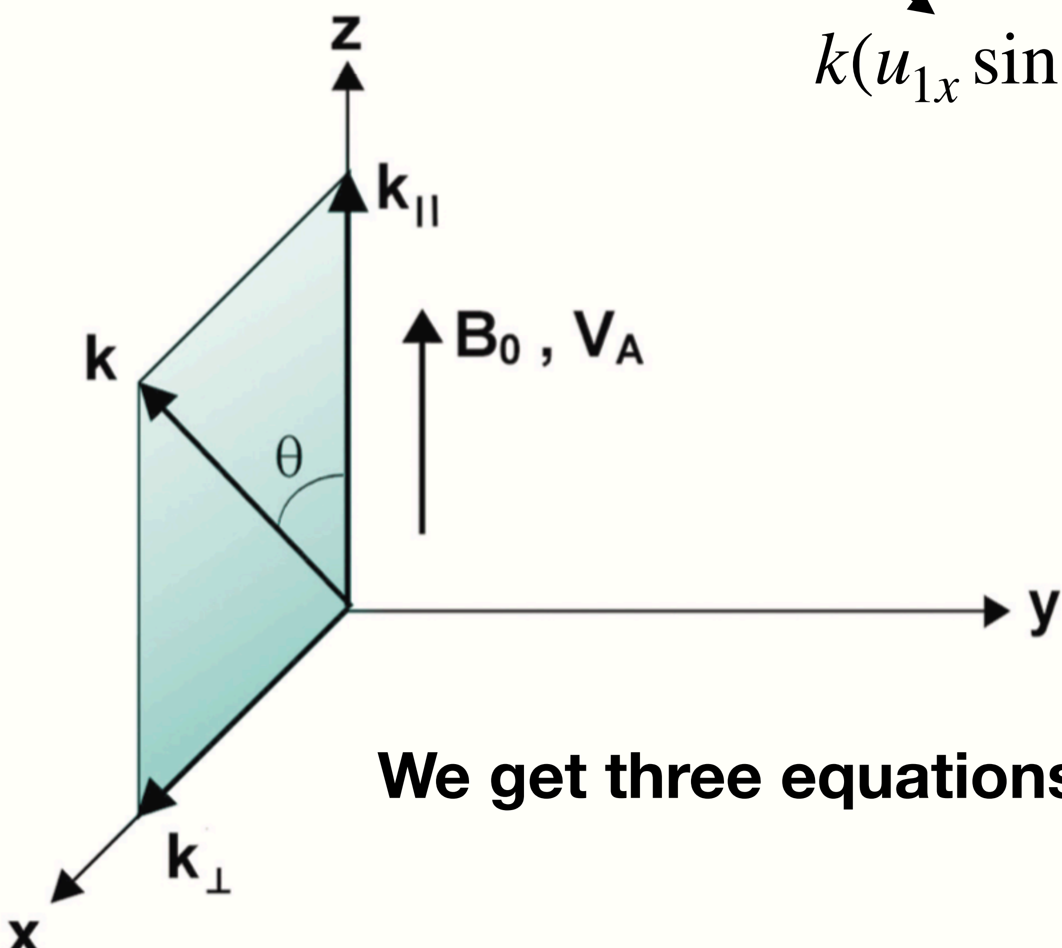
$$(-\omega^2 + k^2V_A^2\cos^2\theta)\, u_{1y} = 0$$

$$(k^2V_s^2\sin\theta\cos\theta)\, u_{1x} + (-\omega^2 + k^2V_s^2\cos^2\theta)\, u_{1z} = 0$$

Linearized MHD equations

Propagation at arbitrary directions

$$-\omega^2\mathbf{u}_1 + (V_s^2 + V_A^2)\underbrace{(\mathbf{k}\cdot\mathbf{u}_1)}_{k(u_{1x}\sin\theta + u_{1z}\cos\theta)}\mathbf{k} + \underbrace{(\mathbf{k}\cdot\mathbf{V}_A)}_{kV_A\cos\theta}[(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{u}_1 - \underbrace{(\mathbf{V}_A\cdot\mathbf{u}_1)}_{V_A u_{1z}}\mathbf{k} - (\mathbf{k}\cdot\mathbf{u}_1)\mathbf{V}_A] = 0$$



Define

$$\mathbf{k} = k(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)$$

$$\mathbf{V}_A = V_A\hat{\mathbf{z}}$$

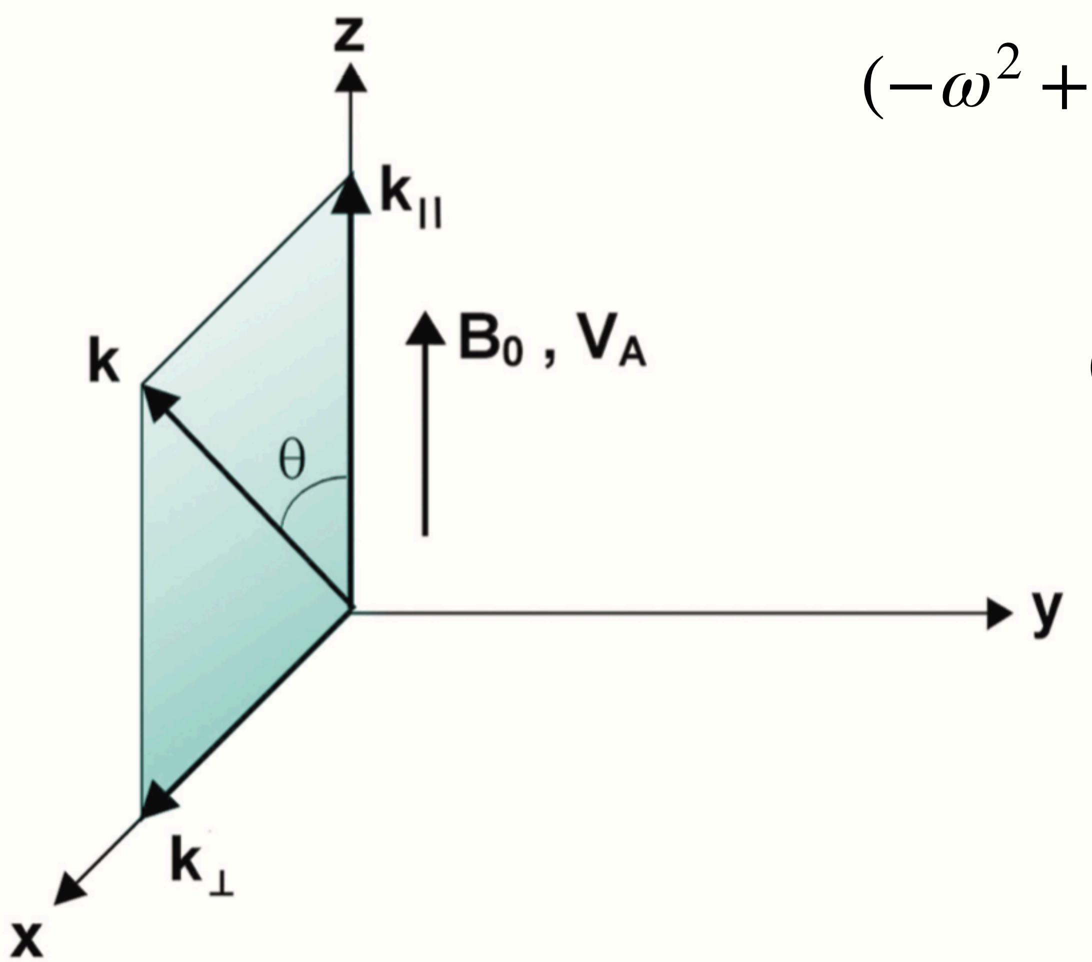
$$\mathbf{u}_1 = u_{1x}\hat{\mathbf{x}} + u_{1y}\hat{\mathbf{y}} + u_{1z}\hat{\mathbf{z}}$$

