MHD Waves 2

Non-Ideal MHD Waves

Ideal MHD Waves

Primitive form

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

$\rho = \rho_0 + \rho_1, \ |\rho_1| \ll \rho_0$
$\mathbf{u} = \mathbf{u}_1, \ |\mathbf{u}_1|$ small

1

$$\boxed{\frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 = 0}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0 \xrightarrow[\text{Adiabatic}]{\nabla p = V_s^2\nabla\rho} \rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \underbrace{V_s^2\nabla\rho}_{\text{Sound waves}} - \underbrace{\mathbf{J}\times\mathbf{B}}_{\text{Alfvén waves}} = 0$$

$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \ |\mathbf{B}_1|$ small
$\rho = \rho_0 + \rho_1, \ |\rho_1| \ll \rho_0$
$\mathbf{u} = \mathbf{u}_1, \ |\mathbf{u}_1|$ small

2

$$\boxed{\rho_0\frac{\partial \mathbf{u}_1}{\partial t} + V_s^2\nabla\rho_1 - \mathbf{B}_0\times\nabla\times\mathbf{B}_1 = 0}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B})$$

$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \ |\mathbf{B}_1|$ small
$\mathbf{u} = \mathbf{u}_1, \ |\mathbf{u}_1|$ small

3

$$\boxed{\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{u}_1\times\mathbf{B}_0)}$$

Linearized equations

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) + \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2V_s^2\cos^2\theta}$$ **Fast**

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) - \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2V_s^2\cos^2\theta}$$ **Slow**

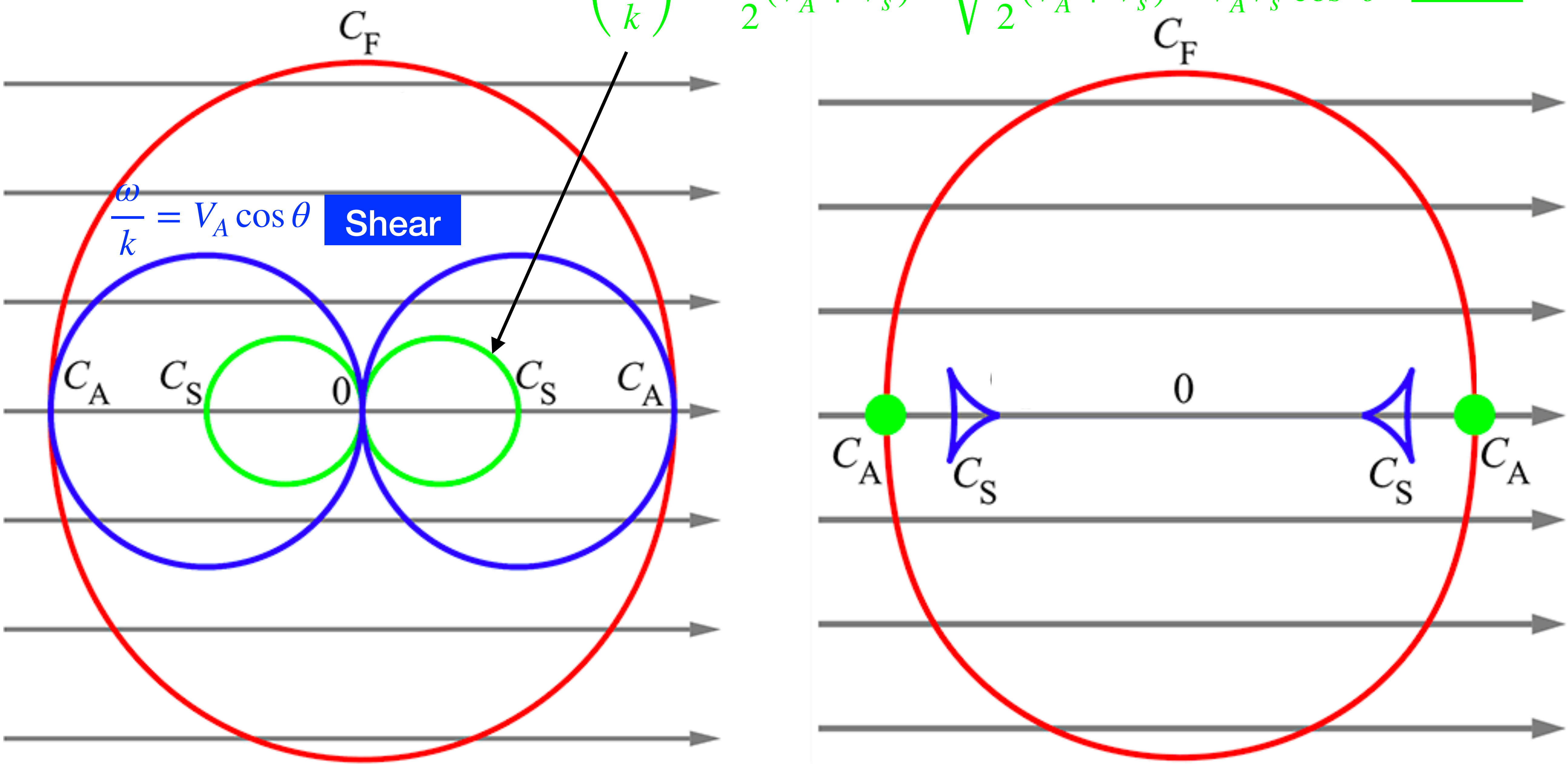
$$\frac{\omega}{k} = V_A\cos\theta$$ **Shear**

Phase and Group Velocity Diagram

Ideal MHD Modes

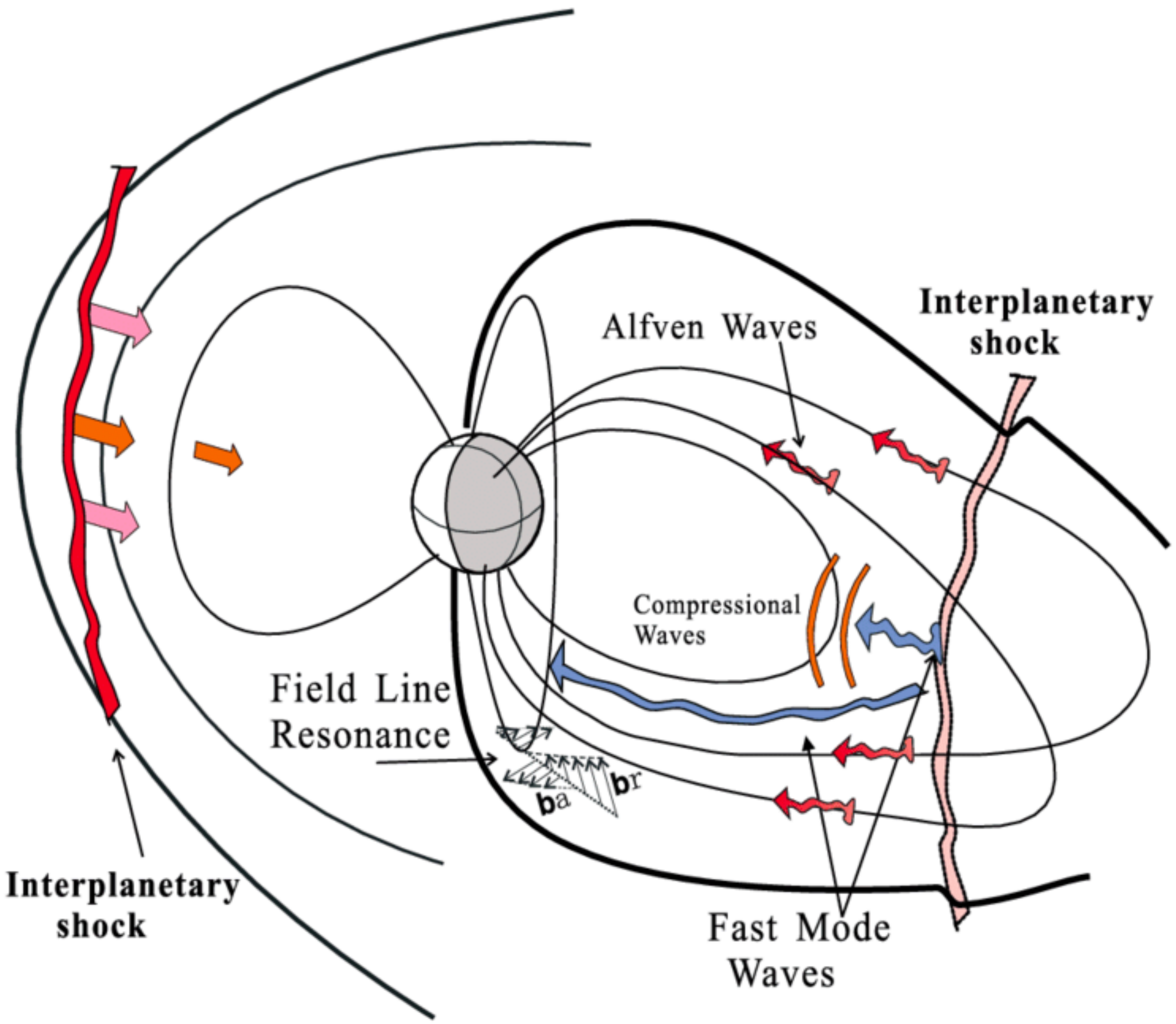
$$v_{ph} = \frac{\partial \omega}{\partial k}\mathbf{k} \qquad\qquad v_g = \frac{\partial \omega}{\partial \mathbf{k}}$$

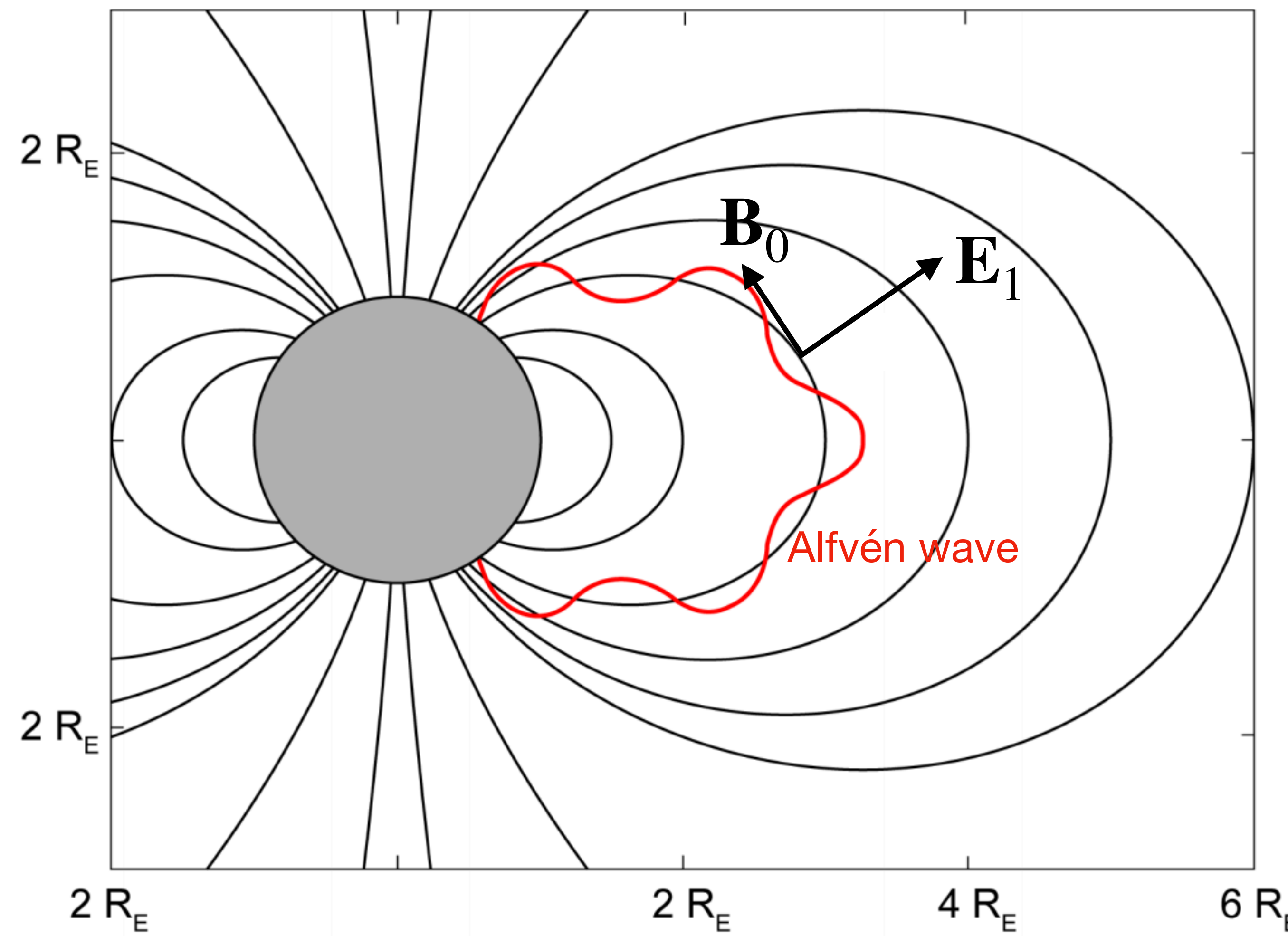
$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) - \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$ Slow



Fast $$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(V_A^2 + V_s^2) + \sqrt{\frac{1}{2}(V_A^2 + V_s^2) - V_A^2 V_s^2 \cos^2\theta}$$

Equal-potential magnetic field lines





$$\mathbf{E}_1 = -\,\mathbf{u}_1 \times \mathbf{B}_0 \longrightarrow \mathbf{E}_1 \perp \mathbf{B}_0$$

Which means in a quasi-steady state:

$$\Delta\Phi = \int_{South}^{North} \mathbf{E}_1 \cdot d\mathbf{l} = \int_{South}^{North} \mathbf{E}_1 \cdot \mathbf{B}_0 / B_0 ds \equiv 0$$

Equal-potential magnetic field lines

- Ideal MHD and electrostatic ionosphere
  => foot-points of a field line in each hemisphere are at the same potential (Hesse et al., JGR 1997)
- How well do global MHD simulations meet this equi-potential condition?

