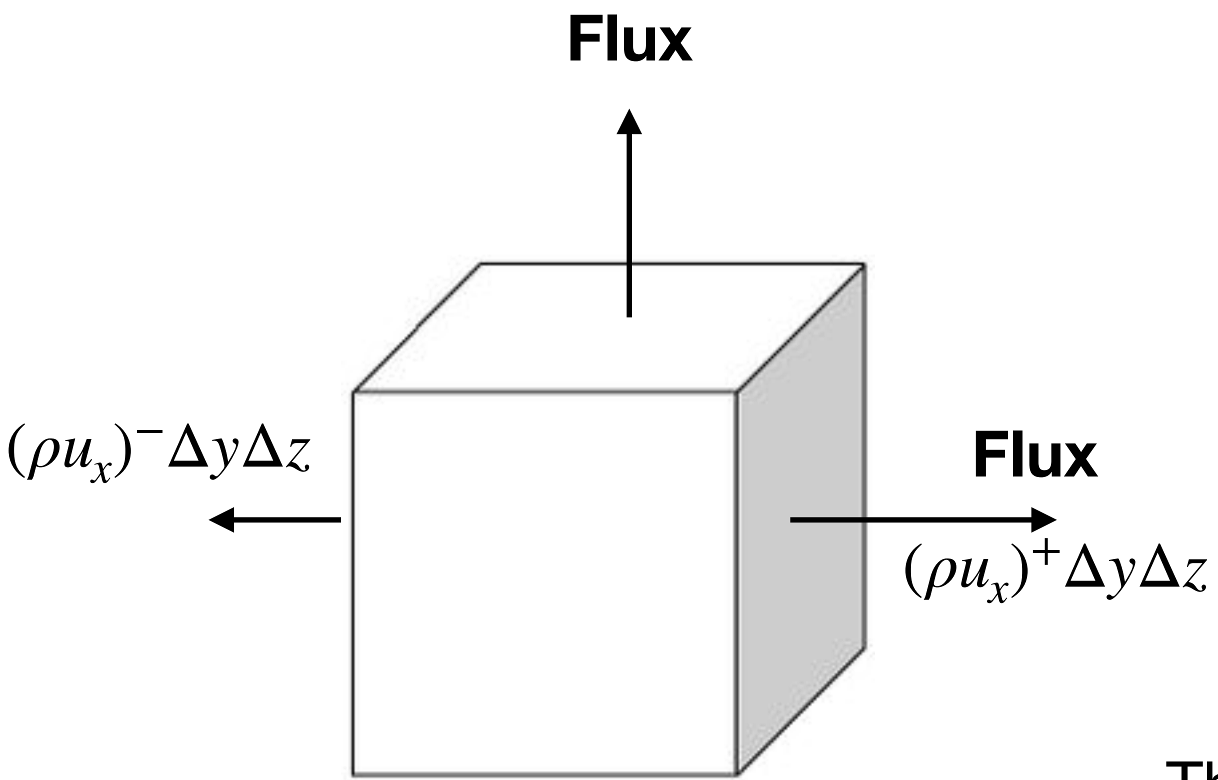
Fluid dynamics

A first step before MHD

- **Mass equation**
  - Conservation laws
  - Convective derivative
- **Momentum equation**
  - Categories of Fluid Flow
  - Viscous stress
  - Navier-Stokes equation
  - Vorticity and angular momentum
- **Energy equation**
  - Ideal gas
  - conduction, radiation and heating
- **Euler’s equations**

Conservation laws



$$Vol = \Delta x \Delta y \Delta z$$

A general conservation law goes as

$$\text{Changes in Mass} = \Sigma_{Flux}\left(\text{in} - \text{out}\right) + P - L$$

Mass Flux through interface

Local Production and lost

The integral form is written as

$$\frac{\partial}{\partial t}\int_{CV} \rho dV = -\oint \rho \mathbf{u} \cdot d\mathbf{S}$$

$$\downarrow \qquad\qquad \downarrow \tfrac{1}{V}$$

$$V \cdot \frac{\partial \bar{\rho}}{\partial t} \qquad \frac{\partial}{\partial x}\rho u_x + \frac{\partial}{\partial y}\rho u_y + \frac{\partial}{\partial z}\rho u_z$$

$$\frac{1}{V}\oint \rho \mathbf{u} \cdot d\mathbf{S}\,\Big|_{x-dir}$$

$$\downarrow$$

$$\frac{(\rho u_x)^{+}\Delta y \Delta z - (\rho u_x)^{-}\Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$\downarrow$$

$$\frac{(\rho u_x)^{+} - (\rho u_x)^{-}}{\Delta x} \longrightarrow \frac{\partial}{\partial t}\rho u_x$$

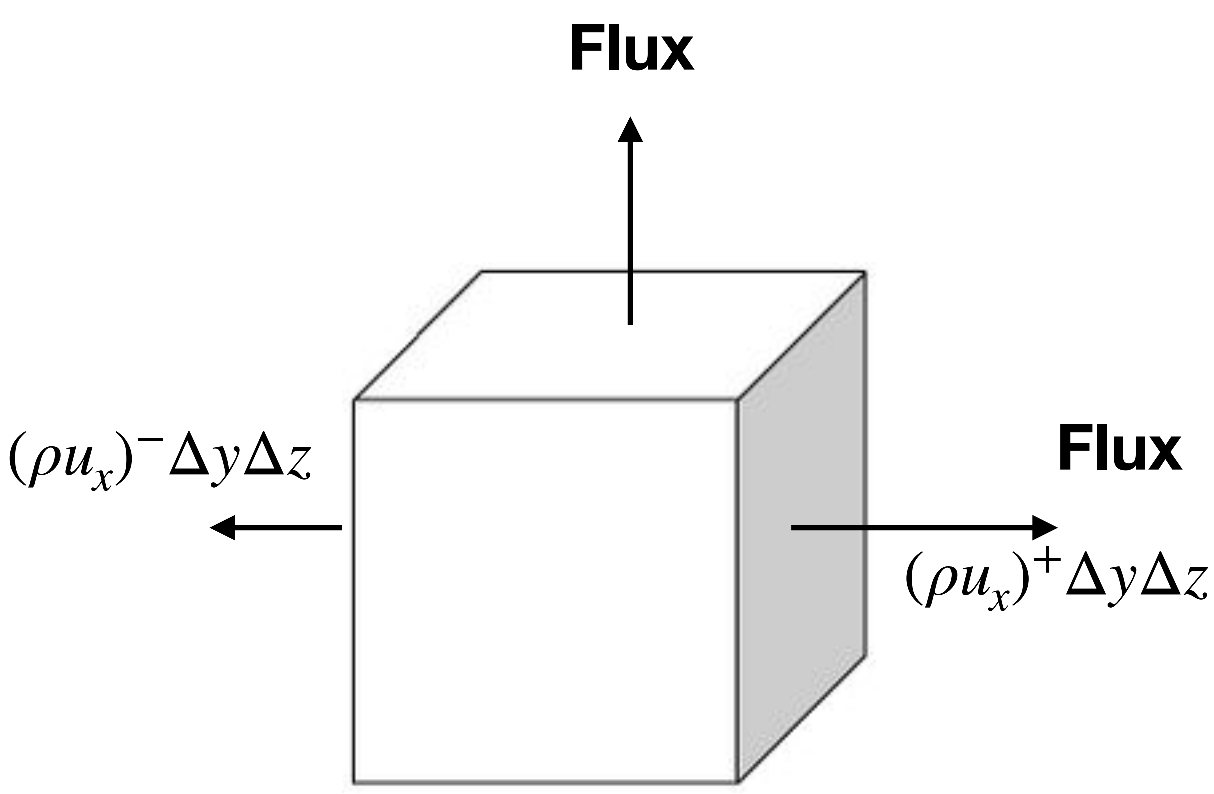
Differential form

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} = -\nabla \cdot \left(\rho \mathbf{u}\right)}$$

Conservation form of differential equations

$$\boxed{\frac{\partial}{\partial t}(\ldots) + \nabla \cdot (\ldots) = 0}$$

Conservation law



$Vol = \Delta x \Delta y \Delta z$

The use of such a form of the equations is that one can obtain *local and global conservation laws* and *jump conditions* from them. Moreover, powerful numerical algorithms exist for the solution of such equations.

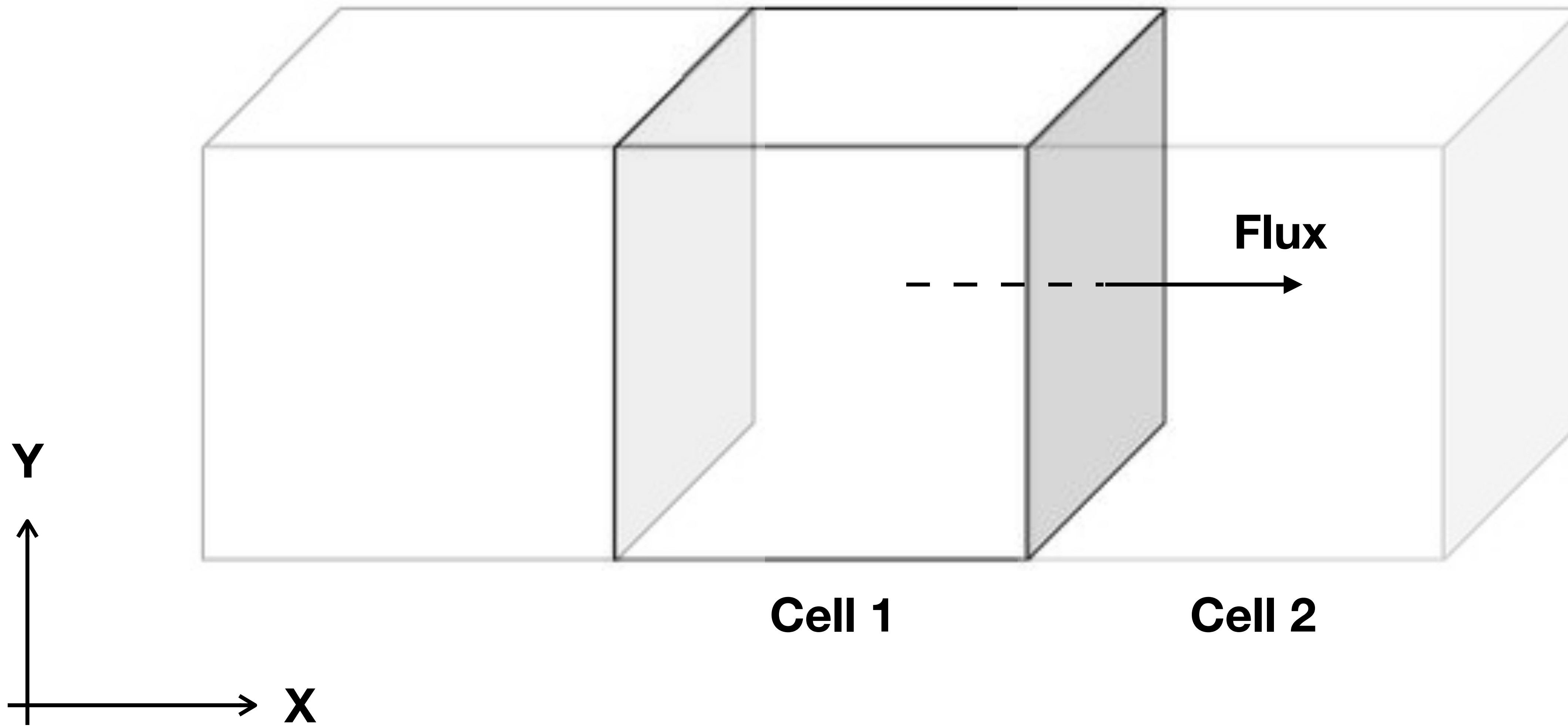
For example mass conservation:

$$\frac{\partial \bar{\rho}}{\partial t} = -\nabla \cdot \left(\rho \mathbf{u}\right)$$

Changes in mass density

Mass fluxes through volume interface

Conservation of mass



- For “Cell 1”, the flux is outward (in the x-direction)
- For “Cell 1”, the flux is the same magnitude but inward (in the negative x-direction)
- When summing the two cells up, the internal flux cancels out
- Mass is conserved through $\dfrac{\partial}{\partial t}\int_{CV} \rho dV = -\oint \rho \mathbf{u} \cdot d\mathbf{S}$

Mass Conservation

For fluid density in a control volume, with velocity field $\mathbf{u}$:

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

Conservation form

Or slightly differently

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0}$$

Primitive form

Eulerian form

Move the divergence term to the RHS:

$$\frac{\partial \bar{\rho}}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

Compressibility

$$\underbrace{\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)}_{\text{Convective Derivative}} \rho \equiv \boxed{\frac{D}{Dt}\rho = -\rho \nabla \cdot \mathbf{u}}$$

Convective Derivative

Lagrangian form

If $\nabla \cdot \mathbf{u} = 0$, then the equation for mass density ρ is decoupled from the flow

If $\nabla \cdot \mathbf{u} = 0$, then the equation for mass density ρ is constant along flow lines

Convective Derivative

$$\frac{D}{Dt}\rho = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\rho$$

mathematically, it's know as the total derivative if $\rho \sim \rho(x, y, z, t)$

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial\rho}{\partial t} + \mathbf{u} \cdot \nabla\rho$$

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \equiv \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \qquad \nabla\rho = \frac{d\rho}{d\mathbf{x}} \equiv \frac{\partial\rho}{\partial x}\hat{x} + \frac{\partial\rho}{\partial y}\hat{y} + \frac{\partial\rho}{\partial z}\hat{z}$$

In fluid dynamics, the convective derivative basically describes the time rate of change of a fluid blob along its trajectory, which is also known as the "material derivative" - very useful in considering terms such as acceleration

Categories of Fluid Flow

Viscous vs. Inviscid

Viscosity	Inviscid	Viscous
ν	Zero	Non-zero

Laminar vs. Turbulent

Reynolds Number	Laminar	Turbulent
Re	Low	High

Potential vs. Rotational

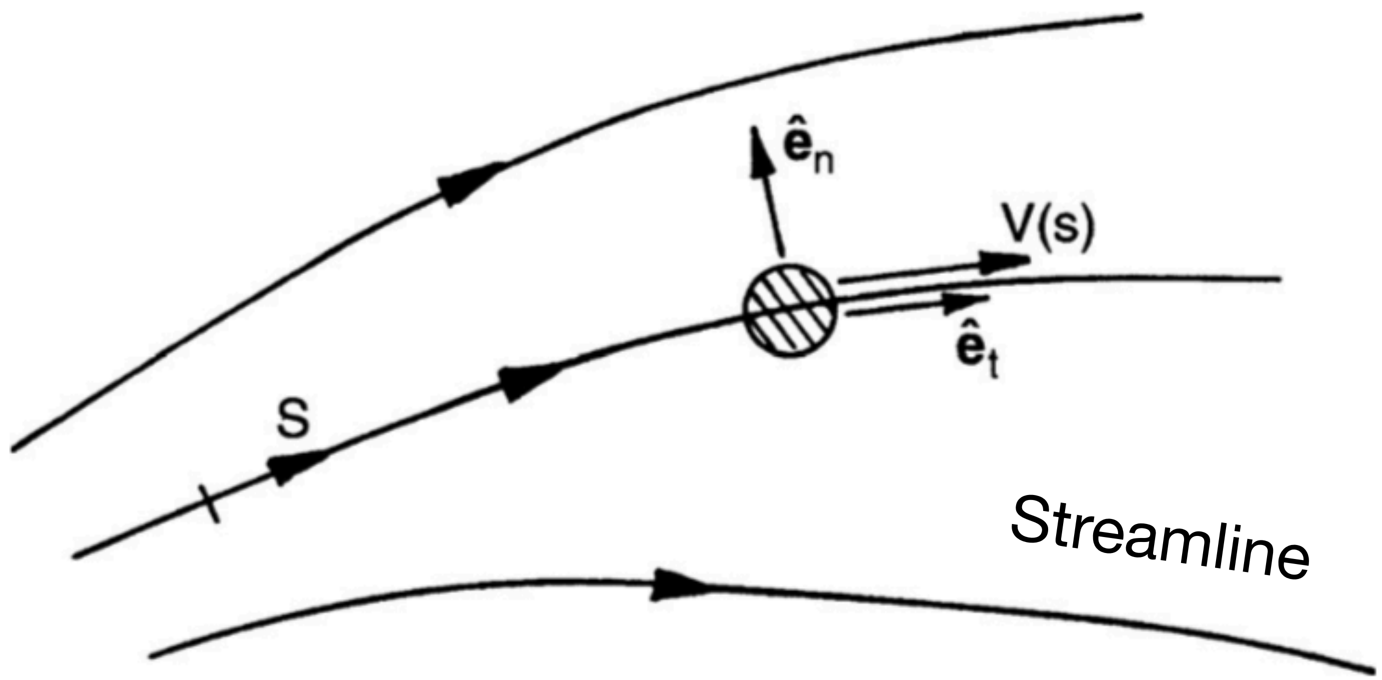
Vorticity	Potential	Rotational
$\nabla \times \mathbf{u}$	Zero	Non-zero