We get three equations

$$\begin{bmatrix} -\omega^2 + k^2V_A^2 + k^2V_s^2\sin^2\theta & 0 & k^2V_s^2\sin\theta\cos\theta \\ 0 & -\omega^2 + k^2V_A^2\cos^2\theta & 0 \\ k^2V_s^2\sin\theta\cos\theta & 0 & -\omega^2 + k^2V_s^2\cos^2\theta \end{bmatrix} \cdot \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \end{bmatrix} = 0$$

Linearized MHD equations

Pure Alfven Mode



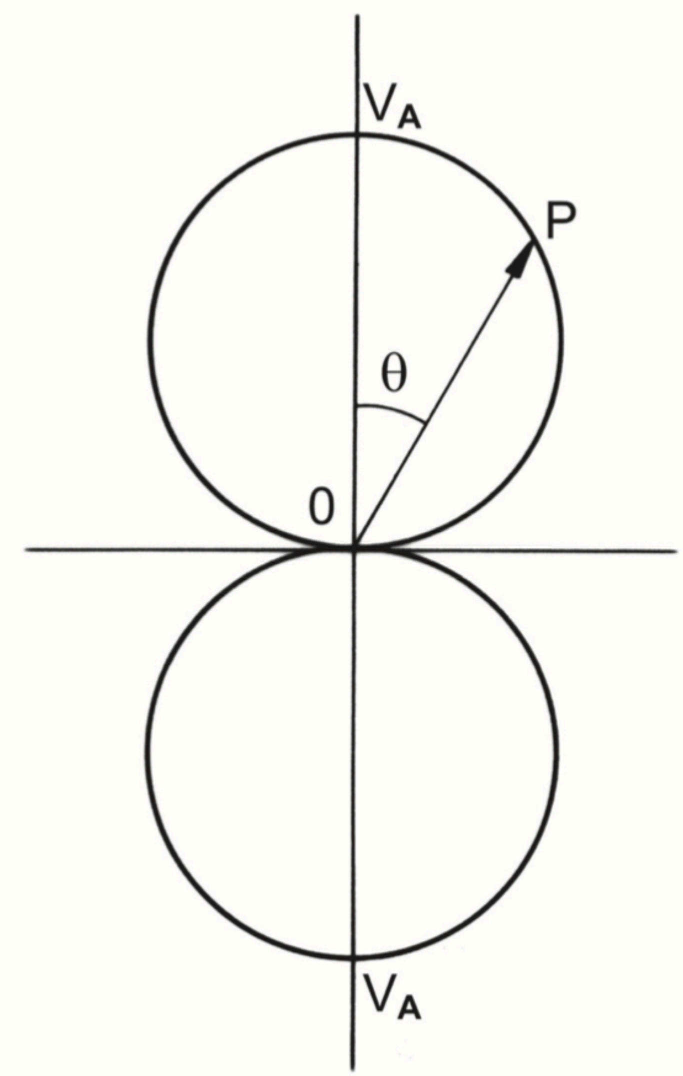
$$(-\omega^2 + k^2V_A^2 + k^2V_s^2\sin^2\theta)\,u_{1x} + (k^2V_s^2\sin\theta\cos\theta)\,u_{1z} = 0$$

$$(-\omega^2 + k^2V_A^2\cos^2\theta)\,u_{1y} = 0$$

$$(k^2V_s^2\sin\theta\cos\theta)\,u_{1x} + (-\omega^2 + k^2V_s^2\cos^2\theta)\,u_{1z} = 0$$