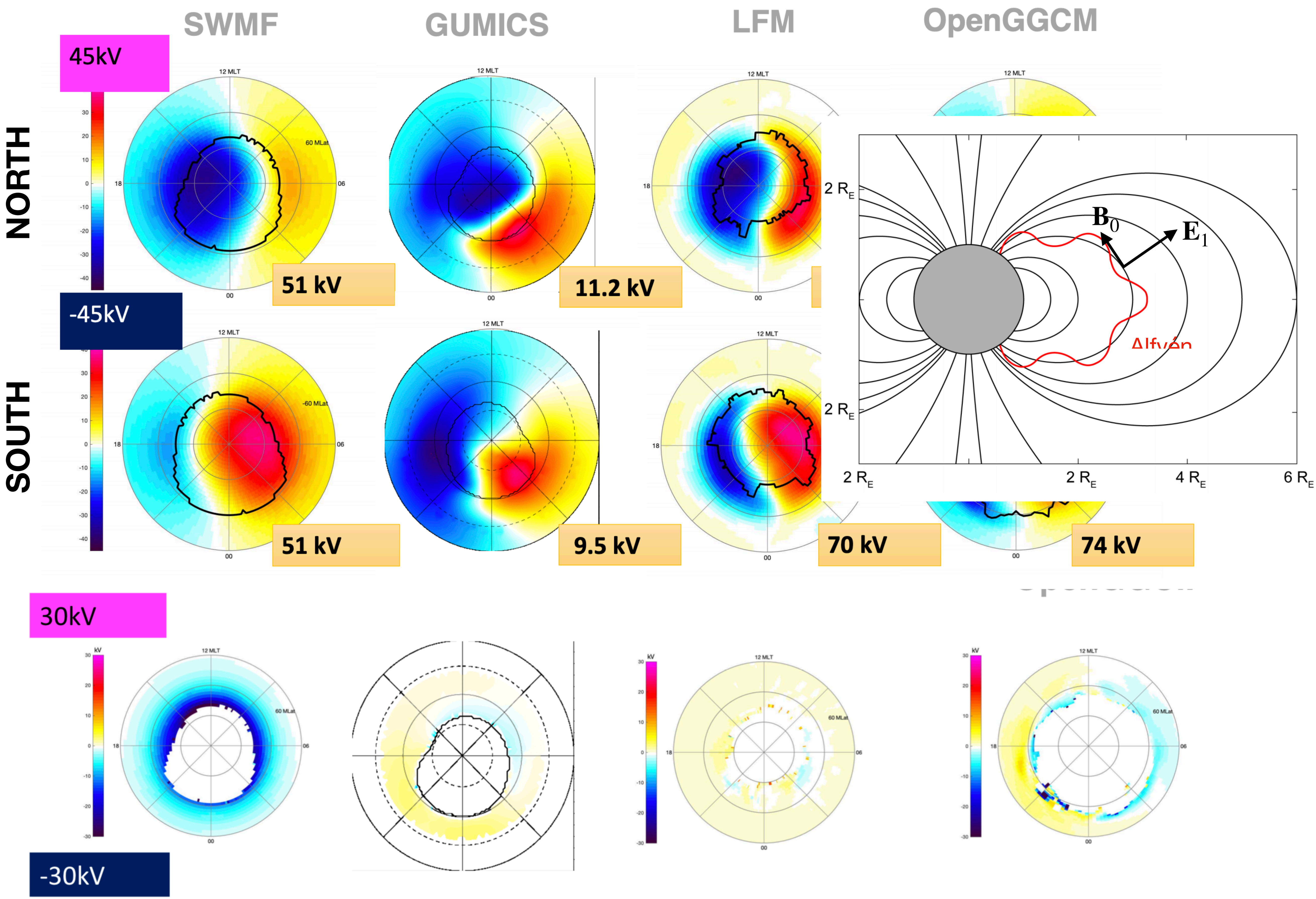
BATS-R-US/SWMF	GUMICS	LFM	Open-GGCM
• Gombosi et al. • U. Michigan	• Janhunen et al. • FMI	• Lyon et al. • Dartmouth College	• Raeder et al. • U. New Hampshire

4 models @ CCMC

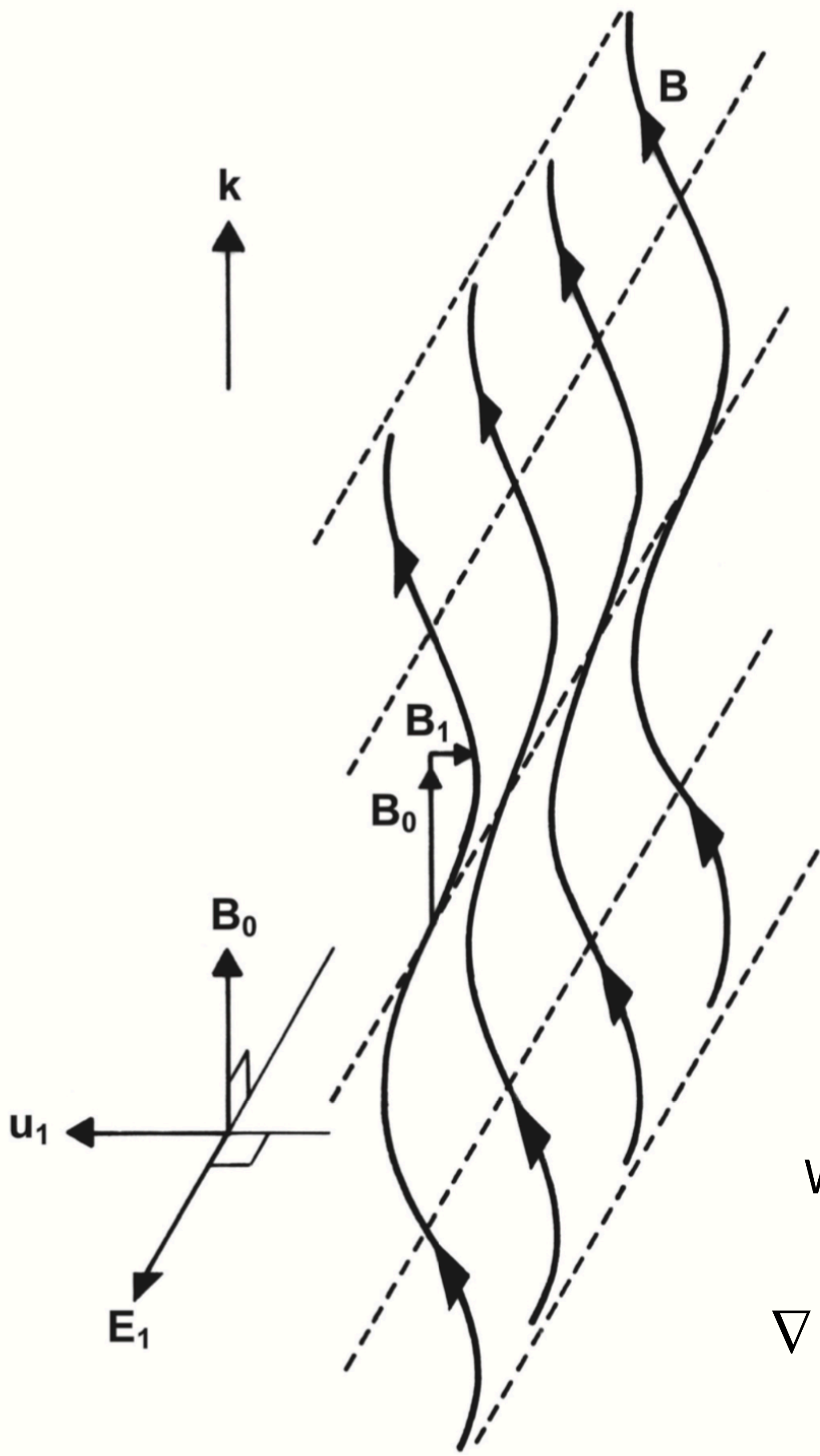
Simplified, constant driving conditions:
IMF By=5nT; Bz=0nT; Bx=0nT
Homogeneous conductance: Σ_P=5mho; Σ_H=0
No dipole tilt
All other parameters default

Rastaetter et al. [2019]

Equal-potential magnetic field lines



Reflection of Alfvén waves



recall: The magnetic perturbation in shear Alfvén wave is

$$\mathbf{B}_1 = -\frac{B_0}{(\omega/k)}\mathbf{u}_1 \longrightarrow \mathbf{u}_1 = -\frac{V_A}{B_0}\mathbf{B}_1$$

And the electric field is

$\hat{\mathbf{b}}_0 = \mathbf{B}_0/B_0$

$$\mathbf{E}_1 = -\mathbf{u}_1 \times \mathbf{B}_0 = \pm\frac{V_A}{B_0}\mathbf{B}_1 \times \mathbf{B}_0 = \pm V_A \mathbf{B}_1 \times \hat{\mathbf{b}}_0$$

Then the perpendicular current is calculated as

$$\mathbf{J}_\perp = \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1)_\perp = \pm\frac{1}{\mu_0}\frac{\partial}{\partial z}(\hat{\mathbf{b}}_0 \times \mathbf{B}_1)$$

Substitute E_1:

$$\mathbf{J}_\perp = -\frac{1}{\mu_0 V_A}\frac{\partial}{\partial z}\mathbf{E}_1 = \mp\Sigma_A\frac{\partial}{\partial z}\mathbf{E}_1$$

Which means in slab geometry, we have the parallel current written as

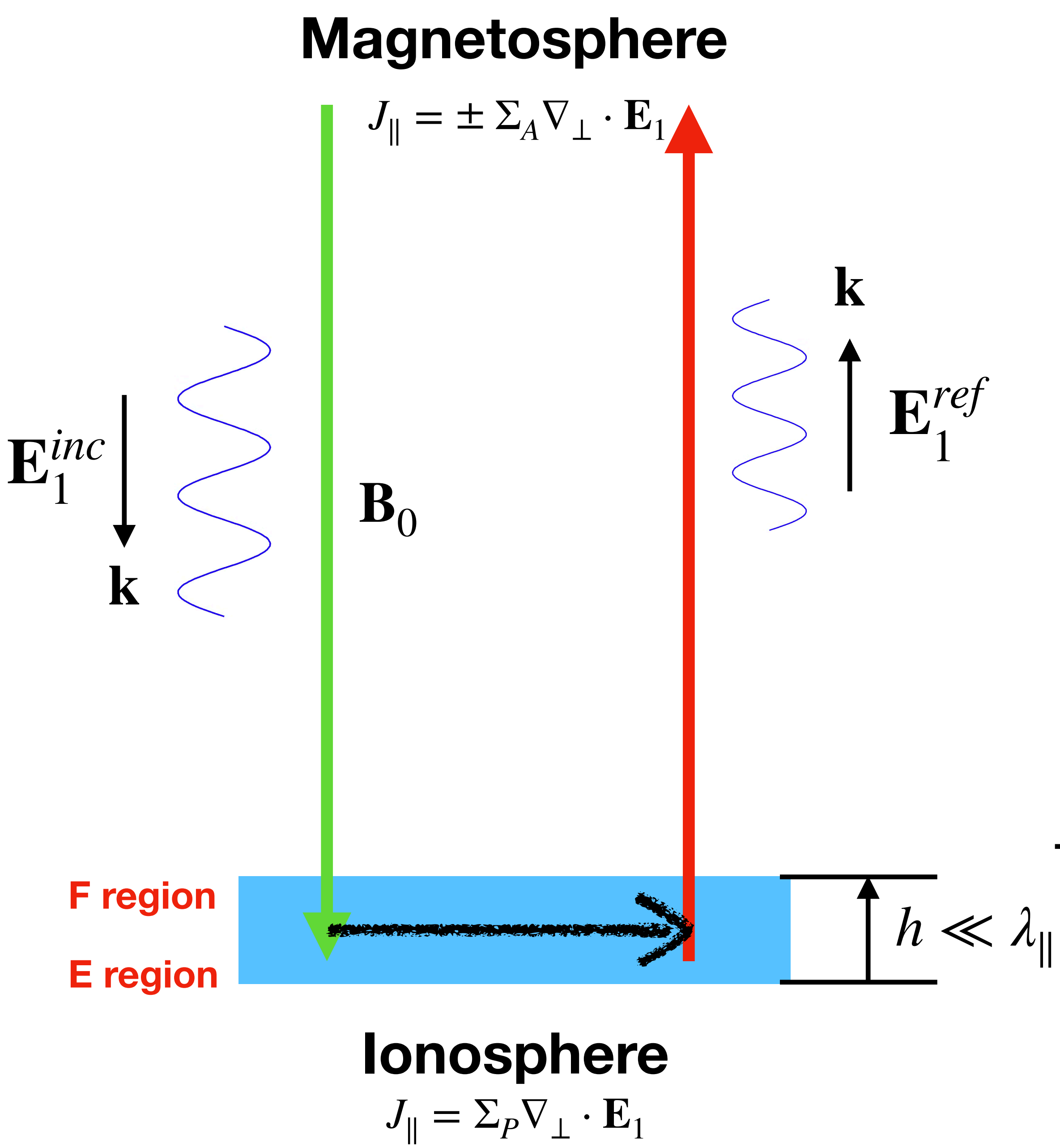
$$\nabla \cdot \mathbf{J} = 0 \longrightarrow J_\parallel = \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1)_\parallel = \boxed{\pm}\Sigma_A \nabla_\perp \cdot \mathbf{E}_1$$

Direction of propagation

Reflection of Alfvén waves above Ionosphere

Electrostatic Magnetosphere-Ionosphere Coupling $\nabla \cdot \mathbf{J} = 0$

Slab geometry



Above the ionosphere

$$J_{\parallel} = J_{\parallel}^{inc} + J_{\parallel}^{ref} = \Sigma_A \nabla_\perp \cdot (\mathbf{E}_1^{inc} - \mathbf{E}_1^{ref})$$

At the ionosphere

$$J_{\parallel} = J_{\parallel}^{inc} + J_{\parallel}^{ref} = \Sigma_P \nabla_\perp \cdot (\mathbf{E}_1^{inc} + \mathbf{E}_1^{ref})$$

So we get

$$\Sigma_A \nabla_\perp \cdot (\mathbf{E}_1^{inc} - \mathbf{E}_1^{ref}) - \Sigma_P \nabla_\perp \cdot (\mathbf{E}_1^{inc} + \mathbf{E}_1^{ref}) = 0$$

$$\longrightarrow \quad \nabla_\perp \cdot [(\Sigma_A - \Sigma_P)\mathbf{E}_1^{inc} - (\Sigma_A + \Sigma_P)\mathbf{E}_1^{ref})] = 0$$

Reflection Coefficient

$$R \equiv \frac{E_1^{ref}}{E_1^{inc}} = \frac{\Sigma_A - \Sigma_P}{\Sigma_A + \Sigma_P}$$

So Reflection of Alfvén wave happens when: 1) going through the ionosphere;
2) sigma_A changes

The Alfvén Waves model for Saturation

Saturation of the polar cap potential: Inference from Alfvén wing arguments

Margaret G. Kivelson[1,2] and Aaron J. Ridley[3]