Incompressible vs. compressible

Compressibility	imcompressible	Compressible
$\nabla \cdot \mathbf{u}$	Zero	Non-zero

Inertial forces vs. Shear stresses

Inertial forces



$$\text{Acceleration} = V\frac{dV}{ds}\hat{\mathbf{e}}_t - \frac{V^2}{R_C}\hat{\mathbf{e}}_n$$

Parallel Centrifugal

Recall high-school physics, the **acceleration** of a fluid element is of order

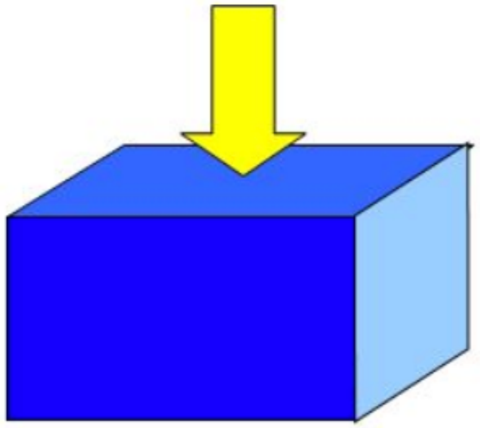
$$\frac{|\mathbf{u}^2|}{l_0}$$

The concept of Stress

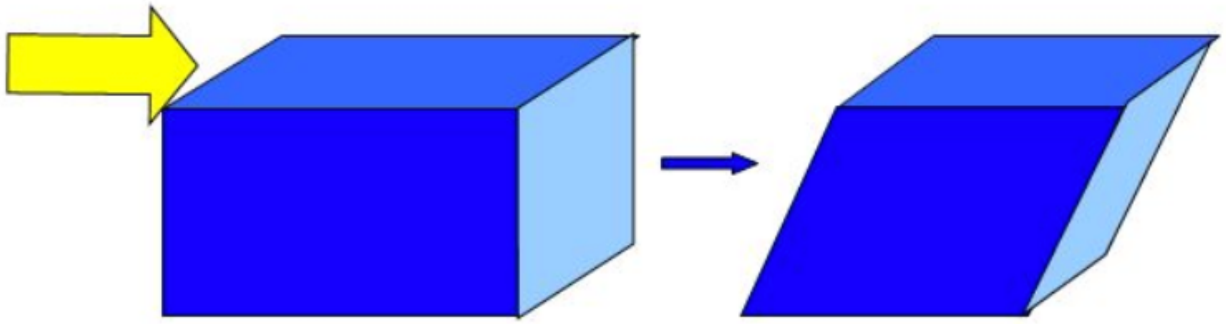
Definition: Stress = force per area

$$\text{Stress } (\tau) = \frac{\text{Force}}{\text{Cross}-\text{sectional area}} \quad \left[\frac{N}{m^2}\right]$$

Normal (or tensile) stress = perpendicular to surface



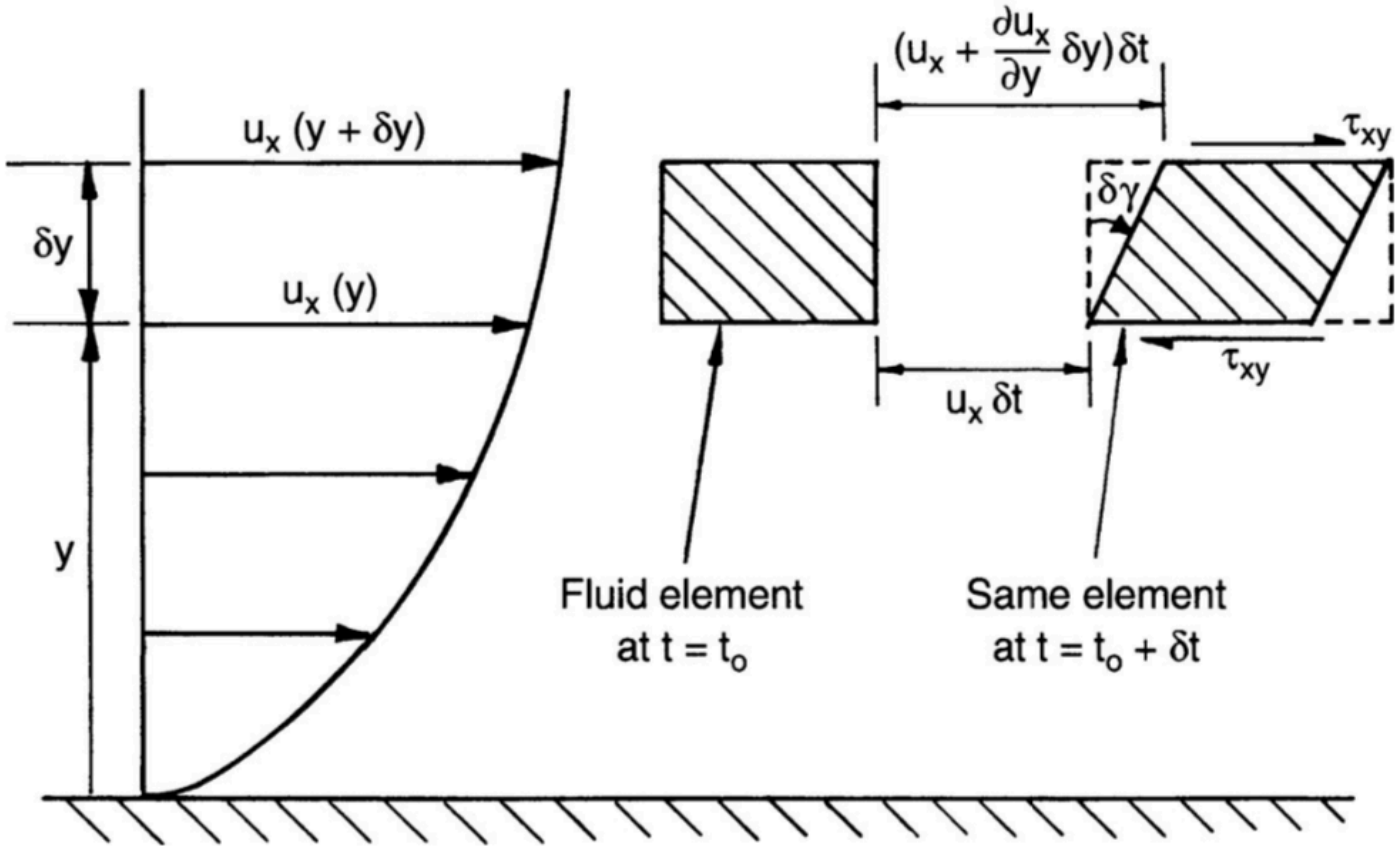
Shear stress = parallel to surface



Difference in surface stress gives force

Shear Force and Shear stresses

Shear forces



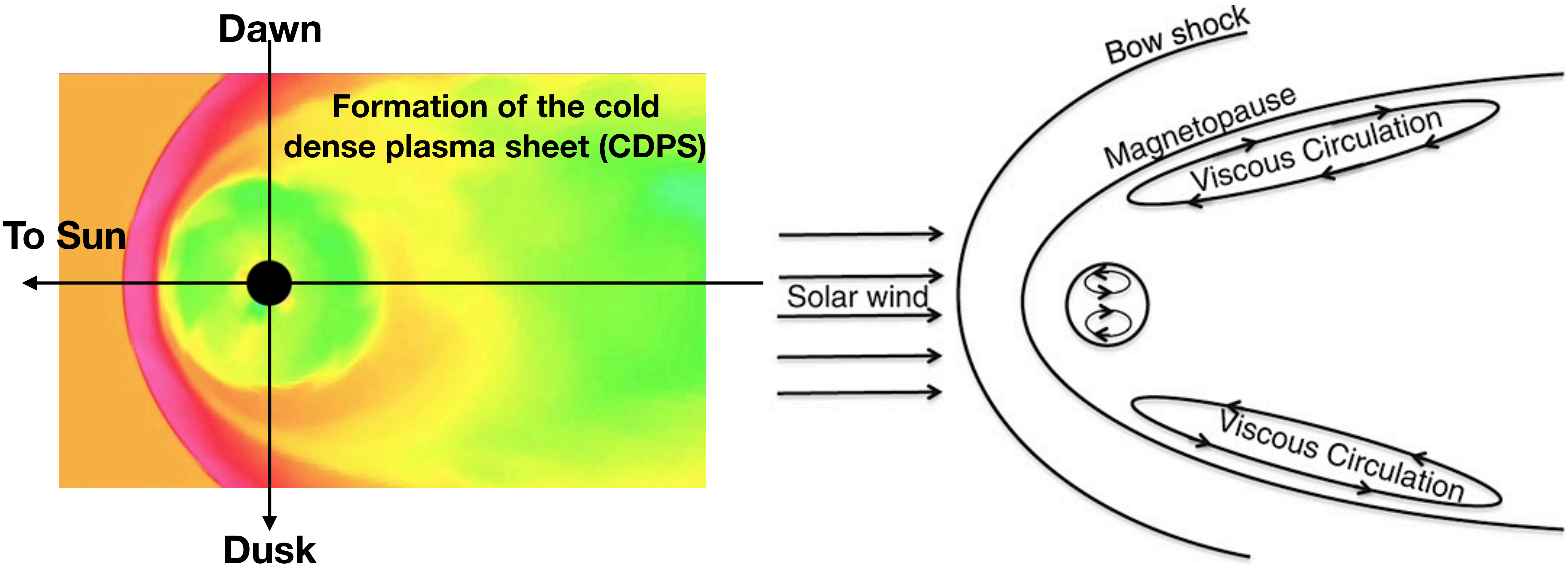
One measure of rate of sliding is the angular distortion rate, **dγ/dt**, of an initially rectangular element

Question: if viscosity is zero, do we still have the shear force?

Newton's law of viscosity says that a shear stress, τ, is required to cause the relative sliding of the fluid layers. Moreover, it states that τ is proportional to **dγ/dt**:

$$\tau = \rho\nu\frac{\partial u_x}{\partial y} \xrightarrow{\text{the net horizontal shear force}} \mathbf{f}_x = \frac{\partial \tau}{\partial y} = \rho\nu\frac{\partial^2 u_x}{\partial y^2}$$

Example: Magnetospheric "Viscous" Interaction



When the interplanetary magnetic field (IMF) interacts with the Earth's magnetosphere, near the boundary where the two types of magnetic fields interact, the shear-force like, "viscous" interaction occurs - magnetospheric plasma starts to circulate within the closed magnetosphere. For northward IMF driving, viscous interaction if one of the key processes.

Question: Magnetospheric plasmas are collisionless, which means they are regarded as inviscid, where is the "viscous" force coming from?

Example: Viscous Interaction

Maxwell Stress

Recall that in the electrodynamics lecture, the Lorentz force per volume acting on charge is

$$\mathbf{F} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Use Maxwell's equations $\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon}$ $\quad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \right)$

Substitution gives

$$\mathbf{F} = \epsilon \mathbf{E}(\nabla \cdot \mathbf{E}) - \frac{1}{\mu_0} \mathbf{B} \times \nabla \times \mathbf{B} - \epsilon \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}$$

Note that $\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} \times \mathbf{E}$

$$\longrightarrow \quad \mathbf{F} = \epsilon \mathbf{E}(\nabla \cdot \mathbf{E}) - \frac{1}{\mu_0} \mathbf{B} \times \nabla \times \mathbf{B} + \epsilon \frac{\partial \mathbf{B}}{\partial t} \times \mathbf{E} - \epsilon \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

$$\xrightarrow{\text{Replace } \frac{\partial \mathbf{B}}{\partial t}} \quad \mathbf{F} = \epsilon \mathbf{E}(\nabla \cdot \mathbf{E}) - \frac{1}{\mu_0} \mathbf{B} \times \nabla \times \mathbf{B} - \epsilon \mathbf{E} \times \nabla \times \mathbf{E} - \epsilon \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

$$\boxed{\epsilon \nabla \cdot \left(\mathbf{E}\mathbf{E} - \frac{1}{2} \mathbf{I} E^2\right)} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \boxed{\frac{1}{\mu_0} \nabla \cdot \left(\mathbf{B}\mathbf{B} - \frac{1}{2} \mathbf{I} B^2\right)}$$

So the Lorentz force equation becomes $\mathbf{F} = \epsilon \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \nabla \cdot \bar{\mathbf{T}}$

$$\bar{\mathbf{T}} = \frac{1}{2} \left(\cancelto{0}{\epsilon E^2} + \frac{B^2}{\mu_0} \right) \bar{\mathbf{I}} - \cancelto{0}{\epsilon \mathbf{E}\mathbf{E}} - \frac{1}{\mu_0} \mathbf{B}\mathbf{B} = \text{ Maxwell Stress tensor}$$

MHD **MHD**

Example: Viscous Interaction

Maxwell Stress tensor in MHD

The full Maxwell stress tensor

$$\bar{\mathbf{T}} = \frac{1}{2}\left(\underbrace{\epsilon E^2}_{1} + \underbrace{\frac{B^2}{\mu_0}}_{2}\right)\bar{\mathbf{I}} - \epsilon \mathbf{E}\mathbf{E} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}$$

$$\frac{1}{2} \sim \frac{\epsilon E^2}{\frac{1}{\mu_0}B^2} = \frac{\left(\frac{E}{B}\right)^2}{\frac{1}{\epsilon \mu_0}} \xrightarrow{E \approx uB} \frac{u^2}{c^2} \xrightarrow{\text{Slow flow}} \ll 1$$

So the electric field terms are much smaller than magnetic field terms in MHD

The Maxwell stress tensor for MHD goes

$$\boxed{\bar{\mathbf{T}} = \frac{1}{2}\frac{B^2}{\mu_0}\bar{\mathbf{I}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}}$$

$$\underbrace{\frac{1}{2}\frac{B^2}{\mu_0}\bar{\mathbf{I}}}_{\text{Magnetic Pressure}} \qquad \underbrace{\frac{1}{\mu_0}\mathbf{B}\mathbf{B}}_{\text{Magnetic Tension}}$$

Lorentz force $\quad \mathbf{F} = \nabla \cdot \bar{\mathbf{T}} = \mathbf{J} \times \mathbf{B}$

Example: Viscous Interaction

Through Maxwell Stress

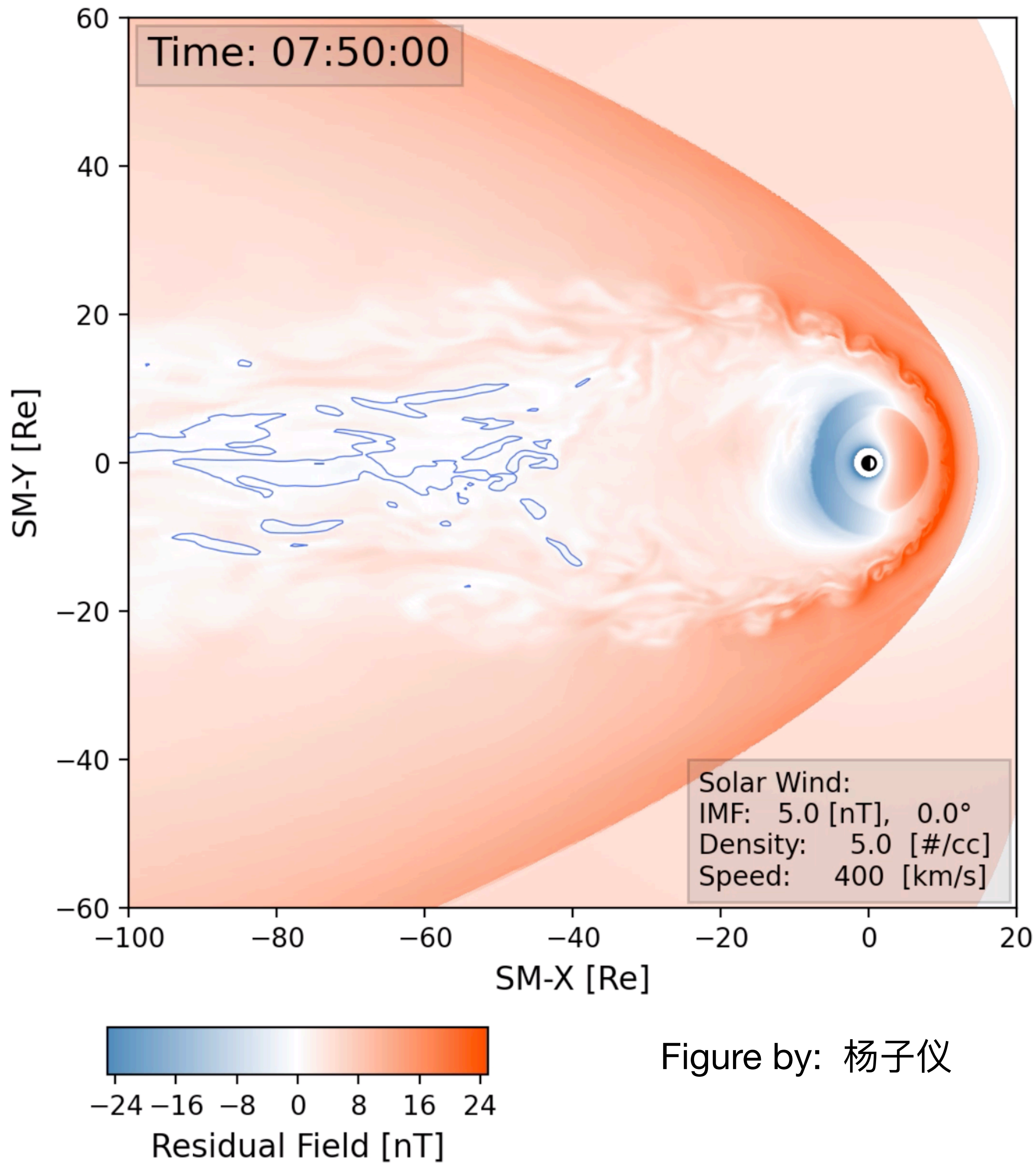
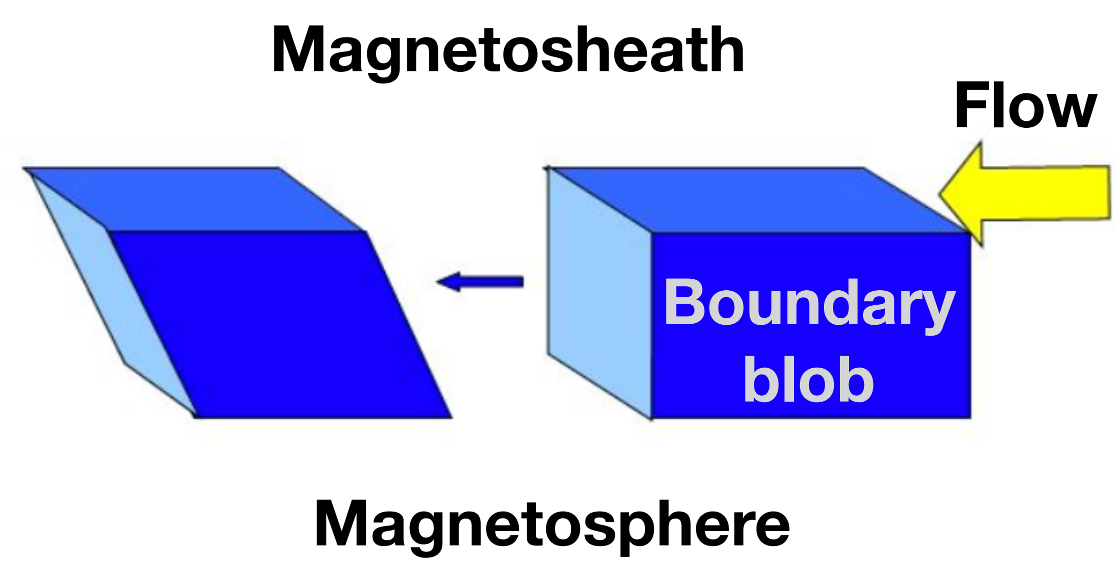