or $v_{1y} \neq 0$ we have a linearly polarized mode

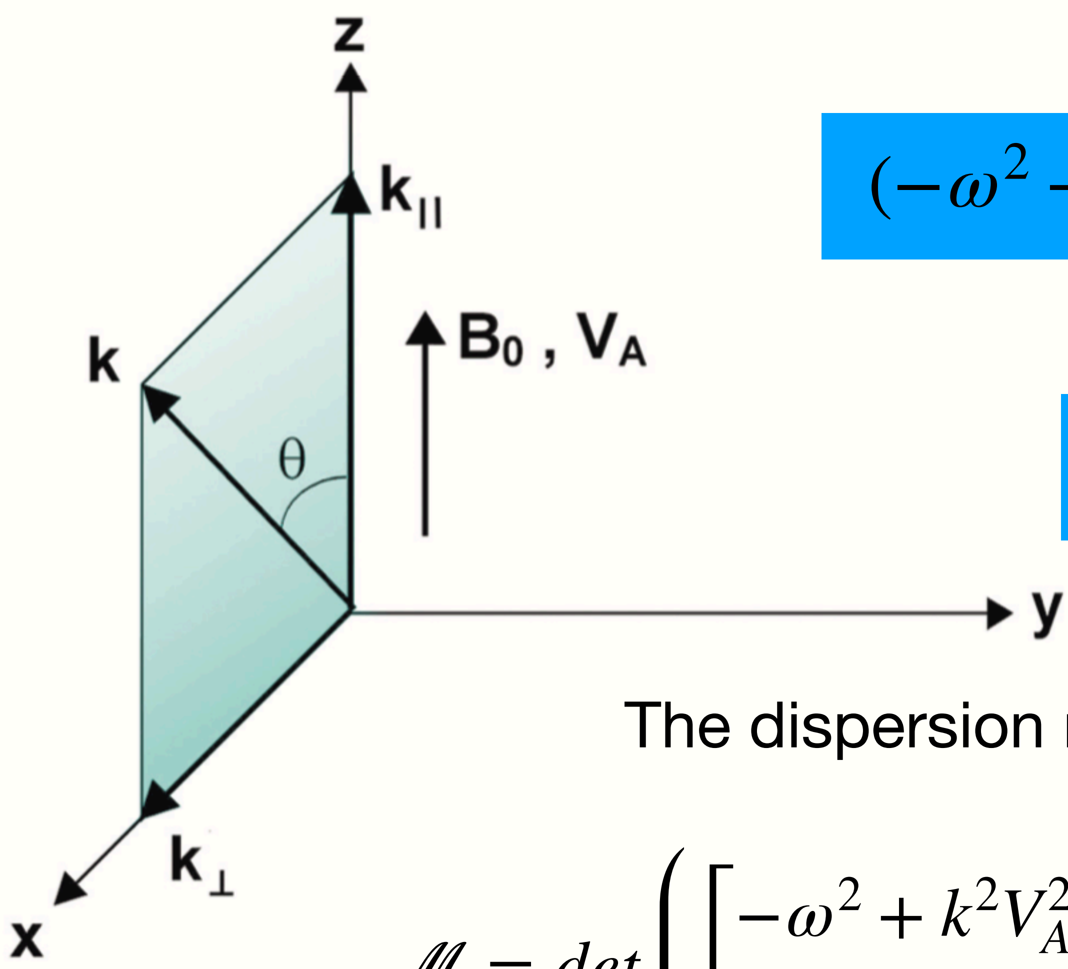
$$\frac{\omega}{k} = V_A\cos\theta$$



- The field components associated with this wave can be seen to be B_{1y}, u_{1y}, E_{1x}, so that it is a transverse Alfvén wave.
- This wave is generally referred to as the pure Alfvén wave;
- Theta = 0, propagate along the magnetic field - shear Alfven wave
- Theta = 90, does not exist
- Theta others: oblique Alfvén wave

Linearized MHD equations

Fast and Slow Mode



$$\left(-\omega^2 + k^2V_A^2 + k^2V_s^2 \sin^2\theta\right) u_{1x} + \left(k^2V_s^2 \sin\theta\cos\theta\right) u_{1z} = 0$$

$$\left(-\omega^2 + k^2V_A^2 \cos^2\theta\right) u_{1y} = 0$$

$$\left(k^2V_s^2 \sin\theta\cos\theta\right) u_{1x} + \left(-\omega^2 + k^2V_s^2 \cos^2\theta\right) u_{1z} = 0$$

The dispersion relation is found as setting the determinant zero:

$$\mathscr{M} = det\left(\begin{bmatrix} -\omega^2 + k^2V_A^2 + k^2V_s^2\sin^2\theta & k^2V_s^2\sin\theta\cos\theta \\ k^2V_s^2\sin\theta\cos\theta & -\omega^2 + k^2V_s^2\cos^2\theta \end{bmatrix}\right) = 0$$

$$\longrightarrow \quad \left(\frac{\omega}{k}\right)^4 - (V_A^2 + V_s^2)\left(\frac{\omega}{k}\right)^2 + V_A^2V_s^2\cos^2\theta = 0$$

$$\longrightarrow \quad \left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) \pm \frac{1}{2}\sqrt{(V_A^2 + V_s^2) - 2V_A^2V_s^2\cos^2\theta}$$

Phase Velocity Diagram

Fast and Slow Mode

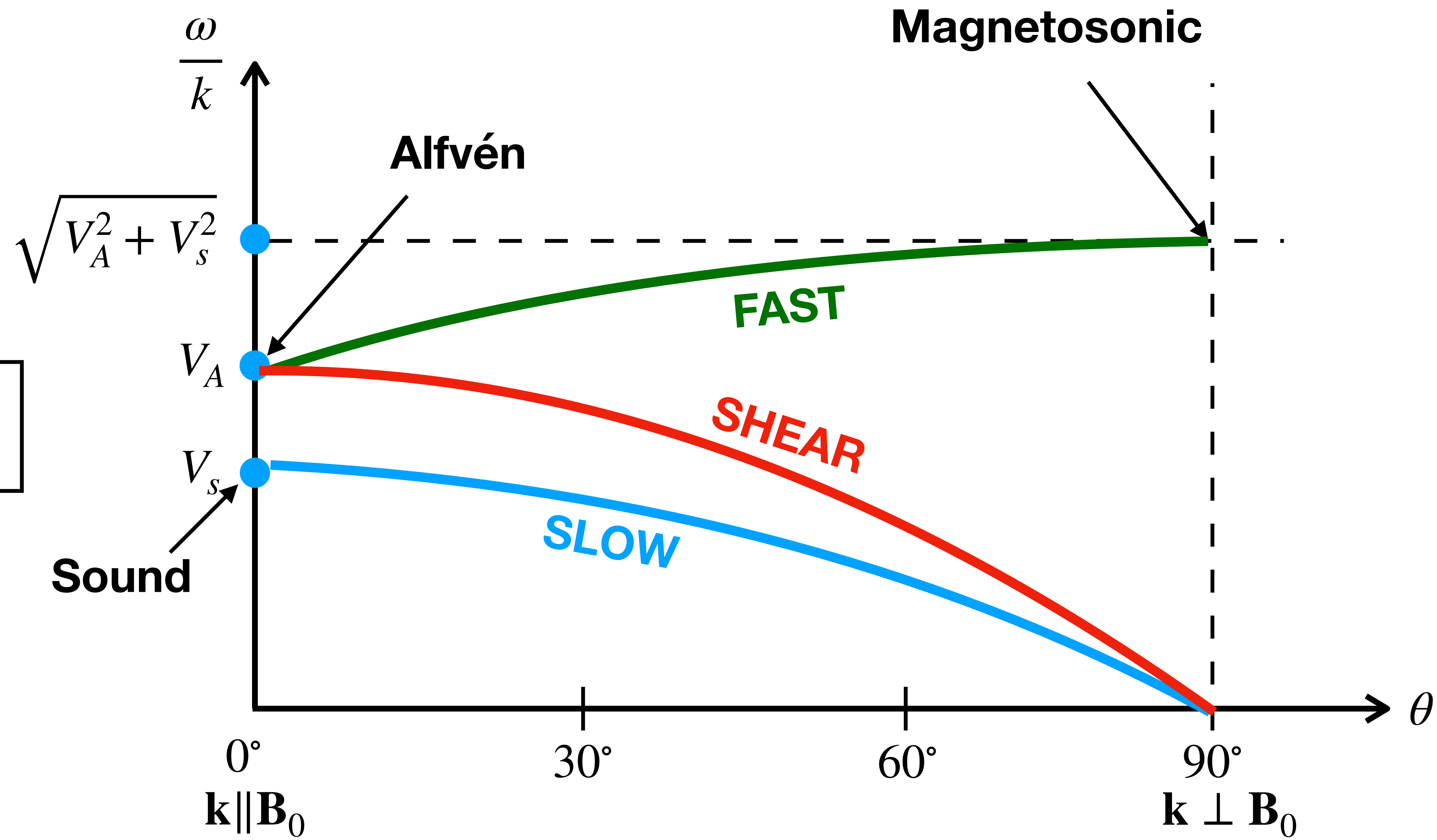
$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) + \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$ Fast