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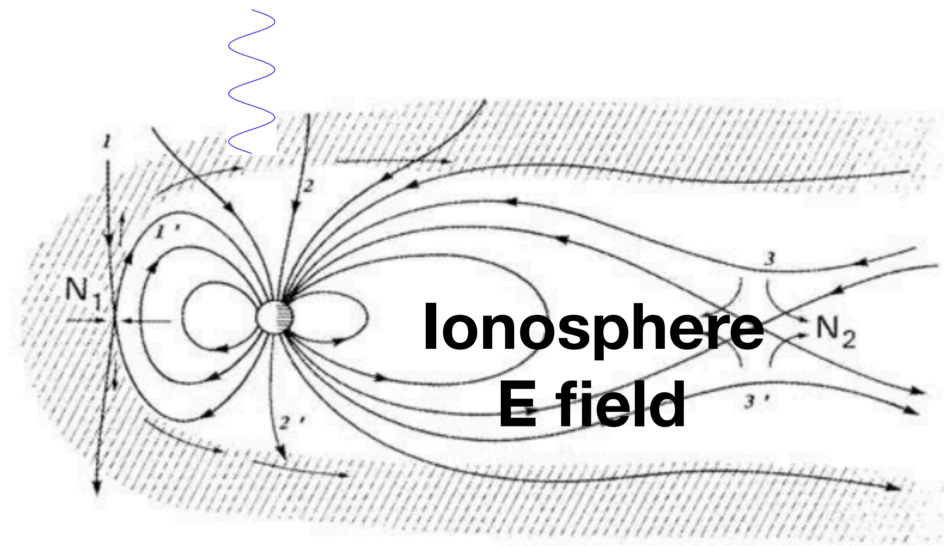
[1] The cross polar cap potential varies roughly linearly with the solar wind electric field for nominal conditions but asymptotes to a constant value of order 200 kV for large electric field. When the impedance of the solar wind across open polar cap field lines dominates the impedance of the ionosphere, Alfvén waves incident from the solar wind are partially reflected, reducing the signal in the polar cap. Thus, the ratio of the cross polar cap potential to the potential imposed by the solar wind is $2\Sigma_A/(\Sigma_P + \Sigma_A)$, where Σ_A is the Alfvén conductance of the solar wind ($= (\rho_{sw}/\mu_o)^{1/2}/B_{sw}$) to within a density-dependent factor on average of order 1, Σ_P is the Pedersen conductance of the ionosphere, and ρ_{sw} (B_{sw}) is the density (magnetic field magnitude) of the solar wind. For small B_{sw}, the response is proportional to B_{sw}. For large B_{sw}, the cross polar cap potential depends only on the solar wind dynamic pressure (with small viscous and density-dependent corrections). Quantitative estimates require knowledge of Σ_P and the dependence of the potential imposed by the solar wind on its measured properties; standard assumptions yield saturation levels consistent with observations made during 13 storm intervals. Previous explanations of saturation have invoked changing reconnection efficiency, specific characteristics of the Region 1 current system, or the effect of the bow shock on the reconnecting plasma. Although our relation is mathematically similar to some previously proposed, our arguments place no constraints on reconnection efficiency or on magnetospheric geometry.

Citation: Kivelson, M. G., and A. J. Ridley (2008), Saturation of the polar cap potential: Inference from Alfvén wing arguments, *J. Geophys. Res.*, *113*, A05214, doi:10.1029/2007JA012302.

Alfvén wave E-field



$$\frac{\phi_{ion}}{\phi_{SW}} \sim \frac{2\Sigma_A}{\Sigma_A + \Sigma_P}$$

Issue: From SW to the ionosphere, Alfven conductance changes orders of magnitude

e.g., in SW, V_A ~ 40 km/s
Above the ionosphere, V_A ~ 4000 km/s

Non-Ideal MHD Waves

- **The effects of the displacement current -** *Boris correction*
- **Resistive damping -** *Alfvenic heating of the thermosphere*
- **Hall MHD waves -** *low-frequency whistler and drift waves*
- **Dispersive Alfvén waves -** *parallel electric field/acceleration*

Effects of the Displacement Current

Recall the Ampere's law used in ideal MHD: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

While the Ampere-Maxwell's law gives $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ $\qquad \frac{|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}|}{|\nabla \times \mathbf{B}|} \sim \frac{1}{c^2}\frac{L_0 E}{t_0 B} \sim \frac{V_A^2}{c^2} \ll 1$

Which means when $V_A \sim c$ the displacement current cannot be ignored

Now the MHD current becomes

$$\mathbf{J} = \frac{1}{\mu_0}\left[\nabla \times \mathbf{B} + \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{u} \times \mathbf{B})\right]$$

$$\frac{\partial}{\partial t}\rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \longrightarrow \boxed{\frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla \cdot \mathbf{u}_1 = 0}$$

$\perp \mathbf{B}_0$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + V_s^2 \nabla \rho - \mathbf{J} \times \mathbf{B} = 0$$

$$\longrightarrow \boxed{\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + V_s^2 \nabla \rho_1 - \mathbf{B}_0 \times \left(\nabla \times \mathbf{B}_1 + \boxed{\frac{1}{c^2}\frac{\partial \mathbf{u}_1}{\partial t} \times \mathbf{B}_0}\right) = 0}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \longrightarrow \boxed{\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)}$$

Effects of the Displacement Current

1 $$\frac{\partial}{\partial t}\rho_1 + \rho_0 \nabla \cdot \mathbf{u}_1 = 0$$

2 $$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} + V_s^2 \nabla \rho_1 - \mathbf{B}_0 \times \left(\nabla \times \mathbf{B}_1 + \frac{1}{c^2}\frac{\partial \mathbf{u}_1}{\partial t} \times \mathbf{B}_0 \right) = 0$$

3 $$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0)$$

$$\frac{\partial}{\partial t}\, 2 \longrightarrow \rho_0 \frac{\partial^2 \mathbf{u}_1}{\partial t^2} + V_s^2 \nabla \frac{\partial \rho_1}{\partial t} + \mathbf{B}_0 \times \left(\nabla \times \frac{\partial \mathbf{B}_1}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{u}_1}{\partial t^2} \times \mathbf{B_0} \right) = 0$$

$$\longrightarrow \frac{\partial^2 \mathbf{u}_1}{\partial t^2} - V_s^2 \nabla(\nabla \cdot \mathbf{u}_1) + \mathbf{V}_A \times \{\nabla \times [\nabla \times (\mathbf{u}_1 \times \mathbf{V}_A)]\} + \frac{1}{c^2}\mathbf{V}_A \times \left(\frac{\partial^2 \mathbf{u}_2}{\partial t^2} \times \mathbf{V}_A \right) =$$

Use 1 3

$$\frac{1}{c^2}\mathbf{V}_A \times \left(\frac{\partial^2 \mathbf{u}_2}{\partial t^2} \times \mathbf{V}_A \right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\mathbf{V}_A \times \mathbf{u}_1 \times \mathbf{V}_A\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{u}_{1\perp} V_A^2$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[V_A^2 \mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_a)\mathbf{V}_A]$$

Effects of the Displacement Current

$$\frac{\partial^2 \mathbf{u}_1}{\partial t^2} - V_s^2 \nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A \times \{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\} + \frac{1}{c^2}\mathbf{V}_A \times \left(\frac{\partial^2 \mathbf{u}_2}{\partial t^2}\times\mathbf{V}_A\right) = 0$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[V_A^2\mathbf{u}_1 - (\mathbf{V}_A\cdot\mathbf{u}_1)\mathbf{V}_A]$$

$$\frac{\partial^2}{\partial t^2}\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_1 - \underbrace{(\mathbf{V}_A\cdot\mathbf{u}_1)\frac{\mathbf{V}_A}{c^2}}_{\frac{V_A^2}{c^2}\mathbf{u}_{1\parallel}}\right] - V_s^2 \nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A \times \{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\} = 0$$

$$\frac{\partial^2}{\partial t^2}\underbrace{\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_{1\perp} + \mathbf{u}_{1\parallel}\right]}_{\textbf{Perp u is modified}} \underbrace{- V_s^2 \nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A \times \{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\}}_{\textbf{The same terms as the ideal MHD modes}} = 0$$

w/o dE/dt: $$\frac{\partial^2}{\partial t^2}\left[\mathbf{u}_{1\perp} + \mathbf{u}_{1\parallel}\right] - V_s^2 \nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A \times \{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\} = 0$$

Effects of the Displacement Current

$$\frac{\partial^2}{\partial t^2}\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_1-(\mathbf{V}_A\cdot\mathbf{u}_1)\frac{\mathbf{V}_A}{c^2}\right]-V_s^2\nabla(\nabla\cdot\mathbf{u}_1)+\mathbf{V}_A\times\{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A)]\}=0$$

Recall $\frac{\partial}{\partial t}\sim -i\omega$ $\quad\nabla\sim i\mathbf{k}$ **Convert to algebra equations**

$$-\omega^2\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_1-(\mathbf{V}_A\cdot\mathbf{u}_1)\frac{\mathbf{V}_A}{c^2}\right]-V_s^2\mathbf{k}(\mathbf{k}\cdot\mathbf{u}_1)+\mathbf{V}_A\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{u}_1\times\mathbf{V}_A)]\}=0$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

bac-cab law

$$-\omega^2\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_1-(\mathbf{V}_A\cdot\mathbf{u}_1)\frac{\mathbf{V}_A}{c^2}\right]+(V_s^2+V_A^2)(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k}$$

$$-(\mathbf{V}_A\cdot\mathbf{u}_1)\mathbf{k}(\mathbf{k}\cdot\mathbf{V})_A[(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{u}_1-(\mathbf{V}_A\cdot\mathbf{u}_1)\mathbf{k}-(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{V}_A]=0$$

Now let's take a look at MHD waves perp and para to B_0

Effects of the Displacement Current

$$\mathbf{k} \perp \mathbf{V_A}$$

$$-\omega^2 \left[\left(1 + \frac{V_A^2}{c^2} \right) \mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_1) \frac{\mathbf{V}_A}{c^2} \right] + (V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k}$$

$$-(\mathbf{k} \cdot \mathbf{V})_A [(\mathbf{k} \cdot \mathbf{V}_A)\mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_1)\mathbf{k} - (\mathbf{k} \cdot \mathbf{u}_1)\mathbf{V}_A] = 0$$

$$\mathbf{k} \perp \mathbf{V_A} \longrightarrow \mathbf{k} \cdot \mathbf{V_A} = 0 \longrightarrow -\omega^2 \left(1 + \frac{V_A^2}{c^2} \right) \mathbf{u}_1 + (V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k}$$

Recall the longitudinal magneto sonic wave: $\mathbf{u}_1 = \frac{1}{\omega^2}(V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k}$

$$\longrightarrow \mathbf{u}_1 \parallel \mathbf{k} \quad \text{longitudinal wave} \quad \xrightarrow{\mathbf{u}_1 \cdot \mathbf{k} = u_1 k} 1 = \frac{k^2}{\omega^2}(V_s^2 + V_A^2) \longrightarrow \frac{\omega}{k} = \sqrt{V_s^2 + V_A^2} = V_F$$

Magnetosonic wave

The new dispersion relation is simply

$$\longrightarrow \frac{\omega}{k} = \sqrt{\frac{V_s^2 + V_A^2}{1 + V_A^2/c^2}} = \frac{V_F}{\sqrt{1 + V_A^2/c^2}}$$

Effects of the Displacement Current

$\mathbf{k} \| \mathbf{V_A}$

$$-\omega^2 \left[\left(1 + \frac{V_A^2}{c^2} \right) \mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_1) \frac{\mathbf{V}_A}{c^2} \right] + (V_s^2 + V_A^2)(\mathbf{k} \cdot \mathbf{u}_1)\mathbf{k}$$

$$-(\mathbf{k} \cdot \mathbf{V}_A)[(\mathbf{k} \cdot \mathbf{V}_A)\mathbf{u}_1 - (\mathbf{V}_A \cdot \mathbf{u}_1)\mathbf{k} - (\mathbf{k} \cdot \mathbf{u}_1)\mathbf{V}_A] = 0$$

$\mathbf{k} \| \mathbf{V_A}$

$\mathbf{u}_1 \| \mathbf{k}$:

$$-\omega^2 + k^2 V_s^2 = 0 \longrightarrow \frac{\omega}{k} = \pm V_s$$

Longitudinal mode unaffected by $\dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t}$

$\mathbf{u}_1 \perp \mathbf{k}$:

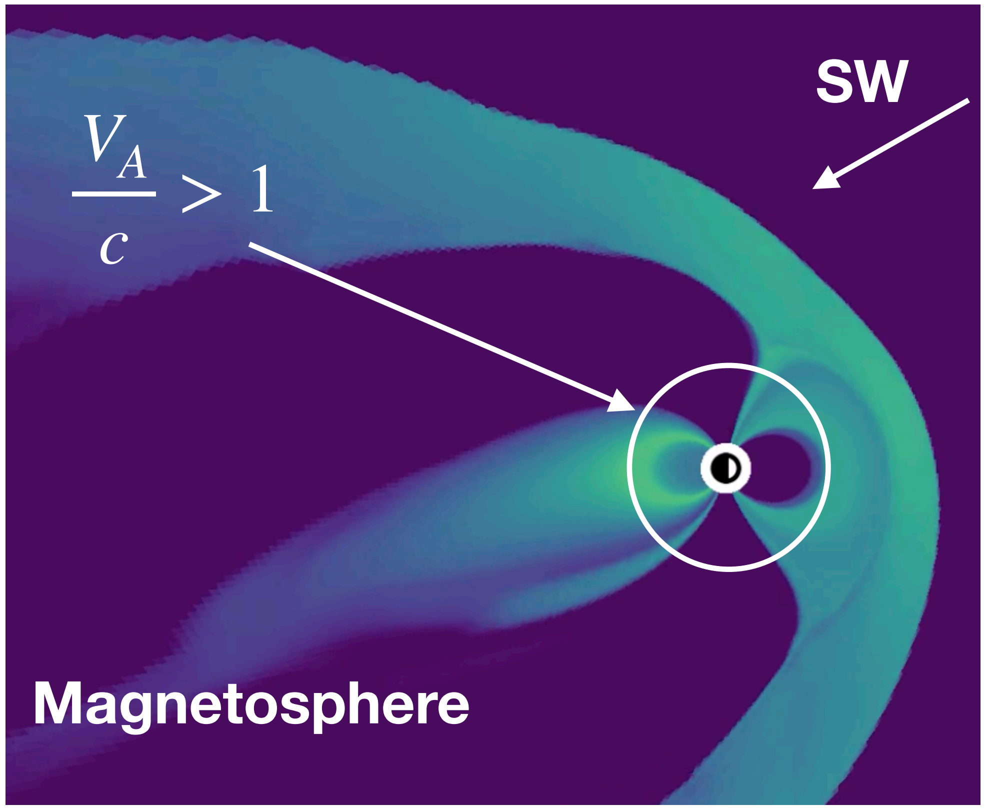
$$-\omega^2 \left(1 + \frac{V_A^2}{c^2} \right) \mathbf{u}_1 + k^2 V_A^2 \mathbf{u} = 0 \longrightarrow \frac{\omega}{k} = \pm \frac{V_A}{\sqrt{1 + V_A^2/c^2}}$$

Transverse mode Reduced speed by $\dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t}$

In the usual limit of $(V_A/c)^2 \ll 1$, the transverse mode reduces to pure Alfvén and the effect of the displacement current is unimportant. When using these results, however, it must be kept in mind that they are **valid only for frequencies such that charge separation effects are negligible**, since the electric force term has been neglected in the equation of motion

Effects of the Displacement Current

The Boris Correction



typically, inside 6-8 RE, the phase speed of pure Alfvén mode approaches the speed of light c

$$\frac{V_A}{c} > 1$$

Which limits the timestep of an explicit solver fo the MHD equations

How to resolve this?