Figure by: 杨子仪

Shear stress through the Lorentz force



Maxwell's stress tensor for MHD:

$$\bar{\mathbf{T}} = \frac{1}{2}\frac{B^2}{\mu_0}\bar{\mathbf{I}} - \frac{1}{\mu_0}\mathbf{B}\mathbf{B}$$

Lorentz force

$$\mathbf{F} = \nabla \cdot \bar{\mathbf{T}}$$

Flow Reynolds Number

Inertial force term goes like $\rho \frac{|\mathbf{u}^2|}{l_0}$ Acceleration

Viscous force term goes like $\rho \nu \frac{|\mathbf{u}|}{l_0^2}$ Viscous drag

$\xrightarrow{\text{Ratio}}$ Reynolds number $\mathrm{Re} = \frac{|u|\, l_0}{\nu}$

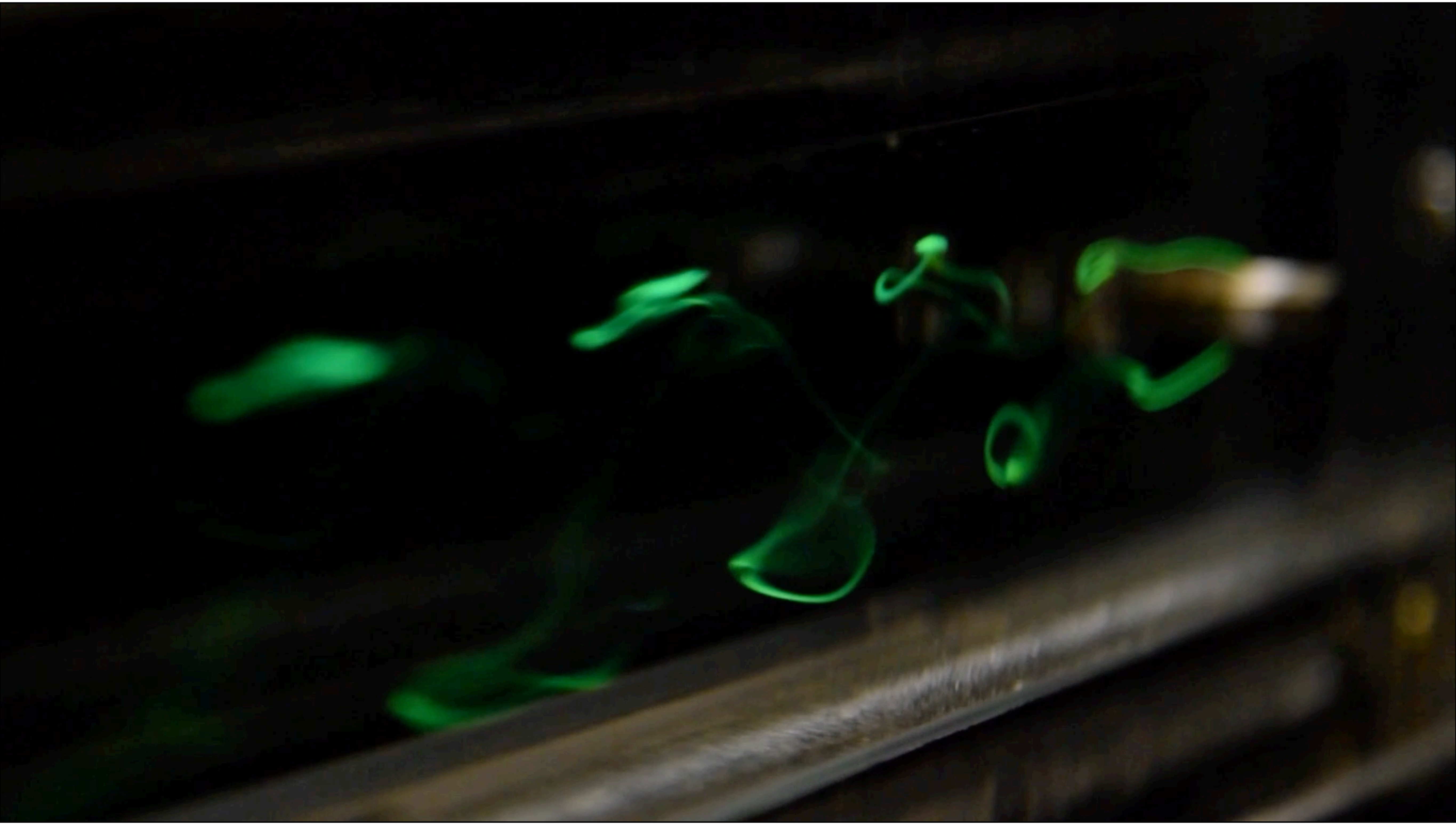
Several important notes about the Reynolds number:

- When Re is small, viscous forces outweigh inertial forces, and when Re is large viscous forces are relatively small.
- When Re is calculated using some characteristic geometric length, it is always very large (think about the magnetosphere: $|u|$ = 400 km/s, $l_0 \sim$ RE = 6380 km, viscosity = 10^{-9}
- When Re is small, flow is laminar (smooth without structures)
- When Re is large, flow is turbulent (chaotic with random behavior)

Question: what is the flow Reynolds number for ionospheric plasma?

Flow Reynolds Number

Laminar and turbulent flows



Flow Reynolds Number

Laminar to turbulent transition

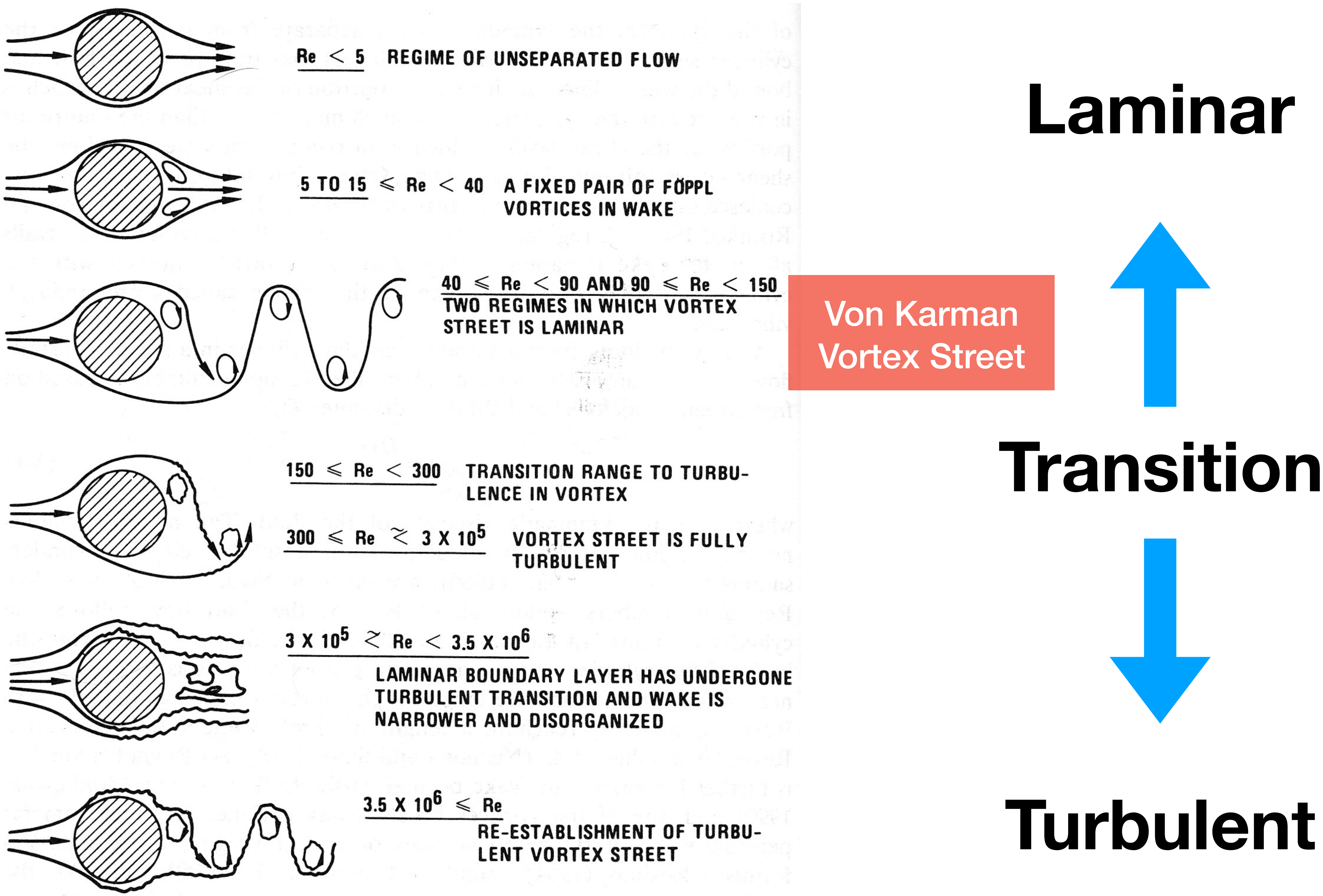


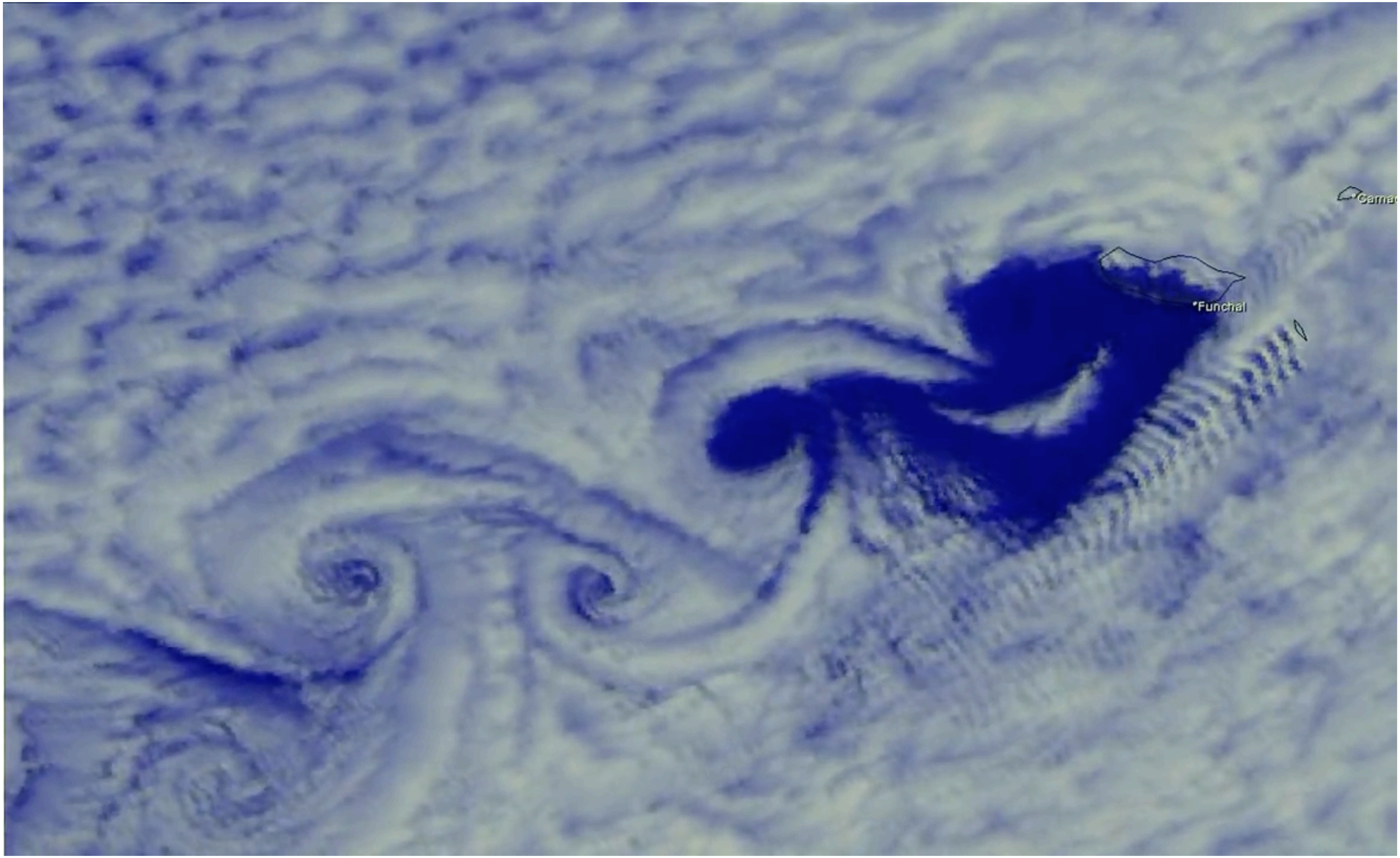
Fig. 3-2 Regimes of fluid flow across smooth circular cylinders (Lienhard, 1966).

