Equations of MHD 3

Single-fluid, Ideal MHD equations

Single-fluid MHD equations

Ideal MHD collisionless, isotropic, Maxwellian, equilibrium - five moment

Mass conservation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum conservation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

Energy conservation

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Ampere's law

$$\mathbf{J} = \nabla \times \mathbf{B}$$

Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

Magnetic Gauss

$$\nabla \cdot \mathbf{B} = 0$$

Resistive MHD

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

Resistive tearing mode

Hall MHD

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{ne}\mathbf{J} \times \mathbf{B}$$

(low-freq) Whistler mode

Conservative versus Primitive

Conservative (semi) form

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial \mathscr{E}}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E} + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Primitive form

$$\frac{\partial}{\partial t}\rho + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\frac{\partial}{\partial t} p + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \mathbf{J} = \nabla \times \mathbf{B} \qquad \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

Primary variables: $\rho, \mathbf{u}, p, \mathbf{B}$

Conserved variables: $\rho, \rho\mathbf{u}, \mathscr{E}, \Phi$

$\mathscr{E} = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}$ Plasma energy

Secondary/Derived variables: $\mathbf{E}, \mathbf{J}$

Mass Conservation

For fluid density in a control volume, with velocity field $\mathbf{u}$:

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

Conservation form

Or slightly differently

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0}$$

Primitive form

Eulerian form

Move the divergence term to the RHS:

$$\frac{\partial \bar{\rho}}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

Compressibility

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho \equiv \boxed{\frac{D}{Dt} \rho = -\rho \nabla \cdot \mathbf{u}}$$

Convective Derivative

Lagrangian form

If $\nabla \cdot \mathbf{u} = 0$, then the equation for mass density ρ is decoupled from the flow

If $\nabla \cdot \mathbf{u} = 0$, then the equation for mass density ρ is constant along flow lines

Momentum conservation

The primitive form goes $\boxed{\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}}+\nabla p\boxed{-\mathbf{J}\times\mathbf{B}}=0$

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\xrightarrow{\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{u})=0}\frac{\partial \rho\mathbf{u}}{\partial t}+\mathbf{u}\nabla\cdot\rho\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\xrightarrow{\nabla\cdot(\mathbf{AB})=\mathbf{A}\cdot\nabla\mathbf{B}+\mathbf{B}\cdot\nabla\mathbf{A}}\frac{\partial \rho\mathbf{u}}{\partial t}+\nabla\cdot\rho\mathbf{uu}$$

$$\nabla p\xrightarrow{\text{Unit tensor}}\nabla\cdot\mathbf{I}p\qquad \mathbf{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$-\mathbf{J}\times\mathbf{B}=\mathbf{B}\times\nabla\times\mathbf{B}\xrightarrow[\substack{\nabla\cdot(\mathbf{AB})=\nabla\mathbf{A}\cdot\mathbf{B}+\nabla\mathbf{B}\cdot\mathbf{A}\\ \nabla\cdot\mathbf{B}\equiv 0}]{\mathbf{A}\times\nabla\times\mathbf{B}=\nabla\mathbf{B}\cdot\mathbf{A}-\nabla\cdot(\mathbf{AB})-\mathbf{B}\nabla\cdot\mathbf{A}}\nabla\left(\frac{1}{2}B^2\right)-\nabla\cdot\mathbf{BB}=\nabla\cdot\left(\mathbf{I}\frac{1}{2}B^2-\mathbf{BB}\right)$$

So adding the terms together:

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p-\mathbf{J}\times\mathbf{B}=0\longrightarrow\frac{\partial \rho\mathbf{u}}{\partial t}+\nabla\cdot\underbrace{(\rho\mathbf{uu}+\mathbf{I}p)}_{\text{Fluid Stress}}+\nabla\cdot\underbrace{\left(\mathbf{I}\frac{1}{2}B^2-\mathbf{BB}\right)}_{\text{Maxwell Stress}}=0$$

Or sometimes you would see this form:

$$\frac{\partial \rho\mathbf{u}}{\partial t}+\nabla\cdot\left[\rho\mathbf{uu}+\underbrace{\mathbf{I}(p+\frac{1}{2}B^2)}_{\text{Total pressure}}-\mathbf{BB}\right]=0$$

Magnetic tension

Maxwell stress and Faraday's Tension

$$\underbrace{-\mathbf{J}\times\mathbf{B}}_{\text{Lorentz Force}} = \mathbf{B}\times\nabla\times\mathbf{B} = \underbrace{\nabla\left(\frac{1}{2}B^2\right)}_{\text{Magnetic pressure}} - \underbrace{\nabla\cdot\mathbf{BB}}_{\text{Magnetic Tension}} = \underbrace{\nabla\cdot\left(\mathbf{I}\frac{1}{2}B^2 - \mathbf{BB}\right)}_{\text{Maxwell Stress tensor}}$$

If div B isn't zero, the Lorentz force becomes:

$$-\mathbf{J}\times\mathbf{B} = = \nabla\cdot\left(\mathbf{I}\frac{1}{2}B^2 - \mathbf{BB}\right) - \underbrace{\mathbf{B}(\nabla\cdot\mathbf{B})}_{\text{Unphysical Parallel Acceleration}}$$

This is why div B = 0 is crucial to MHD calculations

Recall Faraday's law:

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \xrightarrow{\text{take }\nabla\cdot} \frac{\partial\nabla\cdot\mathbf{B}}{\partial t} = -\nabla\cdot\nabla\times\mathbf{E} \equiv 0$$

Which means div B = 0 should be an initial condition for MHD flows

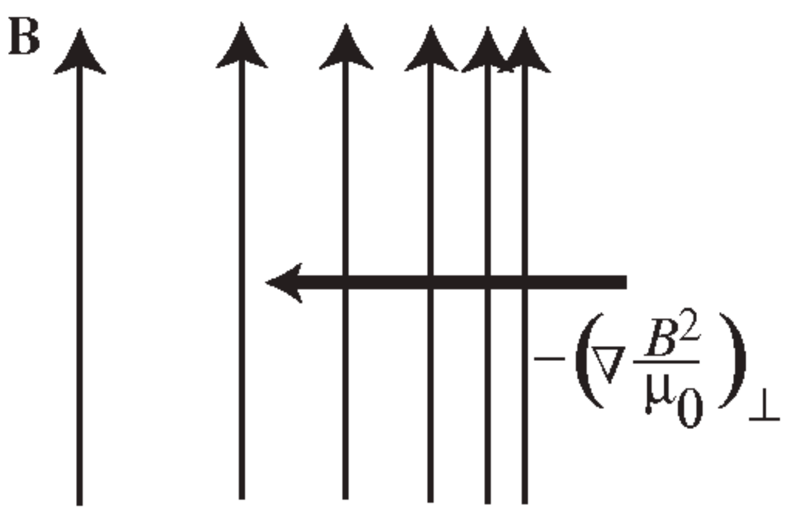
Maxwell stress and Faraday's Tension

$$\underbrace{-\mathbf{J}\times\mathbf{B}}_{\text{Lorentz Force}} = \mathbf{B}\times\nabla\times\mathbf{B} = \underbrace{\nabla\left(\frac{1}{2}B^2\right)}_{\text{Magnetic pressure}} \underbrace{-\nabla\cdot\mathbf{BB}}_{\text{Magnetic Tension}} = \nabla\cdot\underbrace{\left(\mathbf{I}\frac{1}{2}B^2-\mathbf{BB}\right)}_{\text{Maxwell Stress tensor}}$$

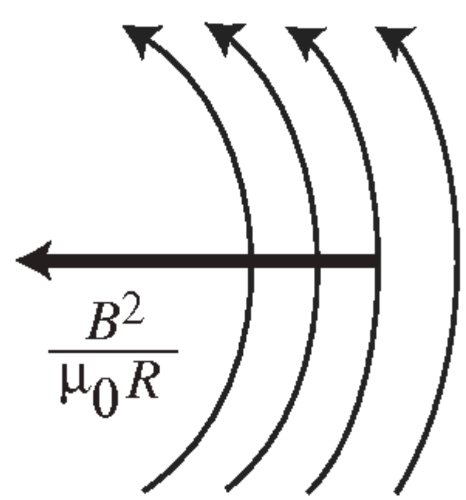
$$\mathbf{J}\times\mathbf{B} = -\nabla\left(\frac{1}{2}B^2\right) + (\mathbf{B}\cdot\nabla)\mathbf{B}$$

Let's use a coordinate system along field lines:

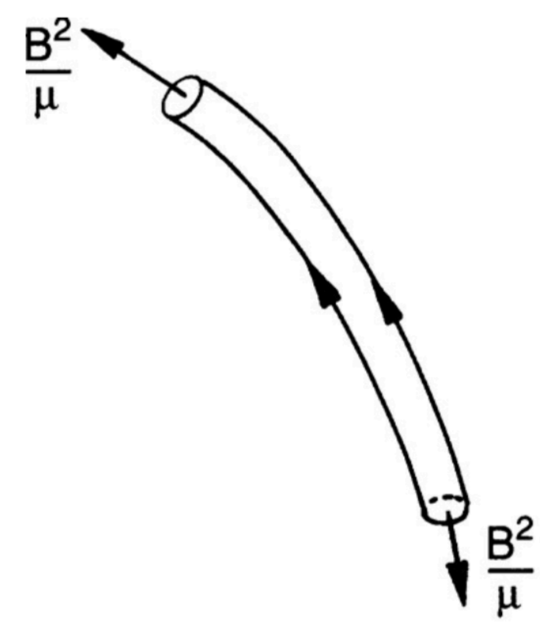
$$\xrightarrow{\text{Ignore } B^2} \quad (\mathbf{B}\cdot\nabla)\mathbf{B} = B\frac{\partial B}{\partial s}\hat{\mathbf{e}}_t - \frac{B^2}{R_C}\hat{\mathbf{e}}_n$$

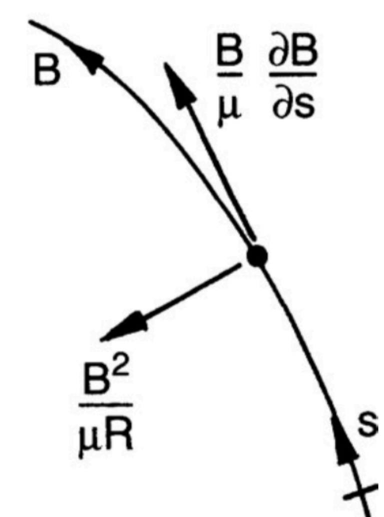


Magnetic pressure



Magnetic Tension



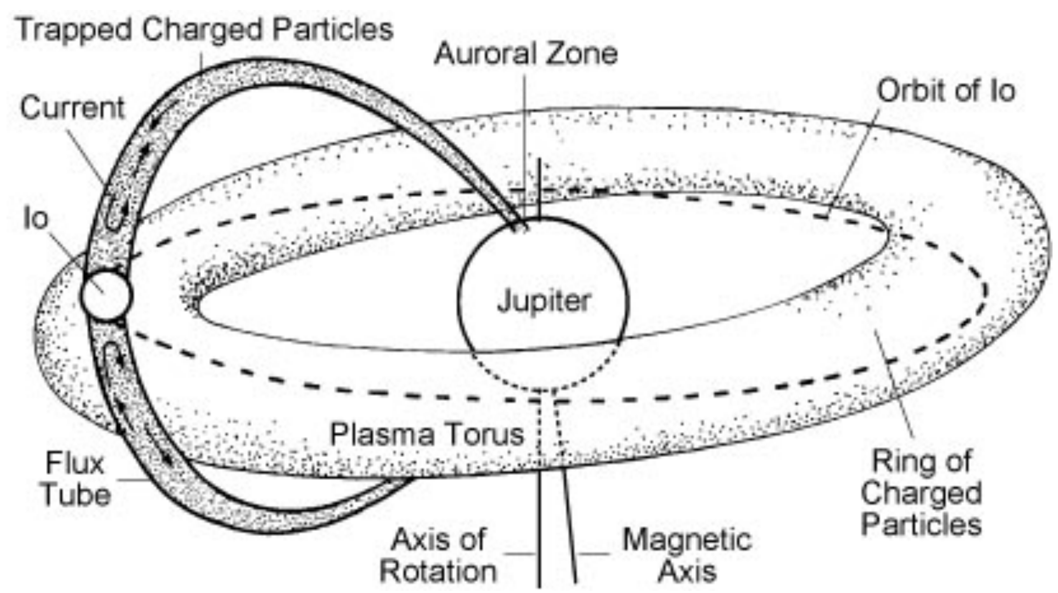


Q: dipole field lines are curved. Does that mean dipole field lines have non-zero Lorentz force?