Momentum eq:

$$\frac{\partial}{\partial t}\rho \mathbf{u} = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \mathbf{J}\times\mathbf{B}$$

Consider displacement current

$$\mathbf{J} = \nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$$

The modified momentum equation is now

$$\frac{\partial}{\partial t}\rho \mathbf{u} = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \left(\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}\right)\times\mathbf{B}$$

Since

$$\frac{\partial \mathbf{E}}{\partial t}\times\mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{\partial \mathbf{B}}{\partial t}\times\mathbf{E} = \frac{\partial}{\partial t}(\mathbf{B}\times\mathbf{u}\times\mathbf{B}) + \mathbf{E}\times\nabla\times\mathbf{E}$$

The momentum equation becomes

$$\frac{\partial}{\partial t}\left(\rho\mathbf{u} + \frac{\mathbf{B}\times\mathbf{u}\times\mathbf{B}}{c^2}\right) = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \nabla\times\mathbf{B}\times\mathbf{B} + \overbrace{\cancel{\frac{1}{c^2}\nabla\times\mathbf{E}\times\mathbf{E}}}^{\mathbf{0}}$$

$$\rho\left(\mathbf{u} + \frac{V_A^2}{c^2}\mathbf{u}_\perp\right) \quad \sim \mathcal{O}\left(\frac{V_A^2}{c^2}\right) \qquad \sim \mathcal{O}\left(\frac{u^2}{c^2}\right)$$

Effects of the Displacement Current

The Alfvén (Boris) Correction

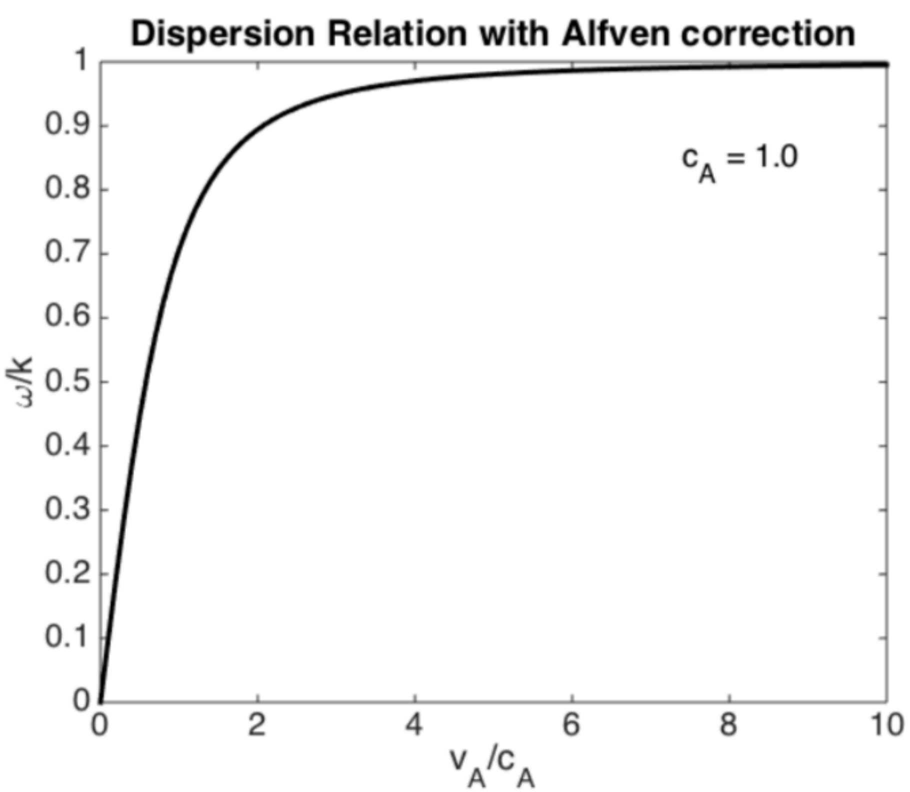
The momentum equation w/ displacement current:

$$\frac{\partial}{\partial t}\left(\rho\mathbf{u} + \frac{\mathbf{B}\times\mathbf{u}\times\mathbf{B}}{c^2}\right) = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \nabla\times\mathbf{B}\times\mathbf{B}$$

$$\xrightarrow{\mathbf{B}\times\mathbf{u}\times\mathbf{B} = B^2\mathbf{u}_\perp} \quad \frac{\partial}{\partial t}\left[\rho\mathbf{u}\left(1+\frac{V_A^2}{c^2}\right) - \frac{(\mathbf{B}\cdot\mathbf{u}\mathbf{B})}{c^2}\right] = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \nabla\times\mathbf{B}\times\mathbf{B}$$

$$= B^2(\mathbf{u} - \mathbf{u}_\parallel)$$
$$= B^2\mathbf{u} - (\mathbf{B}\cdot\mathbf{u})\mathbf{B}$$

$$\longrightarrow \quad \frac{\partial}{\partial t}\left[\rho\mathbf{u}\left(1+\frac{V_A^2}{c^2}\right)\right] = -\nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}P) + \nabla\times\mathbf{B}\times\mathbf{B} + \frac{\partial}{\partial t}\left(\frac{V_A^2}{c^2}\rho\mathbf{u}_\parallel\right)$$



- The Alfvén wave phase is limited to *c*
- The perpendicular momentum is increased by a factor of (Va/c)^2 - magnetic mass kicks in
- In global magnetosphere simulations we usually use a “reduced speed of light” - c = 1000-2000 km/s
- Need to be careful when analyzing Alfvén wave propagation in global MHD models!

Effects of Dissipation Terms

resistive/viscous damping

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0$$

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + V_s^2\nabla\rho - \mathbf{J}\times\mathbf{B} - \rho\eta_k\nabla^2\mathbf{u} = 0$$

Fluid viscosity

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta_m\nabla^2\mathbf{B} = 0$$

Magnetic viscosity

Linearized velocity equation

$$\boxed{\begin{aligned}&-\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right)\mathbf{u}_1\\ &+\left(1+\frac{i\eta_m k^2}{\omega}\right)V_s^2((\mathbf{k}\cdot\mathbf{u_1})\mathbf{k}\\ &-\mathbf{V_A}\times[\mathbf{k}\times(\mathbf{k}\times(\mathbf{u_1}\times\mathbf{V_A})]] = 0\end{aligned}}$$

Linearize the equations w/ plane wave assumption

$$\frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 = 0 \longrightarrow \rho_1 = \frac{\mathbf{k}\cdot\mathbf{u_1}}{\omega}\rho_0$$

$$\rho_0\frac{\partial \mathbf{u}_1}{\partial t} + V_s^2\nabla\rho_1 - \mathbf{B}_0\times\nabla\times\mathbf{B}_1 - \rho_0\eta_k\nabla^2\mathbf{u}_1 = 0 \longrightarrow \omega\mathbf{u}_1 = \frac{\rho_1}{\rho_0}V_s^2\mathbf{k} + \frac{1}{\rho_0}\mathbf{B_0}\times(\mathbf{k}\times\mathbf{B_1}) - i\eta_k k^2\mathbf{u}_1$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{u}_1\times\mathbf{B}_0) + \eta_m\nabla^2\mathbf{B}_1 \longrightarrow \mathbf{B}_1 = -\frac{1}{\omega + i\eta_m k^2}\mathbf{k}\times(\mathbf{u_1}\times\mathbf{B_0})$$

Effects of Dissipation Terms

resistive/viscous damping

The eigenvalue problem is now

$$-\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right)\mathbf{u}_1+\left(1+\frac{i\eta_m k^2}{\omega}\right)V_s^2((\mathbf{k}\cdot\mathbf{u_1})\mathbf{k}+\mathbf{V_A}\times[\mathbf{k}\times(\mathbf{k}\times(\mathbf{u_1}\times\mathbf{V_A})]]=0$$

Recall for the ideal MHD equations: $$-\omega^2\mathbf{u}_1-V_s^2(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k}+\mathbf{V}_A\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{u}_1\times\mathbf{V}_A)]\}=0$$

for the ideal MHD equations w/ displacement currents:

$$-\omega^2\left[\left(1+\frac{V_A^2}{c^2}\right)\mathbf{u}_1-(\mathbf{V}_A\cdot\mathbf{u}_1)\frac{\mathbf{V}_A}{c^2}\right]-V_s^2\mathbf{k}(\mathbf{k}\cdot\mathbf{u}_1)+\mathbf{V}_A\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{u}_1\times\mathbf{V}_A)]\}=0$$

Let's consider the pure Alfvén mode: $\mathbf{k}\|\mathbf{V_A}$ $\quad\mathbf{u}_1\perp\mathbf{k}$

$$-\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right)+k^2V_A^2=0$$

$$\frac{\omega}{k}=V_A+\mathcal{O}(V_A)$$

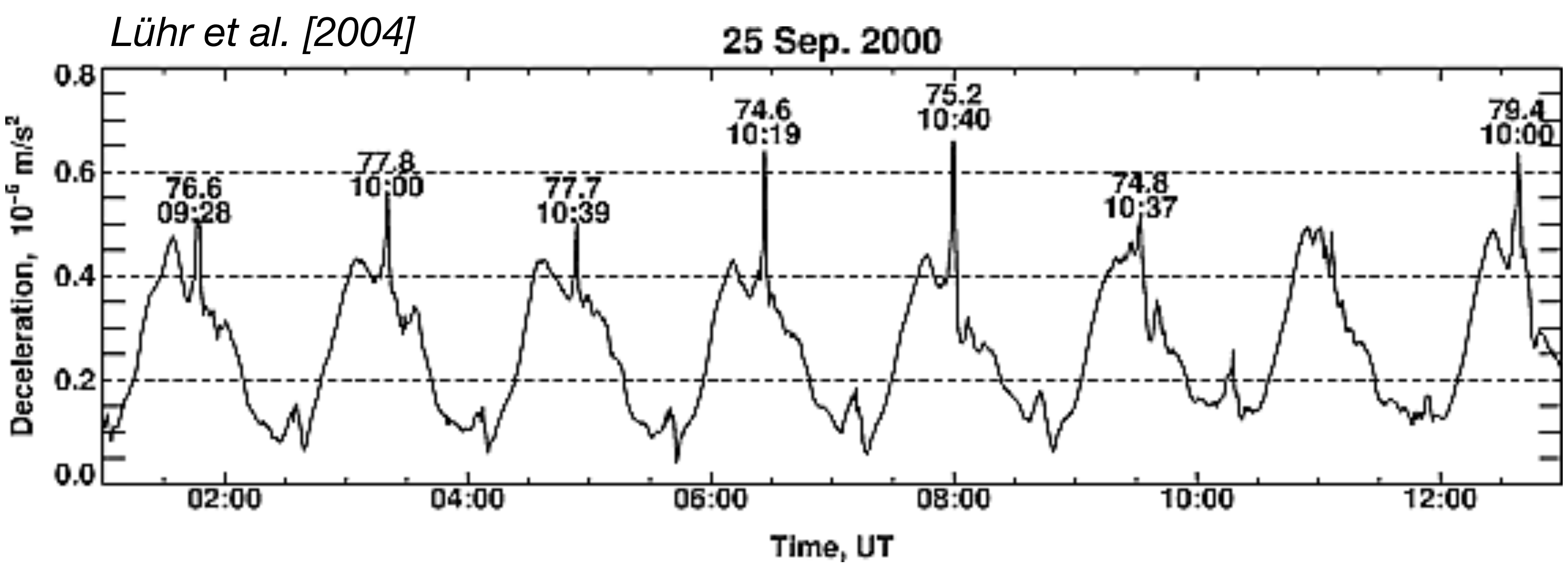
$$k^2V_A^2=\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right)=\omega^2\left[1+\frac{i(\eta_k+\eta_m)}{\omega}k^2-\cancelto{0}{\frac{\eta_k\eta_m}{\omega^2}k^4}\right]\approx\omega^2\left[1+\frac{i(\eta_k+\eta_m)}{V_A^2}\omega\right]$$

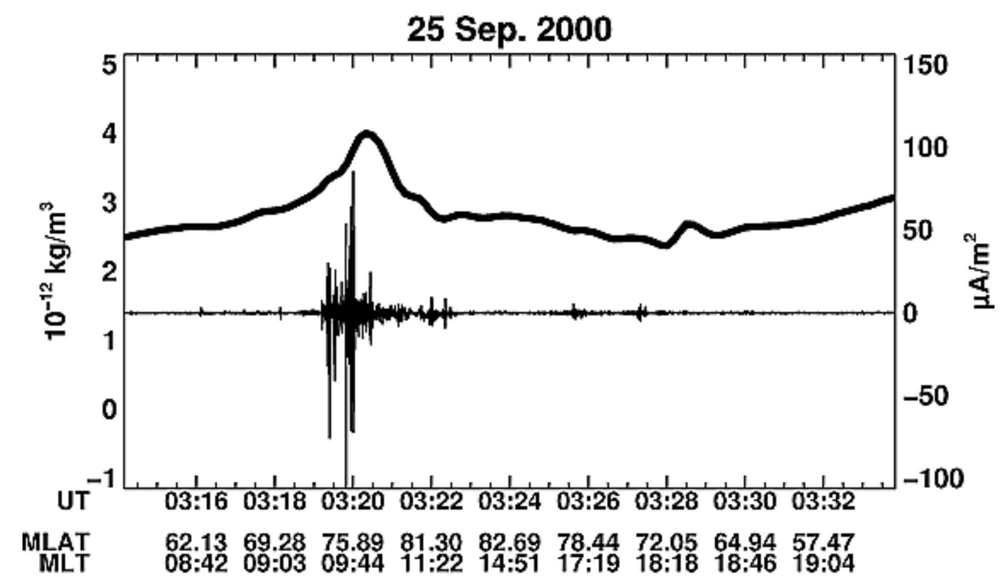
$$\xrightarrow{(1+x)^2\approx 1+x/2}\quad k\approx\frac{\omega}{V_A}+\boxed{\frac{i(\eta_k+\eta_m)}{2V_A^3}\omega^2}$$