Properties of Turbulent flow

- **Unpredictable**
- **Multi scale**
- **Diffusive**
- **High Reynolds Number**
- **Dissipative**

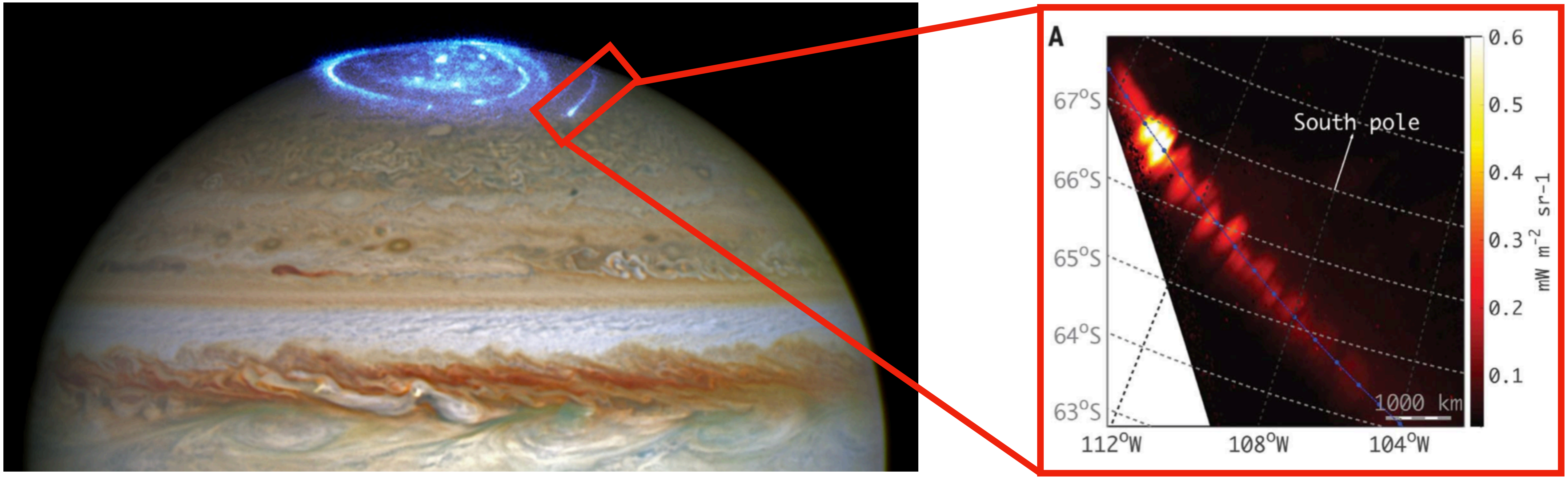
Von Karman Vortex Street

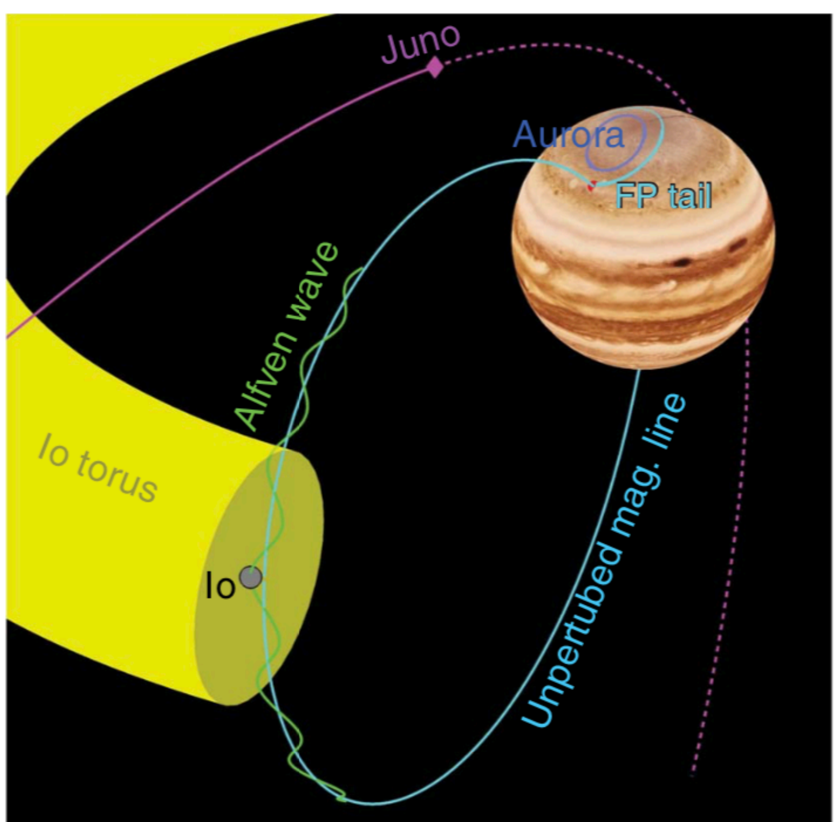
Earth's Atmosphere

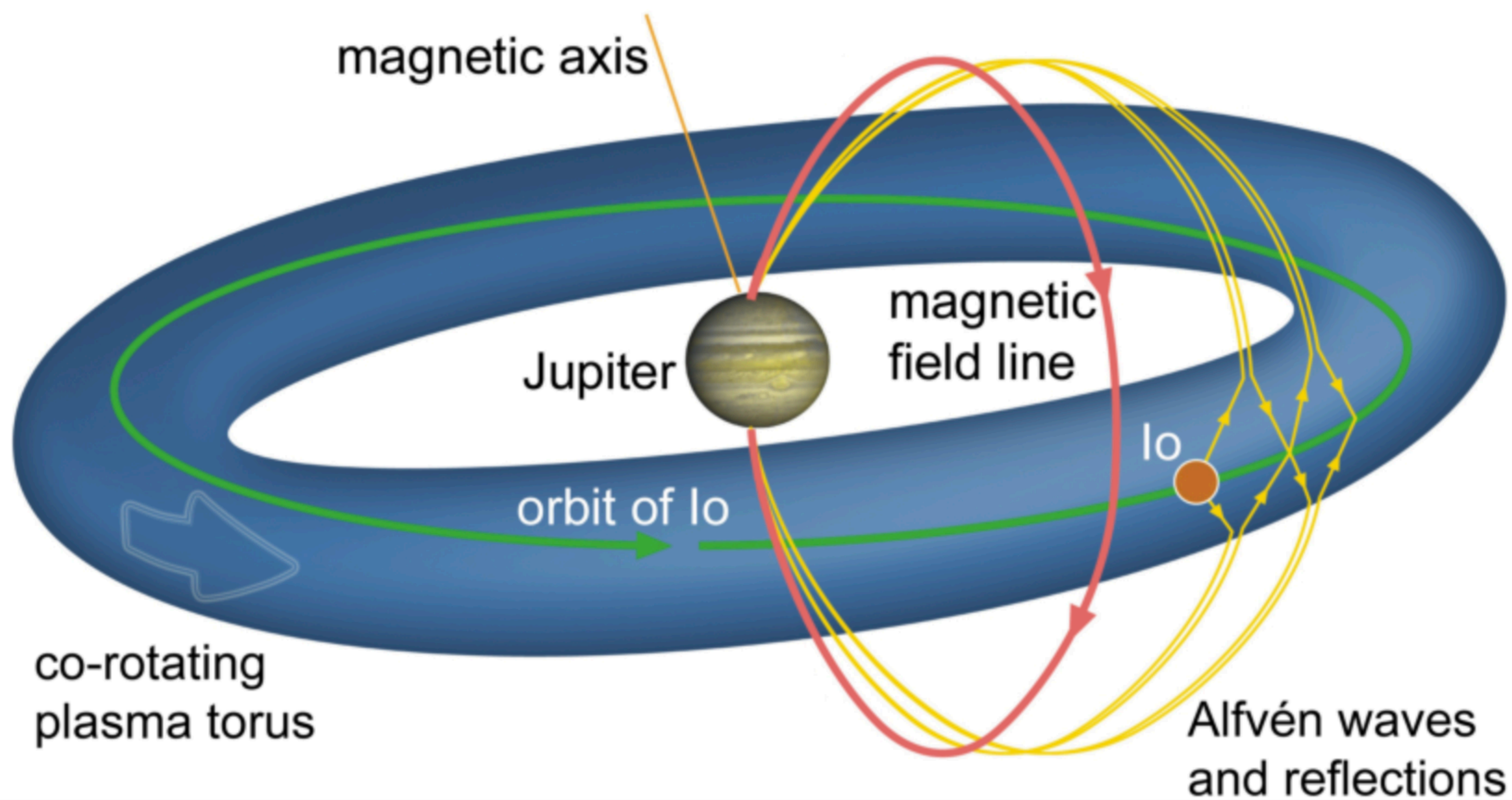


Von Karman Vortex Street

Jupiter's moon







Flow Reynolds Number

Inertial force term goes like $\rho \dfrac{|\mathbf{u}^2|}{l_0}$

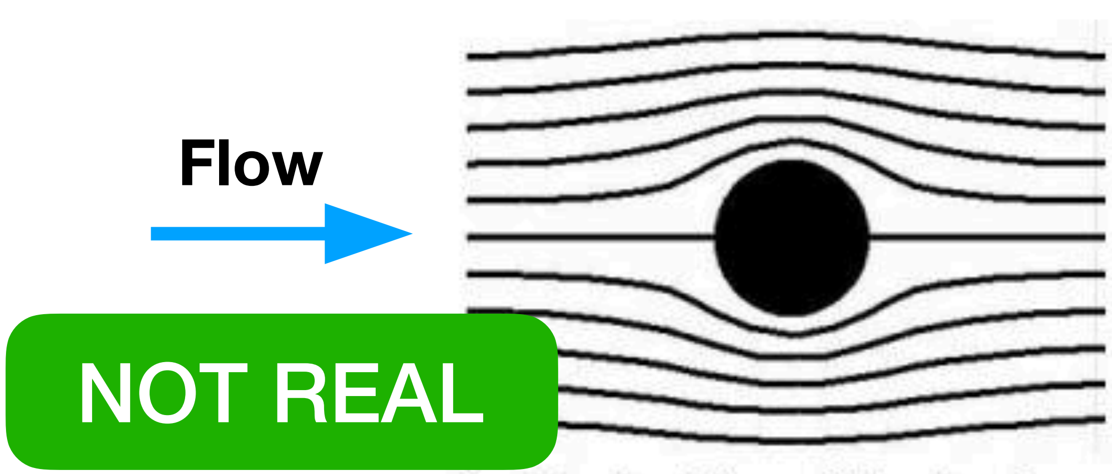
Viscous force term goes like $\rho\nu \dfrac{|\mathbf{u}|}{l_0^2}$

$\xrightarrow{\text{Ratio}}$ Reynolds number $\dfrac{|\mathrm{u}|\, l_0}{\nu}$

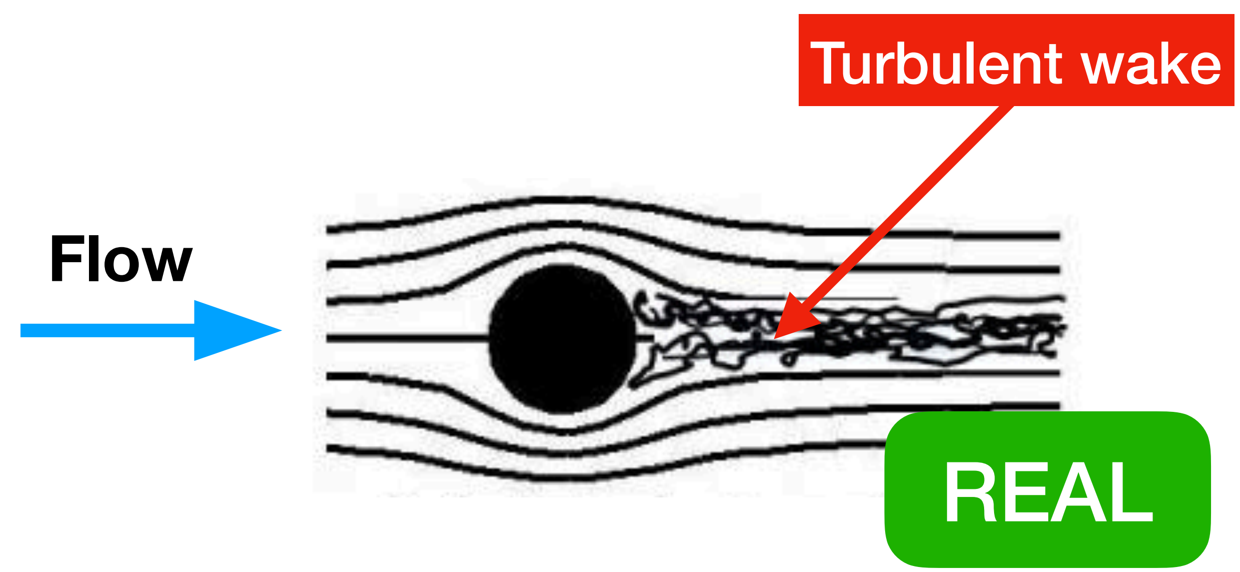
Several important notes about the Reynolds number:

- When Re is calculated using some characteristic geometric length, it is always very large (think about the magnetosphere: $|u| = 400$ km/s, $l_0 \sim$ RE = 6380 km, viscosity = 10^{-9}

Question: Because Re is usually very large, does that mean we can discard viscosity in fluid dynamics completely?



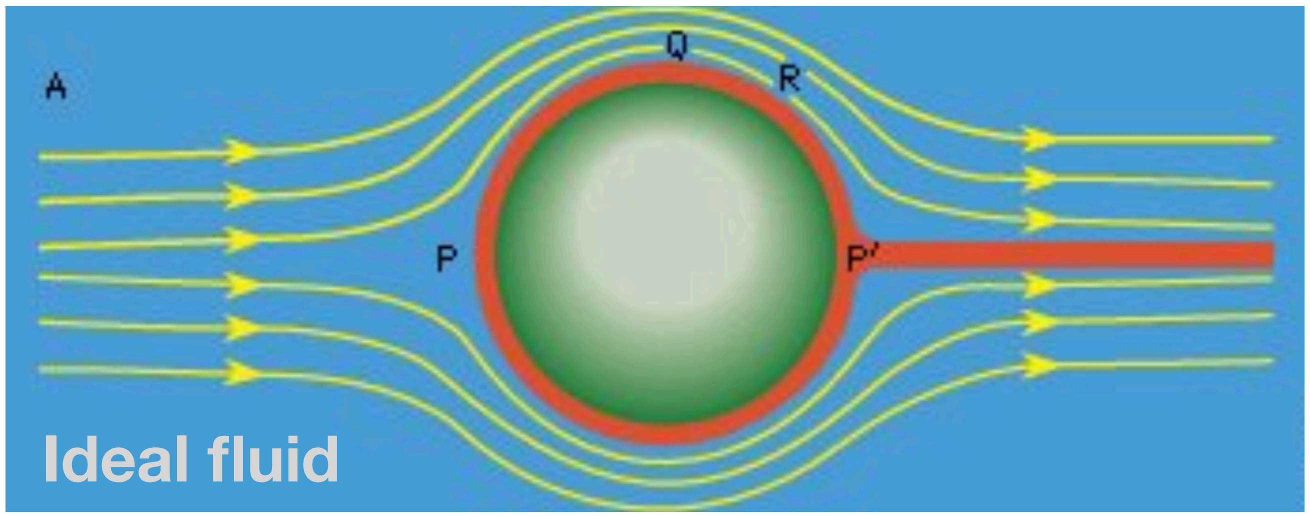
Inviscid $\nu = 0$

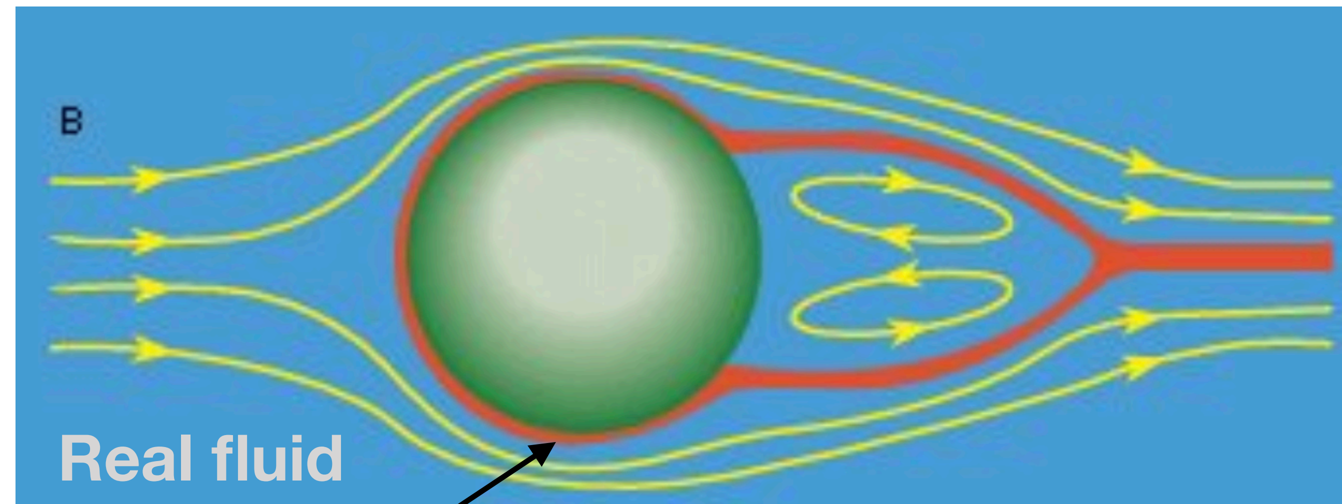


Viscous ν small but non − zero

Boundary Layer and Flow Separation

The problem is that, no matter how small ν might be, there are always **thin regions** near surfaces where the shear stresses are important, i.e., of the order of the inertial forces. This layer is called a **Boundary Layer**





Boundary Layers

Why? - the "no-slip" condition, which means the fluid "sticks" to the surface; so the velocity drops from it's free-stream value to zero on the surface

Where? - a transition region near the surface where velocity increases from zero to external flow speed

