Stat510(Section001): Homework #3

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Instructor:Brandon Legried

Tiejin Chen **tiejin@umich.edu**

Problem 1

we know that the mean of e^{-x} is 1, the variance of it is also 1.Hence,we get the skewness:

$$E(\frac{(x-1)^3}{\sigma^3}) = \int_0^\infty (x-1)^3 e^{-x} dx = \int_0^\infty (x^3 - 3x^2 + 3x - 1)e^{-x}$$

We have:

$$\int_0^\infty xe^{-x} dx = (-xe^{-x} + e^{-x})\Big|_{x=0}^\infty = 1$$

$$\int_0^\infty x^2 e^{-x} dx = (-x^2 e_{-x})\Big|_{x=0}^\infty + 2\int_0^\infty xe^{-x} dx = 2$$

$$\int_0^\infty x^3 e^{-x} dx = (-x^3 e_{-x})\Big|_{x=0}^\infty + 3\int_0^\infty x^2 e^{-x} dx = 6$$

Hence:

$$E(\frac{(x-1)^3}{\sigma^3}) = 6 - 6 + 3 - 1 = 2$$

The skewness of e^{-x} is 2.

Problem 2

We can know that:

$$Y = tan(x)d$$

Hence:

$$P(Y \leq y) = P(tan(x)d \leq y) = P(X \leq arctan(\frac{y}{d})) = \frac{arctan(y/d)}{\pi/2} = \frac{2arctan(y/d)}{\pi}$$

Hence:

$$f_Y(y) = F'_Y(y) = \frac{2}{\pi}\frac{d^2}{d^2 + y^2}$$

is the pdf of Y.

$$E(Y) = \frac{2}{\pi}\int_{y=0}^\infty \frac{d^2 y}{d^2 + y^2} dy = \frac{2}{\pi}\frac{d^2}{2} log(d^2 + y^2)\Big|_{y=0}^\infty$$

We can see that E(Y) does not exists.

Problem 3

Part a

Proof. We get:

$$E(|x-a|) = \int_{x=-\infty}^\infty |x-a| f_X(x) dx = \int_{x=-\infty}^a (a-x) f_X(x) dx + \int_{x=a}^\infty (x-a) f_X(x) dx$$

To get the min of this function. We calculate $\frac{d(E(|x-a|))}{d(a)} = 0$:

$$\frac{d(E(|x-a|))}{d(a)} = (a-x) f_X(a)\Big|_{x=a} + \int_{x=-\infty}^a f_X(x) dx + (x-a) f_X(a)\Big|_{x=a} - \int_{x=a}^\infty f_X(x) dx = 0$$

Thus,

$$\int_{x=-\infty}^a f_X(x) dx = \int_{x=a}^\infty f_X(x) dx \to F_X(a) = 1 - F_X(a)$$

Problem 3 continued on next page...

We have:

$$F_X(a) = \frac{1}{2} \rightarrow P(X \leq a) = \frac{1}{2}$$

Thus, we prove that $E(|x-a|)$ is minimized at median. □

Part b

Proof. We have:

$$\begin{aligned}
&= \int_{x=-\infty}^{\infty} (x-m)^2 f_X(x)dx - \int_{x=-\infty}^{\infty} (x-E(x))^2 f_X(x)dx \\
&= -2m\int_{x=-\infty}^{\infty} xf_X(x)dx + m^2\int_{x=-\infty}^{\infty} f_X(x)dx + 2E(x)\int_{x=-\infty}^{\infty} xf_X(x)dx - E(x)^2\int_{x=-\infty}^{\infty} f_X(x)dx \\
&= -2mE(x) + m^2 + 2E(x)^2 - E(x)^2 \\
&= 2E(x)(E(x)-m) + m^2 - E(x)^2
\end{aligned}$$

Now we need to prove this formula is greater than 0. We have:

$$\begin{aligned}
2E(x)(E(x)-m) + m^2 - E(x)^2 &= 2E(x)(E(x)-m) - (E(x)^2 - m^2) \\
&= 2E(x)(E(x)-m) - (E(x)-m)(E(x)+m) \\
&= (E(x)-m)(E(x)-m) \\
&= (E(x)-m)^2 \geq 0
\end{aligned}$$

Thus we prove the conclusion. □

Problem 4

we have:

$$E(Y^n) = \int_{y=x}^{\infty} \frac{\alpha x^\alpha}{y^{\alpha+1}} y^n dy = \int_{y=x}^{\infty} \alpha x^\alpha y^{n-\alpha-1}$$

Now we have three situaion:

1. $n > \alpha$:

$$\begin{aligned}
E(Y^n) &= \int_{y=x}^{\infty} \alpha x^\alpha y^{n-\alpha-1} dy \\
&= \frac{1}{n-\alpha}\alpha x^\alpha y^{n-\alpha}\Big|_{y=x}^{\infty} = \infty
\end{aligned}$$

   Hence, this situaion, the expection does not exists.