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) - \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$ Slow

$$\frac{\omega}{k} = V_A \cos\theta$$

Shear

CASE I: $V_A > V_s$



Phase Velocity Diagram

Fast and Slow Mode

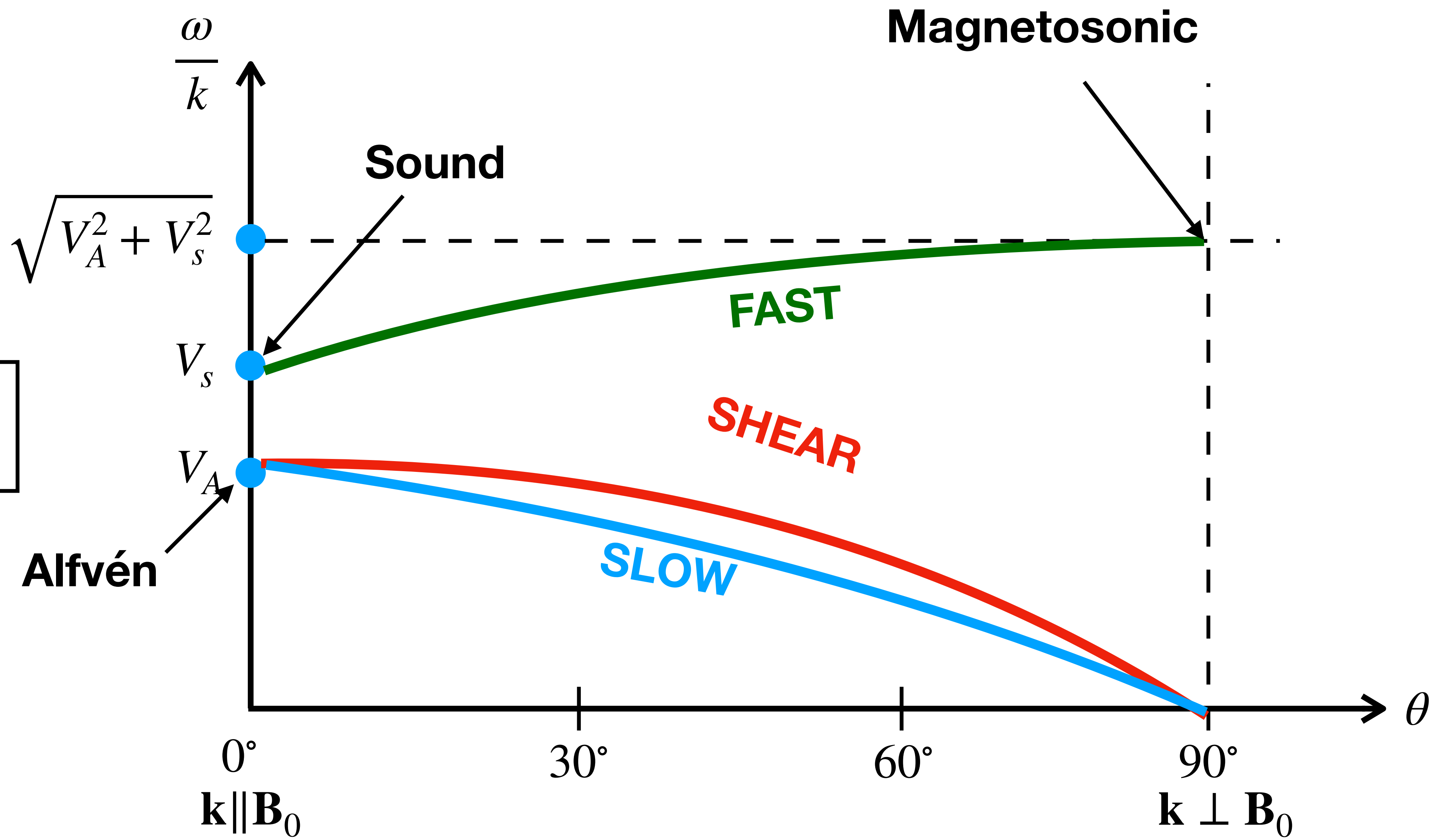
$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) + \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$ **Fast**

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) - \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$ **Slow**