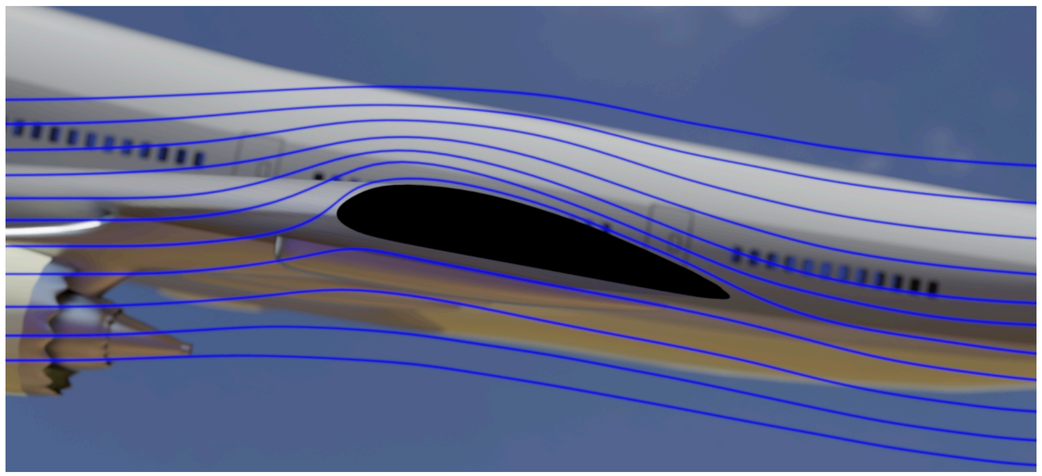
How? - inertial forces similar order of magnitude to viscous forces:

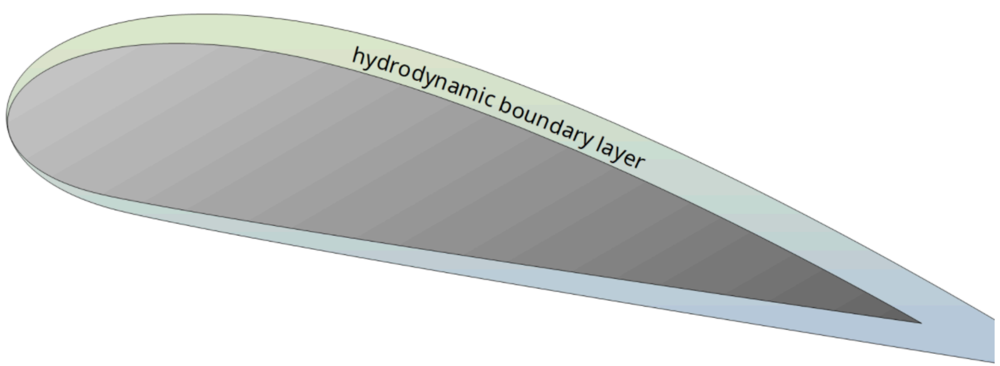
$$\underbrace{\rho\frac{|\mathbf{u}^2|}{l_0}}_{\text{inertial}} \approx \underbrace{\rho\nu\frac{|\mathbf{u}|}{\delta^2}}_{\text{viscous}} \longrightarrow \underbrace{\frac{\delta}{l_0} \sim \sqrt{\frac{\nu}{|\mathbf{u}|\,l_0}}}_{\text{Thickness of B. L.}}$$

Boundary Layer and Flow Separation

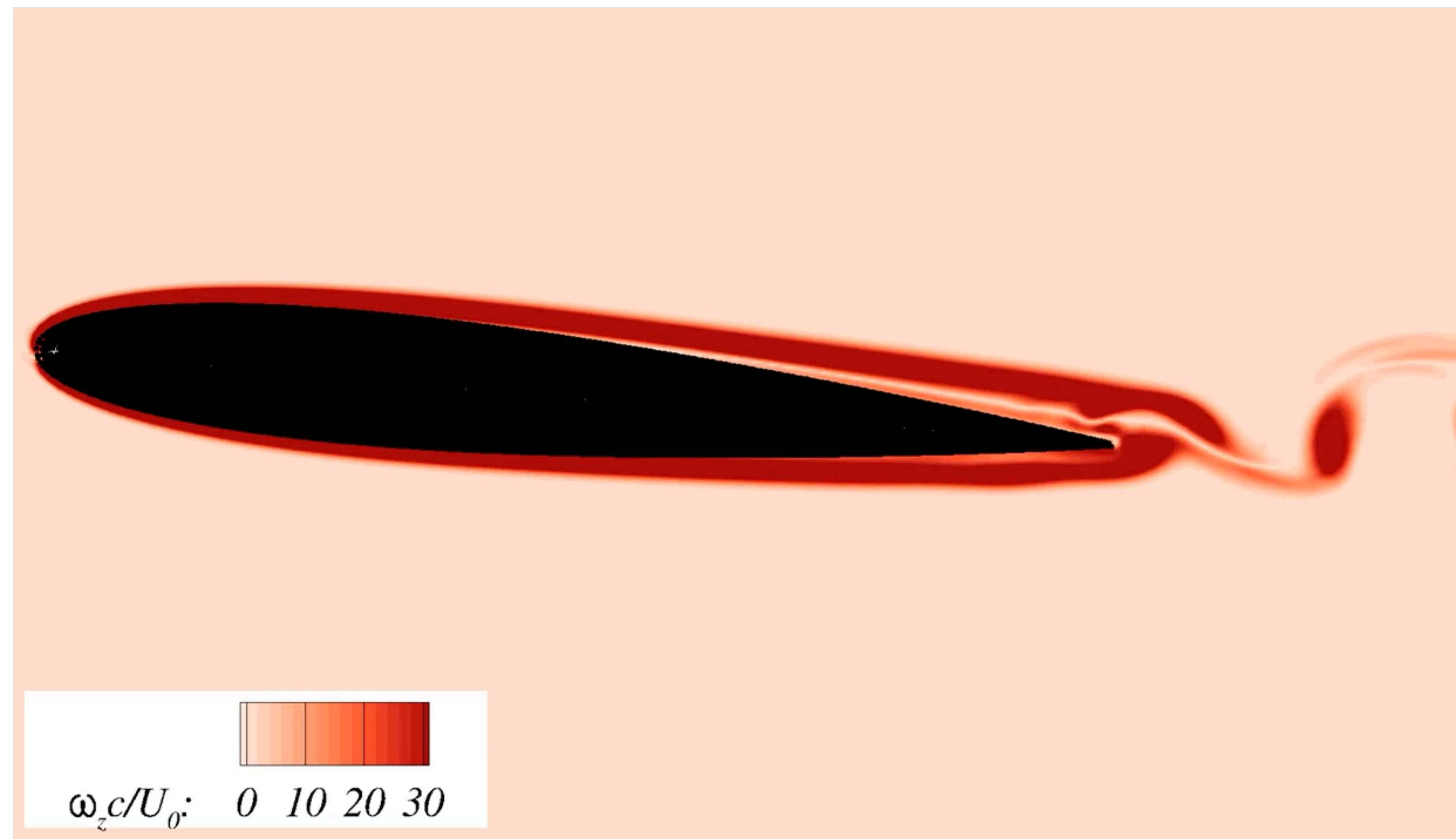
Flow separation

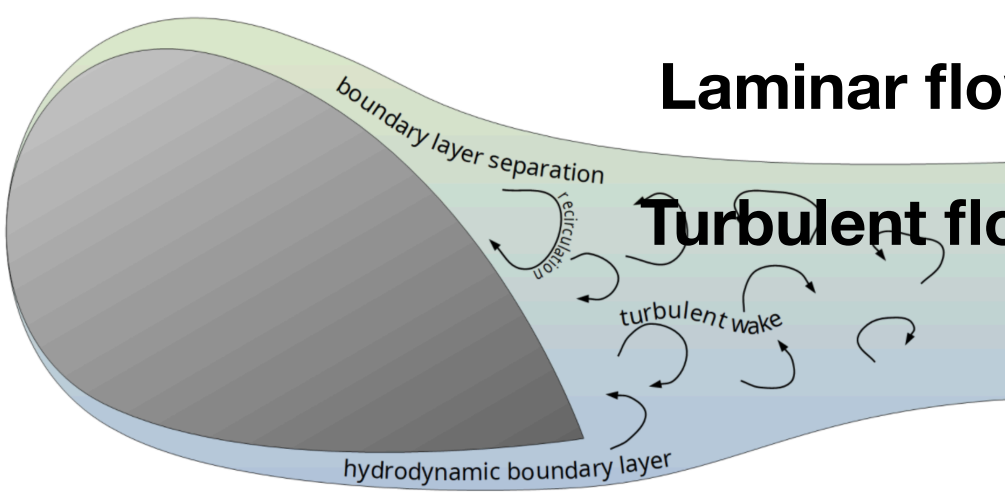
When flowing around a body, the fluid tries to follow the profile of the surface!





Finite viscosity -> boundary layer development





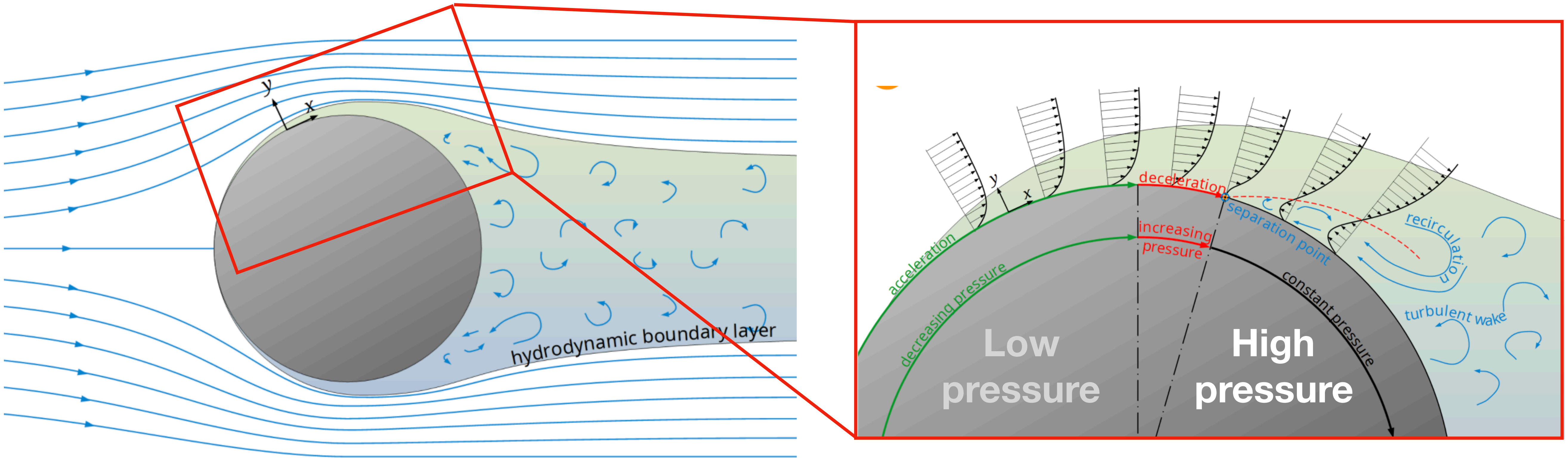
Laminar flow

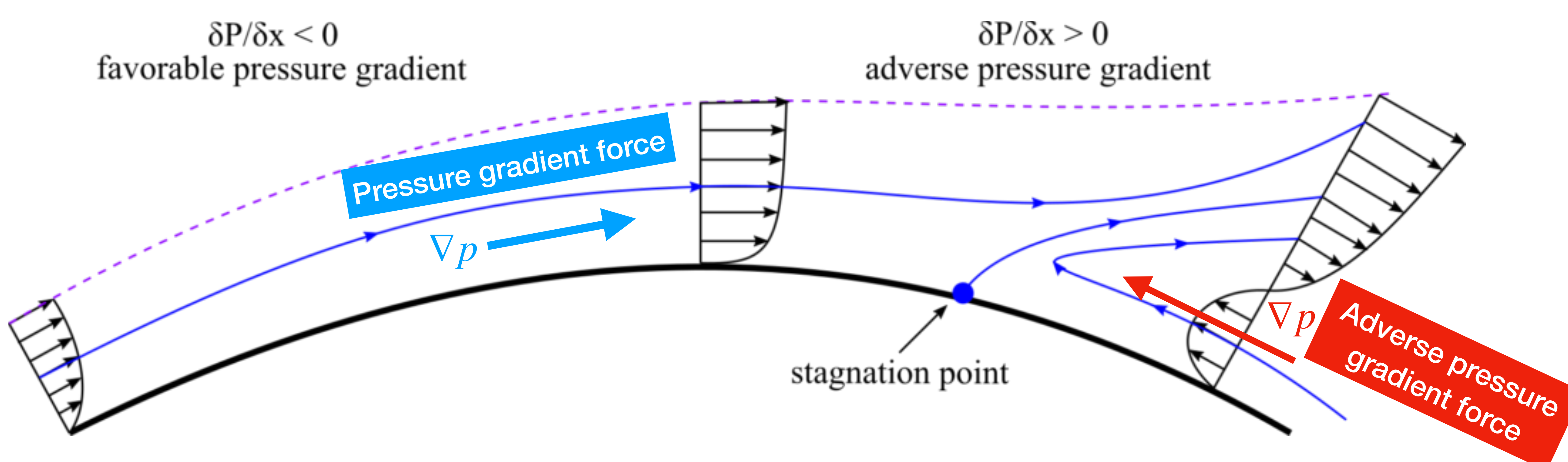
Turbulent flow

In the case of a boundary layer separation (flow separation), the flow can no longer follow the profile of the body around which it flows and separates turbulently from it.

Why Flow Separates?