2. $n = \alpha$:

$$\begin{aligned}
E(Y^n) &= \int_{y=x}^{\infty} \frac{\alpha x^\alpha}{y} dy \\
&= \alpha x^\alpha log(y)\Big|_{y=x}^{\infty} = \infty
\end{aligned}$$

   This situaion, the expection does not exists.

3. $n < \alpha$:

$$\begin{aligned}
E(Y^n) &= \int_{y=x}^{\infty} \alpha x^\alpha y^{n-\alpha-1} dy \\
&= \frac{1}{n-\alpha}\alpha x^\alpha y^{n-\alpha}\Big|_{y=x}^{\infty} = \frac{\alpha x^n}{\alpha - n}
\end{aligned}$$

Hence , when $n < \alpha$, we have $E(Y^n)$ exists.Then we consider the skewness. If the skweness exists, then $E(Y^2)$ must exist. Hence $\alpha > 2$.And also $E(Y - \frac{\alpha x}{\alpha-1})^3$ need to exist. We have:

$$\begin{aligned}
E(Y - \frac{\alpha x}{\alpha - 1})^3 &= \int_{y=x}^{\infty} \frac{\alpha x^\alpha}{y^{\alpha+1}}(Y - E(y))^3 dy \\
&= E(Y^3) - 3E(y)E(Y^2) + 2(E(y))^3 \\
&= \frac{\alpha}{\alpha-3}x^3 - \frac{3\alpha^2}{(\alpha-1)(\alpha-2)}x^3 + \frac{2\alpha^3}{(\alpha-1)^3}x^3
\end{aligned}$$

Thus,$\alpha > 3$, skweness exists, We also have:

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{\alpha}{\alpha-2}x^2 - \frac{\alpha^2}{(\alpha-1)^2}x^2$$

Hence:

$$skweness = \frac{E(Y - \frac{\alpha x}{\alpha-1})^3}{Var(Y)^{3/2}}$$

And, we have mgf:

$$E[e^{ty}] = \int_{y=x}^{\infty} \alpha x^\alpha y^{-\alpha-1} e^{ty} dy$$

we can know, for any n:

$$\lim_{y\to\infty} \frac{e^{ty}}{y^n} = \infty$$

Which shows that $y^{-\alpha-1}e^{ty}$ does not converge. Hence $E[e^{ty}] = \infty$. mgf does not exist.

Problem 5

Part i

We have:

$$\begin{aligned}
E(\frac{1}{1+x}) &= \sum_{x=0}^{n} \frac{1}{1+x}\binom{n}{x}p^x(1-p)^{n-x} \\
&= \sum_{x=0}^{n} \frac{1}{1+x}\frac{n!}{(n-x)!x!}p^x(1-p)^{n-x} \\
&= \frac{1}{n+1}\sum_{x=0}^{n} \frac{(n+1)!}{(n-x)!(x+1)!}p^x(1-p)^{n-x} \\
&= \frac{1}{(n+1)p}\sum_{k=1}^{n} \frac{(n+1)!}{(n-x)!k!}p^k(1-p)^{n+1-k} \\
&= \frac{1}{(n+1)p}(1-(1-p)^{n+1})
\end{aligned}$$

Part ii

Problem 5 continued on next page...

$$
\begin{aligned}
E(\frac{1}{1+x}) &= \sum_{x=0}^{\infty} \frac{1}{1+x} \frac{\lambda^x}{x!} e^{-\lambda} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1}}{(x+1)!} \\
&= \frac{e^{-\lambda}}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \\
&= \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1) \\
&= \frac{1 - e^{-\lambda}}{\lambda}
\end{aligned}
$$

Now let us consider the relationship between them. We know that, when n is large enough,Poisson can be viewed as a binomial disturbtion with paramater np. Now, let us consider:

$$\lim_{n\to\infty} \frac{1-(1-p)^{n+1}}{(n+1)p} = \lim_{n\to\infty} \frac{1-(1-\frac{\lambda}{n})^{n+1}}{(n+1)p}$$

we have:

$$\lim_{n\to\infty} (n+1)p = \lambda$$

$$\lim_{n\to\infty} (1-\frac{\lambda}{n})^{n+1} = e^{-\lambda}$$

Hence:

$$\lim_{n\to\infty} \frac{1-(1-\frac{\lambda}{n})^{n+1}}{(n+1)p} = \frac{1-e^{-\lambda}}{\lambda}$$

Which is to say, the relationship bewteen Poisson and Binomial disturbition is the same relationship bewteen their $\frac{1}{1+x}$.

