Stat510(Section001): Homework #2

Due on Spet. 29, 2021 at 11:59pm

Instructor:Brandon Legried

Tiejin Chen **tiejin@umich.edu**

Problem 1

We can know that:

$$\begin{aligned} P(head) = P(head|head\ in\ both\ side\ coins\ selected) \times P(head\ in\ both\ side\ coins\ selected) \\ +P(head|head\ in\ one\ side\ coins\ selected) \times P(head\ in\ one\ side\ coins\ selected) \\ +P(head|no\ head\ coins\ selected) \times P(no\ head\ coins\ selected) \end{aligned}$$

Hence we can get:

$$P(head) = \frac{3}{9} \times 1 + \frac{2}{9} \times \frac{1}{2} + \frac{4}{9} \times 0 = \frac{4}{9}$$

Problem 2

We use X_1 to present the result of first observation. And X_2 present the result of second observation. Our goal is to compute:

$$P(X_2 = green|X_1 = green)$$

We use Z to present the card we choose. And we number the card. One card is red on both sides is 1, one is green on both sides is 2, and one is red on one side and green on the other is 3. We have:

$$P(X_2 = green|X_1 = green) = \sum_{i=1}^{3} P(X_2 = green|Y = i, X_1 = green)P(Y = i|X_1 = green)$$

of course,we get:

$$P(Y = 1|X_1 = green) = 0, P(X_2 = green|Y = 3, X_1 = green) = 0$$

Then, we calculate when $i = 2$. We use Bayes' Rule to get:

$$P(Y = 2|X_1 = green) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{2}{3}$$

Thus:

$$P(X_2 = green|X_1 = green) = P(X_2 = green|Y = 2, X_1 = green)P(Y = 2|X_1 = green) = 1 \times \frac{2}{3} = \frac{2}{3}$$

Problem 3

Part a

We use Y as a r.v to present the coin we select. Use X_i as a r.v to present the result of i-th toss. In this part, we calculate$P(Y = i|X_1 = head$ for every i. First, we can know that:

$$P(Y = 1|X_1 = head) = 0$$

Now, Using Bayes' Rule. we have:

$$P(Y = i|X_1 = head) = \frac{P(X_1 = head|Y = i)P(Y = i)}{P(X_1 = head)} = \frac{p_i \times \frac{1}{5}}{P(X_1 = head)}$$

We also have:

$$P(X_1 = head) = \frac{1}{5}(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1) = \frac{1}{2}$$

Problem 3 continued on next page. . .

Thus:

$$P(Y = i|X_1 = head) = \frac{2}{5}p_i$$

Then:

$$P(Y = 2|X_1 = head) = \frac{1}{10}, P(Y = 3|X_1 = head) = \frac{1}{5}, P(Y = 4|X_1 = head) = \frac{3}{10}, P(Y = 5|X_1 = head) = \frac{2}{5}$$

Part b
We have:

$$P(X_2 = head|X_1 = head) = \sum_{i=1}^{5} P(X_2 = head|Y = i, X_1 = head)P(Y = i|X_1 = head) = \frac{1}{4}\frac{1}{10}+\frac{1}{2}\frac{1}{5}+\frac{3}{4}\frac{3}{10}+\frac{2}{5} = \frac{3}{4}$$

Hence the probability of obtaining a second head is $\frac{3}{4}$.

Problem 4

R_n is a uniform disturbution on 1,..,n+1. That means,for every $i \in 1, ...n+1, P(R_n = i) = \frac{1}{n+1}$. Now we prove this conclusion with induction.

Proof. First, we prove the situaion of n = 1.we know that:

$$P(R_n = 1) = P(select\ white\ ball) = \frac{1}{2} = P(select\ red\ ball) = P(R_n = 2)$$

That proves the situation of n =1.
Now Assumed that the conclusion established when $n = x - 1$, now we prove the situation of $n = x$.We consider $P(R_x = 1)$ first. That is to say, every time I choose the ball, I choose the white ball, Hence:

$$P(R_x = 1) = \frac{1}{2} \times \frac{2}{3} \times ... \times \frac{x}{x+1} = \frac{1}{x+1}$$

And for $i = n + 1$, we have the same logic to get:

$$P(R_x = x + 1) = \frac{1}{x+1}$$

Now, for any $i \in \{2, ..., x\}$, we have:

$$\begin{aligned} P(R_x = i) &= P(R_{x-1} = i-1)P(select\ red\ when\ x-th\ selection) + P(R_{x-1} = i)P(select\ white\ when\ x-th\ selection) \\ &= \frac{1}{x} \times \frac{i}{x-1+2} + \frac{1}{x} \times \frac{x-1+2-i-1}{x-1+2} \\ &= \frac{1}{x}\frac{x}{x+1} \\ &= \frac{1}{x+1} \end{aligned}$$

Hence for any $i \in \{1, ..., x+1\}$, we have:

$$P(R_x = i) = \frac{1}{x+1}$$

That proves our conclusion with induction. □

Problem 5

Part a

We use Bayes' Rule to compute $P(X = n + k|X > n)$,we have:

$$P(X = n + k|X > n) = \frac{P(X > n|X = n + k)P(X = n + k)}{P(X > n)}$$

we can know that $P(X > n|X = n + k) = 1$ immediately. And with pdf, we can know:

$$P(X = n + k|X > n) = \frac{p(1 - p)^{n+k-1}}{\sum_{i=n}^{\infty} p(1 - p)^i}$$

The denominator is a sum of geometric series, we have:

$$\sum_{i=n}^{\infty} p(1 - p)^i = \lim_{i\to\infty} \frac{p(1 - p)^n(1 - p(1 - p)^i)}{1 - (1 - p)} = \frac{p(1 - p)^n}{p} = (1 - p)^n$$

Thus, we have:

$$P(X = n + k|X > n) = \frac{p(1 - p)^{n+k-1}}{(1 - p)^n} = p(1 - p)^{k-1} = P(X = k)$$

That ends of our prove.