Magnetic flux conservation

Flux tubes

Planetary magnetospheres

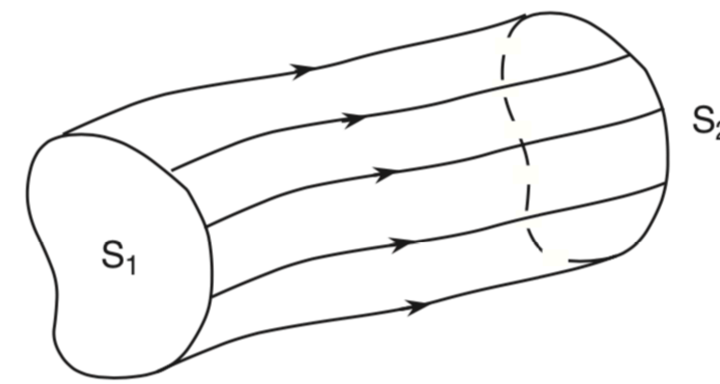


Definition:

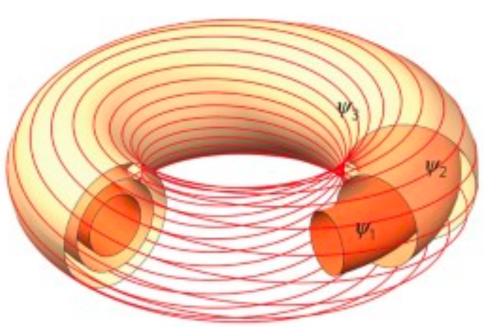
magnetic fields confining plasmas are basically *tubular structures*.

Physical interpretation

$$\nabla \cdot \mathbf{B} \equiv 0$$



Tokamak

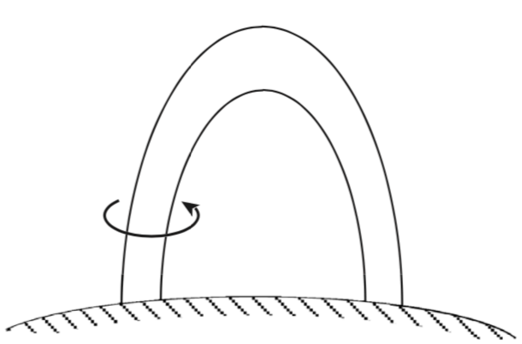


$$\int_V \nabla \cdot \mathbf{B} dV = \oint_S \mathbf{B} \cdot \mathbf{n} dS = -\int_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1 dS_1 + \int_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2 dS_2$$

Define magnetic flux

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n} dS$$

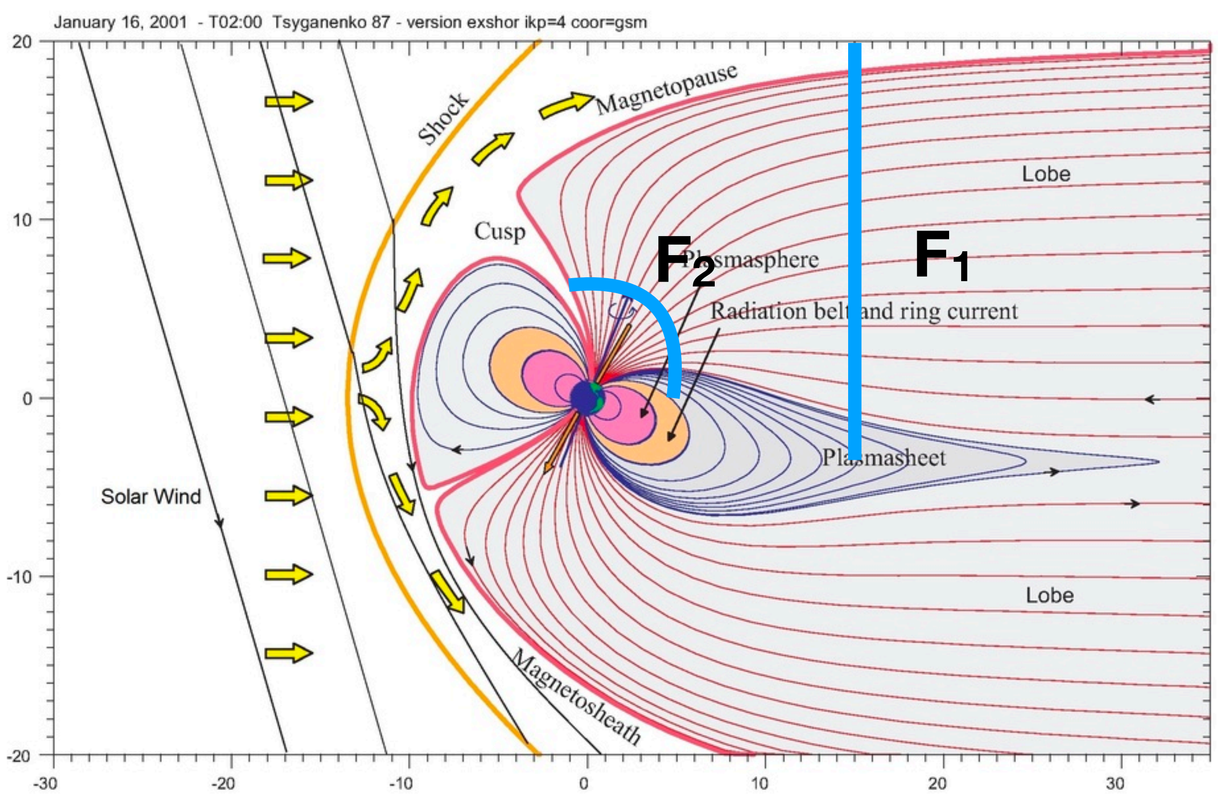
Solar corona

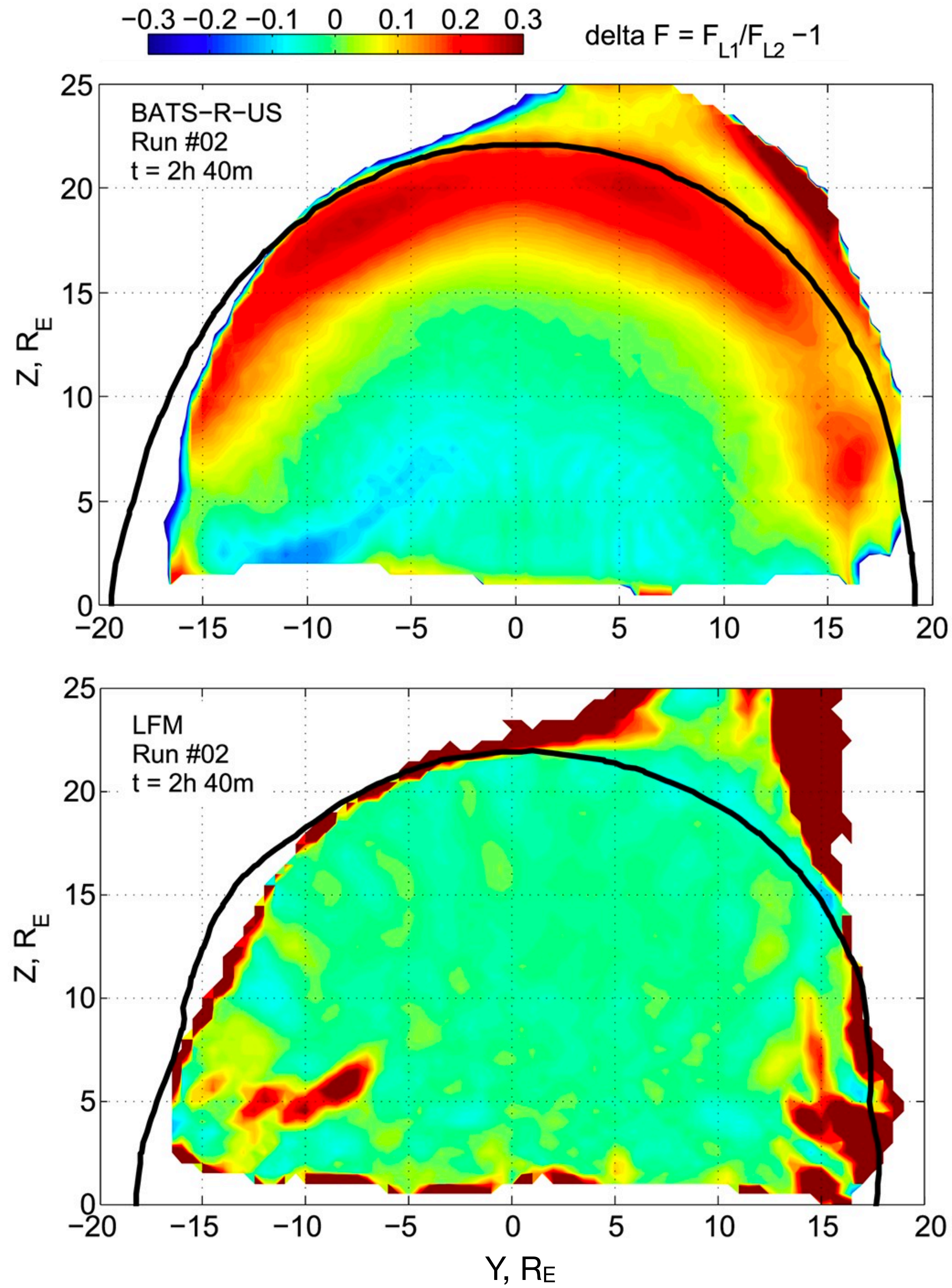


through an arbitrary cross-section of the magnetic flux tube is a well-defined quantity, i.e. it does not depend on how S is taken.

