Stat510(Section001): Homework #4

Due on Nov.3, 2021 at 11:59pm

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Problem 1

Part a

Let $X_i, i = 1, 2, 3$ be uniform distribution on (1,10), We have:

$$E(sum\ of\ three\ numbers\ shown) = E(X_1 + X_2 + X_3) = 3 \times E(X_1) = 16.5$$

Part b

All three cards are drawn randomly without re-placement. All the cards should have same probability to be choosen. And we can know from that the probability of any card been choosen is $\frac{3}{10}$. Hence we have:

$$E(sum\ without\ replacement) = \frac{3}{10}(1 + ... + 10) = 3 \times 5.5 = 16.5$$

Problem 2

Part 1

We have:

$$P(X_T = x) = \frac{P(X = x)}{1 - P(X = 0)} = \frac{e^{-\lambda}\lambda^x}{x!(1 - e^{-\lambda})}, x = 1, ...$$

$$E(X_T) = \sum_{x=1} x \frac{e^{-\lambda}\lambda^x}{x!(1 - e^{-\lambda})} = \frac{1}{(1 - e^{-\lambda})} \sum_{x=1} x \frac{e^{-\lambda}\lambda^x}{x!} = \frac{\lambda}{1 - e^{-\lambda}}$$

$$E(X_T^2) = \sum_{x=1} x^2 \frac{e^{-\lambda}\lambda^x}{x!(1 - e^{-\lambda})} = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}}$$

$$Var(X_T) = E(X_T^2) - E(X_T)^2 = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} - \frac{\lambda^2}{(1 - e^{-\lambda})^2}$$

Part 2

We have:

$$P(X_T = x) = \frac{\binom{x+r-1}{x} p^r (1-p)^x}{1 - p^r}, x = 1, ...$$

$$E(X_T) = \frac{1}{1 - p^r} E(X) = \frac{r(1-p)}{(1 - p^r)p}$$

$$E(X_T^2) = \frac{E(X^2)}{1 - p^r} = \frac{r(1-p) + r^2(1-p)^2}{p^2(1 - p^r)}$$

$$Var(X_T) = \frac{r(1-p) + r^2(1-p)^2}{p^2(1 - p^r)} - \frac{r^2(1-p)^2}{(1 - p^r)^2 p^2}$$

Problem 3

It is very similar to Hypergeometric distribution, However, we can know that the last capture must be marked animal. Hence we have:

$$P(X = x) = \frac{\binom{M}{K-1}\binom{N-M}{x-K}}{\binom{N}{x-1}} \frac{M - K + 1}{N - x + 1}, x = K, ..., N$$

$$\begin{aligned}
E(X) &= \sum_{x=K}^{N} \frac{\binom{M}{K-1}\binom{N-M}{x-K}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1} x \\
&= \sum_{x=K}^{N} \frac{\binom{M}{K-1}\binom{N-M}{x-K}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1} (x-k+k) \\
&= K + (N-M) \sum_{x=K}^{N} \frac{\binom{M}{K-1}\binom{N-M-1}{x-K-1}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1} \\
&= K + \frac{(N-M)K}{M+1} \sum_{x=K+1}^{N} \frac{\binom{M+1}{K}\binom{N-M-1}{x-K-1}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1}
\end{aligned}$$

We can see that $\frac{\binom{M+1}{K}\binom{N-M-1}{x-K-1}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1}$ is the pmf of the situation of N total animals, M+1 marked animals and K+1 obtained marked animals($M \leq N-1$). Hence:

$$E(X) = \frac{(N-M)K}{M+1} + K$$

And if $M+1 > N$, we have $M = N$, And to obtain K marked animals, we only need to capture K animals. Hence, this expection formula also work when $M = N$. Hence, this is the formula.
For $Var(X)$, we consider $E(X^2)$ First

$$\begin{aligned}
E(X^2) &= \sum_{x=K}^{N} \frac{\binom{M}{K-1}\binom{N-M}{x-K}}{\binom{N}{x-1}} \frac{M-K+1}{N-x+1} x^2 \\
&= (N+1) \sum_{x=K}^{N} \frac{\binom{M}{K-1}\binom{N-M}{x-K}}{\binom{N+1}{x}} \frac{M-K+1}{N-x+1} x \\
&= \frac{(N+1)K}{M+1} \sum_{y=K+1}^{N+1} \frac{\binom{M+1}{K}\binom{N-M}{y-K-1}}{\binom{N+1}{y-1}} \frac{M-K+1}{N+1-y+1} (y-1) \\
&= \frac{(N+1)K}{M+1} (E(X_{N+1,M+1,K+1}) - 1) \\
&= \frac{(N+1)K}{M+1} (\frac{(N-M)(K+1)}{M+2} + K)
\end{aligned}$$

