Statistics 510, Homework 3

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Directions: Do all the problems. The total grade is 40 points, with all problems being 5 points each. You may work with others in the class or use any sources you like to study these problems, but when you go to write the solutions you should be able to write it using only the textbook and course notes. If you obtain facts, advice, or inspiration from any source outside the textbook or course notes, you must indicate in your solution what the source is. In all cases, you must make it clear to the reader what logic/work was needed to reach your solution. Solutions must be posted to Canvas – any legible version, handwritten or typed, is acceptable.

#1

Compute the skewness for the random variable with pdf $f_X(x) = e^{-x}, x \geq 0$, a pdf that is skewed to the right.

#2

Construct a random right triangle as follows. Let X be a random variable whose distribution is uniform on $(0, \frac{\pi}{2})$. For each X, select a point (d, y) in Quadrant I of the plane with terminal angle X (that is, the angle between the horizontal axis and the line joining the origin with (d, y)) and vertical line segment at y. Here, Y is the height of the random triangle. For a fixed constant $d > 0$, write the pdf of Y and $\mathbf{E}[Y]$.

#3

This exercise concerns the median of a distribution. Assume that X is continuous.

(a) Show that $\mathbf{E}|X-a|$ is minimized at $a=m$ where m is the median of X. Recall that the median of a random variable X is the value such that $\mathbf{P}(X \leq m) = 1/2$.

(b) Let $m(X)$ denote the median of X. Explain why

$$\mathbf{E}(X - m(X))^2 \geq \mathrm{Var}[X].$$

#4

This exercise concerns the Type I Pareto distribution. For any positive real numbers $x, \alpha > 0$, we say that Y is $Pareto(x, \alpha)$ if it has pdf

$$f_Y(y) = \frac{\alpha x^\alpha}{y^{\alpha+1}}, y \geq x.$$

For each integer $n \geq 1$, determine when $\mathbf{E}[Y^n]$ exists and compute where applicable. Also, determine when the skewness of Y exists and compute it. Explain whether the mgf exists.

#5

Consider the random variable $Y = \frac{1}{1+X}$. Compute $\mathbf{E}[Y]$ when (i) X is a binomial distribution with parameters $n \geq 1$ and $p \in (0,1)$ and (ii) X is a Poisson distribution with $\lambda > 0$. Explain the relationship between your answers.

#6

The log-normal distribution from the example on nonunique moments has an interesting property. If we have the pdf

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} e^{-(\log x)^2/2}, x \geq 0,$$

then we showed that all the moments exist and are finite. Prove that this distribution does not have an mgf, i.e.

$$M_X(t) = \int_{x=0}^{\infty} e^{tx} \frac{1}{x\sqrt{2\pi}} e^{-(\log x)^2/2}\, dx$$

does not even exist for any $t > 0$.

#7

This exercise concerns *maximizing expected utility.* A common assumption in gambling is that there is a utility function U and when the person must choose between any two gambles X and Y, they prefer X to Y if $\mathbf{E}[U(X)] > \mathbf{E}[U(Y)]$ and will be indifferent between X and Y if $\mathbf{E}[U(X)] = \mathbf{E}[U(Y)]$.

(a) A decision maker has a utility function of the form

$$U(x) = x^c, \ x \geq 0.$$

This person has two choices. They may either earn 1 dollar guaranteed or use that dollar to enter a lottery that pays 500 with probability 0.001 and pays 0 with probability 0.999. For what values of c does the person prefer to play the lottery?

(b) A decision maker has the same utility function as before, but the game is as follows. They may either earn Z dollars guaranteed or use that money to take a gamble whose payout is as follows. Flip a fair coin repeatedly until the time T when the first tail appears. The payout is then $P(T) = 2^{T-1}$ for all $T \geq 1$ (note $T - 1$ is the number of heads, not the number of flips). For what values of c and Z does this person prefer to take the gamble?

#8

Suppose X is a continuous random variable with cdf F_X and suppose $M(c) = \mathbf{E}[e^{cX}]$ exists for some distinguished $c \in \mathbb{R}$. Now define a random variable Y with the cdf

$$F_Y(y) = \frac{1}{M(c)} \int_{x=-\infty}^{y} e^{cx} f_X(x)\ dx.$$

(a) Write the distribution of Y when X has the exponential distribution with parameter $\lambda > 0$.

(b) Under what conditions is the mgf of Y defined in a neighborhood of 0? Under these conditions, write the mgf of Y.