Magnetic flux conservation

Flux tubes





$$F_1 = \int_{S_1} \mathbf{B}_1 \cdot \mathbf{n}_1 dS_1$$ **x = -15 RE**

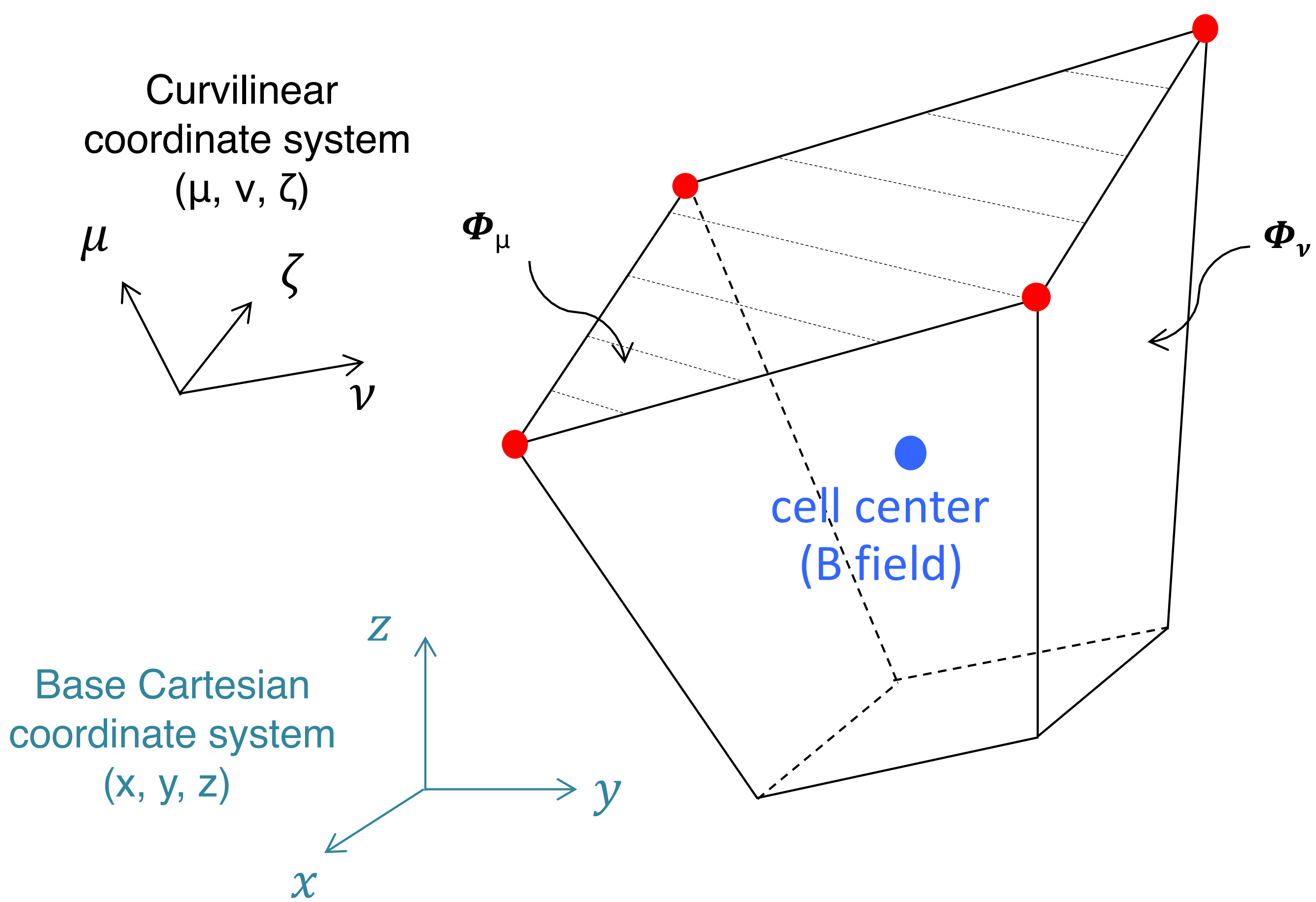
$$F_2 = \int_{S_2} \mathbf{B}_2 \cdot \mathbf{n}_2 dS_2$$ **r = 5 RE**

Since $\nabla \cdot \mathbf{B} \equiv 0$

$$\frac{F1}{F2} - 1 \approx 0$$

Magnetic flux conservation

Relationship between flux and field



Consider the following volume integral:

$$\int_V \nabla \cdot (\mathbf{r}\mathbf{B}) dV = \int_V (\mathbf{B} \cdot \nabla)\mathbf{r} dV + \int_V \overset{0}{\cancel{(\nabla \cdot \mathbf{B})}}\mathbf{r} dV$$

$$= \int_V \mathbf{B} dV = \bar{\mathbf{B}} V$$

Gauss'

$$\oint_S (\mathbf{r}\mathbf{B}) \cdot d\mathbf{S} = \oint_S \mathbf{r}(\mathbf{B} \cdot d\mathbf{S})$$

$$= \sum_{S=\mu,\nu,\zeta} \mathbf{r}^\mathbf{S} \Phi^S + \mathcal{O}(\Delta r^2)$$

Problem: in solving the Maxwell's equations, the primary variable evolved is the magnetic flux on cell interfaces (Φ). How do we get the magnetic field vector at cell center?

Answer: use the concept of Magnetic flux

So the (average) magnetic field vector is

$$\boxed{\mathbf{B} = \frac{1}{V} \sum_{S=\mu,\nu,\zeta} \mathbf{r}^\mathbf{S} \Phi^S}$$

Eqn used in GAMERA

Magnetic flux conservation

Conservative form

$$\nabla \times \mathbf{E} = -\nabla \times (\mathbf{u} \times \mathbf{B}) \xrightarrow{\nabla \times (\mathbf{A} \times \mathbf{B}) = \nabla \cdot (\mathbf{AB} - \mathbf{BA})} \nabla \cdot (\mathbf{uB} - \mathbf{Bu})$$

So the Faraday's law can also be cast into the conservative form:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \longrightarrow \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{uB} - \mathbf{Bu}) = 0$$

Useful in studying shocks

Considering Resistivity - resistive MHD:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J}$$

So the Faraday's law becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\nabla \times (-\mathbf{u} \times \mathbf{B} + \eta \mathbf{J}) \xrightarrow{\mathbf{J} = \nabla \times \mathbf{B}} \frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{Convective}} + \underbrace{\eta \nabla^2 \mathbf{B}}_{\text{Diffusive}}$$

MHD parameters

Recall the Navier-Stokes equation: $\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g}$

Let's consider ***gravity*** and ***viscosity*** in the equation of plasma motion (MHD):

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \frac{\partial \mathbf{u}}{\partial t} + \underbrace{\rho \mathbf{u} \cdot \nabla \mathbf{u}}_{1} = \underbrace{- \nabla p}_{2} + \underbrace{\mathbf{J} \times \mathbf{B}}_{3} + \underbrace{\nu \nabla^2 \mathbf{u}}_{4} + \underbrace{\rho \mathbf{g}}_{5}$$

Together with the ***resistive*** term in the Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{6} + \underbrace{\eta \nabla^2 \mathbf{B}}_{7}$$

$\frac{1}{2}$: **Flow Mach Number**

$\frac{2}{3}$: **Plasma beta**

$\frac{3}{7}$: **Lindquist number**

$\frac{1}{3}$: **Alfven Mach Number**

$\frac{2}{5}$: **Pressure scale height**

$\frac{1}{4}$: **Reynolds Number**

$\frac{6}{7}$: **Magnetic Reynolds number**

Applications of the momentum equation

1. Hydrostatic equilibrium - solar atmosphere
2. MHD equilibrium - z pinch
3. Pressure balance 1 - sunspot
4. Pressure balance 2 - stand-off distance
5. Parker's solar wind model - supersonic solar wind
6. * Force balance - cpcp saturation *
7. Field-aligned current generation - earth, Jupiter

Applications of the momentum equation

Hydrostatic equilibrium

$$\boxed{\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g}}$$

The force balance between pressure gradient and gravity

Consider the Earth's atmosphere - a gas layer, assuming ideal gas law - $p = nk_BT = \frac{\rho k_B T}{\langle m \rangle}$

Assume equilibrium, the momentum equation is reduced to: $\nabla p = \rho \mathbf{g}$

Re-write that equation in spherical coordinates (only radial variations): $\frac{\partial p}{\partial r} = -\rho g(r) = -\frac{\langle m \rangle g(r)}{k_B T} p$

A couple of things:

1. dp/dr < 0: pressure must decrease with r
2. The rate of decrease depends on a factor $\xi = \frac{\langle m \rangle g(r)}{k_B T}$

Solve the ODE (quite simple): $p(r) = p_0 e^{-\frac{\langle m \rangle g}{k_B T}(r - R_0)}$ or $p(r) = p_0 e^{-\frac{h}{h_s}}$

Here $h_s = \frac{k_B T}{\langle m \rangle g}$ is called the ***scale height*** Assumption: thin atmosphere - doesn't work for solar atmosphere!

For deep atmospheres (solar), need to solve the full equation such as

$$\frac{\partial p}{\partial r} = -\rho g(r) = -\frac{\langle m \rangle g(r)}{k_B T} p = -\frac{\langle m \rangle GM}{k_B T} \frac{p}{r^2}$$

Applications of the momentum equation

MHD equilibrium

$$\boxed{\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g}}$$

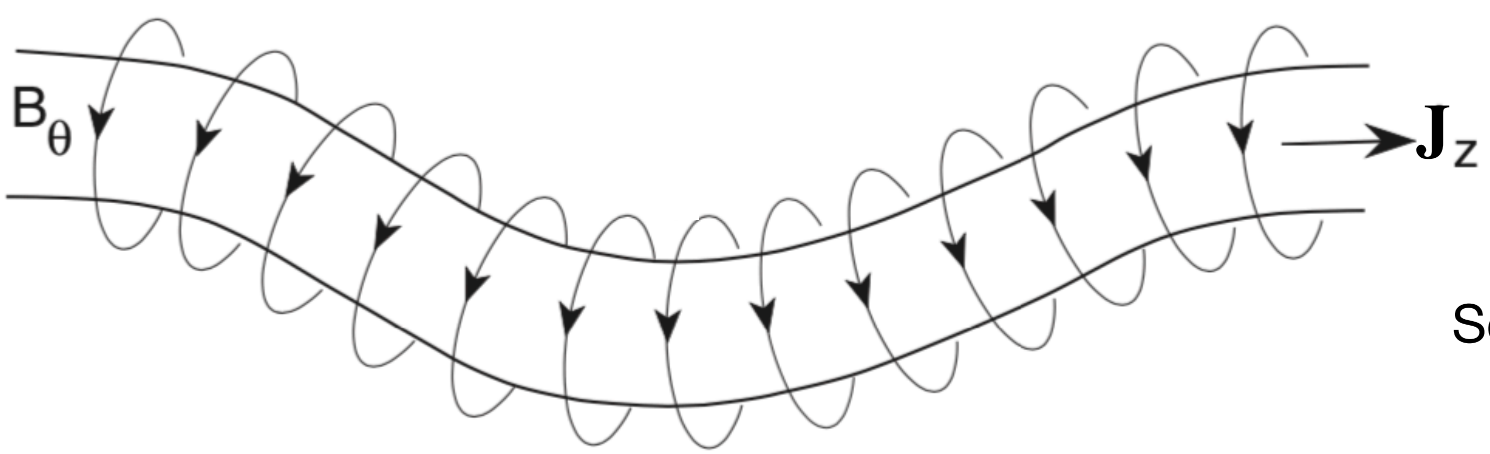
The force balance between pressure gradient and Lorentz

$$\nabla p = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

With $\nabla \cdot \mathbf{B} = 0$

example: z-pinch



In cylindrical geometry along the field, the MHD equations become

$$\frac{dp}{dr} = -J_z B_\theta \qquad J_z = \frac{1}{\mu_0 r} \frac{d}{dr} r B_\theta$$

So we get the following PDE for p and B_theta

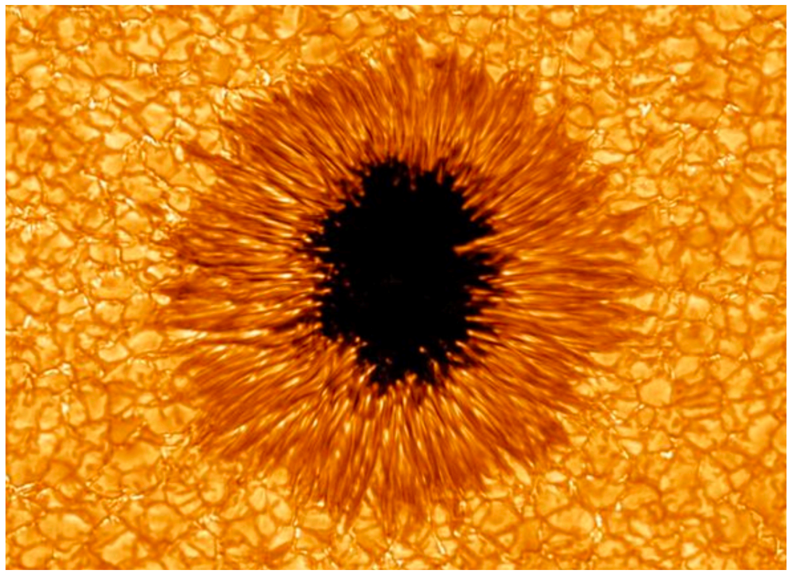
$$\frac{dp}{dr} = -\frac{B_\theta}{\mu_0 r} \frac{d}{dr} r B_\theta$$