Hence, we have:

$$Var(X) = E(X^2) - E(X)^2 = \frac{(N-M)(N+1)K}{(M+2)(M+1)} \frac{M-K+1}{M+1} = \frac{(N-M)(N+1)K}{(M+2)(M+1)} (1 - \frac{K}{M+1})$$

Problem 4

Part a

Proof. If we have a disturbution $Poi(\lambda + \mu)$. We have:

$$P(Poi(\lambda+\mu) = x) = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^x}{x!}$$

Consider the Binomial theorem for any $0 \leq y \leq x$, we have $\lambda^y \mu^{x-y}$ part of $P(Poi(\lambda+\mu) = x)$ is:

$$\frac{\binom{x}{y} e^{-(\lambda+\mu)} \lambda^y \mu^{x-y}}{x!} = \frac{e^{-(\lambda+\mu)\lambda^y \mu^{x-y}}}{y!(x-y)!}$$

And we have:

$$P(H+T=x) = \sum_{i+j=x, ij \geq 0} P(H=i)P(T=j)$$

For any $0 \leq y \leq x$, we have $\lambda^y \mu^{x-y}$ part of $[P(H+T=x)$ is:

$$P(H=y)(T=x-y) = \frac{e^{-\lambda}\lambda^y}{y!}\frac{e^{-\mu}\mu^{x-y}}{(x-y)!} = \frac{e^{-(\lambda+\mu)}\lambda^y\mu^{x-y}}{y!(x-y)!}$$

We find for any y, $\lambda^y \mu^{x-y}$ part of $[P(H+T=x)$ is equal to $\lambda^y \mu^{x-y}$ part of $P(Poi(\lambda+\mu)=x)$.Hence they are the same. We get $H+T \sim Poi(\lambda+\mu)$. □

Part b
We have:

$$P(H=x|H+T=n) = P(H=x, T=n-x)$$

However, now what we consider is conditional distribution instead of conditional probability. We need to generalize this probability. We have:

$$\begin{aligned}
f_n(x) &= \frac{P(H=x, T=n-x)}{P(H+T)=n} \\
&= \frac{P(H=x)P(T=n-x)}{P(H+T)=n} \\
&= \binom{n}{x}(\frac{\lambda}{\lambda+\mu})^x(\frac{\mu}{\lambda+\mu})^{n-x}
\end{aligned}$$

Hence it is a binomial distribution indeed with parameter n and $p = \frac{\lambda}{\lambda+\mu}$.

Problem 5

Part a Let us consider $F_T(t)$ first. We have:

$$F_T(t) = \int_{x=0}^{t} \frac{1}{\beta}e^{-\frac{x}{\beta}}dx = 1 - e^{-\frac{t}{\beta}}$$

Hence we have:

$$h_T(t) = -\frac{d}{dt}log(1-F_T(t)) = -\frac{d}{dt}(-\frac{t}{\beta}) = \frac{1}{\beta}$$

Part b we have:

$$F_T(t) = P(T \leq t) = P(S^{\frac{1}{\gamma}} \leq t) = P(S \leq t^y) = 1 - e^{-\frac{t^\gamma}{\beta}}$$

Hence:

$$h_T(t) = -\frac{d}{dt}(-\frac{t^\gamma}{\beta}) = \frac{\gamma}{\beta}t^{\gamma-1}$$

Thus we prove it.
Part c we have :

$$h_T(t) = -\frac{d}{de^{\frac{-(t-\mu)}{\beta}}}log(\frac{e^{\frac{-(t-\mu)}{\beta}}}{1+e^{\frac{-(t-\mu)}{\beta}}})\frac{de^{\frac{-(t-\mu)}{\beta}}}{dt} = \frac{1}{\beta}(1-\frac{e^{\frac{-(t-\mu)}{\beta}}}{1+e^{\frac{-(t-\mu)}{\beta}}}) = \frac{F_T(t)}{\beta}$$