Part b

Proof. for every integer k, the formula $P(Y = n + k|Y > n) = P(Y = k)$ statisfies, which means we have the following formula:

$$P(Y > n + k|Y > n) = P(Y > k)$$

Using Bayes' Rule, we have:

$$P(Y > n + k) = P(Y > n)P(Y > k)$$

Dor any n and k. Now, We define $R(n) = P(Y > n)$, So we have:

$$R(n + k) = R(n)R(k)$$

And we let $k = 0$, we can get:

$$R(0) = \frac{R(n)}{R(n)} = 1$$

We assume $R(1) = q$, Then we have:

$$R(2) = R(1)R(1) = q^2, R(3) = R(2 + 1) = q^3, ..., R(n) = q^n.$$

And we have:

$$P(Y = n) = P(Y > n - 1) - P(Y > n) = R(n - 1) - R(n) = q^{n-1} - q^n = q^{n-1}(1 - q)$$

let $1 - q = p$, we have:

$$P(Y = n) = p(1 - p)^{n-1}$$

for any n, Which proves that Y has some geometric pmf. □

Problem 6

From the defintion, we can know that:

$$Y = \begin{cases} 0 & X \leq 100 \\ X - 100 & 100 < X < 5100 \\ 5000 & X \geq 5100 \end{cases}$$

Using F_Y to present the cdf of Y. We have

$$F_Y(n) = \begin{cases} 0 & n < 0 \\ P(X \leq n + 100) & 0 \leq n < 5000 \\ 1 & n \geq 5000 \end{cases}$$

We only need to compute $P(X \leq n + 100)$. From the pdf of X, we have:

$$P(X \leq n + 100) = \int_0^{n+100} \frac{1}{(1+x)^2} dx = -\frac{1}{1+x}\bigg|_{x=0}^{n+100} = \frac{100+n}{101+n}$$

Hence,we have:

$$F_Y(n) = \begin{cases} 0 & n < 0 \\ \frac{100+n}{101+n} & 0 \leq n < 5000 \\ 1 & n \geq 5000 \end{cases}$$

Problem 7

It is easy to know that:

$$V = \begin{cases} 5 & T < 3 \\ 2T & T \geq 3 \end{cases}$$

Using F_V to present the cdf of V, we have:

$$F_V(n) = \begin{cases} 0 & n < 5 \\ P(T < 3) & 5 \leq n < 6 \\ P(T \leq \frac{n}{2}) & n \geq 6 \end{cases}$$

Now we calculate $P(T < 3)$ and $P(T \leq \frac{n}{2})$ we have:

$$P(T < 3) = \int_0^3 \frac{2}{3} e^{-2t/3} dt = -e^{-2t/3}\bigg|_{t=0}^{3} = 1 - e^{-2}$$

We also have:

$$P(T \leq \frac{n}{2}) = \int_0^{\frac{n}{2}} \frac{2}{3} e^{-2t/3} dt = 1 - e^{-n/3}$$

Hence, the final result is:

$$F_V(n) = \begin{cases} 0 & n < 5 \\ 1 - e^{-2} & 5 \leq n < 6 \\ 1 - e^{-n/3} & n \geq 6 \end{cases}$$

Problem 8

We use F_Y and F_Z to present the cdf of Y and Z. With the defintion, we have:

$$P(Y \leq a) = P(X < a) + P(X = a) = P(X \leq a)$$

$$P(Y \leq n) = P(Y \leq a) + P(a < Y \leq n) = P(X \leq a) + P(a < X \leq n) = P(X \leq n)$$

for any n thta $a < n < b$.Hence, we have:

$$F_Y(n) = \begin{cases} 0 & n < a \\ F_X(n) & a \leq n < b \\ 1 & n \geq b \end{cases}$$

For Z, we have for any $0 \leq n \leq b$:

$$P(Z \leq n) = P(X \leq n, |X| \leq b) + P(|X| > b) = P(-b \leq X \leq n) + 1 - P(-b \leq X \leq b) = F_X(n) + 1 - F_X(b)$$

for any $n < 0$, we have:

$$P(Z \leq n) = P(X \leq n, |X| \leq b) = P(-b \leq X \leq n) = F_X(n) - P(X < -b) = F_X(n) - \lim_{x \to (-b)^-} F_X(x)$$

Hence, we get:

$$F_Z(n) = \begin{cases} 0 & n < -b \\ F_X(n) - \lim_{x \to (-b)^-} F_X(x) & -b \leq n < 0 \\ F_X(n) + 1 - F_X(b) & 0 \leq n \leq b \\ 1 & n > b \end{cases}$$