Applications of the momentum equation

Static Pressure Balance

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \rho\mathbf{g}$$

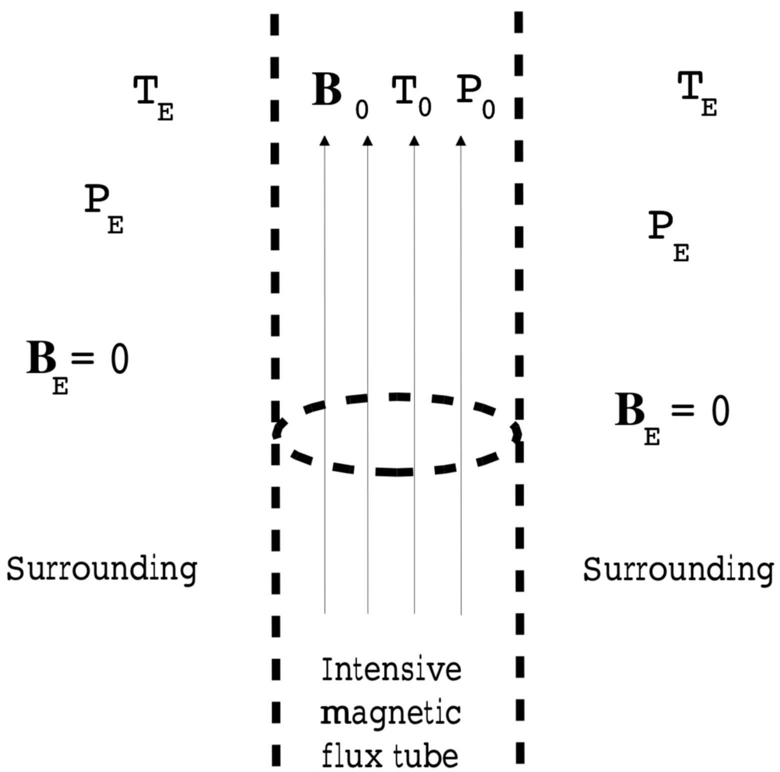
The force balance between gas pressure and magnetic pressure



The two terms are combined such as in the conservative form of the momentum eqn:

$$\cancel{\frac{\partial \rho\mathbf{u}}{\partial t}} + \nabla\cdot\left[\cancel{\rho\mathbf{u}\mathbf{u}} + \mathbf{I}(p + \frac{1}{2}B^2) - \cancel{\mathbf{B}\mathbf{B}}\right] = 0$$

$$\longrightarrow \nabla\left(p + \frac{1}{2}B^2\right) = 0$$



Means total pressure must equal inside and outside: $p_E = p_0 + \frac{1}{2}B^2$

Assume density remains the same inside and outside: $\rho_0 = \rho_E$

Assuming ideal gas law, simple re-arrangement of the equation gives

$$\frac{T_0}{T_E} = 1 - \frac{B_o^2}{2p_E} < 1$$

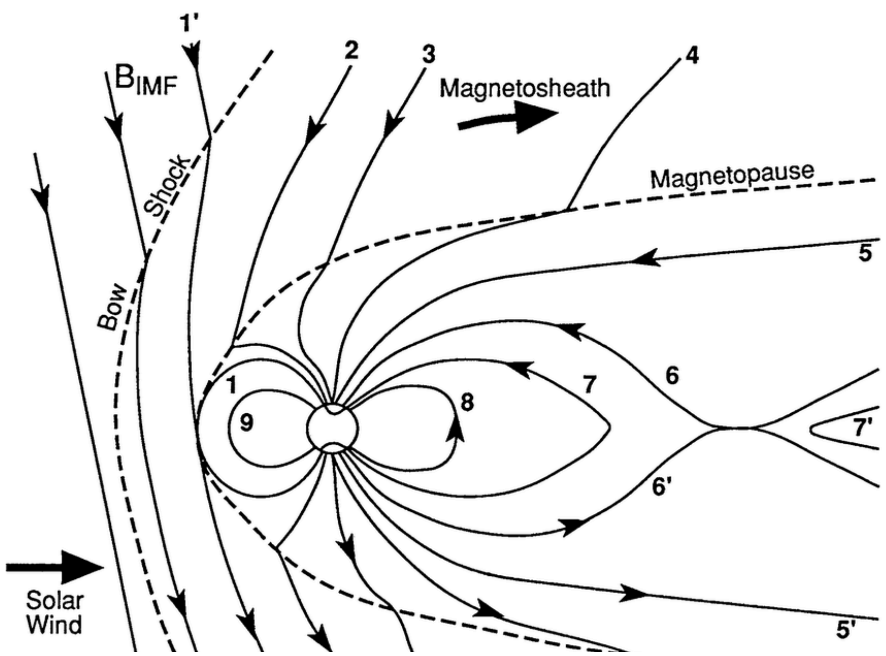
Lower temperature inside a sunspot!

Applications of the momentum equation

Dynamic Pressure Balance

$$\boxed{\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \rho\mathbf{g}}$$

The force balance between dynamic pressure and magnetic pressure



The two terms are combined such as in the conservative form of the momentum eqn:

$$\cancel{\frac{\partial \rho\mathbf{u}}{\partial t}} + \nabla\cdot\left[\rho\mathbf{u}\mathbf{u} + \mathbf{I}(\cancel{p} + \frac{1}{2}B^2) - \cancel{\mathbf{B}\mathbf{B}}\right] = 0$$

In 1-D pressure balance, it's basically

$$\rho_{SW}u_{SW}^2 = \frac{1}{2}B_{msphere}^2$$

Along the sun-planet line, ignore dipole tilt: $B \sim \dfrac{B_E}{r^3}$

We can work out the stand-off distance as: $\rho_{SW}u_{SW}^2 = \left(\dfrac{B_E}{r_{so}^3}\right)^2 \xrightarrow{\text{Complicated Math}} r_{SO} = \left(\dfrac{B_E^2}{\rho_{SW}u_{SW}^2}\right)^{\frac{1}{6}}$

For Earth, $r_{SO} \sim 10\ R_E$, for Jupiter $r_{SO} \sim 42\ R_J$

Applications of the momentum equation

Dynamic Pressure Balance

Now here’s a problem （那么问题来了）

JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 103, NO. A1, PAGES 225–235, JANUARY 1, 1998

A global magnetohydrodynamic simulation of the Jovian magnetosphere

Tatsuki Ogino
Solar Terrestrial Environment Laboratory, Nagoya University, Toyokawa, Aichi, Japan

Raymond J. Walker, Margaret G. Kivelson[1]
Institute of Geophysics and Planetary Physics, University of California, Los Angeles

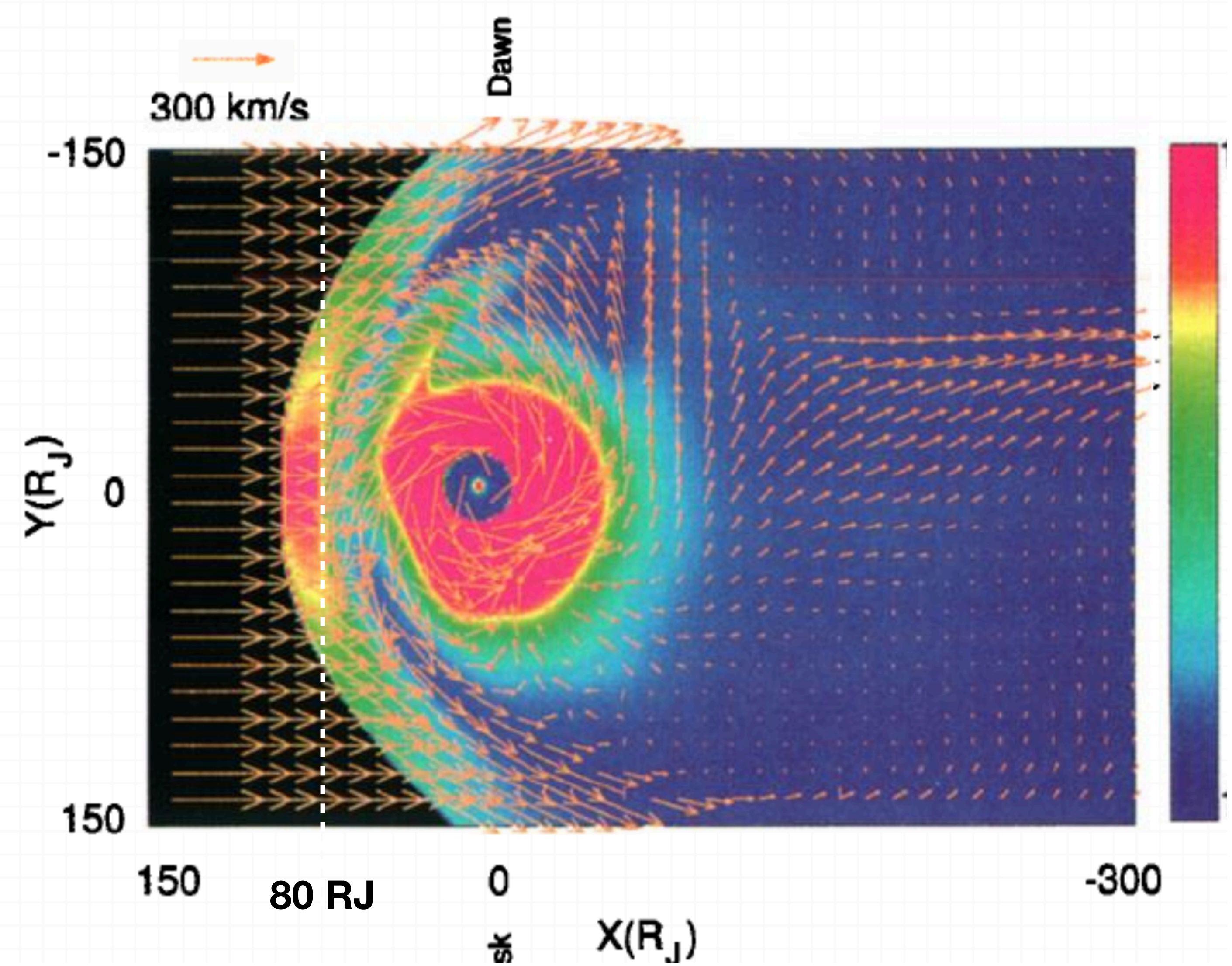
$$\partial\rho / \partial t = -\nabla\cdot(\mathbf{v}\rho) + D\nabla^2\rho$$

$$\partial\mathbf{v} / \partial t = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla P/\rho + (\mathbf{J}\times\mathbf{B})/\rho + \mathbf{g} + \Phi/\rho$$

$$\partial P / \partial t = -(\mathbf{v}\cdot\nabla)P - \gamma P\nabla\cdot\mathbf{v} + D_p\nabla^2 P$$

$$\partial\mathbf{B} / \partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2 B$$