Problem 6

With l'Hopital rule, we can know:

$$\lim_{x\to\infty} e^{tx-log(x)^2} \to \infty$$

That is to say, for any c, we have a k, for every $x > k$, we have:

$$e^{tx-log(x)^2} > c$$

Hence:

$$\frac{1}{\sqrt{2\pi}} \int_{x=k}^{\infty} \frac{e^{tx-log(x)^2}}{x} \geq \frac{1}{\sqrt{2\pi}} \int_{x=k}^{\infty} \frac{c}{x} = \infty$$

Hence:

$$\frac{1}{\sqrt{2\pi}} (\int_{x=0}^{k} \frac{e^{tx-log(x)^2}}{x} + \int_{x=k}^{\infty} \frac{e^{tx-log(x)^2}}{x}) = \infty$$

Thus, the mgf does not exists.

Problem 7

Part a

We let X be always earn 1 dollars, and Y be the gamble. We have:

$$E(U(X)) = 1, E(U(Y)) = 0.001 \times 500^c$$

if c is not even , we have:

$$0.001 \times 500^c > 1 \rightarrow c > log_{500}(1000)$$

Hence:

$$c > log_{500}(1000)$$

Part b

We let X be the earn Z dollars and Y be take the gamble. Then we have:

$$E(U(X)) = Z^c, E(U(Y)) = \sum_{T=1}^{\infty}(2^{T-1})^c \frac{1}{2}^T$$

We have:

$$E(U(Y)) = \sum_{T=1}^{\infty} 2^{(c-1)T-c}$$

if $c \geq 1$, we have $E(U(Y)) = \infty$, Hence we will always be perfer to take the gamble.Now we consider when $n < 1$ we have :

$$E(U(Y)) = \sum_{T=1}^{\infty} 2^{(c-1)T-c} = \frac{\frac{1}{2}}{1-2^{c-1}} = \frac{1}{2-2^c}$$

$$\frac{1}{2-2^c} \geq Z^c \rightarrow 2Z^c - (2Z)^c \leq 1$$

Hence if $c > 1 \ or 2Z^c - (2Z)^c \leq 1$, we perfer to take the gamble.

Problem 8

Part a

We assume the X is an exponential distribution with the following pdf:

$$f_X(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}} \ x > 0$$

We let $\lambda = \frac{1}{\beta}$, we can get a new form of pdf:

$$f_X(x) = \lambda e^{-\lambda x} \ x > 0$$

We will use this form to calculate. Let us consider the M(c) First:

$$M(c) = \lambda \int_0^{\infty} e^{(c-\lambda)x}dx = \frac{\lambda}{c-\lambda}e^{(c-\lambda)x}\Big|_0^{\infty}$$

if $c \geq \lambda(\frac{1}{\beta}$ in other word),M(c) does not exists. and if $c < \lambda(\frac{1}{\beta}$ in other word),we have:

$$M(c) = \frac{\lambda}{\lambda - c}$$

Thus, c must less than λ, then We have:

$$F_Y(y) = \frac{1}{M(c)}(\frac{\lambda}{\lambda-c} - \frac{\lambda}{\lambda-c}e^{-(\lambda-c)y}), y > 0$$

Problem 8 continued on next page. . .

Hence:

$$f_Y(y) = F'_Y(y) = (\lambda - c)e^{-(\lambda-c)y}, y > 0$$

Hence Y is also an exponential distribution with parameter 'λ'(in a form of $f_Y(y) = \lambda e^{-\lambda y}$)as $\lambda - c$. If we write it in a form of 'β' way, we have:

$$f_Y(y) = (\frac{1}{\beta} - c)e^{-(\frac{1}{\beta}-c)y}, y > 0$$

Therefore, in a 'β' way, Y is an exponential distribution with parameter 'β'(in a form of $f_Y(y) = \frac{1}{\beta}e^{-\frac{y}{\beta}}$) as $\frac{\beta}{1-\beta c}$

Part b

We have:

$$f_Y(y) = F'_Y(y) = \frac{1}{M(c)}e^{cy}f_x(y)$$

And the mgf of Y is:

$$M_Y(a) = \frac{1}{M(c)}\int_{-\infty}^{\infty} e^{ay}e^{cy}f_x(y)dy = \frac{1}{M(c)}\int_{-\infty}^{\infty} e^{(a+c)y}f_x(y)dy = \frac{M(a+c)}{M(c)}$$

Hence, the mgf of Y has defintion in a neighborhood of 0 if and only if the mgf of X is defined in a neighborhood of c.