Adverse pressure gradient

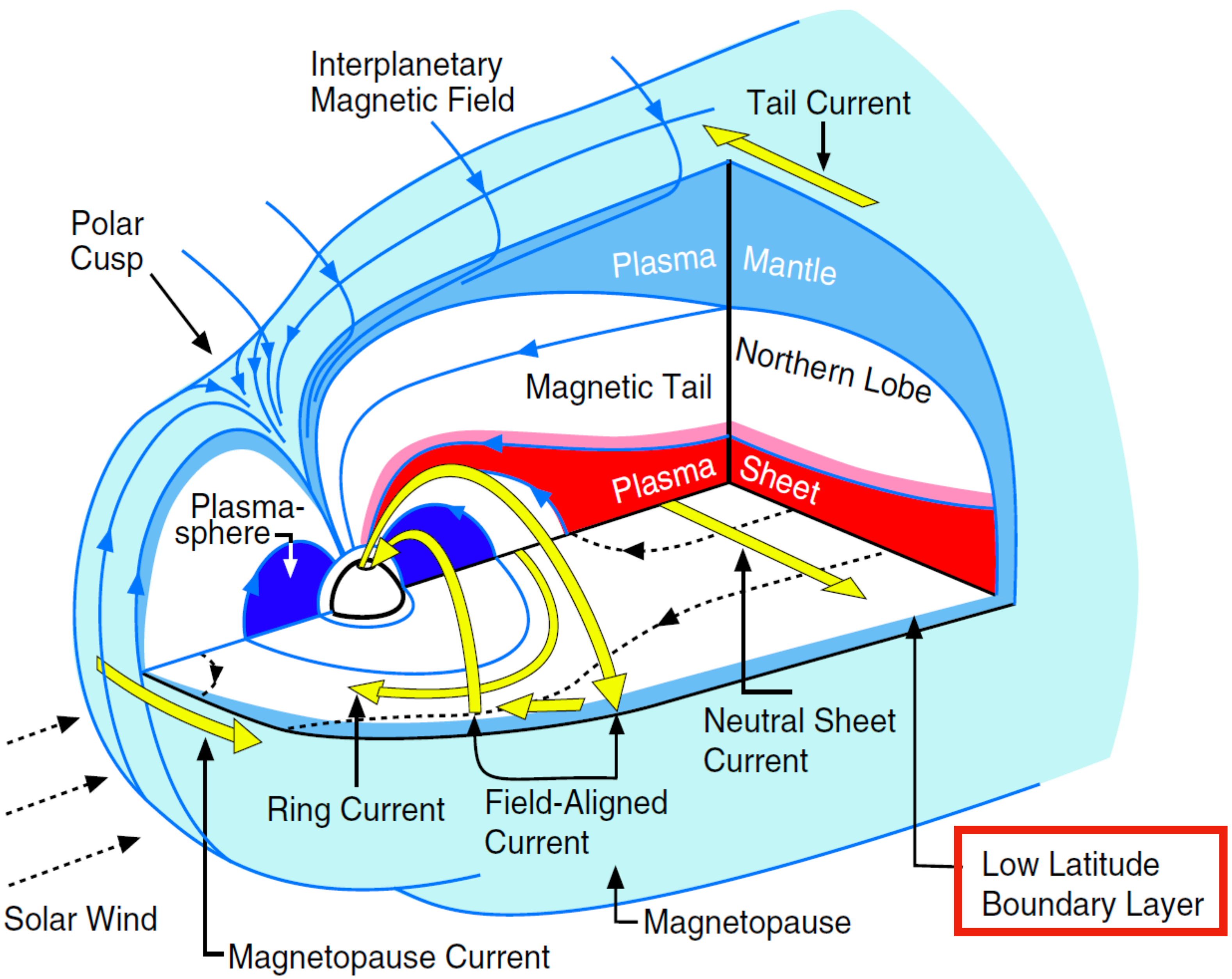




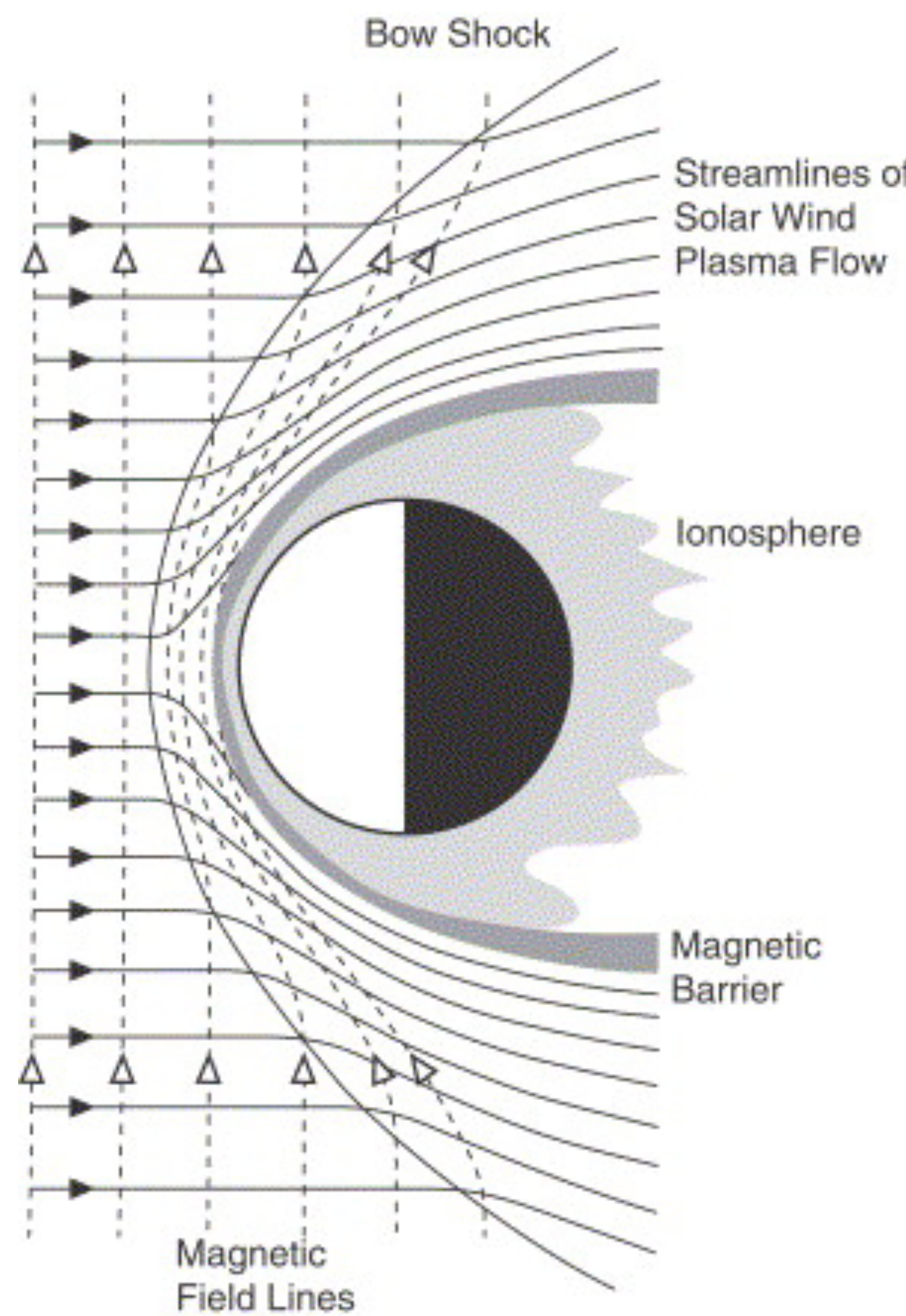
Example: Magnetopause Boundary Layer

Examples

Magnetized Planet (Earth)

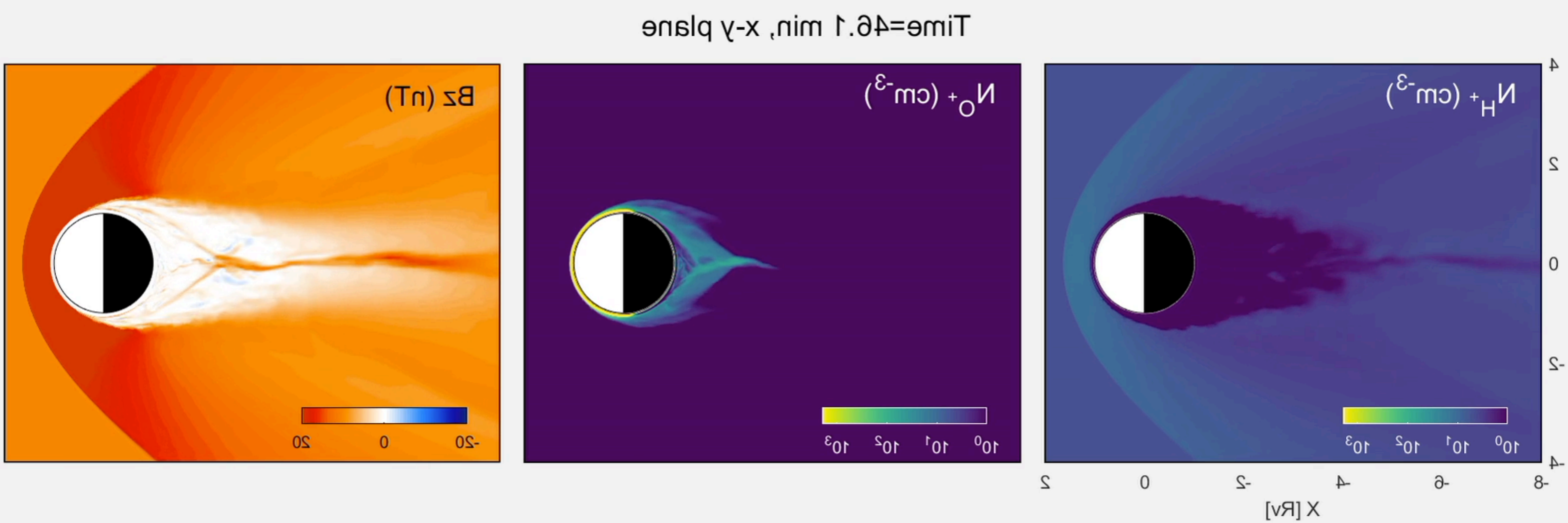


Unmagnetized Planet (Venus)

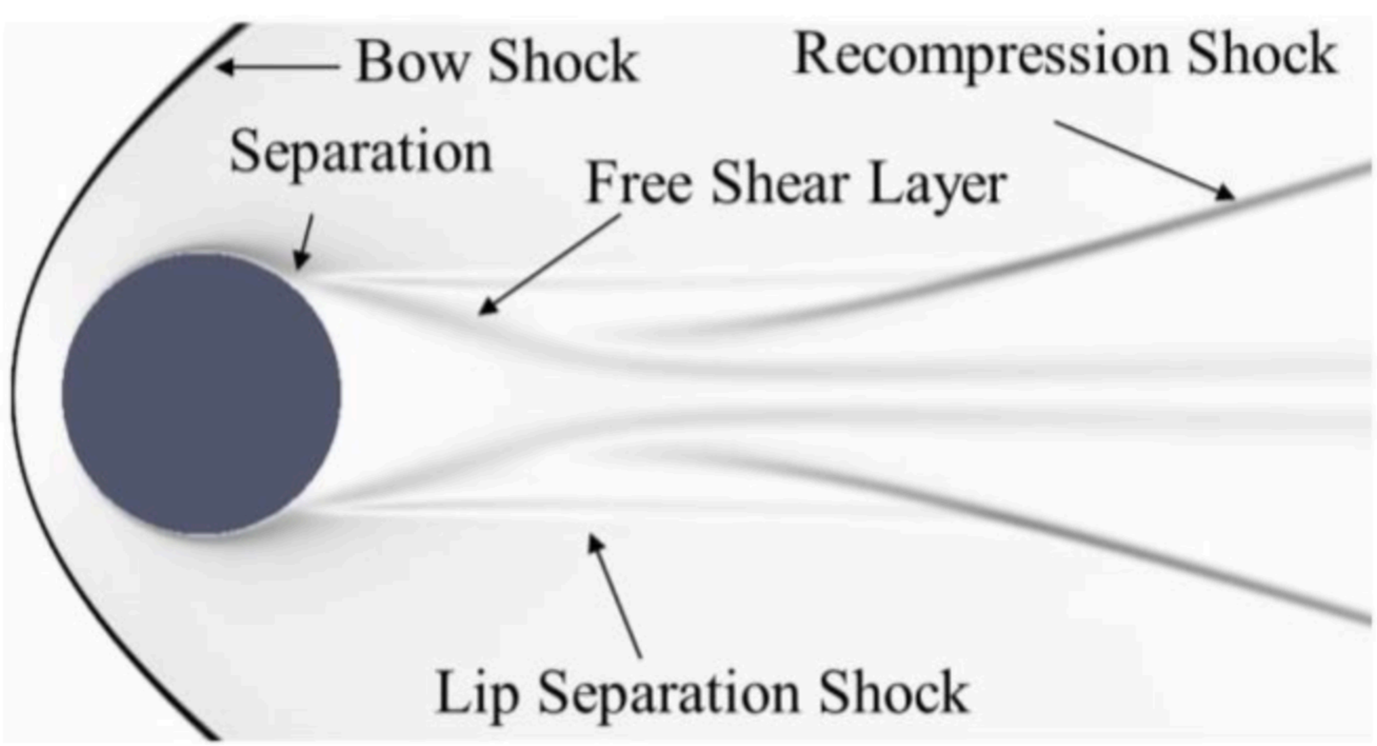


LLBL: Field-aligned current generation (MI coupling)
Plasma mixing (SW-M coupling)

Example: Venus Boundary Layer



Movie by: 党童



Ma=10; Re≈5.5×10 4

Are we doing a good job in resolving boundary layers in the Venus problem?

Example: The Low Latitude Boundary Layer

M-I coupling within the LLBL

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Theory of the Low-Latitude Boundary Layer

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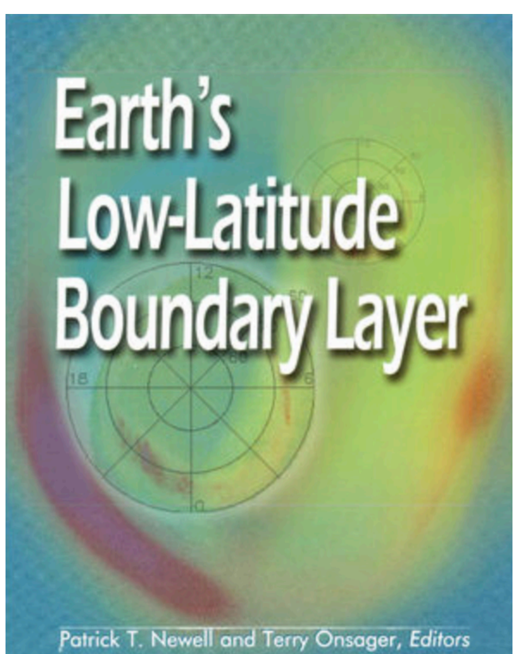
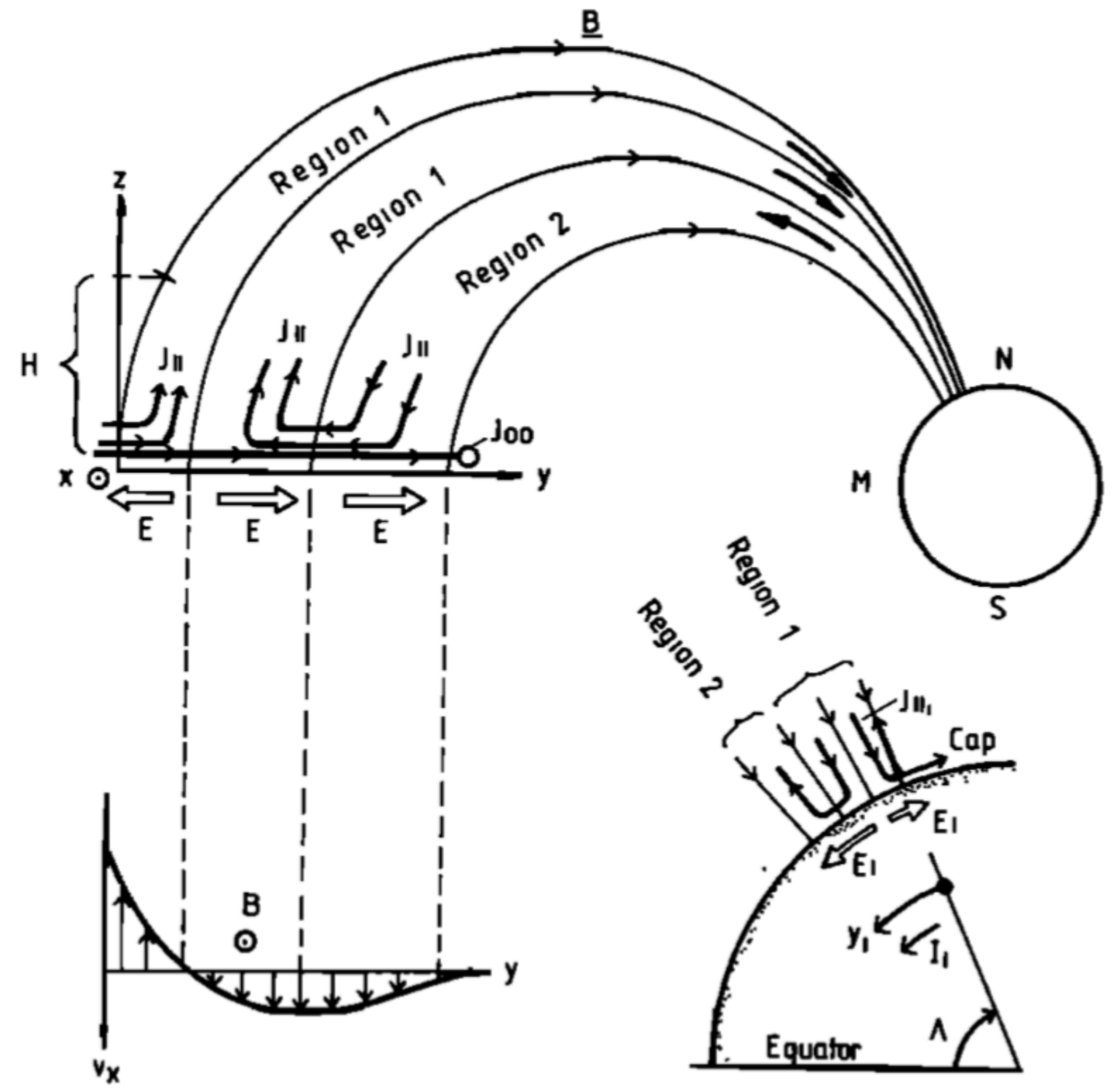
A one-dimensional steady state fluid mechanical model is developed of the low-latitude plasma boundary layer inside the dawn and dusk magnetopause. Momentum transfer in the layer is produced by viscosity and/or mass diffusion. Coupling to the ionosphere is achieved via field-aligned currents, the magnitude of which is limited by parallel potential drops. These currents flow into and out of the ionosphere in the manner described by Iijima and Potemra. The higher-latitude (region 1) currents are associated with the boundary layer proper, while the lower-latitude (region 2) ones are associated with a region of sunward return flow adjacent to the boundary layer. The parallel potential drops have a magnitude of typically 2–3 kV and a north-south extent of 100–200 km. The calculated potential profile corresponds reasonably well to observed inverted V precipitation events.