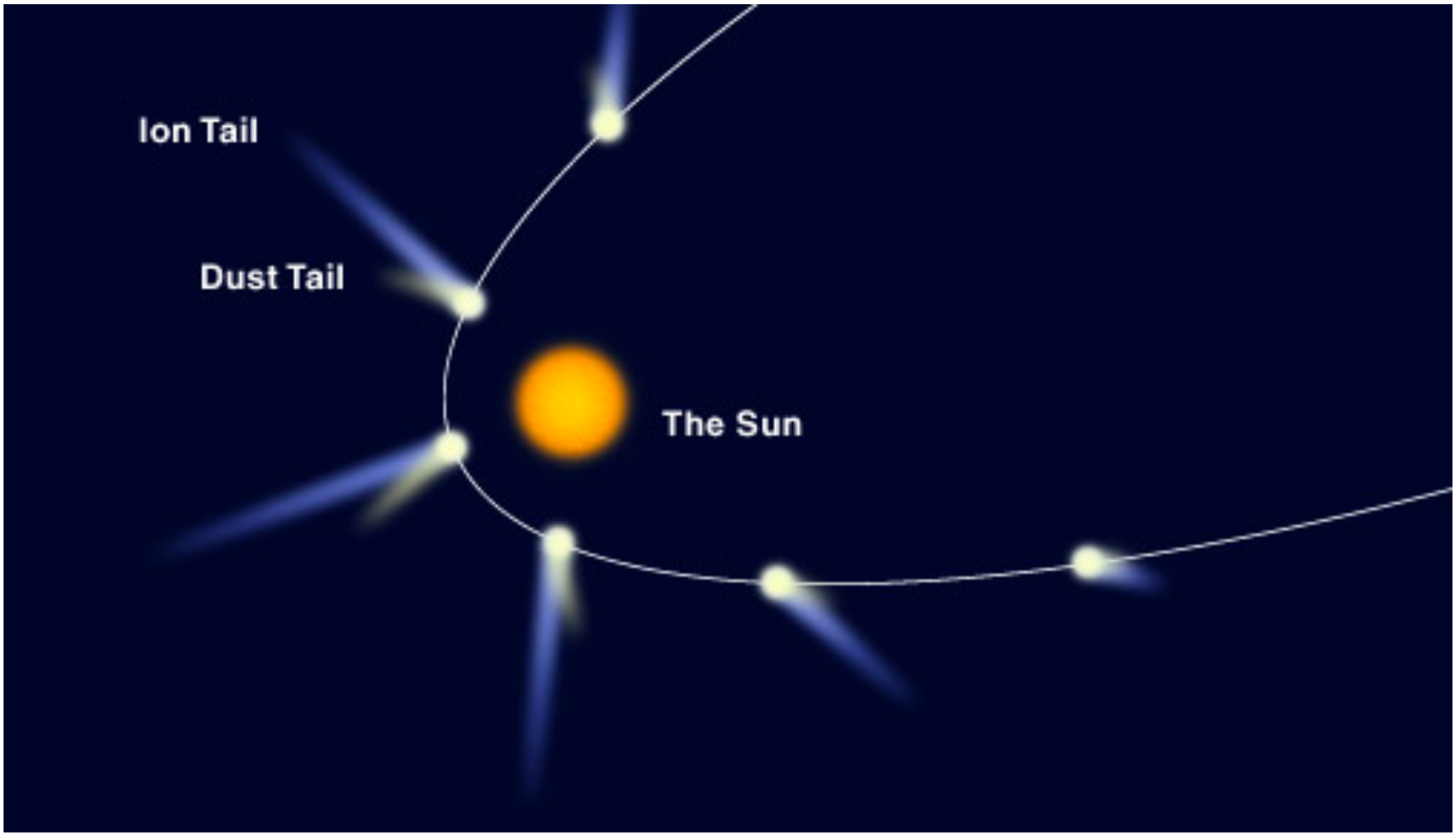
$$\mathbf{J} = \nabla\times(\mathbf{B} - \mathbf{B}_d)$$



The Ogino MHD model for the jovian magnetosphere gives 80 RJ as the stand-off distance - it’s “consistent” with observations!

Applications of the momentum equation

Parker's solar wind solution



Applications of the momentum equation

Parker's solar wind solution

$$\boxed{\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \rho\mathbf{g}}$$

The force balance between convective, pressure gradient and gravity

Assume steady state, isothermal:

$$\nabla\cdot\rho\mathbf{u} = 0,\ p = nk_BT$$

Newton's gravitational law: $g = \dfrac{GM}{r^2}$

Assuming spherical symmetry: $\mathbf{u} \sim u(r)$

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \rho\mathbf{g} \longrightarrow \rho u\frac{du}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho$$

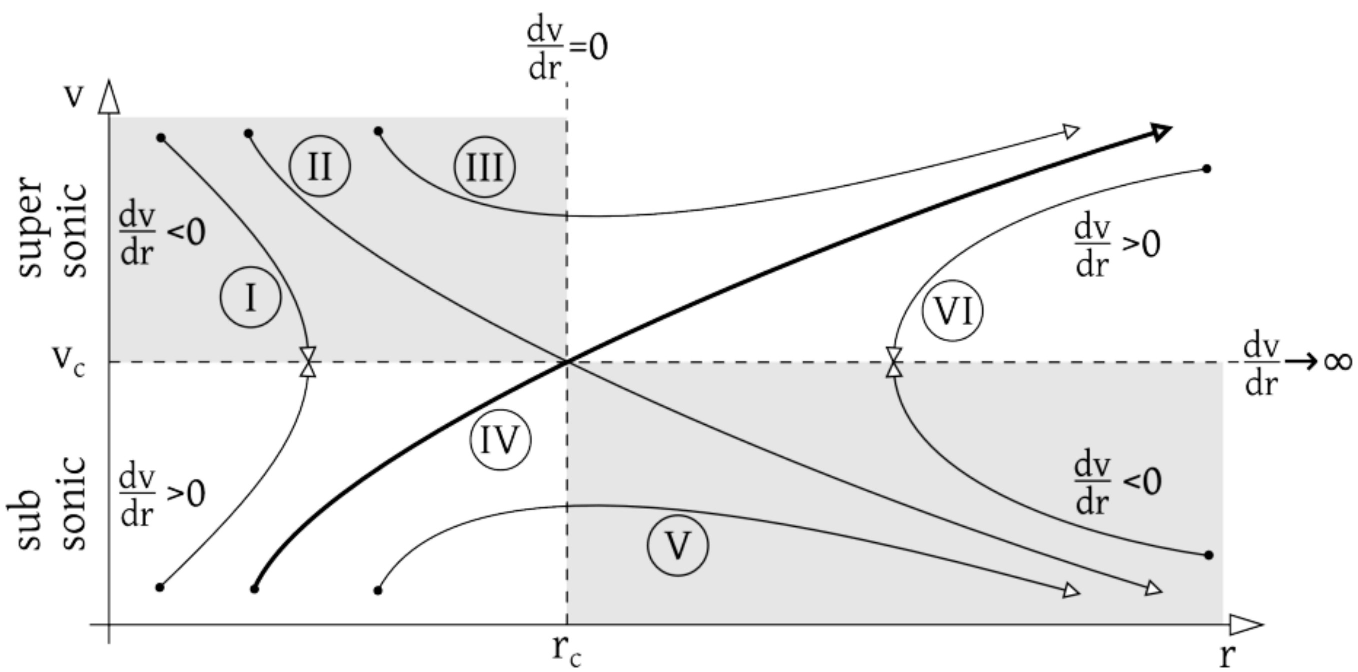
$$\xrightarrow[p = nk_BT]{\text{ideal gas law:}} u\frac{du}{dr} = -\frac{k_BT}{m}\frac{1}{\rho}\frac{d\rho}{dr} - \frac{GM}{r^2} \qquad (1)$$

using the mass continuity: $\nabla\cdot\rho\mathbf{u} = 0$

$$\longrightarrow \frac{1}{\rho}\frac{d\rho}{dr} = -\left(\frac{2}{r} + \frac{1}{u}\frac{\partial u}{\partial r}\right) \qquad (2)$$

$$\xrightarrow{\text{Substitute 2}} \frac{du}{dr} = \frac{u}{r}\,\frac{\frac{2k_BT}{m} - \frac{GM}{r}}{u^2 - \frac{k_BT}{m}}$$

$$\xrightarrow{r_c = \frac{GMm}{2k_BT}} \frac{du}{dr} = \frac{u}{2r}\,\frac{1 - \frac{r_c}{r}}{\frac{u^2}{c_s^2} - 1}$$



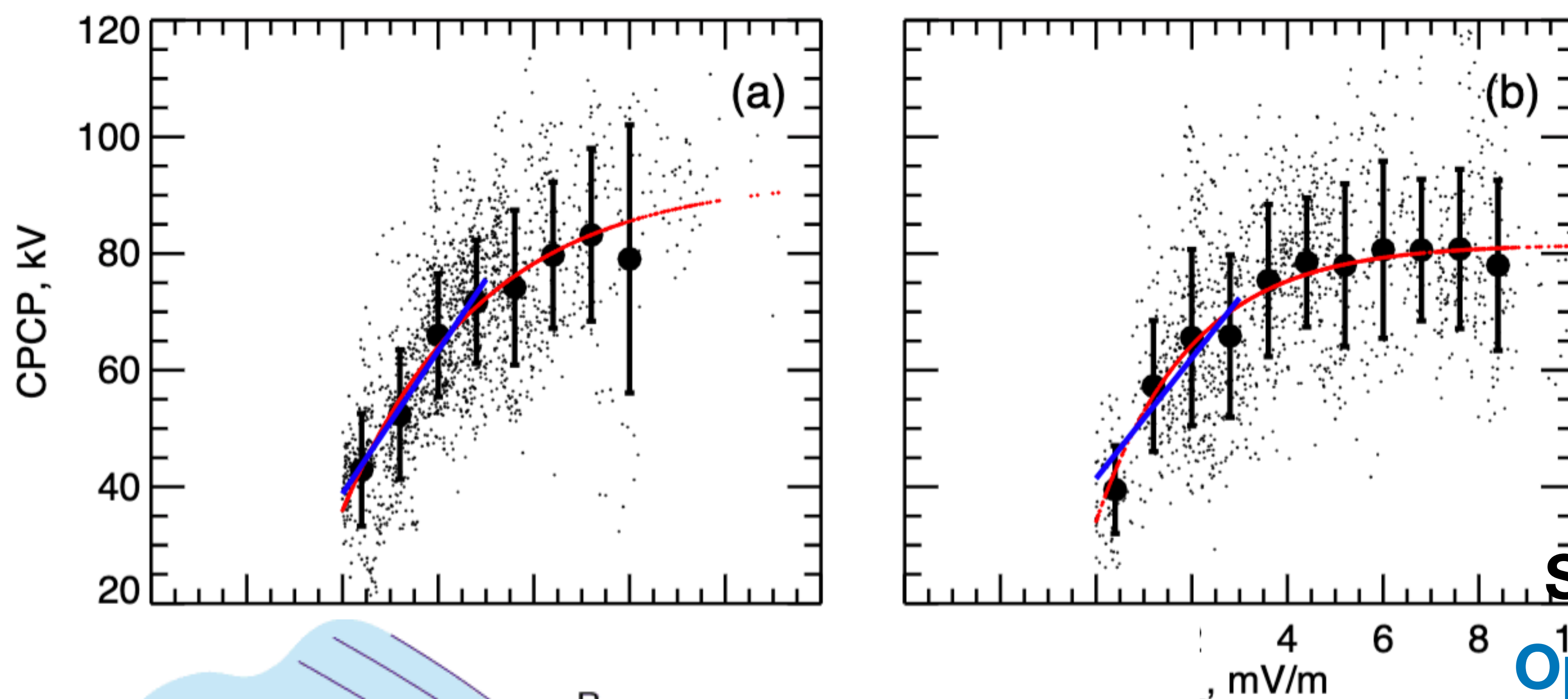
- I: non-physical
- VI: non-physical
- II, III: supersonic at the Sun's surface
- V: not supersonic at the Earth
- **IV: the solar wind solution**