Problem 6

Part i

We have:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f'_X(x) = -x\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x f_X(x) = x\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Hence:

$$f'_X(x) + x f_X(x) = 0$$

Part ii

Proof. we have:

$$1 - F_X(x) = \int_{t=x}^{\infty} f_X(t)dt$$

With the conclusion of part i and partial integration we have:

$$1 - F_X(x) = \int_{t=x}^{\infty} \frac{-f'_X(t)}{t} dt = \frac{-f_X(t)}{t}\Big|_{t=x}^{\infty} - \int_{t=x}^{\infty} \frac{f_X(t)}{t^2} dt$$

We use the conclusion of part i and partial integration again to get:

$$1 - F_X(x) = \frac{f_X(x)}{x} + \int_{t=x}^{\infty} \frac{f'_X(t)}{t^3} dt = \frac{f_X(x)}{x} + \frac{f_X(t)}{t^3}\Big|_{t=x}^{\infty} + 3\int_{t=x}^{\infty} \frac{f_X(t)}{t^4} dt$$

Since $t^4 > 0, f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} > 0$ for every t in (x, ∞), we have:

$$\int_{t=x}^{\infty} \frac{f_X(t)}{t^4} dt > 0$$

We use M to present this integration value, and we know $M > 0$, $\frac{M}{f_X(x)} > 0$. Hence,we have:

$$1 - F_X(x) = \frac{f_X(x)}{x} - \frac{f_X(x)}{x^3} + 3M$$

$$\frac{1 - F_X(x)}{f_X(x)} = x^{-1} - x^{-3} + 3\frac{M}{f_X(x)} > x^{-1} - x^{-3}$$

Which proves the left side of inequality. Now let us prove the right side, we use the conclusion of part i and partial integration once again to get:

$$1 - F_X(x) = \frac{f_X(x)}{x} - \frac{f_X(x)}{x^3} - 3\int_{t=x}^{\infty} \frac{f'_X(t)}{t^5} dt = \frac{f_X(x)}{x} - \frac{f_X(x)}{x^3} - 3\frac{f_X(t)}{t^5}\Big|_{t=x}^{\infty} - 15\int_{t=x}^{\infty} \frac{f_X(t)}{t^6} dt$$

Using the same logic, we use Z to present the $\int_{t=x}^{\infty} \frac{f_X(t)}{t^6} dt$, And with $t^6 > 0, f_X(t) > 0$ for every t in (x, ∞),we have:

$$Z = \int_{t=x}^{\infty} \frac{f_X(t)}{t^6} > 0$$

Hence:

$$1 - F_X(x) = \frac{f_X(x)}{x} - \frac{f_X(x)}{x^3} + 3\frac{f_X(x)}{x^5} - 15Z$$

$$\frac{1 - F_X(x)}{f_X(x)} = x^{-1} - x^{-3} + 3x^{-5} - 15\frac{Z}{f_X(t)} < x^{-1} - x^{-3} + 3x^{-5}$$

Which proves the right side of inequality. $\square$

Problem 7

Part a
We have:

$$E(X) = \int_{x=0}^{\infty} \frac{2}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} (-e^{-\frac{x^2}{2}} \Big|_{x=0}^{\infty}) = \frac{2}{\sqrt{2\pi}}$$

$$E(X^2) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x x e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} (-x e^{-\frac{x^2}{2}} \Big|_{x=0}^{\infty} + \int_{x=0}^{\infty} e^{-\frac{x^2}{2}} dx) = \frac{2}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2} = 1$$

$$Var(X) = E(X^2) - E(X)^2 = 1 - \frac{2}{\pi}$$

Part b
the kernel of X is e^{tx^2}, and the kernel of gamma disturbution is $x^a e^{-by}$. We need to transform the e^{tx^2} to e^{-by}. Hence, Let $Y = X^2$, we have:

$$f_Y(y) = f_X(\sqrt{y}) \Big| \frac{1}{2\sqrt{y}} \Big| = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

It is now a gamma disturbution since we have:

$$f_Y(y) = \frac{1}{\Gamma(\frac{1}{2}) 2^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{-\frac{y}{2}}$$

with $\alpha = \frac{1}{2}$ and $\beta = 2$. Hence $g(X) = X^2$.

Problem 8

We have:

$$f_X(x) = \int_{y=x}^{1} 2(x+y) dy = -3x^2 + 2x + 1, 0 \leq x \leq 1$$

$$f_Y(y) = \int_{x=0}^{y} 2(x+y) dx = 3y^2, 0 \leq y \leq 1$$