Assumptions:

- Equal-potential fieldlines
- Momentum conservation
- Conducting ionosphere
- Current continuity

$$\rho v_y \frac{dv_x}{dy} = j_{00}B - \sigma v_x B^2 + \mu \frac{d^2 v_x}{dy^2}$$



Potential and Rotational Flow

Vorticity

Definition of vorticity:

$$\boxed{\omega = \nabla \times \mathbf{u}}$$

$\omega = 0$ everywhere → **Potential flow**

$\omega \neq 0$ → **Rotational flow**

Stokes theorem

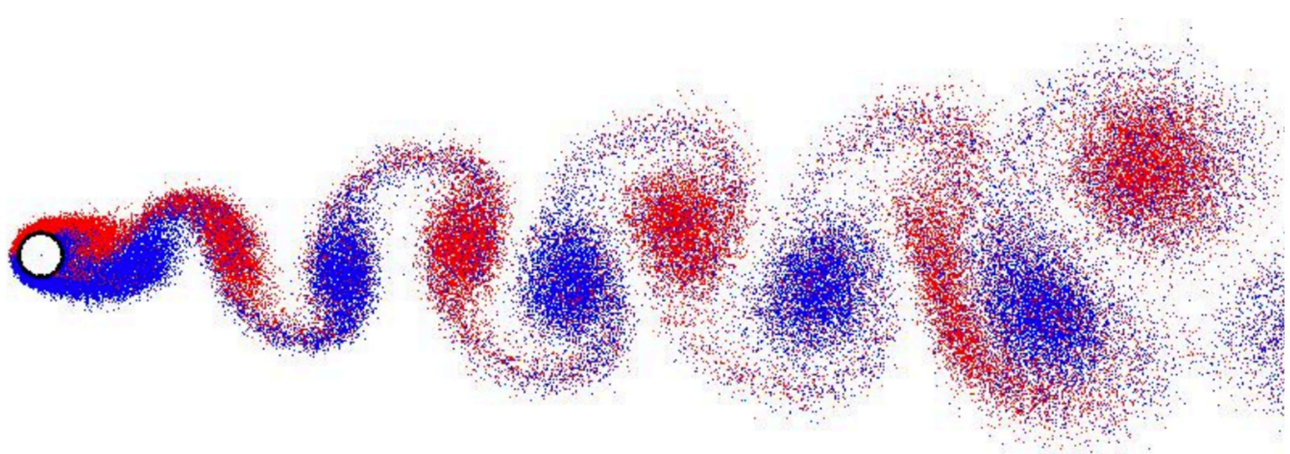
$$\int_S \omega \cdot d\mathbf{S} = \oint_C \mathbf{u} \cdot d\mathbf{l}$$

Physically, vorticity **ω** is a measure of the angular velocity of a fluid element about its center

In fact, one can show that the angular velocity of a fluid element $\mathbf{\Omega} = \frac{1}{2}\omega$

Why is vorticity useful in fluid dynamics? - it's conserved without buoyancy or Lorentz forces!

How do vorticity evolve - through diffusion

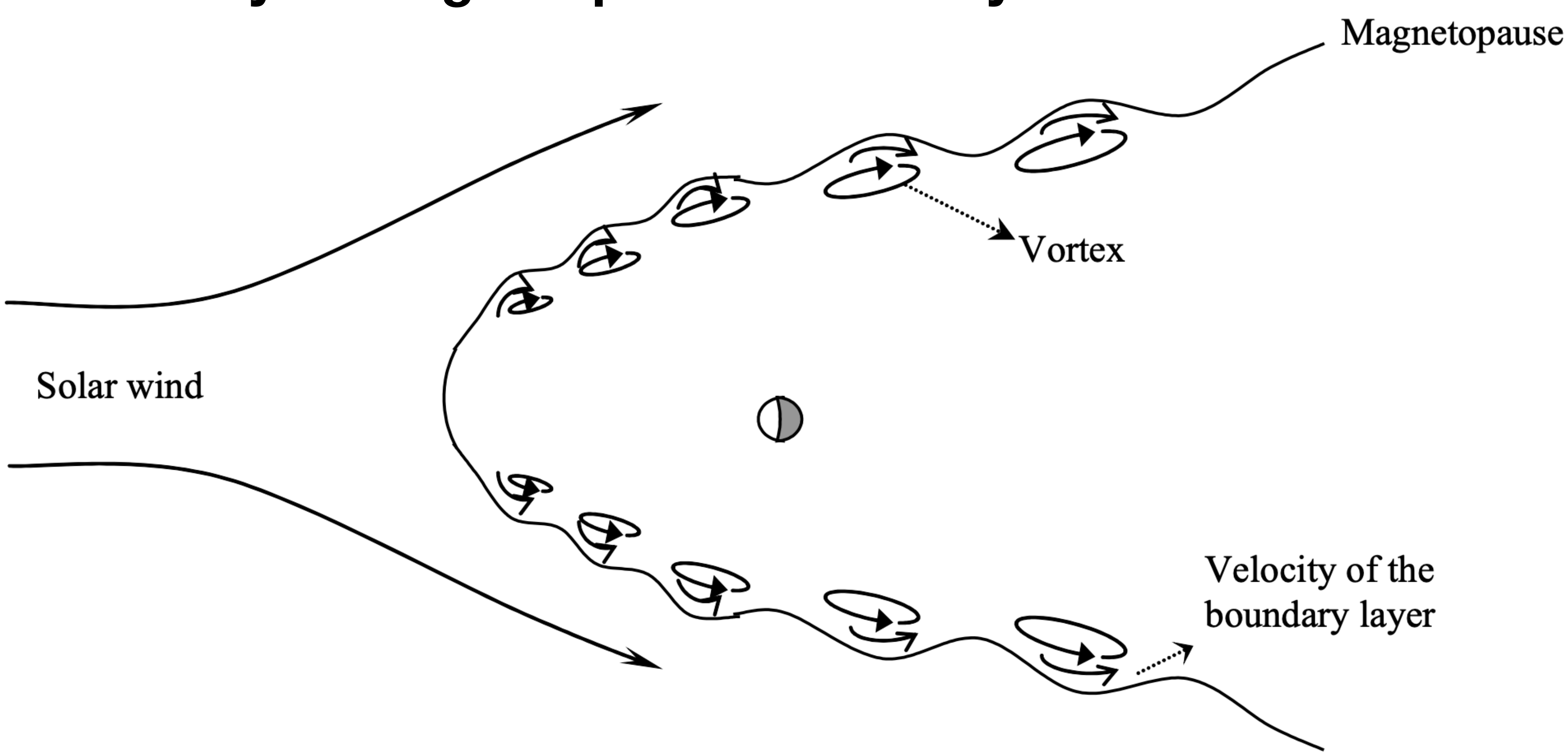


For each vortex, $\int_S \omega \cdot d\mathbf{A}$ is conserved

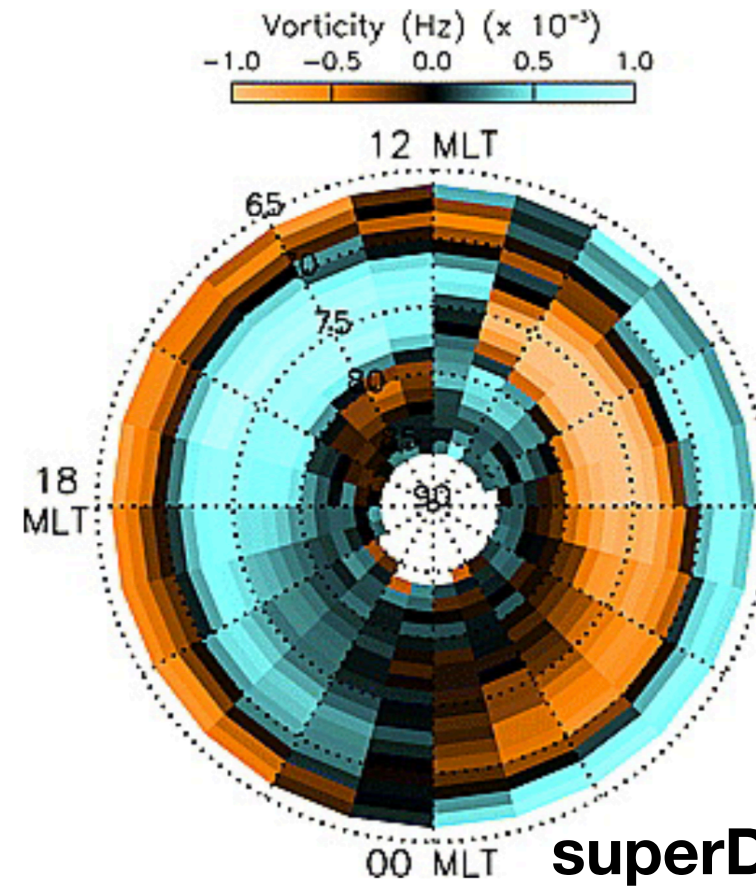
Where dA is the differential size of a vortex

Vorticity

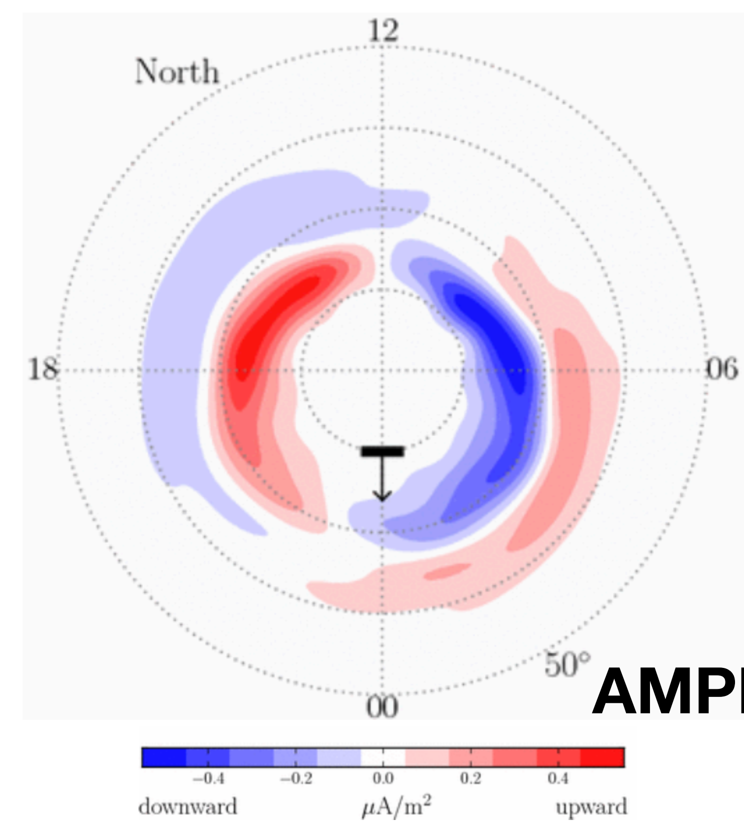
Vorticity at Magnetopause boundary



Vorticity at ionosphere



superDARN



AMPERE

Vorticity ~ Field-aligned currents, why?

$$J_{\parallel} = \nabla_{\perp} \cdot (\Sigma \mathbf{E}) \xrightarrow[\text{Ignore Hall Currents}]{\text{Uniform conductance}} J_{\parallel} = \Sigma_P \nabla \cdot \mathbf{E}$$

Perfect conducting $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$:

$$J_{\parallel} = -\Sigma_P \nabla \cdot (\mathbf{u} \times \mathbf{B}) = -\Sigma_P \mathbf{B} \cdot (\nabla \times \mathbf{u}) = \Sigma_P \mathbf{B} \cdot \omega$$

Potential Flow

Assumptions

1) **flow velocity has zero viscosity** $\nu = 0$ (inviscid)

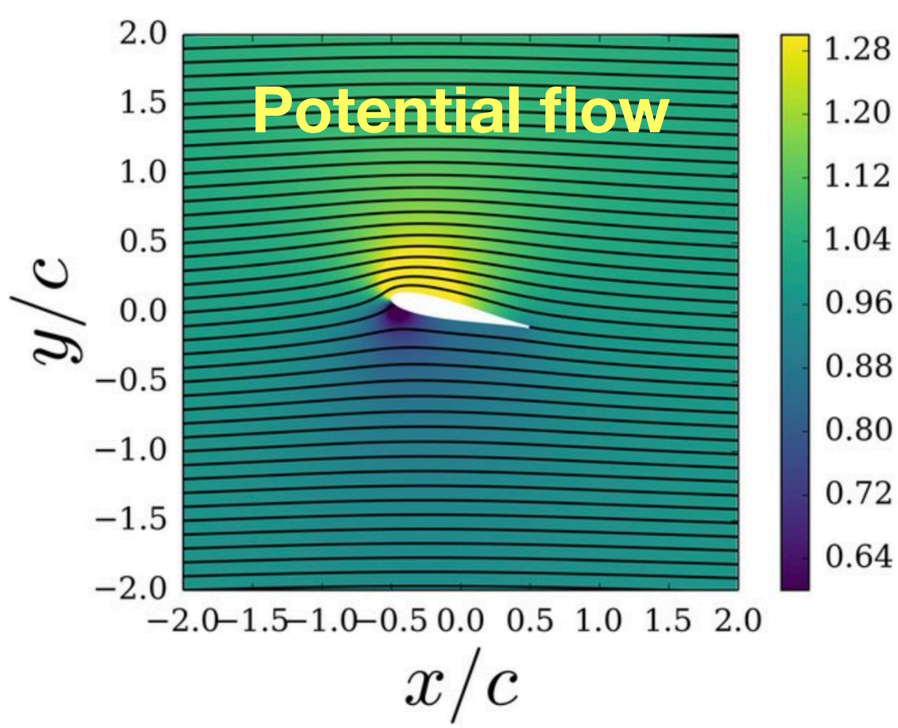
2) **flow velocity has zero vorticity** $\nabla \times \mathbf{u} \equiv 0$ (irrotational)

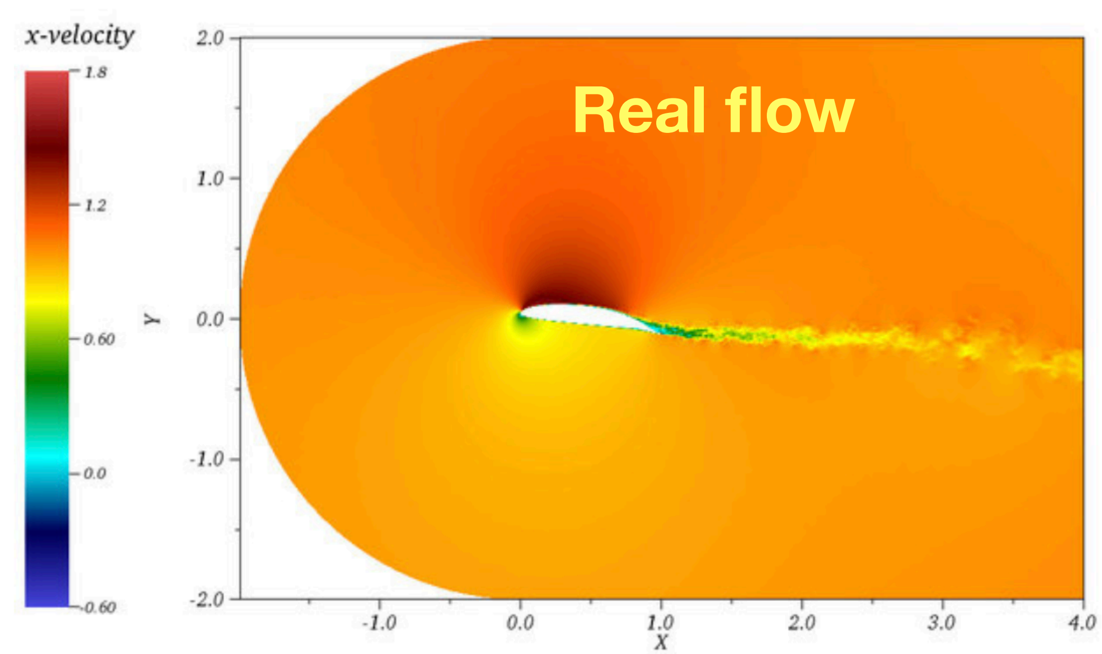
3) **flow velocity is incompressible** $\nabla \cdot \mathbf{u} = 0$ (mass conservation)

Reduced Fluid equation

$$\nabla \times \mathbf{u} \equiv 0 \longrightarrow \mathbf{u} = \nabla \phi \xrightarrow{\nabla \cdot \mathbf{u} = 0} \nabla^2 \phi = 0$$

Now fluid-dynamics becomes fluid-kinematics, nothing to do with Newton's law!