Applications of the momentum equation

Magnetosheath force balance and saturation of CPCP

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

The force balance between pressure gradient and Lorentz force



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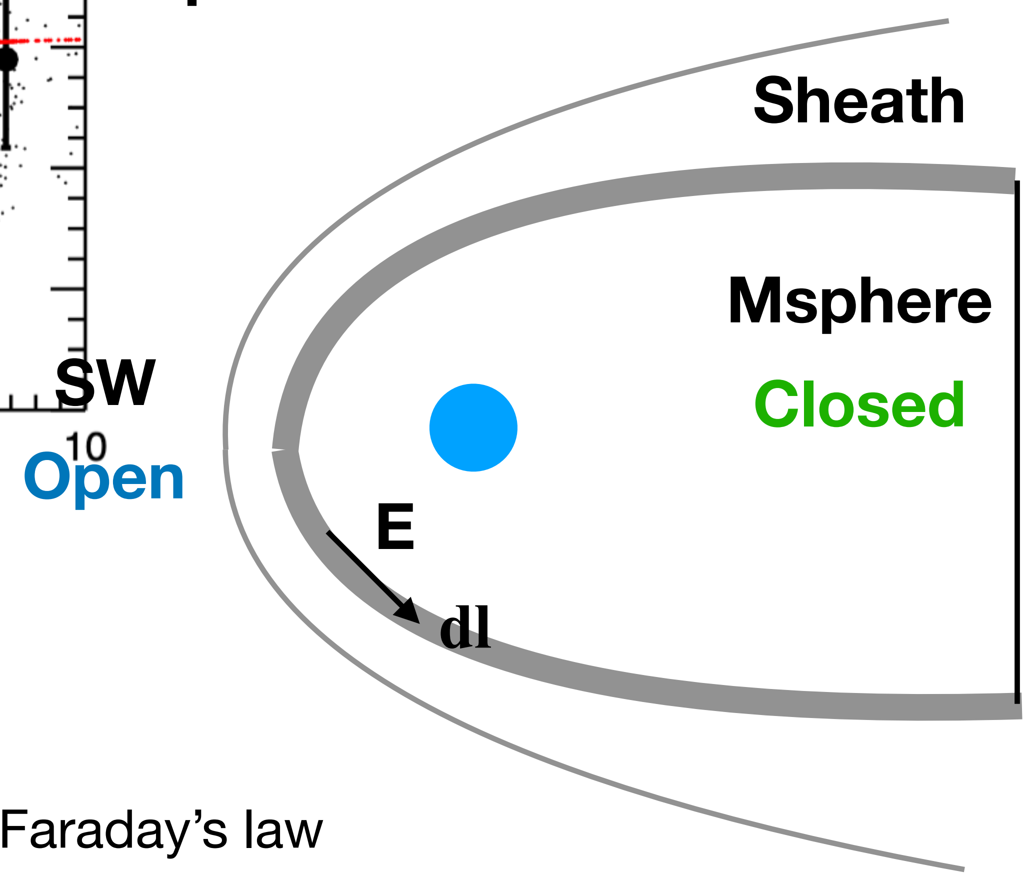
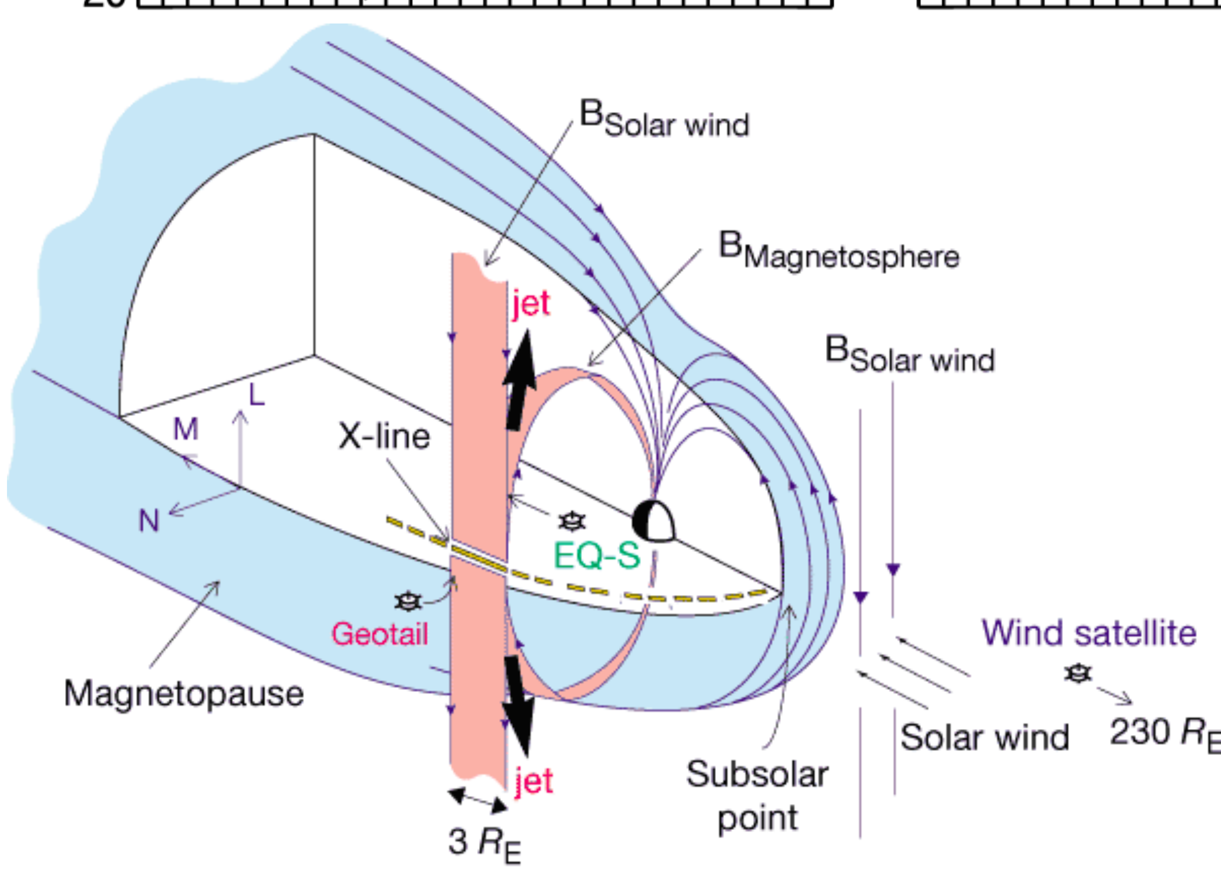
Open Flux Generation





Faraday's law

$$\frac{\partial \Phi}{\partial t} = \mathbf{E} \cdot \mathbf{dl}$$

To zeroth-order, in a quasi-steady/balanced state:

Applications of the momentum equation

The perpendicular current

$$\boxed{\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \nu\nabla^2\mathbf{u} + \rho\mathbf{g}}$$

The force balance between inertial, pressure gradient and Lorentz

$$\longrightarrow \quad \mathbf{B}\times\left(\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{J}\times\mathbf{B} + \cancel{\nu\nabla^2\mathbf{u}} + \cancel{\rho\mathbf{g}}\right)$$

Note that $\mathbf{B}\times\mathbf{J}\times\mathbf{B} = B^2\mathbf{J}_\perp$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$\longrightarrow \quad \mathbf{J}_\perp = -\frac{1}{B^2}\left(\underbrace{\rho\frac{d\mathbf{u}}{dt}\times\mathbf{B}}_{\text{Intertial Current}} + \underbrace{\nabla p\times\mathbf{B}}_{\text{Diamagnetic Current}}\right)$$

"Inertial current" is also known as "polarization current", it is formally equivalent to the polarization drift in a time varying electric field

"diamagnetic current" is a combination of gradient, curvature and magnetization currents, but the corresponding drifts are not in the MHD model:

$$\mathbf{J}_{D\perp} = -\frac{1}{B^2}\nabla p\times\mathbf{B} = \mathbf{J}_{\nabla\mathbf{B}} + \mathbf{J}_C + \mathbf{J}_M$$

Where

$$\mathbf{J}_{\nabla\mathbf{B}} = -\frac{p}{2B^4}\nabla(B^2)\times\mathbf{B} \qquad \mathbf{J}_C = -\frac{p}{B^2}(\mathbf{b}\cdot\nabla\mathbf{b})\times\mathbf{B} \qquad \mathbf{J}_M = -\nabla\times p\frac{\mathbf{B}}{B^2}$$

Applications of the momentum equation

The Vasyliunas equation

With $\mathbf{J}_\perp = -\frac{1}{B^2}\left(\rho\frac{d\mathbf{u}}{dt}\times\mathbf{B} + \nabla p\times\mathbf{B}\right)$, consider $\nabla\cdot\mathbf{J} = 0$ → $\nabla_\perp\cdot\mathbf{J}_\perp + \nabla_\parallel\cdot\mathbf{J}_\parallel = 0$

$$\nabla_\parallel\cdot\mathbf{J}_\parallel = \mathbf{B}\cdot\frac{J_\parallel}{B} = -\nabla_\perp\cdot\left[-\frac{1}{B^2}\left(\rho\frac{d\mathbf{u}}{dt}\times\mathbf{B} + \nabla p\times\mathbf{B}\right)\right]$$

Ignore initial terms → e.g., inner msphere

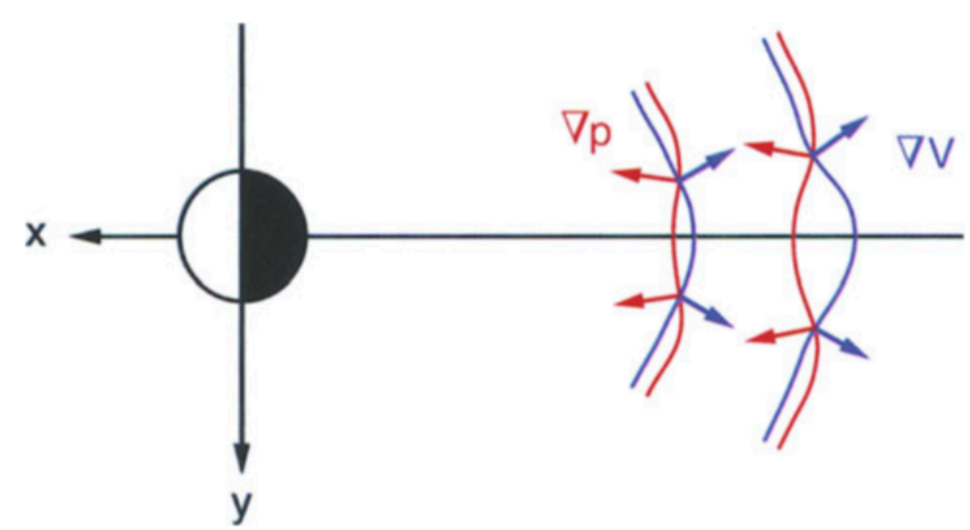
$$\int\left(\nabla_\parallel\cdot\mathbf{J}_\parallel = -\nabla_\perp\cdot\left[-\frac{1}{B^2}\left(\cancel{\rho\frac{d\mathbf{u}}{dt}\times\mathbf{B}}^{\,0} + \nabla p\times\mathbf{B}\right)\right]\right)ds$$

Bounce average →

$$\left(\frac{J_\parallel}{B}\right)_{\mathrm{eq}}^{\mathrm{ion}} = \frac{\mathbf{B}_{eq}}{B_{eq}^2}\cdot\nabla p_{eq}\times\nabla V$$, where $V = \int_{eq}^{ion}\frac{ds}{B}$ Flux tube volume

The Vasyliunas equation for FAC
Usually works for R2 currents

Note: this approach does NOT imply any mechanism of how the currents are generated. It simply addresses diversion from the perpendicular to parallel currents and vice versa.



