Statistics 510, Homework 4

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Directions: Do all the problems. The total grade is 40 points, with all problems being 5 points each. You may work with others in the class or use any sources you like to study these problems, but when you go to write the solutions you should be able to write it using only the textbook and course notes. If you obtain facts, advice, or inspiration from any source outside the textbook or course notes, you must indicate in your solution what the source is. In all cases, you must make it clear to the reader what logic/work was needed to reach your solution. Solutions must be posted to Canvas– any legible version, handwritten or typed, is acceptable.

#1

Consider ten cards in a bag, labeled in order from 1 to 10.

(a) Draw three cards randomly and independently, with replacement. That is, any card drawn is put back in the bag. Write the expectation of the sum of the three numbers shown.

(b) Same problem, except the three cards are drawn randomly without replacement.

#2

This exercise concerns truncated discrete distributions. If the random variable X has range $\{0, 1, 2, ...\}$, we might define the 0-truncated random variable X_T has pmf

$$\mathbf{P}(X_T = x) = \frac{\mathbf{P}(X = x)}{\mathbf{P}(X > 0)}, x = 1, 2, 3, ...$$

Write the pmf, mean, and variance of X_T when (i) $X \sim Poi(\lambda)$ with $\lambda > 0$ and (ii) X has the pmf

$$f_X(x) = \binom{x + r - 1}{x} p^r (1-p)^x, x \in \{0, 1, 2, ...\},$$

with r a positive integer and p a real number in $(0, 1)$.

#3

A population of N animals has had a number M of its members captured, marked, then released into the original population. Let X be the number of animals that are necessary to recapture (without re-release) in order to obtain K marked animals. Write the pmf, expectation, and variance of X.

#4

Let H and T be independent Poisson random variables with parameters $\lambda, \mu > 0$, respectively. (a) Show that $H + T \sim Poi(\lambda + \mu)$. (b) Show that the conditional distribution given any integer $n \geq 1$, given by

$$f_n(x) = \mathbf{P}(H = x | H + T = n), x \in \{0, 1, ..., n\},$$

follows a binomial distribution with parameters n and p, for some p. What is p?

#5

Suppose the random variable T is the length of life of an object. Define the *hazard function* $h_T(t)$ of T to be

$$h_T(t) = \lim_{\delta \to 0} \frac{\mathbf{P}(t \le T < t + \delta | T \ge t)}{\delta}.$$

If T is a continuous random variable, then

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = -\frac{d}{dt} \log(1 - F_T(t)).$$

Verify the following indicated hazard functions.

(a) If $T \sim Exp(\beta)$ for some $\beta > 0$, then $h_T(t) = 1/\beta$.

(b) If $S \sim Exp(\beta)$ and $T = S^{1/\gamma}$ for some $\beta, \gamma > 0$, then $h_T(t) = (\gamma/\beta)t^{\gamma-1}$.

(c) If $T \sim Logistic(\mu, \beta)$ for some $\mu \in \mathbb{R}$ and $\beta > 0$, that is

$$F_T(t) = \frac{1}{1 + e^{-(t-\mu)/\beta}},$$

then $h_T(t) = F_T(t)/\beta$.

#6

Let X be an $N(0,1)$ distribution and let $f_X(x)$ be its pdf and $F_X(x)$ be its cdf. (i) Show that $f_X'(x) + x f_X(x) = 0$. (ii) Show for $x > 0$ that

$$x^{-1} - x^{-3} < \frac{1 - F_X(x)}{f_X(x)} < x^{-1} - x^{-3} + 3x^{-5}.$$

#7

This exercise concerns the folded normal distribution. Let X have pdf

$$f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, x > 0.$$

(a) Find the mean and variance of X.

(b) If X has a folded normal distribution, find the transformation g and values $\alpha, \beta > 0$ such that $Y = g(X)$ has the $Gamma(\alpha, \beta)$ distribution.

#8

Let X, Y be continuous random variables whose joint pdfs is

$$f_{X,Y}(x, y) = 2(x + y), 0 \leq x \leq y \leq 1.$$

Write the marginal pdfs $f_X(x)$ and $f_Y(y)$.