$$\frac{\omega}{k} = V_A \cos\theta$$

Shear

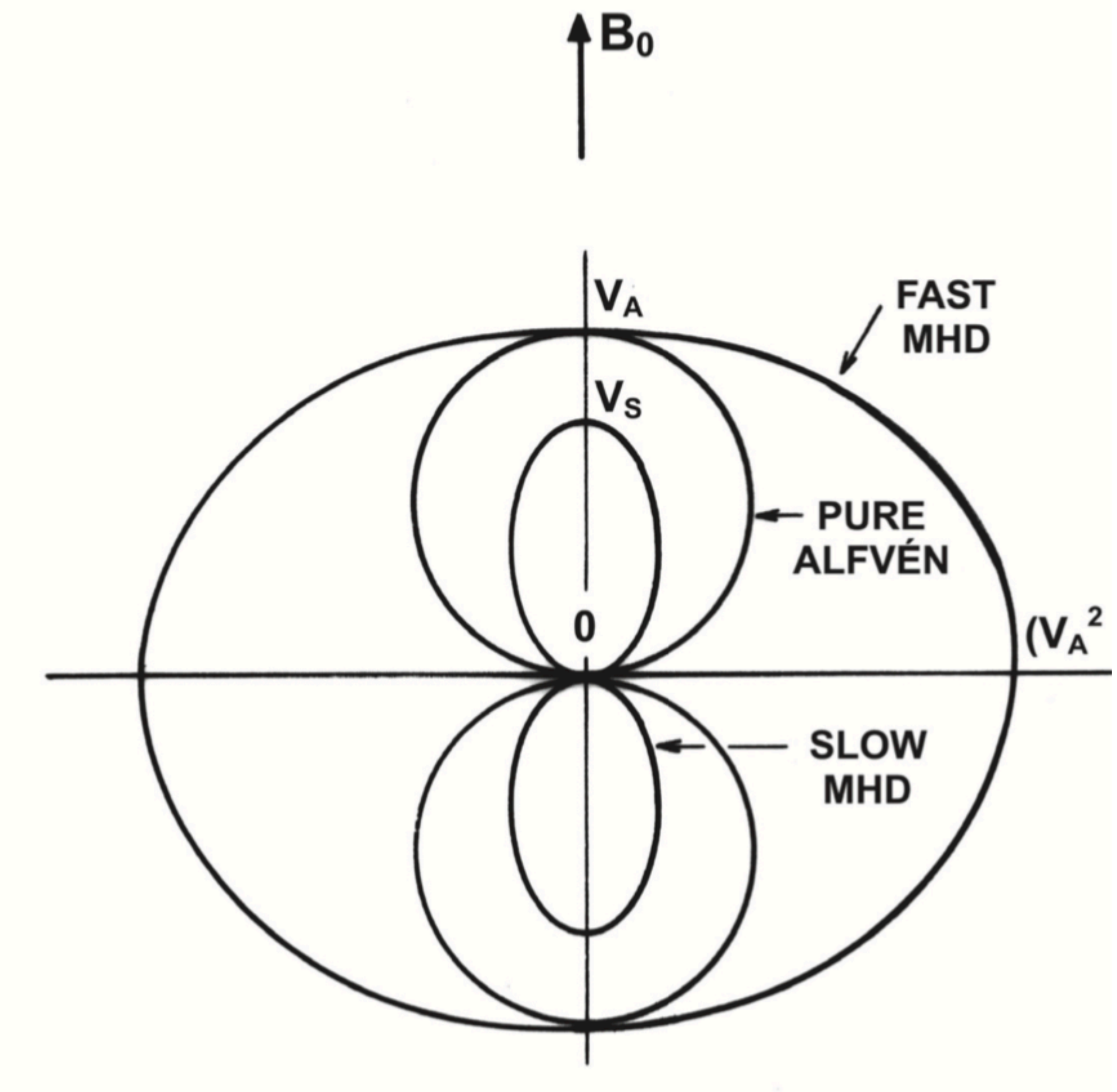
CASE II: $V_A < V_s$



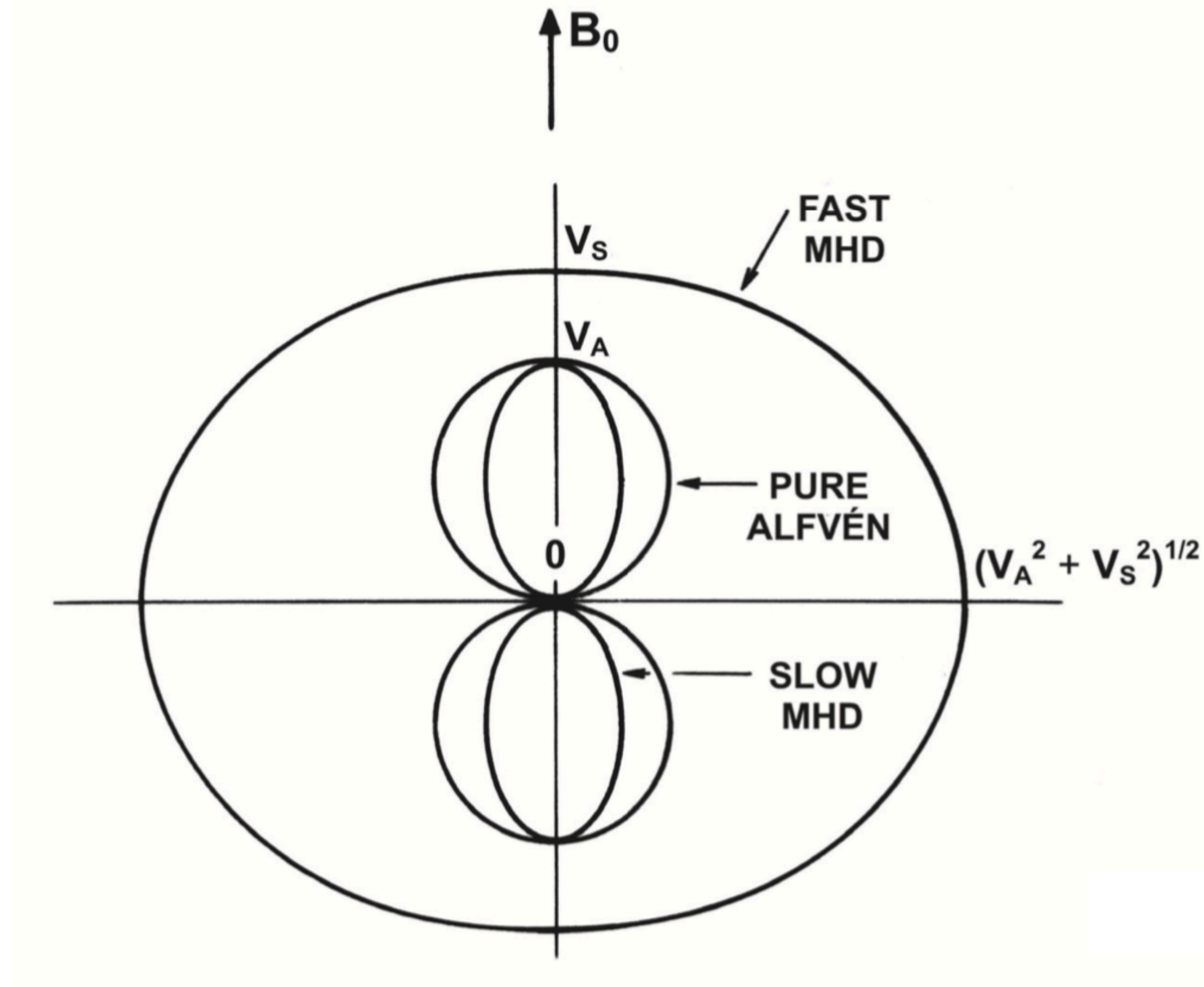
Phase Velocity Diagram

Ideal MHD Modes

$$v_{ph} = \frac{\partial \omega}{\partial k}\mathbf{k}$$



CASE I: $V_A > V_s$



CASE II: $V_A < V_s$

Group Velocity Diagram

Ideal MHD Modes