Applications of the momentum equation

Generation of FAC

Combine the Ampere's and Faraday's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ $\quad \dfrac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$

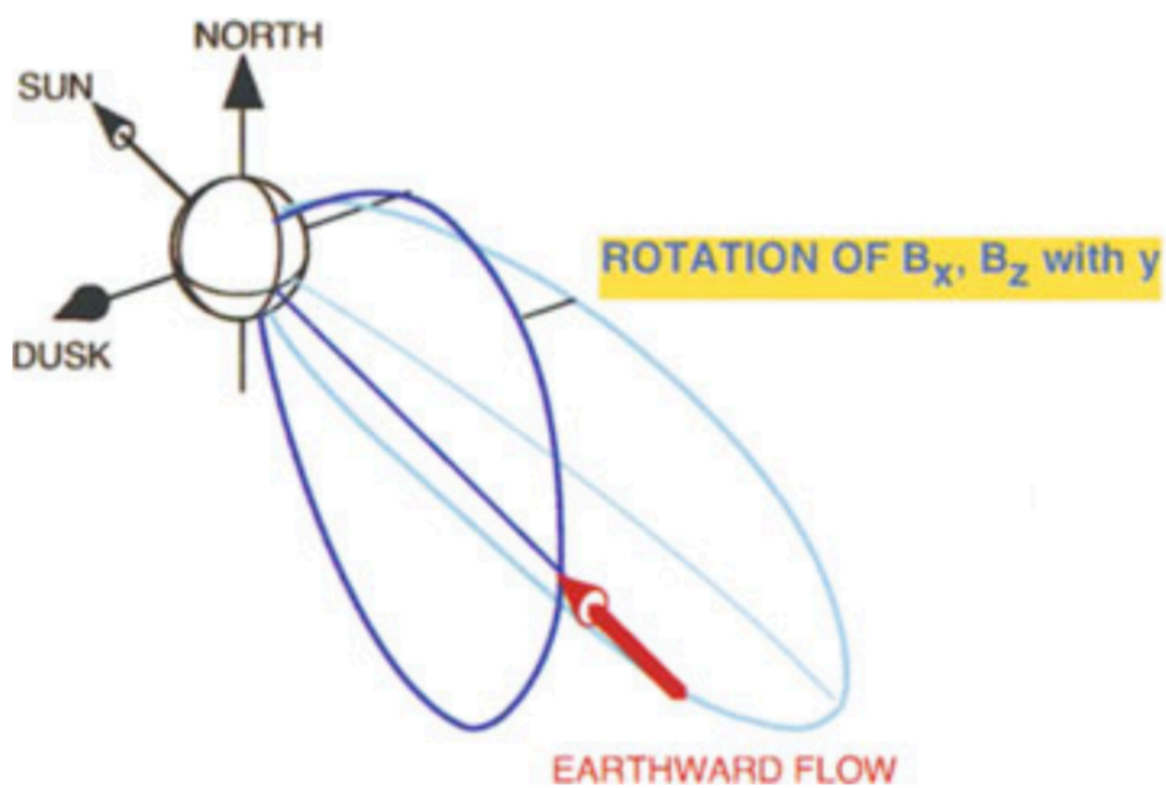
Taking time derivative of the Ampere's law and use the Faraday's law:

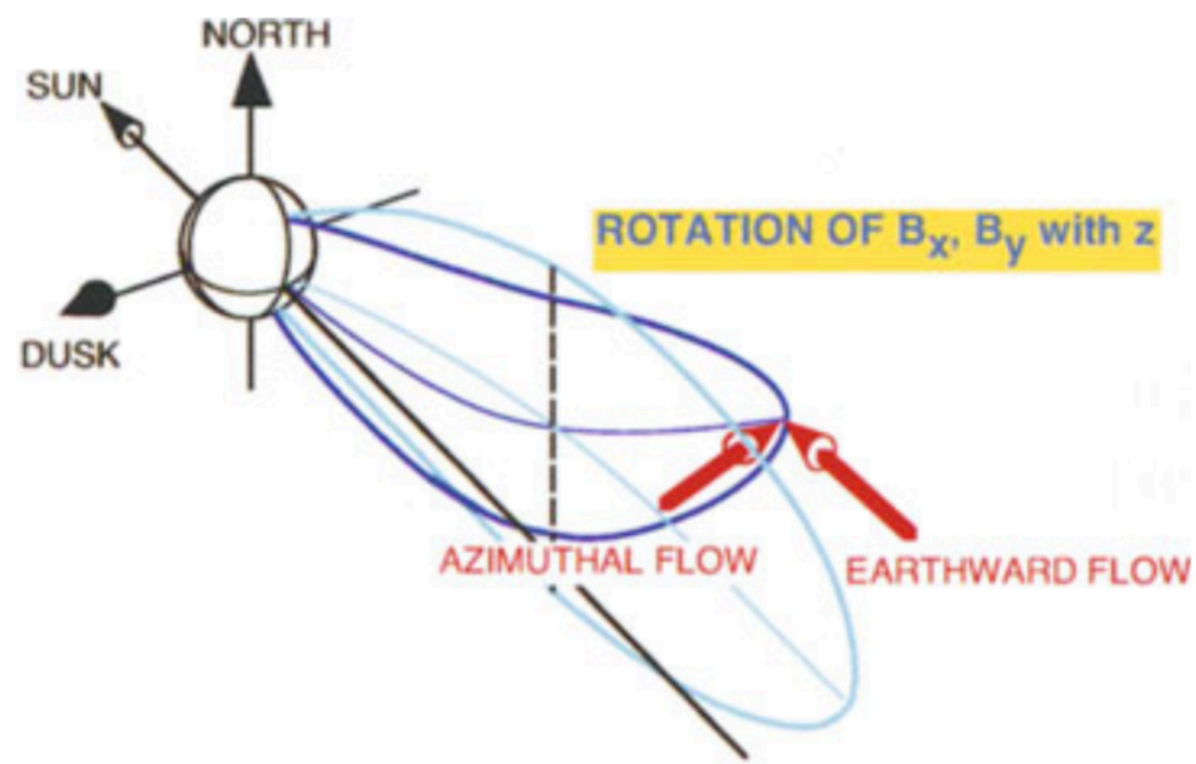
$$\frac{\partial \mathbf{J}}{\partial t} = -\nabla \times \mathbf{E} = -\frac{1}{\mu_0}\nabla \times \left[\mathbf{B}\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u}\right]$$

Assume uniform magnetic field and only look at the parallel term:

$$\frac{\partial \mathbf{J}_{\parallel}}{\partial t} = \frac{1}{\mu_0}\mathbf{B} \cdot \nabla(\nabla \times \mathbf{u}) = \frac{1}{\mu_0}\mathbf{B} \cdot \nabla \Omega_{\parallel}$$

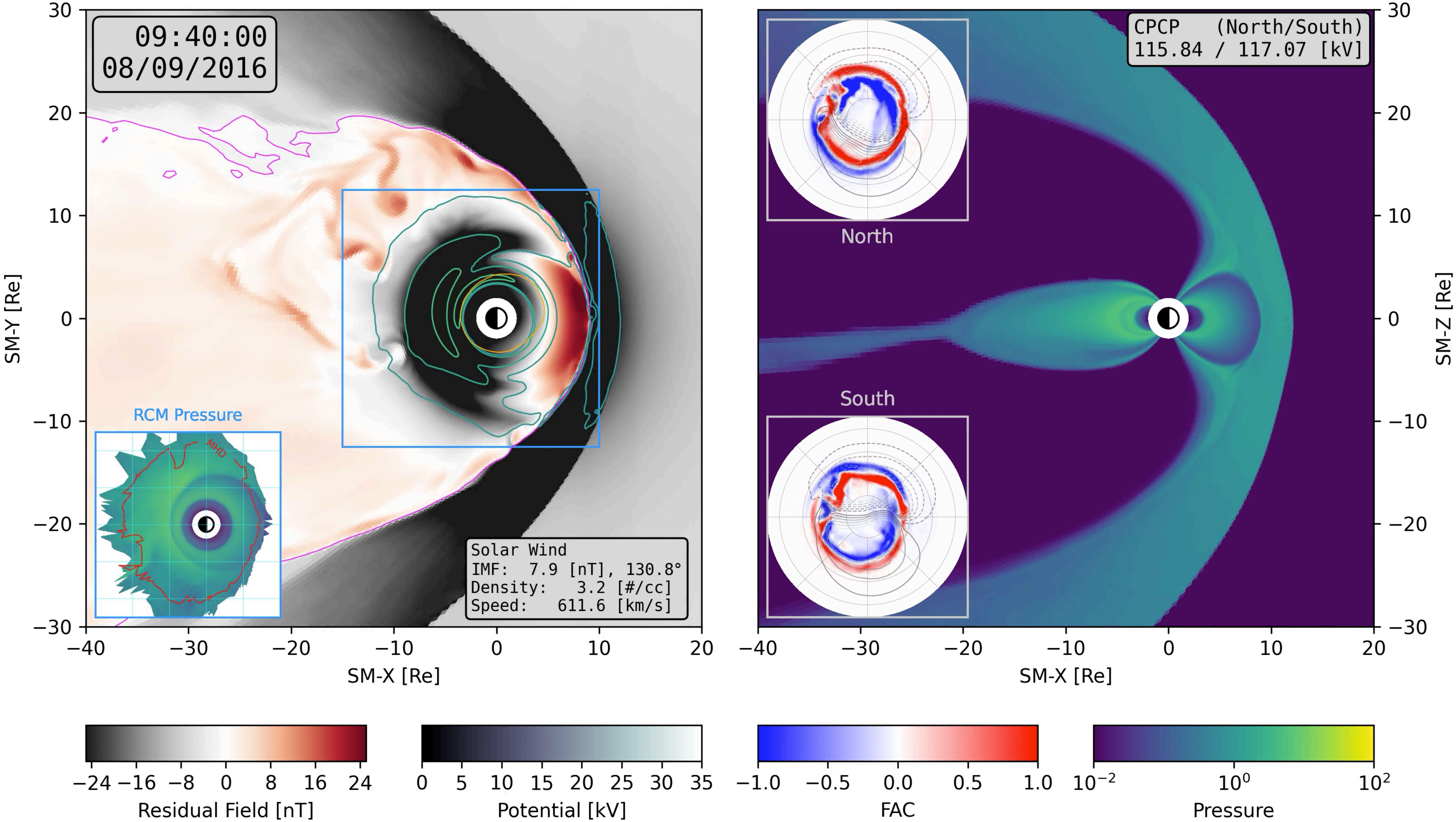
This equation means that field-aligned currents are generated through differential rotation of plasma