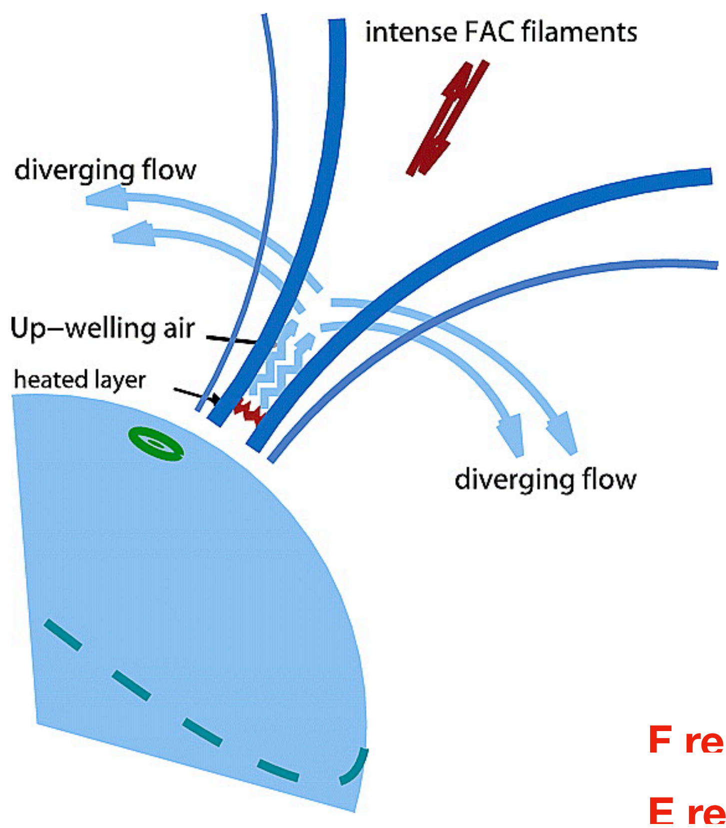
Damping! $\sim e^{ik\zeta}=e^{i\omega V_A\zeta}\cdot e^{-\alpha\zeta}$

Where did the wave energy go?

Alfvenic Heating of the Thermosphere



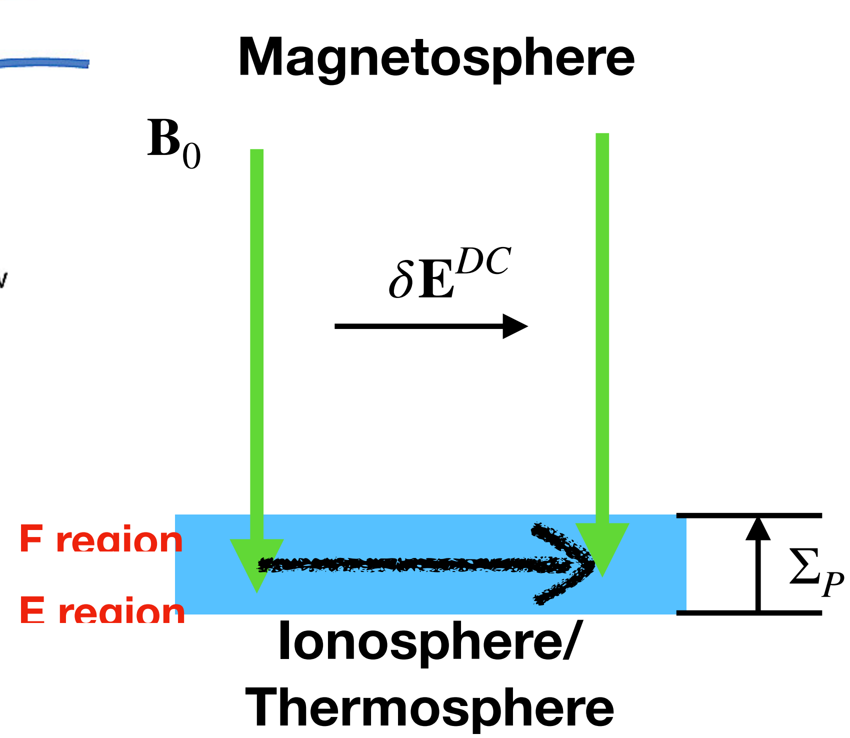




Lühr et al. [2004]

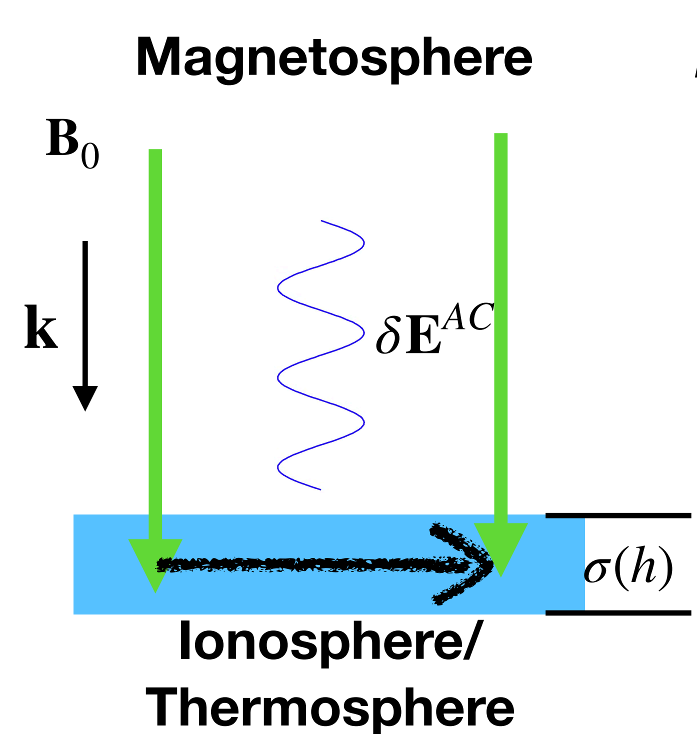
$$\mathbf{j}\cdot\mathbf{E} = \sigma_{||}E_{||}^2 + \sigma_p(E_\perp + \partial E_\perp)^2$$

DC Heating

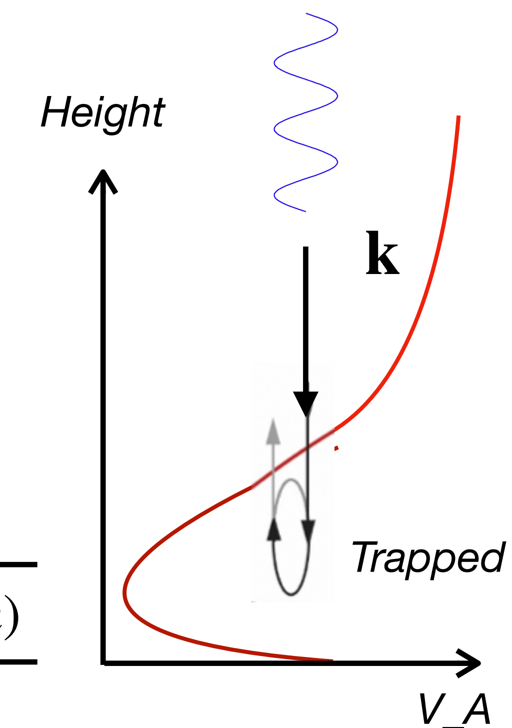


W_{QS} $\Sigma_P(\delta E^{DC})^2$

AC Heating



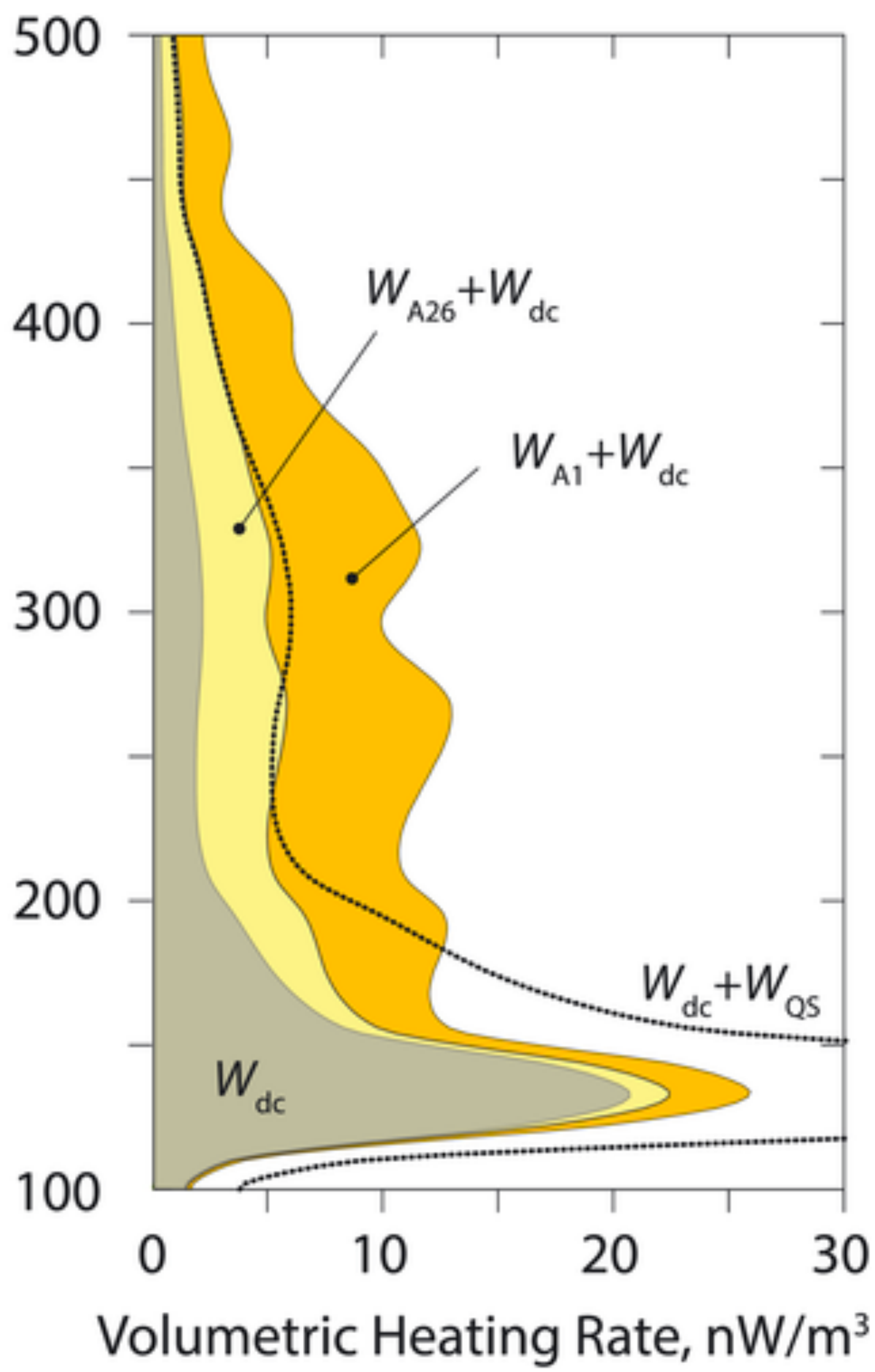
W_A $\iint \sigma_P[\delta E^{AC}(t)]^2 dz\,dt$



F-region resonator

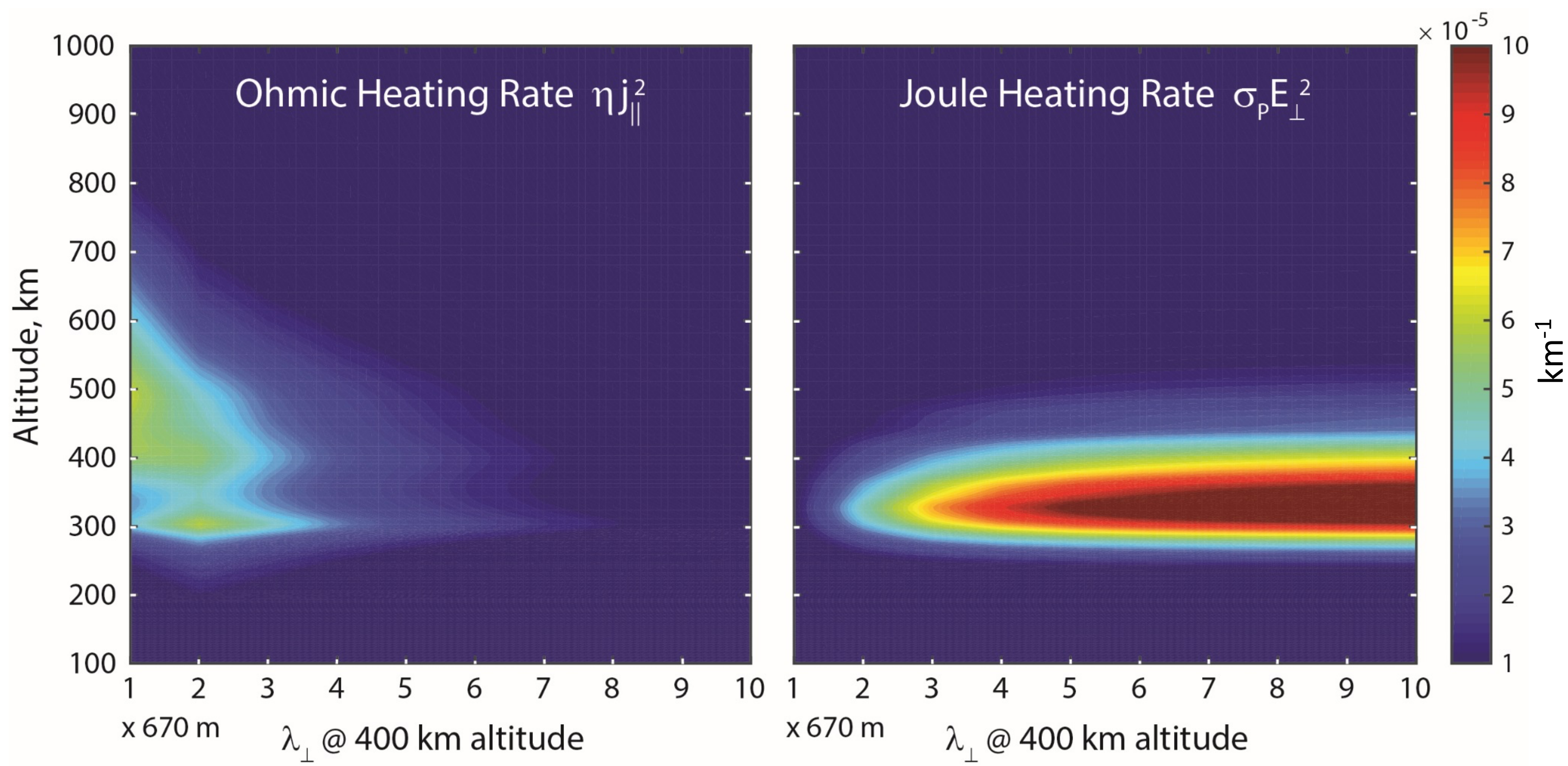
Alfvenic Heating - Model Results

AC heating vs DC heating



Integrated, normalized heating rates $\int_0^{t_\infty} dt \int_0^L dx\, W_H \Big/ \boxed{\int_0^{\tau_w} dt \int_0^L dx\, S_\parallel}$ constant