The Navier-Stokes Equation

Recall Newton's second law: $\mathbf{F} = m\mathbf{a}$ $\quad$ $\mathbf{a} = \dfrac{D}{Dt}\mathbf{u}$ is acceleration

Applying Newton's law to fluid element

N-S equation

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{F} \xrightarrow[\text{External forces}]{\text{Pressure gradient force, viscous force}} \boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_{\text{ext}}}$$

$\dfrac{D\mathbf{u}}{Dt}$ is expressed as $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ Total derivative

Thus the Navier-Stokes equation is also written as

$$\underbrace{\rho \frac{\partial \mathbf{u}}{\partial t}}_{\text{Local}} + \underbrace{\rho \mathbf{u} \cdot \nabla \mathbf{u}}_{\text{Convective}} = \overbrace{\nabla p}^{\text{Pressure}} + \underbrace{\nu \nabla^2 \mathbf{u}}_{\text{Viscous}} + \mathbf{F}_{\text{ext}}$$

External

Recall that in terms of curvilinear coordinates attached to a streamline:

$$\mathbf{u} \cdot \nabla \mathbf{u} = u \frac{du}{ds} \hat{\mathbf{e}}_t - \frac{u^2}{R} \hat{\mathbf{e}}_n$$

Inertial accelerations

The Navier-Stokes Equation

The vorticity equation

It is relatively straightforward to show that the Navier-Stokes equation can be written as:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \omega - \nabla \left(\frac{p}{\rho} + \frac{1}{2} u^2 \right) + \nu \nabla^2 \mathbf{u}$$

Where $\omega = \nabla \times \mathbf{u} \longrightarrow \nabla \cdot \omega \equiv 0$

$\partial / \partial t \equiv 0 \qquad \nu = 0$

1. If the flow is steady-state and inviscid:

$$\mathbf{u} \times \omega - \nabla \left(\frac{p}{\rho} + \frac{1}{2} u^2 \right) = 0 \xrightarrow{\mathbf{u} \cdot} \mathbf{u} \cdot \nabla \left(\frac{p}{\rho} + \frac{1}{2} u^2 \right) = 0 \xrightarrow{\text{along } \mathbf{u}} \frac{p}{\rho} + \frac{1}{2} u^2 = \text{Const}$$

Bernoulli's theorem

2. If we take the curl of the Naiver-Stokes equation:

$$\nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \omega - \nabla \left(\frac{p}{\rho} + \frac{1}{2} u^2 \right) + \nu \nabla^2 \mathbf{u} \right] \longrightarrow \frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega$$

Vorticity equation

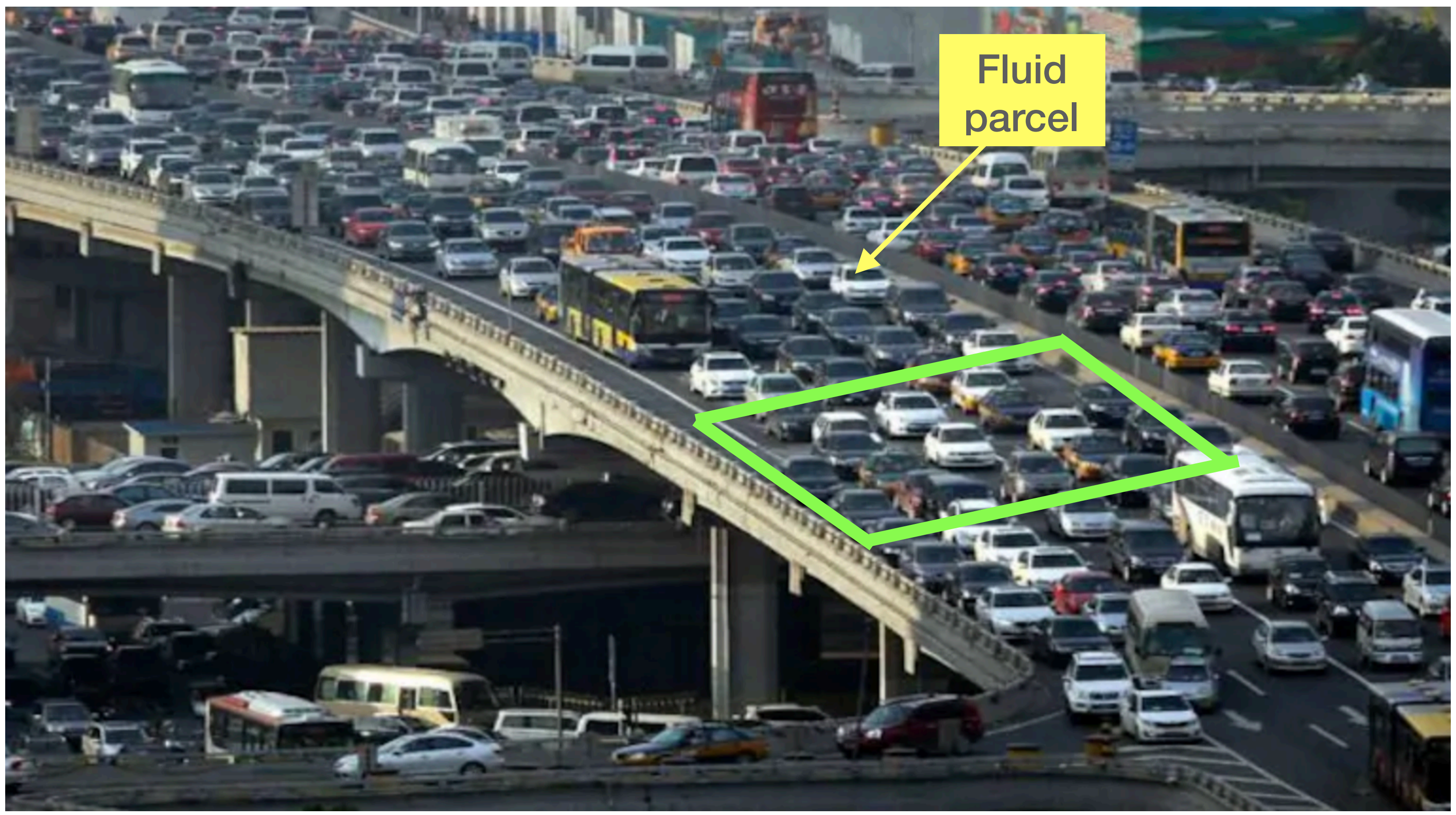
Recall: $$\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu \sigma} \nabla^2 \mathbf{B}$$

Magnetic field transport equation

The Navier-Stokes Equation

The incompressible case

In fluid dynamics, incompressible flow refers to a flow in which the density remains constant in any fluid parcel, i.e. any infinitesimal volume of fluid moving in the flow.



If we regard the traffic jam as a flow of vehicles along roadways, then each vehicle could be regarded as a fluid parcel - and it's mass density remains constant, regardless the vehicle is moving or not - this is called **incompressible**

To change the density of cars within the green rectangle, the only way is to move vehicles around - through convection

mathematically, incompressible flow means the divergence of velocity is zero everywhere

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{and} \qquad \frac{d\mathbf{u}}{dt} = \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_{\text{ext}}$$

And the mass equation is no longer needed (why?)

Thermal Dynamics of Ideal Fluid

γ : ratio of specific heat

ρ : mass density

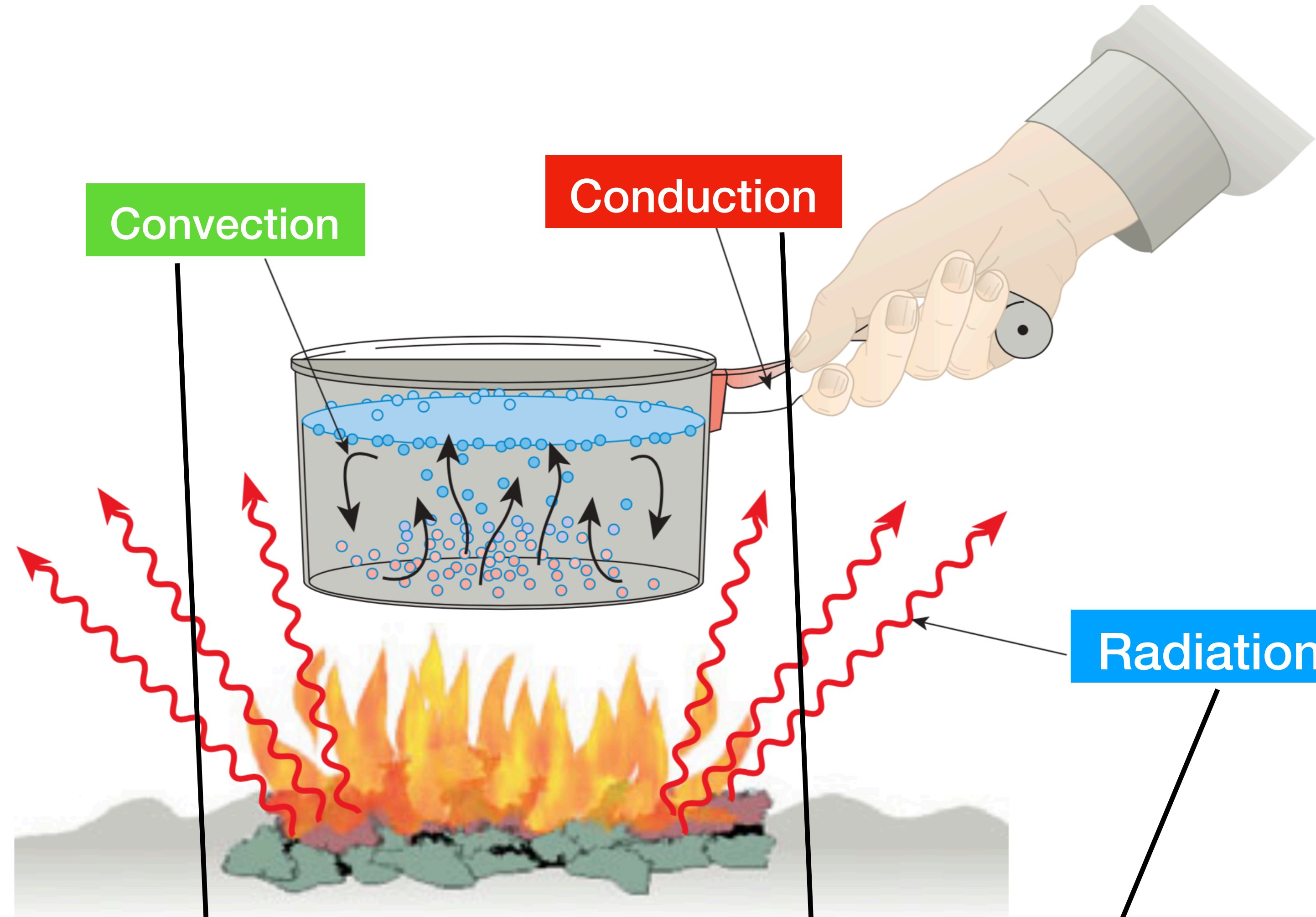
e : internal energy per unit mass

p : pressure

ρe : internal energy

S : entropy

Definition $\rho e = \dfrac{p}{\gamma - 1}$



Governing equation

$$\frac{De}{Dt} + (\gamma - 1)e\nabla\cdot\mathbf{u} = \mathbf{L}$$

$$\frac{\partial e}{\partial t} + \boxed{\mathbf{u}\cdot\nabla e + (\gamma - 1)e\nabla\cdot\mathbf{u}} = \boxed{\nabla\cdot\mathbf{q}} + \boxed{L_R} + F_H$$

Other heating

Source terms

$$\mathbf{L} = \nabla\cdot\mathbf{q} + L_R + F_H$$

The Dissipative Nature of Viscous processes

Consider the kinetic energy fluid

$$E_{kin} = \frac{1}{2}\int_V \rho\,|\mathbf{u}|^2 dV$$

To get the time rate of change in the kinetic energy, use the Naiver-Stokes equation

$$\rho\frac{d\mathbf{u}}{dt} = \nabla p + \nu\,\nabla^2\mathbf{u} + \mathbf{F}_{\text{ext}}$$

$$\int_V \mathbf{u}\cdot\left(\rho\frac{d\mathbf{u}}{dt} = \nabla p + \nu\,\nabla^2\mathbf{u} + \mathbf{F}_{\text{ext}}\right) dV$$

$$\frac{d}{dt}E_{kin}$$

$$\int_V \mathbf{u}\cdot\left(-\nabla p + \nu\,\nabla^2\mathbf{u} + \mathbf{F}_{\text{ext}}\right) dV$$

Some math

$$\frac{d}{dt}E_{kin} = \underbrace{\oint_S \mathbf{u}\cdot(\mathbf{n}\cdot\sigma)dS}_{\text{Work done by hydrodynamic force}} - \underbrace{2\mu\int_V |\nabla\mathbf{u}|^2 dV}_{\text{Energy dissipation through viscosity}} + \underbrace{\int_V \rho\mathbf{F}_{ext}\cdot\mathbf{u}dV}_{\text{Work done by external body force}}$$

Forms of the energy equation (ideal gas)

Internal energy $$\frac{De}{Dt} + (\gamma - 1)e\nabla \cdot \mathbf{u} = 0 \qquad \frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e\mathbf{u}) + p\nabla \cdot \mathbf{u} = 0$$

Pressure $$\frac{Dp}{Dt} + \gamma p\nabla \cdot \mathbf{u} = 0 \qquad \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p\nabla \cdot \mathbf{u} = 0$$

Entropy $$\frac{DS}{Dt} = 0 \quad \textbf{Or} \quad \frac{D}{Dt}\left(p\rho^{-\gamma}\right) = 0$$

Entropy per volume $$\frac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho S\mathbf{u}) = 0$$

Kinetic energy equation $$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2) + \nabla \cdot (\frac{1}{2}\rho u^2\mathbf{u}) + \mathbf{u} \cdot \nabla p = 0$$

Fluid energy equation $$\frac{\partial}{\partial t}E_P + \nabla \cdot ((E_P + p)\mathbf{u}) + \mathbf{u} \cdot \nabla p = 0$$

Forms of the set of Fluid Equations

Total derivative form

Mass

$$\frac{D}{Dt}\rho = -\rho\nabla\cdot\mathbf{u}$$

Velocity

$$\rho\frac{D\mathbf{u}}{Dt} = \nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}_{\mathrm{ext}}$$

Pressure

$$\frac{Dp}{Dt} = -\gamma p\nabla\cdot\mathbf{u}$$

Primitive form

$$\frac{\partial\bar{\rho}}{\partial t} + \mathbf{u}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{u}$$

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \underbrace{\rho\mathbf{u}\cdot\nabla\mathbf{u}}_{1} = \underbrace{\nabla p}_{2} + \underbrace{\nu\nabla^2\mathbf{u}}_{3} + \mathbf{F}_{\mathrm{ex}}$$

$$\frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p = -\gamma p\nabla\cdot\mathbf{u}$$

Conservative form

Mass

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\right) = 0$$

Momentum

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}p) = \nu\nabla^2\mathbf{u} + \mathbf{F}_{\mathrm{ext}}$$

Energy

$$\frac{\partial}{\partial t}E_P + \nabla\cdot((E_P + p)\mathbf{u}) + \mathbf{u}\cdot\nabla p = 0$$

$\frac{1}{2}$ **: Mach Number**

$\frac{1}{3}$ **: Reynolds Number**

Euler's Equations

Mass

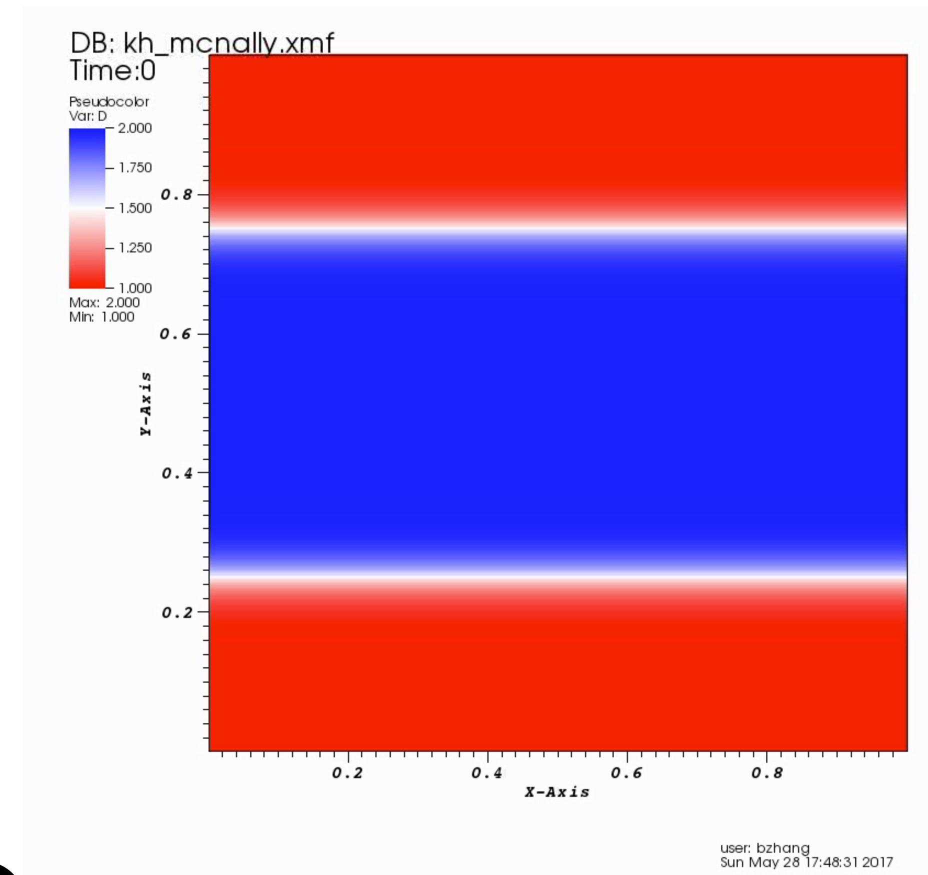
$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

Momentum

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) = 0$$

Energy

$$\frac{\partial}{\partial t} E_P + \nabla \cdot ((E_P + p)\mathbf{u}) = 0$$



Ideal MHD

Mass

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\right) = 0$$

Momentum

Hydrodynamic force

Lorentz force

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u} + \mathbf{I}p) = \mathbf{J} \times \mathbf{B} = \nabla \cdot \left(\frac{1}{2} \frac{B^2}{\mu_0} \bar{\mathbf{I}} - \frac{1}{\mu_0} \mathbf{B}\mathbf{B} \right)$$

Reynolds Stress

Maxwell Stress

Energy

$$\frac{\partial}{\partial t} E_P + \nabla \cdot ((E_P + p)\mathbf{u}) = \mathbf{u} \cdot \mathbf{J} \times \mathbf{B}$$