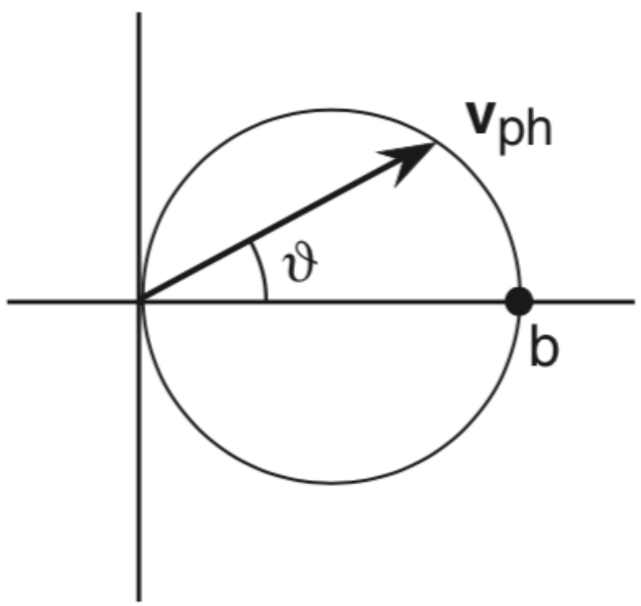
$$v_g = \frac{\partial \omega}{\partial \mathbf{k}}$$

For example the shear mode:

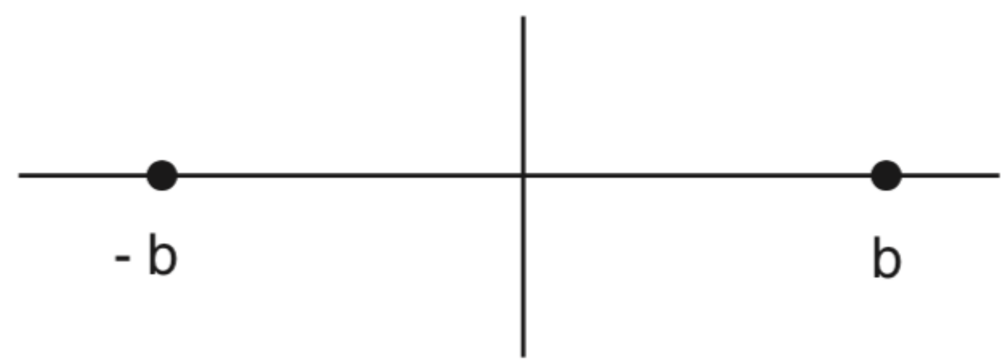
$$v_{ph} = \frac{\omega}{k} = V_A \cos\theta$$

$$v_g = \frac{\partial}{\partial \mathbf{k}}(V_A \cos\theta) = \frac{\partial}{\partial \mathbf{k}}(V_A \mathbf{k}\cdot\mathbf{b})$$