$\tau_w = 0.5$ s , $L = \lambda_\perp$



W_{dc}: the heating rate from the DC (convection) electric field
W_{QS}: the heating rate from the AC electric field treated in a DC way
W_{Ax}: the heating rate from the AC electric field treated as wave

conclusion:

W_{QS} over estimates the heating in the E region, underestimates F region

Waves in Hall MHD

Recall the single-fluid Hall MHD equations:

Mass conservation $$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation $$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

Energy conservation $$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Faraday's law $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Ampere's law $$\mathbf{J} = \nabla \times \mathbf{B}$$

Hall MHD

Ohm's law $$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{ne}\mathbf{J} \times \mathbf{B}$$

Magnetic Gauss $$\nabla \cdot \mathbf{B} = 0$$

$\mathbf{E} \times \mathbf{B}$ drift

$\frac{\mathbf{J}}{ne} = \mathbf{u}_+$; ion drift

The additional Hall Physics is in the Ohm's law! Electrons: fronzen-in; Ions: gyration

Linearize the Hall MHD equations

$$\frac{\partial}{\partial t}\rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0 \longrightarrow \boxed{\frac{\partial}{\partial t}\rho_1 + \rho_0\nabla\cdot\mathbf{u}_1 = 0} \quad 1$$

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + V_s^2\nabla\rho - \mathbf{J}\times\mathbf{B} = 0 \longrightarrow \boxed{\rho_0\frac{\partial\mathbf{u}_1}{\partial t} + V_s^2\nabla\rho_1 - \mathbf{B}_0\times\left(\nabla\times\mathbf{B}_1\right) = 0} \quad 2$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B} - \frac{1}{ne}\nabla\times\mathbf{B}\times\mathbf{B}) \longrightarrow \boxed{\frac{\partial\mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{u}_1\times\mathbf{B}_0 - \frac{1}{n_0 e}\nabla\times\mathbf{B_1}\times\mathbf{B_0})} \quad 3$$

*Ignore the Hall drift wave from ∇n_0

$$\frac{\partial}{\partial t}\ 2 \longrightarrow \rho_0\frac{\partial^2\mathbf{u}_1}{\partial t^2} + V_s^2\nabla\frac{\partial\rho_1}{\partial t} + \mathbf{B}_0\times\left(\nabla\times\frac{\partial\mathbf{B}_1}{\partial t}\right) = 0 \xrightarrow[\text{Substitute}]{1 \quad 3}$$

$$\frac{\partial^2\mathbf{u}_1}{\partial t^2} - V_s^2\nabla(\nabla\cdot\mathbf{u}_1) + \mathbf{V}_A\times\{\nabla\times[\nabla\times(\mathbf{u}_1\times\mathbf{V}_A - \frac{1}{n_0 e}\nabla\times\mathbf{B_1}\times\mathbf{B_0})]\} = 0$$

Errr… someone try out the math?

But, Hall term is not a resistivity! $\omega^2 \sim k^3$

A smart way to think about Hall MHD

Hall MHD **Ohm's law**

$$\mathbf{E} + \underbrace{\mathbf{u} \times \mathbf{B}}_{\text{① ExB drift, Ideal MHD scale}} = \frac{1}{ne}\mathbf{J} \times \mathbf{B} = \underbrace{\frac{1}{\mu_0 ne}\nabla \times \mathbf{B} \times \mathbf{B}}_{\text{② Hall drift, Ion scale}}$$

Compare the two terms:

$$\frac{2}{1} = \frac{\frac{1}{\mu_0 ne}\nabla \times \mathbf{B} \times \mathbf{B}}{\mathbf{u} \times \mathbf{B}} \xrightarrow[u \sim u_0]{\nabla \sim \frac{1}{L_0}} \frac{\frac{1}{\mu_0 ne}\frac{1}{L_0}B^2}{u_0 B} \longrightarrow \frac{\frac{B}{\mu_0 ne u_0}}{L_0} = \frac{L_{Hall}}{L_0}$$

L_Hall is the characteristic scale of the Hall drift:

$$L_{Hall} = \frac{B}{\mu_0 ne u_0} = \frac{B}{\sqrt{\mu_0 n M_i}} \cdot \frac{1}{\sqrt{\mu_0} \cdot \frac{ne}{\sqrt{nM_i}}} \cdot \frac{1}{u_0} = \frac{V_A}{u_0} \cdot \frac{1}{\sqrt{\mu_0 \epsilon} \cdot \sqrt{\frac{ne^2}{\epsilon_0 M_i}}} \sim \frac{c}{\omega_{pi}}$$

$\frac{V_A}{u_0}$: $\mathcal{O}(1)$

$\frac{c}{\omega_{pi}}$: d_i Ion skin depth

So

$$\frac{\frac{1}{\mu_0 ne}\nabla \times \mathbf{B} \times \mathbf{B}}{\mathbf{u} \times \mathbf{B}} \sim \frac{d_i}{L_0}$$

d_i: Ion inertial length

L_0: Ideal MHD characteristic length

- The Hall drift term has a much smaller scale compared to the ExB drift
- When consider the effect of Hall drift, the MHD ExB drift is basically uniform
- Then Hall physics decouples from the fluid dynamics locally

Whistler Mode in Hall MHD

Faraday's law $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} = -\nabla\times(\cancel{-\mathbf{u}}\times\mathbf{B} + \frac{1}{ne}\mathbf{J}\times\mathbf{B})$$

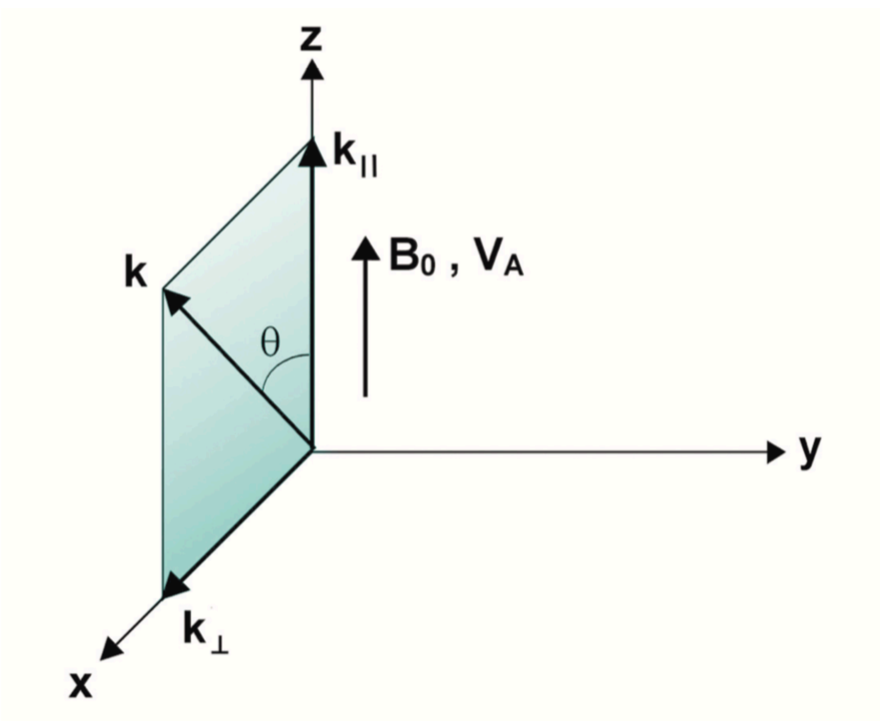
u ~ uniform

The Ohm's law has two additional terms:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times(\frac{1}{ne}\mathbf{J}\times\mathbf{B}) \quad = \underbrace{-\frac{1}{ne}\nabla\times(\mathbf{J}\times\mathbf{B})}_{\text{Whistler waves}} + \underbrace{\frac{1}{n^2 e}\nabla n\times(\mathbf{J}\times\mathbf{B})}_{\text{Hall drift waves}}$$

Let's take a look at the whistler mode:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{ne}\nabla\times(\mathbf{J}\times\mathbf{B}) \xrightarrow[\mathbf{J}=\nabla\times\mathbf{B}_1]{\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1} \frac{\partial \mathbf{B}_1}{\partial t} = -\frac{1}{ne}\nabla\times(\nabla\times\mathbf{B}_1\times\mathbf{B}_0)$$



$$\xrightarrow{\text{Plane wave}} -i\omega\mathbf{B}_1 = -\frac{1}{ne}\mathbf{k}\times(\mathbf{k}\times\mathbf{B}_1\times\mathbf{B}_0)$$

$$\mathbf{k} = k_z\hat{\mathbf{e}}_z$$

Only consider propagation along B_0: $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z \quad \mathbf{B}_1 = B_{1x}\hat{\mathbf{e}}_x + B_{1y}\hat{\mathbf{e}}_y$

$$\longrightarrow \begin{aligned} \omega B_{1x} &= -i\frac{k_z^2 B_0}{ne}B_{1y} \\ \omega B_{1y} &= i\frac{k_z^2 B_0}{ne}B_{1x} \end{aligned} \longrightarrow \frac{\omega}{k_z} = V_A\left(\frac{k_z c}{\omega_{pi}}\right) = V_A\left(k_z d_i\right)$$

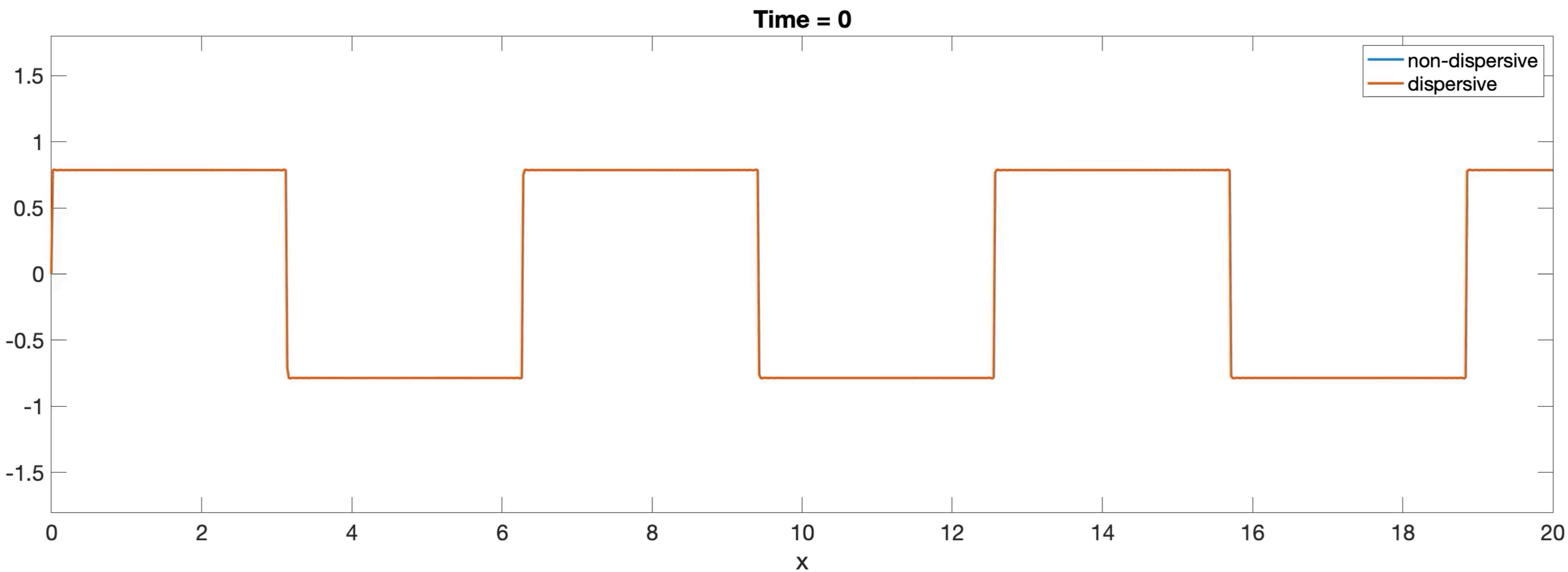
Whistler mode

Phase speed varies with Kz - dispersive

What is Dispersion

Non-dispersive : waves with different k travels at the same speed

dispersive : waves with different k travels at different speed



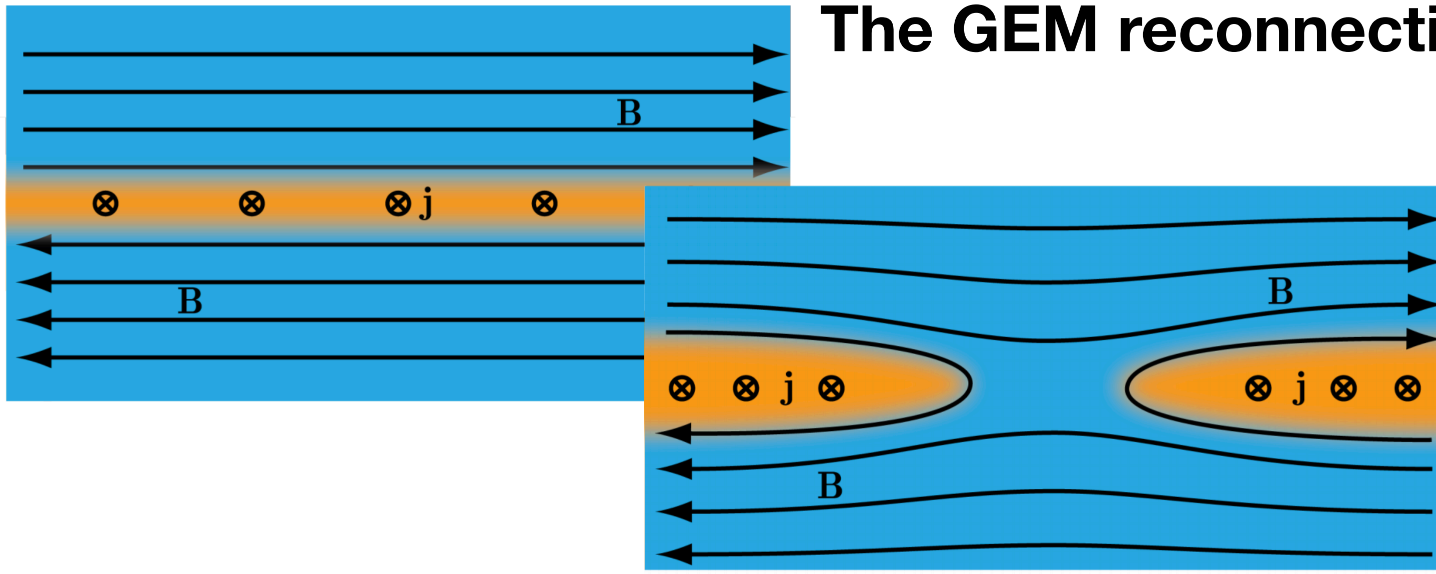
$$f(x - vt) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)(x - vt)]$$ Non-dispersive — "waveform" kept the same

$$= \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)(x - v(n)t)]$$ dispersive — "waveform" oscillatory