Applications of the momentum equation

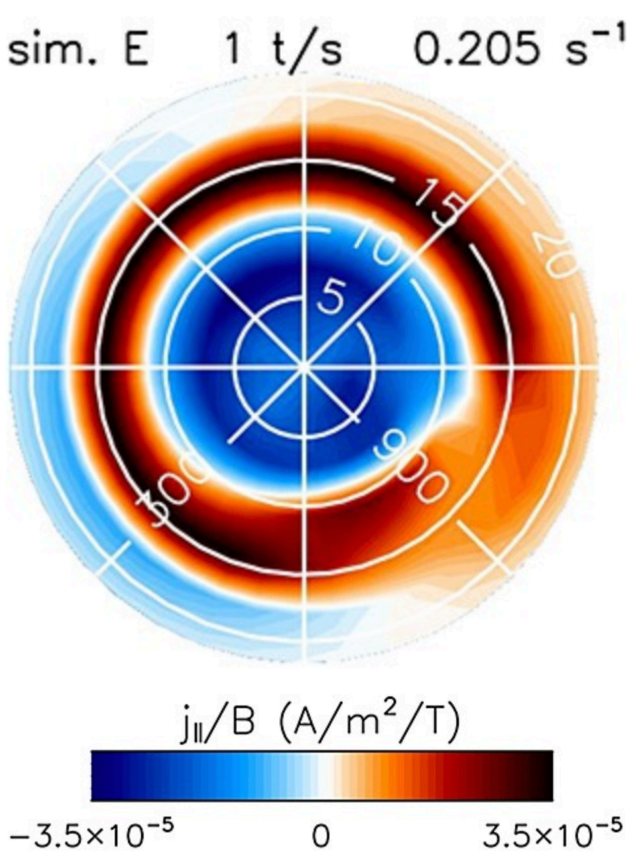
Generation of FAC - Gamera MHD



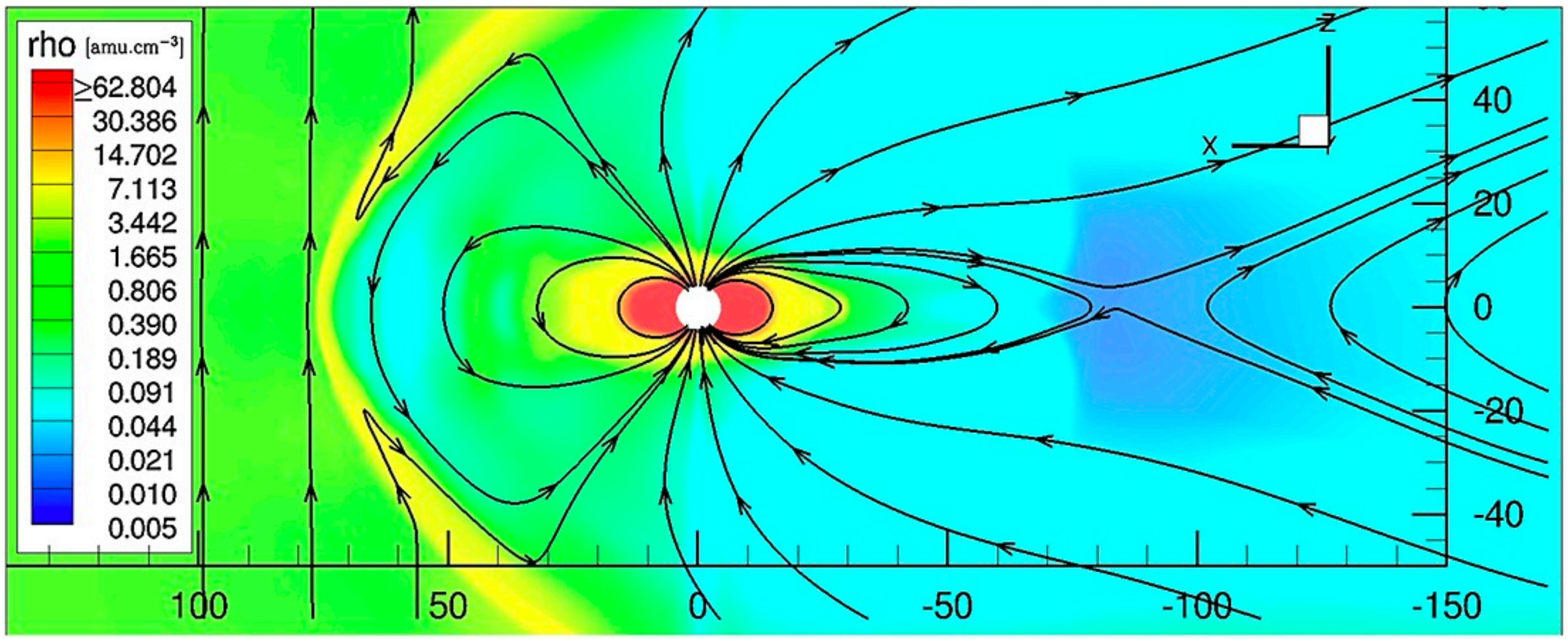
Applications of the momentum equation

Now here’s a problem （那么问题又来了）

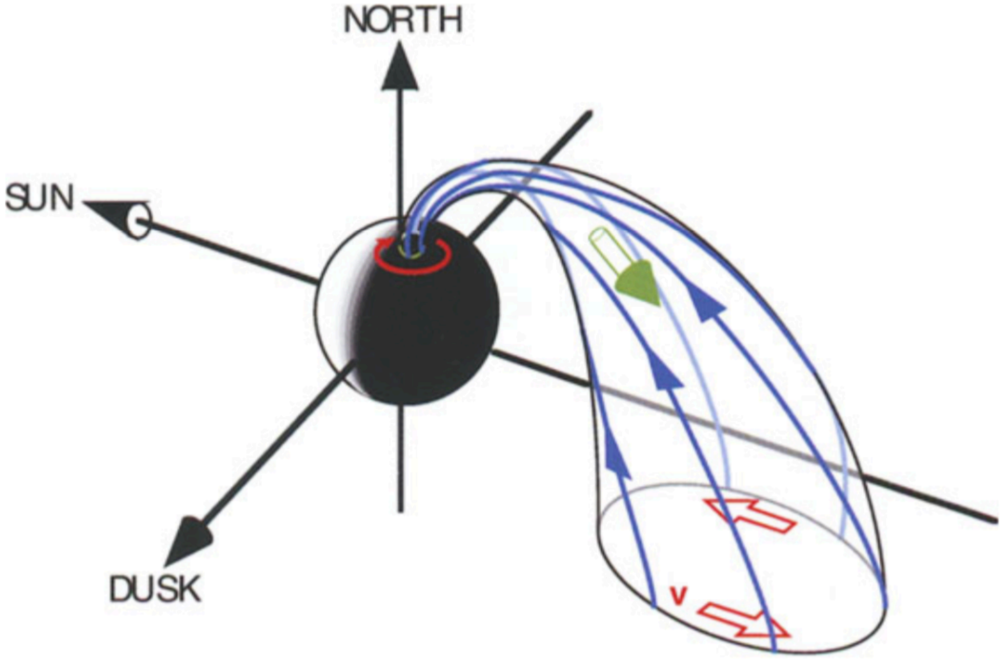
FAC at ionosphere



Field lines in the Magnetosphere



Q: What’s the problem here?



Differential rotation

$$\frac{\partial \mathbf{J}_{\parallel}}{\partial t} = \frac{1}{\mu_0}\mathbf{B}\cdot\nabla\Omega_{\parallel}$$

Energy equations

Ideal fluid

Recall the pressure equation from the moment transport equation: $\frac{\partial}{\partial t}p + \mathbf{u}\cdot\nabla p + \gamma p \nabla\cdot\mathbf{u} = 0$

Note the relationship between pressure and internal energy: $\rho e = p/(\gamma - 1)$, we get the internal energy eqn:

$$\frac{\partial}{\partial t}e + \mathbf{u}\cdot\nabla e + (\gamma - 1)e\nabla\cdot\mathbf{u} = 0$$

$e = \frac{p}{\rho(\gamma - 1)}$ Internal energy per mass

Or the evolution of internal energy ρe

$$\frac{\partial}{\partial t}\rho e + \nabla\cdot\rho e\mathbf{u} + p\nabla\cdot\mathbf{u} = 0$$

Try the temperature eqn!

For adiabatic processes of ideal gases, which is used in ideal MHD, the above equations are equivalent to the conservation of entropy:

$$\frac{DS}{Dt} = \frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

$S = p\rho^{-\gamma}$

Neglecting thermal conduction, heat flow

The above equation means the entropy convected by the fluid (w/o dissipation, heating) is constant

The entropy per unit volume ρS can be cast into a conservative form

$$\frac{\partial}{\partial t}\rho S + \nabla\cdot\rho S\mathbf{u} = 0$$

Energy equations

Kinetic energy

Starting from the equation of motion

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0 \xrightarrow{\mathbf{u}\,\cdot} \mathbf{u}\cdot\left(\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = 0\right)$$

$$\longrightarrow \frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) - \frac{1}{2}u^2\frac{\partial \rho}{\partial t} + \frac{1}{2}\rho\mathbf{u}\cdot\nabla u^2 + \mathbf{u}\cdot\nabla p - \mathbf{u}\cdot\mathbf{J}\times\mathbf{B} = 0$$

Mass eqn

$$\longrightarrow \boxed{\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \nabla\cdot(\frac{1}{2}\rho u^2\mathbf{u}) + \mathbf{u}\cdot\nabla p - \mathbf{u}\cdot\mathbf{J}\times\mathbf{B} = 0}$$

Kinetic energy equation

Magnetic energy

Starting from the Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \xrightarrow{\mathbf{B}\,\cdot} \mathbf{B}\cdot\left(\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\right) \xrightarrow{\mathbf{E}=-\mathbf{u}\times\mathbf{B}} \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B}\cdot\nabla\times\mathbf{u}\times\mathbf{B}$$

$$\xrightarrow{\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}} \frac{\partial}{\partial t}(\frac{1}{2}B^2) + \nabla\cdot\left[\mathbf{B}\times(\mathbf{u}\times\mathbf{B})\right] - (\mathbf{u}\times\mathbf{B})\cdot\nabla\times\mathbf{B} = 0$$

$$\xrightarrow{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}} \boxed{\frac{\partial}{\partial t}(\frac{1}{2}B^2) + \nabla\cdot\left[B^2\mathbf{u} - \mathbf{u}\cdot\mathbf{B}\mathbf{B}\right] + \mathbf{u}\cdot\mathbf{J}\times\mathbf{B} = 0}$$

Magnetic energy equation

Energy equations

Kinetic energy

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \nabla \cdot (\frac{1}{2}\rho u^2 \mathbf{u}) + \mathbf{u} \cdot \nabla p - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Magnetic energy

$$\frac{\partial}{\partial t}(\frac{1}{2}B^2) + \nabla \cdot \left[B^2 \mathbf{u} - \mathbf{u} \cdot \mathbf{B}\mathbf{B}\right] + \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Internal energy

$$\frac{\partial}{\partial t}\rho e + \nabla \cdot \rho e \mathbf{u} + p \nabla \cdot \mathbf{u} = 0$$

K+M+I

$$\frac{\partial}{\partial t}\underbrace{(\frac{1}{2}\rho u^2 + \rho e + \frac{1}{2}B^2)}_{\text{Total energy}} + \nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + p + B^2\right)\mathbf{u} - \mathbf{u} \cdot \mathbf{B}\mathbf{B}\right] = 0$$

$p_{tot} = p + \frac{1}{2}B^2$ Total pressure

Total energy equation

$$\boxed{\frac{\partial \mathscr{E}_T}{\partial t} + \nabla \cdot \left[\left(\mathscr{E} + p_{tot}\right)\mathbf{u} - \mathbf{u} \cdot \mathbf{B}\mathbf{B}\right] = 0}$$

Fully-conservative

K+I

$$\frac{\partial}{\partial t}\underbrace{(\frac{1}{2}\rho u^2 + \rho e)}_{\text{Plasma energy}} + \nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\right] - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Plasma energy equation

$$\boxed{\frac{\partial \mathscr{E}_P}{\partial t} + \nabla \cdot \mathbf{u}\left(\mathscr{E}_P + p\right) - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0}$$

Semi-conservative

More on Energy equations

Kinetic energy $$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \nabla \cdot (\frac{1}{2}\rho u^2 \mathbf{u}) + \mathbf{u} \cdot \nabla p - \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Magnetic energy $$\frac{\partial}{\partial t}(\frac{1}{2}B^2) + \nabla \cdot \left[B^2\mathbf{u} - \mathbf{u} \cdot \mathbf{B}\mathbf{B}\right] + \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = 0$$

Internal energy $$\frac{\partial}{\partial t}\rho e + \nabla \cdot \rho e \mathbf{u} + p\nabla \cdot \mathbf{u} = 0$$

A couple of notes:

$$\mathbf{u} \cdot \mathbf{J} \times \mathbf{B} \xrightarrow{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})} \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = \mathbf{J} \cdot \mathbf{B} \times \mathbf{u} \xrightarrow{\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0} \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = \mathbf{J} \cdot \mathbf{E}$$

- In ideal MHD w/o dissipation, J dot E is the work done by the Lorentz force u dot F
- In ideal MHD w/o dissipation, J dot E is NOT heating (don’t call it Joule heating!)
- The Lorentz force term does not show up in the internal energy equation (why?)
- The semi-conservative form is very useful in numerical solving of the MHD equations