$$= \nabla_k (V_A \mathbf{k}\cdot\mathbf{b}) = V_A \mathbf{b}$$



$(\omega - k)$ coordinate

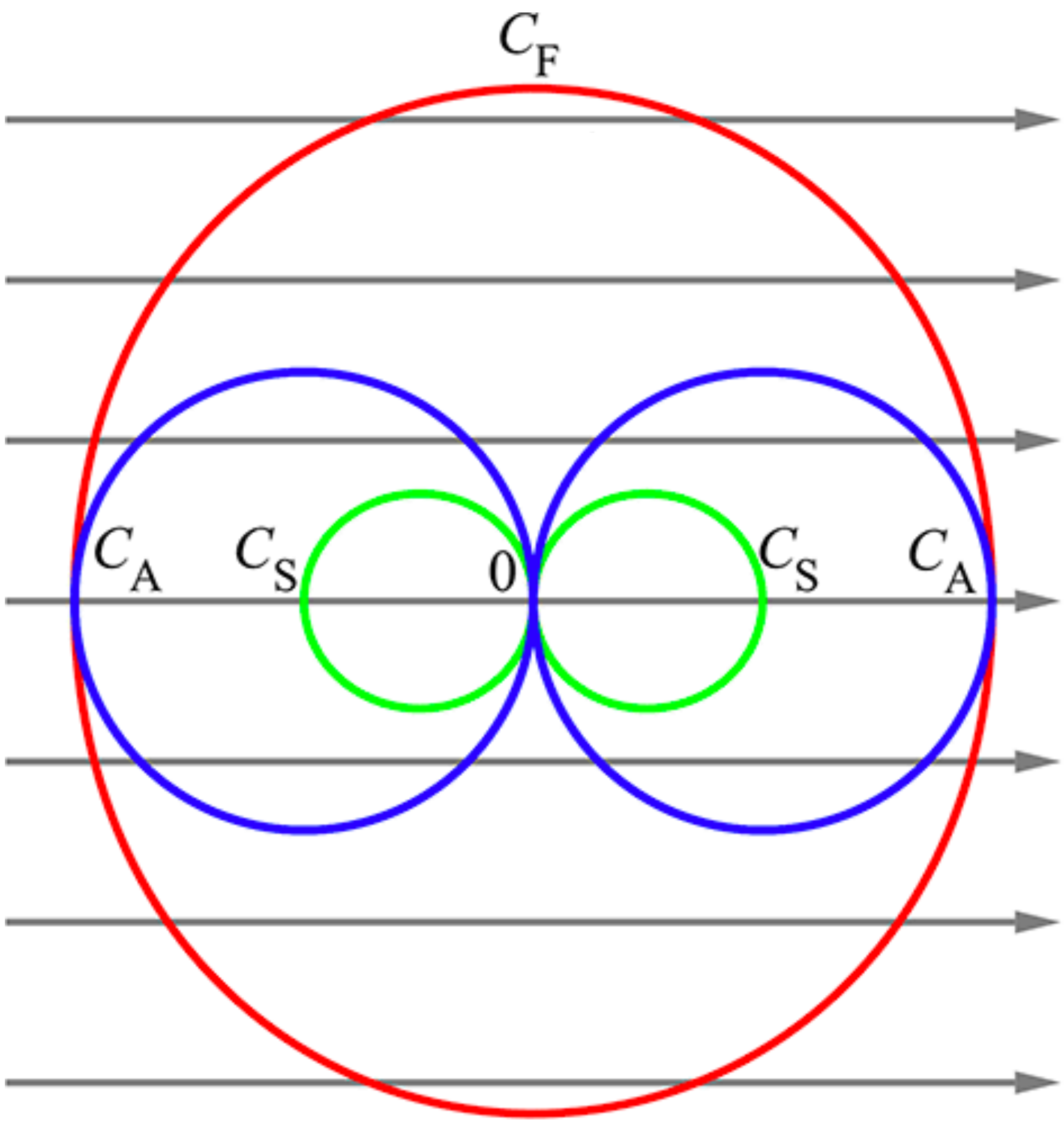


Group velocity diagram

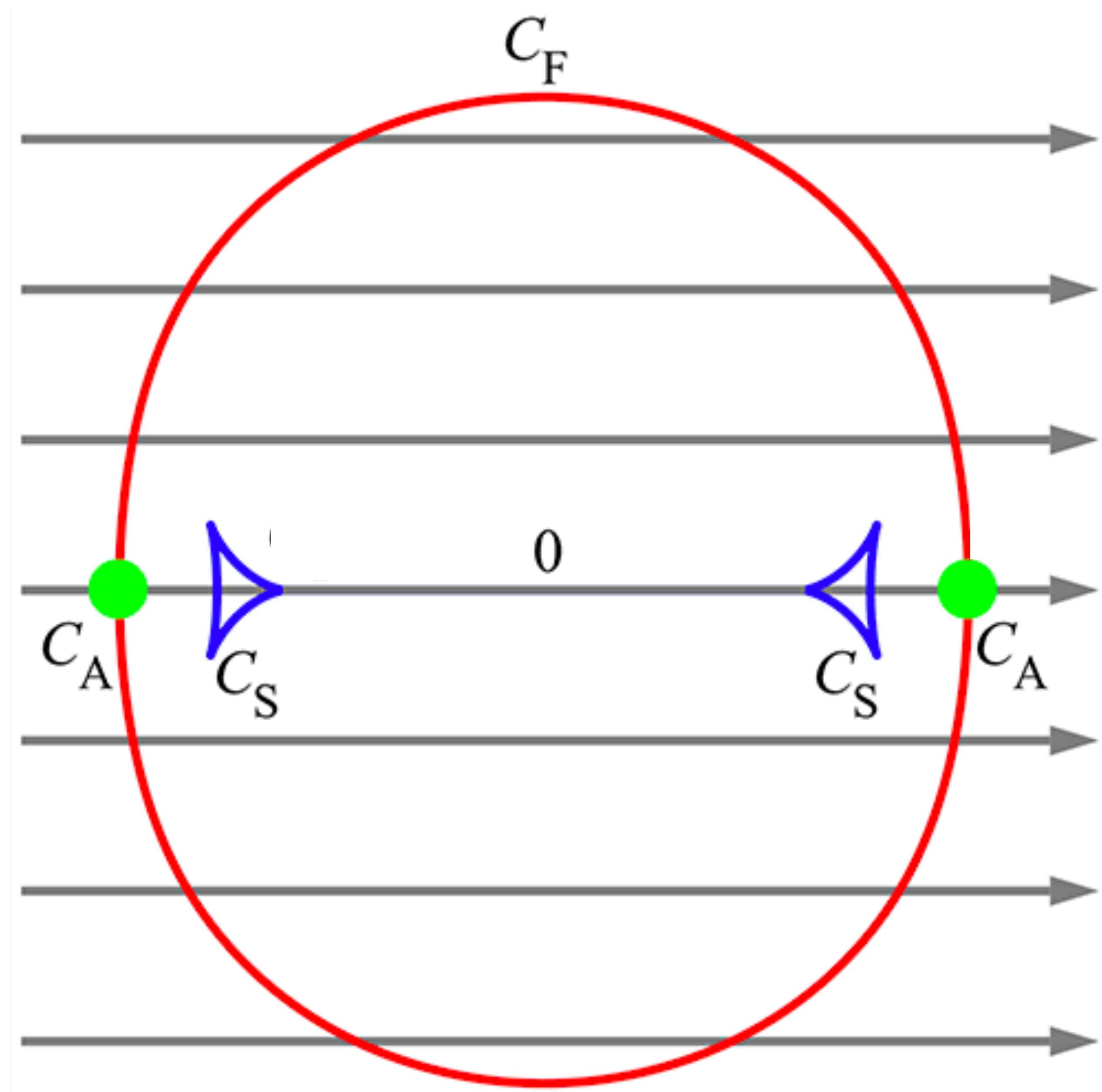
Group Velocity Diagram

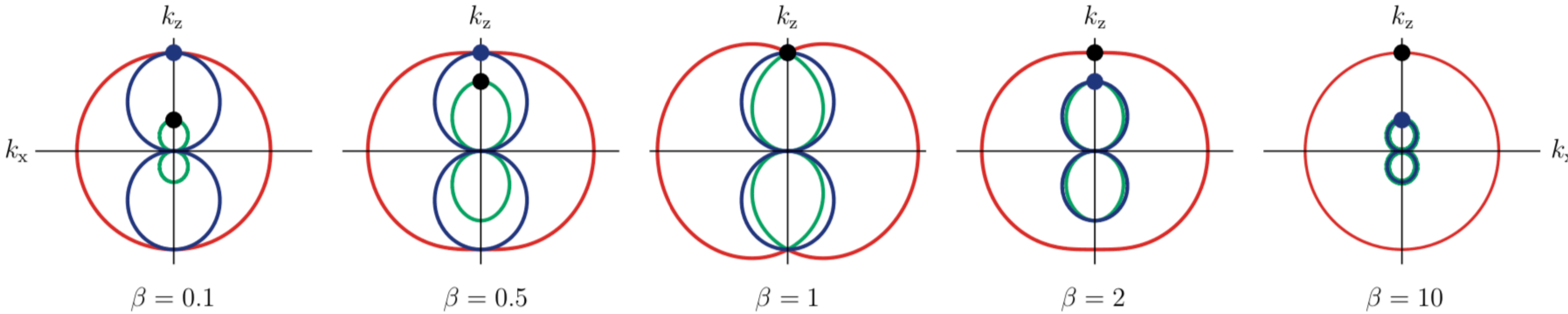
Ideal MHD Modes

$$v_{ph} = \frac{\partial \omega}{\partial k}\mathbf{k}$$



$$v_g = \frac{\partial \omega}{\partial \mathbf{k}}$$





"Friedrichs diagrams" for MHD waves: Phase speed plotted as radial distance, with the angle between **k** and $\mathbf{B}_0$ shown as the angle away from the y–axis. Here, $\beta = (c_s/V_{\rm A})^2$. Blue point: Alfvén speed. Black point: sound speed. Curve color-codes shown below.