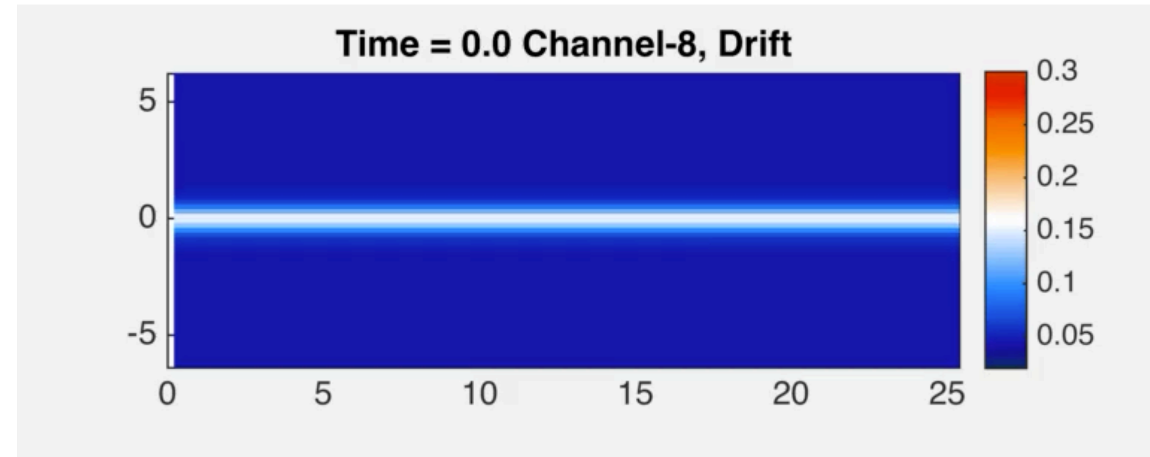
$$v(n) \sim \sqrt{2n+1}$$

Why Hall MHD?

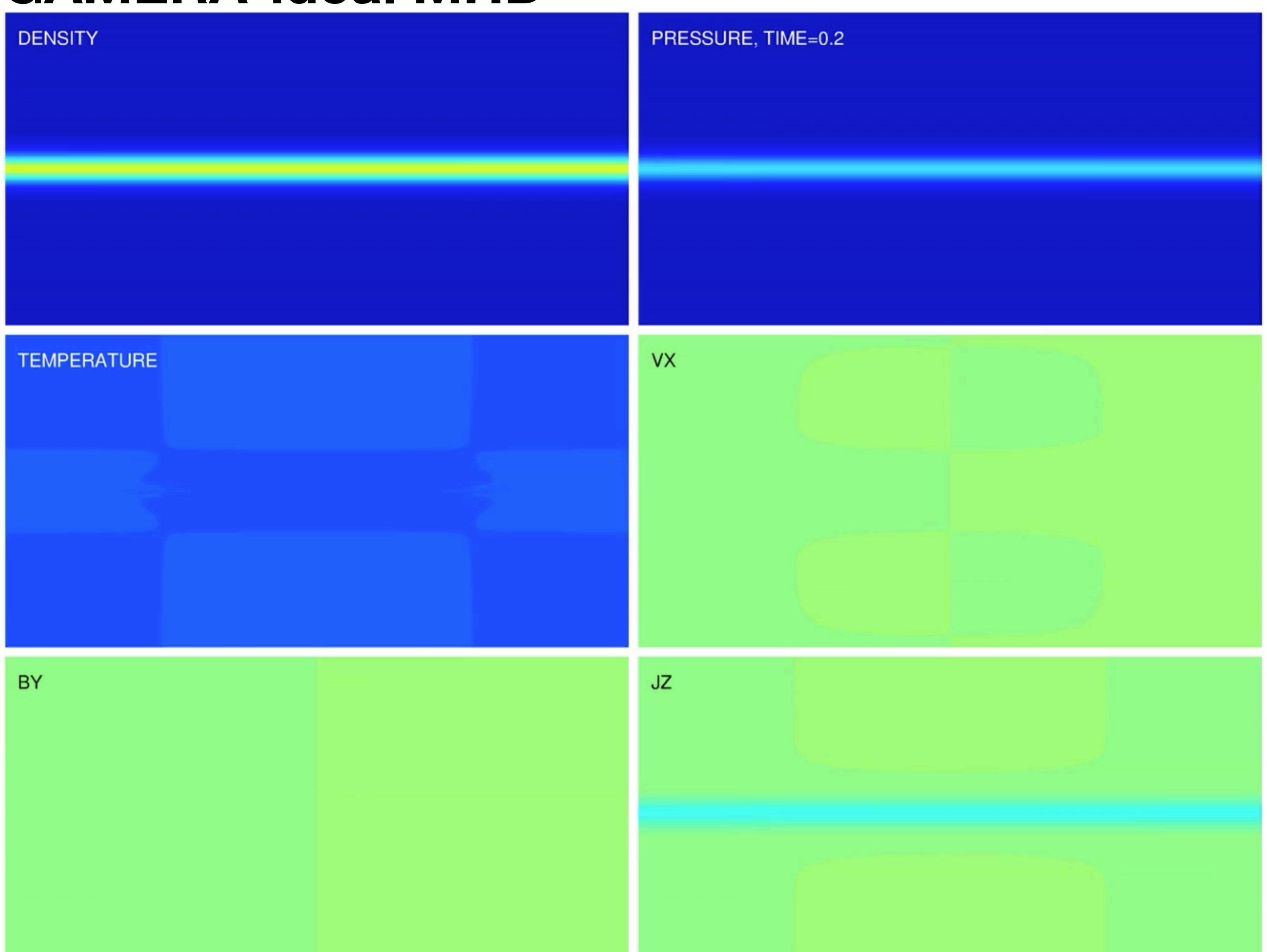
The GEM reconnection challenge

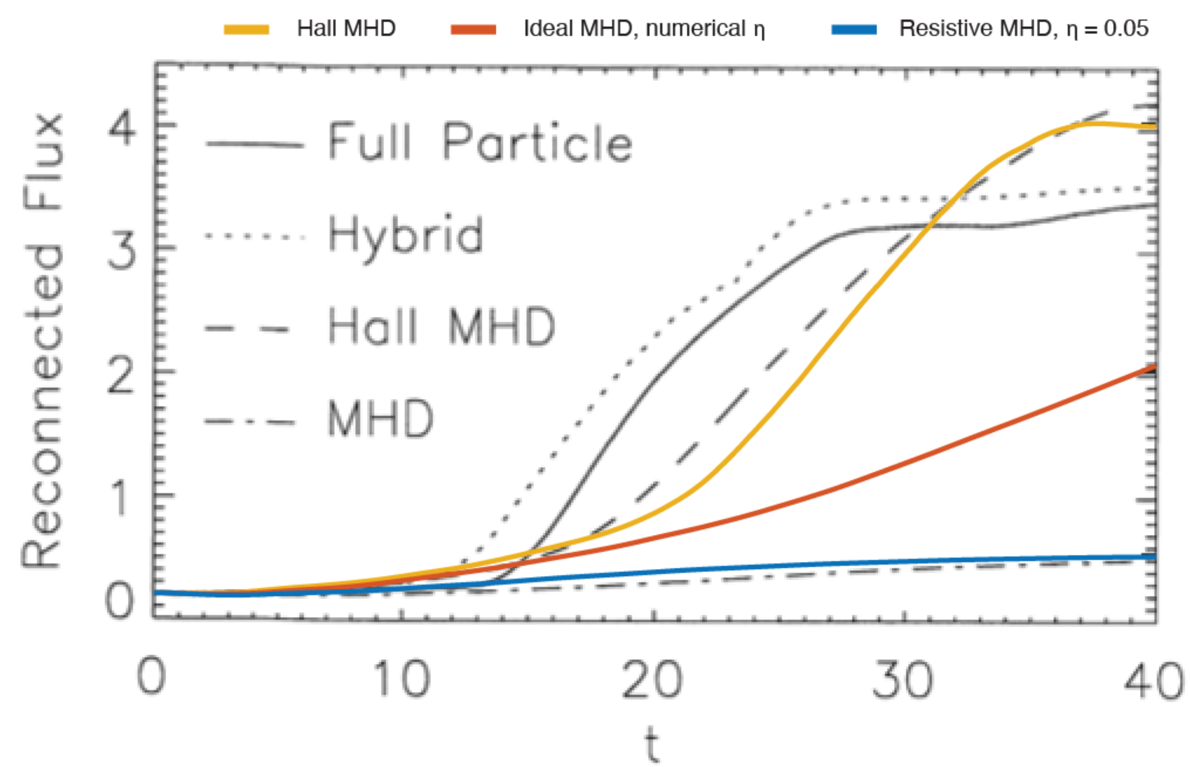


GAMERA-Hall MHD



GAMERA-ideal MHD





blue: resistive MHD - very slow

Red: numerical resistivity - 0.1

Yellow: Hall MHD - 0.2

Whistler Modes in Hall MHD vs Two-Fluid MHD

Two-fluid MHD: electron inertia is considered:

Two-fluid MHD

$$n_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$$

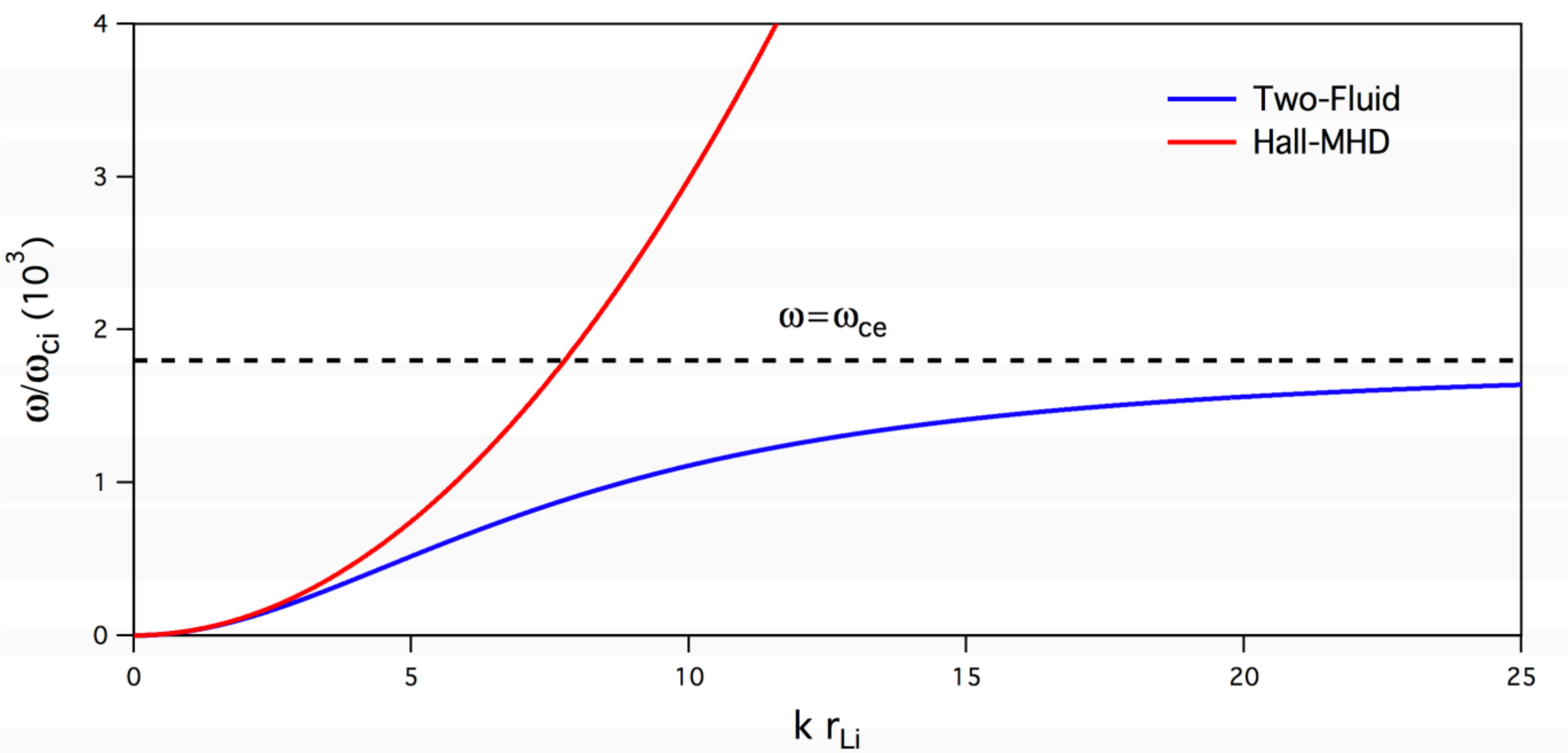
Electron motion

Hall MHD: electron is mass-less m_e -> 0:

Hall MHD

$$0 = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$$

Ohm's law



- At low frequency, long wavelength, Hall MHD matches two-fluid whistler waves (R mode*)
- At high frequency, Hall MHD misses resonance and diverges from two-fluid MHD, system becomes mathematically stiff (hard to solve)

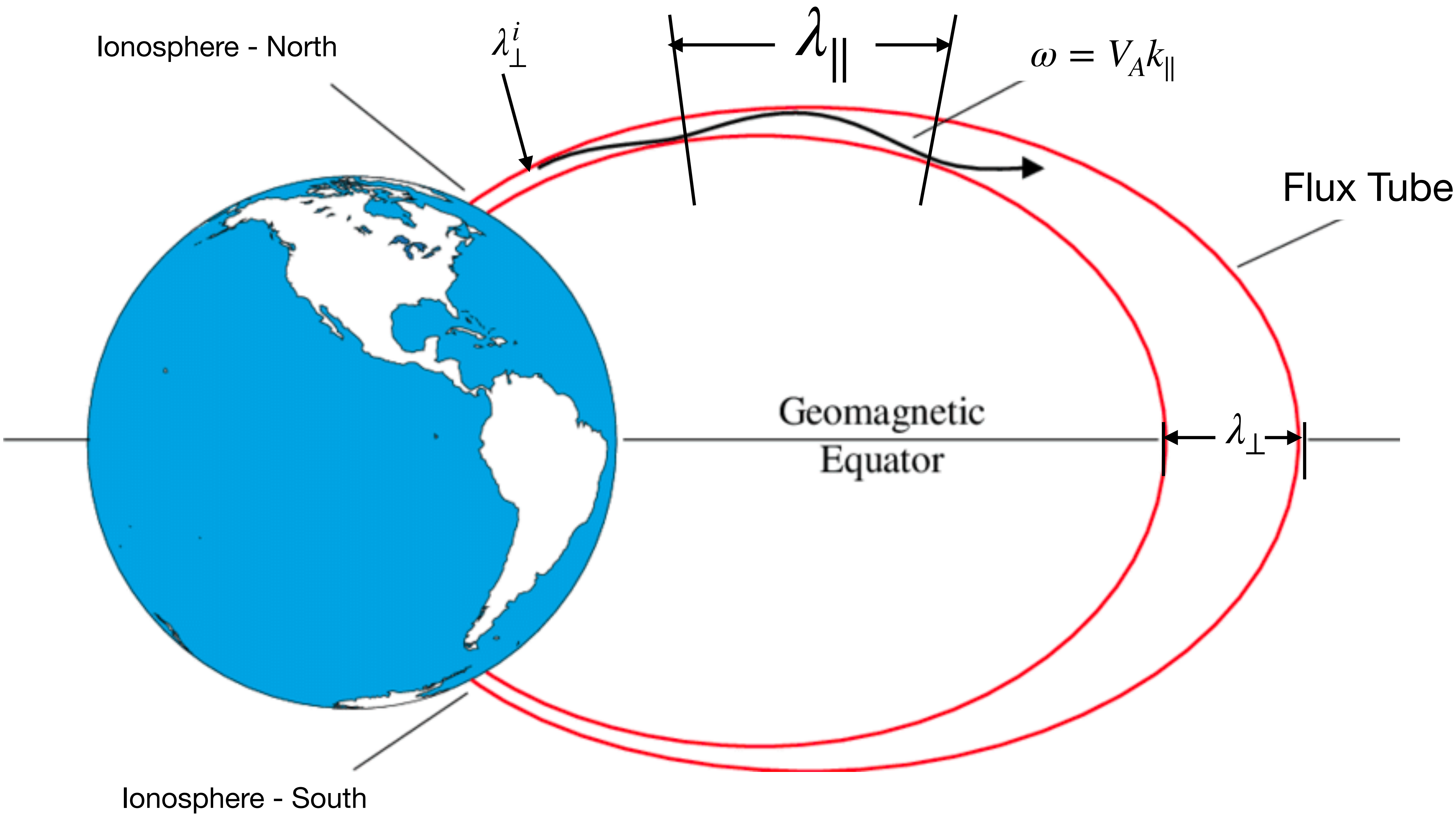
Two fluid MHD resolves:
Plasma frequency
Light wave

$m_e \to 0$

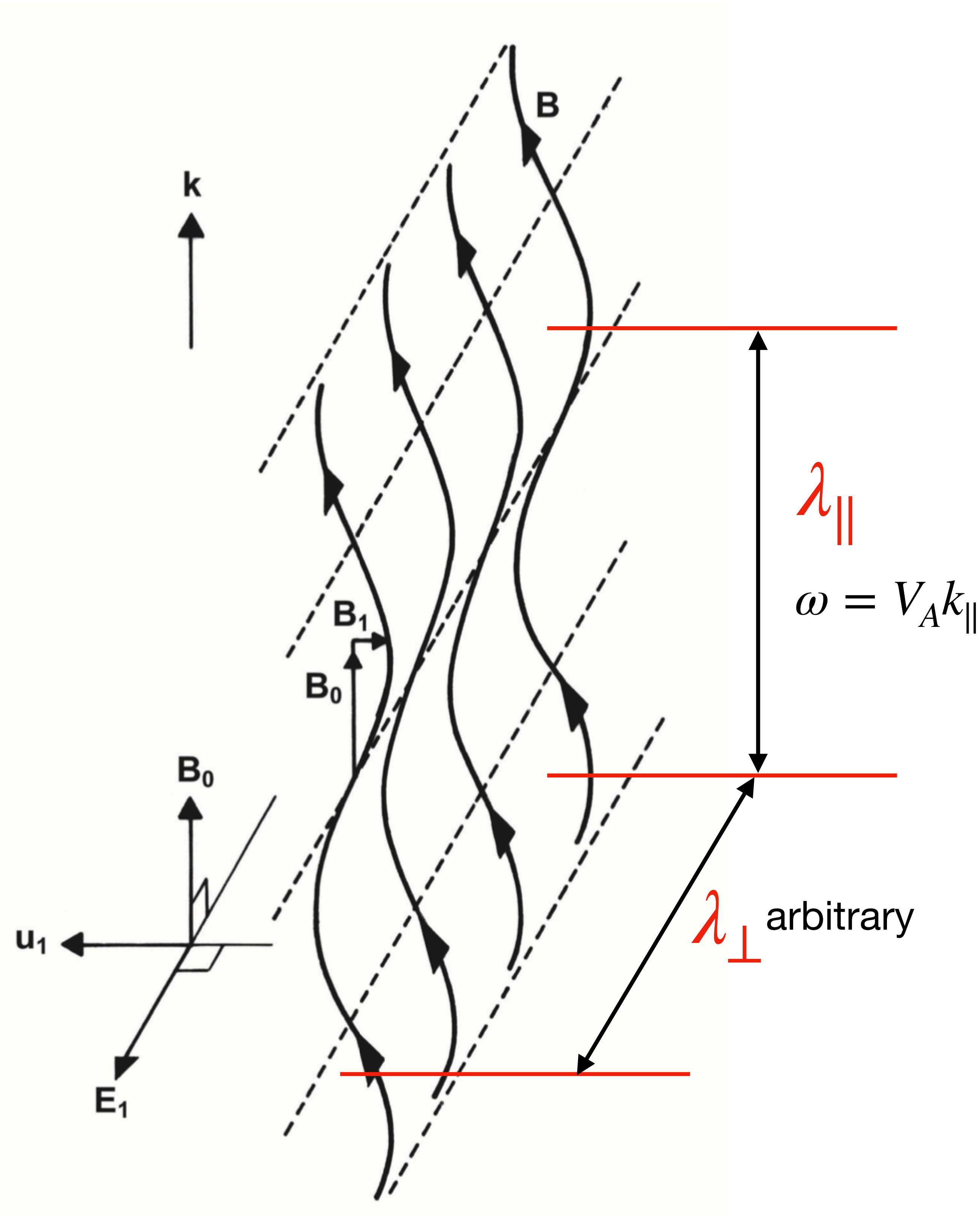
$c \to \infty$

Hall MHD resolves:
Low-frequency whistler

Perpendicular Scale Variation of Alfvén waves



Recall Alfvén waves



recall: The magnetic perturbation in shear Alfvén wave is

$$\mathbf{B}_1 = -\frac{B_0}{(\omega/k)}\mathbf{u}_1 \qquad \mathbf{u}_1 = -\frac{V_A}{B_0}\mathbf{B}_1$$

And the electric field is

$$\mathbf{E}_1 = -\mathbf{u}_1 \times \mathbf{B}_0 = -\frac{V_A}{B_0}\mathbf{B}_1 \times \mathbf{B}_0$$

So in the pure Alfvén mode, the plasma undergoes ExB drift only

For $\omega \ll \omega_{ci}$, the zeroth order plasma drift is **ExB**

the first order plasma drift is **polarization drift**

$$\mathbf{u}_d = \frac{M_i}{eB^2}\frac{d}{dt}\mathbf{E}_\perp \qquad \frac{u_d}{u_1} \sim \frac{\omega}{\omega_{ci}}$$

Now, this drift is species dependent - resulting a net perpendicular current:

$$\mathbf{J}_\perp = ne\mathbf{u}_d = \frac{nM_i}{B^2}\frac{d}{dt}\mathbf{E}_\perp = \frac{1}{\mu_0 V_A^2}\frac{d\mathbf{E}_\perp}{dt}$$

Dispersive Alfvén waves

think: like in the Hall MHD analysis, the additional (interesting) term takes place in the Ohm's law, which means the fluid dynamics (bulk motion) is decoupled from the Maxwell's equations; dispersive Alfvén waves occur when the perpendicular scale is small (approaching inertial length), so the fluid motion can also be ignored in the analysis and we only need the Maxwell's equations for deriving the dispersion relation

Trick: we use the vector and scalar potential when analysis electromagnetic fields:

$$\mathbf{E} = -\nabla\phi - \frac{\partial A_z}{\partial t}\hat{\mathbf{e}}_z$$

low-beta approximation ($\beta \ll 1$), neglected the perpendicular component of the vector potential.

$$\longrightarrow \quad E_{\parallel} = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \qquad \mathbf{E}_{\perp} = -\nabla_{\perp}\phi$$

$$\mu_0 \mathbf{J} = -\nabla \times \nabla \times (A_z \hat{\mathbf{e}}_z) = \nabla\nabla\cdot(A_z\hat{\mathbf{e}}_z) - \nabla^2 A_z \hat{\mathbf{e}}_z$$

$$\longrightarrow \quad \mu_0 J_{\parallel} = -\nabla_{\perp}^2 A_z \qquad \mu_0 \mathbf{J}_{\perp} = \nabla_{\perp}^2 \frac{\partial A_z}{\partial z}$$

Physical origin of parallel electric field: electron motion $n_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$

$\perp \mathbf{B}$

$E_{\parallel} \sim \partial u_{e\parallel}/\partial t$

Inertial Alfvén wave (IAW)

$E_{\parallel} \sim \partial p_e/\partial z$

Kinetic Alfvén wave (KAW)

Inertial Alfvén waves

low-beta approximation (β ≪ 1), neglected the perpendicular component of the vector potential.

Parallel dynamics:

$$\mathbf{E} = -\nabla\phi - \frac{\partial A_z}{\partial t}\hat{\mathbf{e}}_z \longrightarrow E_{\parallel} = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \quad (1)$$

$$\mu_0 \mathbf{J} = -\nabla\times\nabla\times(A_z\hat{\mathbf{e}}_z) \longrightarrow \mu_0 J_{\parallel} = -\nabla_{\perp}^2 A_z \quad (2)$$

$$n_e m_e \frac{D\mathbf{u}_e}{Dt} = -\nabla p_e - n_e e(\mathbf{E} + \mathbf{u}_e\times\mathbf{B}) \longrightarrow m_e \partial u_{e\parallel}/\partial t = eE_{\parallel} \quad (3)$$

Parallel current (electron motion): $\longrightarrow J_{\parallel} = -neu_{e\parallel}$ (4)

Substitute (3) (4) **into** (1) $\longrightarrow$ $\boxed{(1 - \frac{c^2}{\omega_{pe}^2}\nabla_{\perp})\frac{\partial A_z}{\partial t} = -\frac{\partial \phi}{\partial z}}$

Electron skin depths $\frac{c}{\omega_{ce}} \equiv \lambda_e$

Perpendicular dynamics:

$$\mathbf{E} = -\nabla\phi - \frac{\partial A_z}{\partial t}\hat{\mathbf{e}}_z \longrightarrow \mathbf{E}_{\perp} = -\nabla_{\perp}\phi \quad (5)$$

$$\mu_0 \mathbf{J} = -\nabla\times\nabla\times(A_z\hat{\mathbf{e}}_z) \longrightarrow \mu_0 \mathbf{J}_{\perp} = \nabla_{\perp}^2 \frac{\partial A_z}{\partial z} \quad (6)$$

Polarization drift current $\longrightarrow \mathbf{J}_{\perp} = \frac{1}{\mu_0 V_A^2}\frac{d\mathbf{E}_{\perp}}{dt}$ (7)

Combine (5) (6) (7) $\longrightarrow$ $\boxed{\nabla_{\perp}\frac{\partial A_z}{\partial z} = -\frac{1}{V_A^2}\nabla_{\perp}\frac{\partial \phi}{\partial t}}$

Inertial Alfvén waves

Now we get the dispersion relation of IAW:

Recall: pure Alfvén mode: $\frac{\omega}{k_{\|}} = V_A$

$$(1 - \lambda_e^2 \nabla_\perp)\frac{\partial^2 A_z}{\partial t^2} = \frac{1}{V_A^2}\frac{\partial^2 A_z}{\partial z^2} \xrightarrow{\text{Plane wave}} \boxed{\left(\frac{\omega}{k_{\|}}\right)^2 = \frac{V_A^2}{1 + k_\perp^2 \lambda_e^2}}$$

A couple of things:

- IAWs are always sub-Alfvénic - phase speed slower than V_A
- IAWs are important when k lambda_perp >>1 - electron inertial length (~1km auroral zone)

Why E|| develops in the auroral acceleration region?

- Fieldlines converge as r^3 - perpendicular scale length decreases fast
- Parallel current increases when polarization drift becomes important - need parallel acceleration to ensure current continuity:

$$J_{\|} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1)_{\|} = \Sigma_A \nabla_\perp \cdot \mathbf{E}_1 \sim \Sigma_A k_\perp E_1$$

- The electric field associated with IAW is both electromagnetic and electrostatic (potential drop):

$$E_{\|} = -\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t}$$

Kinetic Alfvén waves

Leave as HW exercise

Summary

- The displacement current term dE/dt modifies the shear mode by reducing its phase speed by a factor of $\sqrt{1 + V_A^2/c^2}$
- In solar wind - magnetosphere - ionosphere coupling, Alfvén waves reflect through the changes in the wave conductance (Pedersen conductance if ionosphere)
- Dissipation terms (viscosity and/or magnetic diffusion) causes damping of propagating Alfvén waves
- Hall MHD is important when the scale size of the problem approaching ion inertial length (skin depth), which introduces dispersion
- Parallel electric field is generated when the perpendicular scale of the Alfvén wave approaches electron inertial scale (IAW) or thermal scale (KAW), due to the current continuity constraint.

Now think: What’s carrying the currents in MHD?