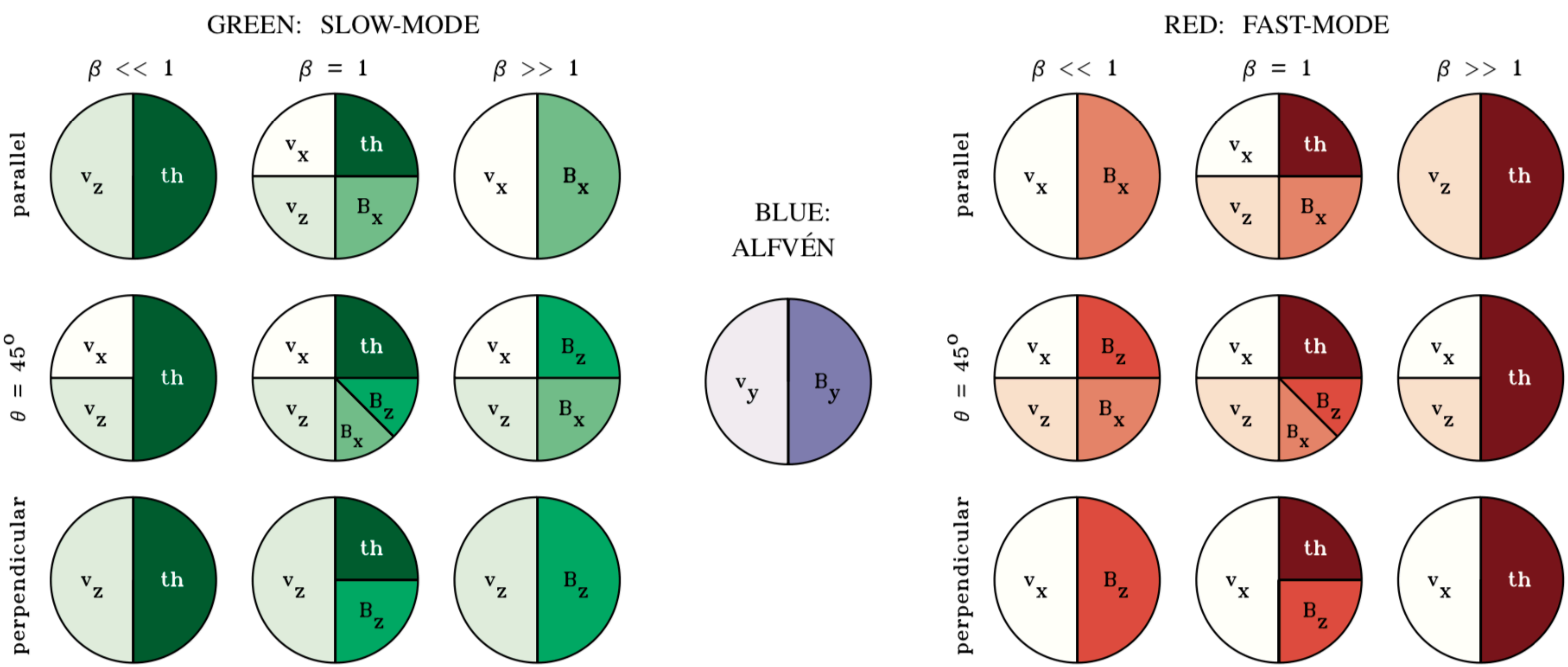


Illustration of how MHD waves **partition their total fluctuation energy into kinetic, magnetic, and thermal energy** in various regimes: wavevectors parallel to $\mathbf{B}_0$ (top row), an isotropic distribution of wavevectors (middle row), wavevectors perpendicular to $\mathbf{B}_0$ (bottom row); columns denote plasma β regimes. Kinetic energy fractions are denoted v_i, magnetic energy fractions are denoted B_i, and the thermal energy fraction is denoted 'th'.

Summary of ideal MHD waves

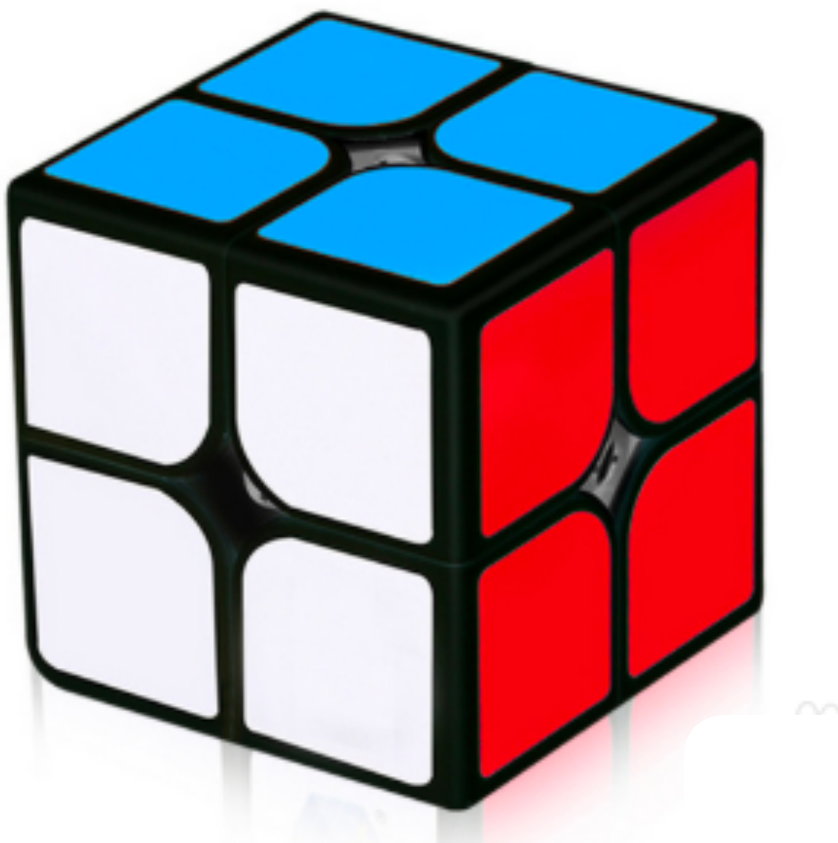
- Waves are ubiquitous in astrophysical, laboratory, space, and heliospheric plasmas
- The three principal wave modes for ideal MHD are the shear Alfvén wave, the slow magnetosonic wave, and the fast magnetosonic wave
- The shear Alfvén wave is a transverse wave that propagates along the magnetic field
- Slow and fast magnetosonic waves are longitudinal waves that may propagate obliquely
- Plasma waves are well-studied in solar, space, and laboratory plasmas and play important roles in a variety of astrophysical plasmas

Limitation of the linear analysis

We linearized the equations of ideal MHD and combined them to derive the dispersion relationship for shear Alfvén waves, fast magnetosonic waves, and slow magnetosonic waves for a uniform, static, and infinite background

Think: In what ways do our assumptions limit the applicability of these results?
What are some situations where these assumptions are invalid?

MHD waves I learned from class



MHD waves I see in research