Stat510(Section001): Homework #5

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Problem 1

For uniform distribution, we can use the area of whole (X,Y) and the area of condition expression to calculate the probability. We have the total area of whole(X,Y) is $2 \times 2 = 4$.Hence we have:

1. $P(X^2 + Y^2 < 1) = \frac{\pi}{4}$
2. $P(2x - y > 0) = \frac{1}{2}$
3. $P(|X + Y| < 2) = 1$
4. $P(|X - Y| > \frac{1}{2}) = \frac{9/4}{4} = \frac{9}{16}$
5. The area of $(max(X,Y) > \frac{1}{2})$ is $\frac{1}{2}2 \times 2 - \frac{1}{2}\frac{1}{2} = \frac{7}{4}$, Hence,$P(max(X,Y) > \frac{1}{2}) = \frac{7}{16}$

Problem 2

First, we consider the cdf of X+Y, if $0 \leq n \leq 1$, we have:

$$P(X + Y \leq n) = \int_{x=0}^{n/2} \int_{y=x}^{n-x} 2(x+y)dydx = \int_{x=0}^{n/2} -4x^2 + n^2 dx = \frac{n^3}{3}$$

if $1 < n \leq 2$, we have:

$$P(X + Y \leq n) = \int_{x=n-1}^{n/2} \int_{y=x}^{n-x} 2(x+y)dydx + \int_{x=0}^{n-1} \int_{y=x}^{1} 2(x+y)dydx = -\frac{n^3}{3} + n^2 - \frac{1}{3}$$

Hence we have:

$$F_{X+Y}(n) = \begin{cases} n^2 & 0 \leq n \leq 1 \\ -n^2 + 2n & 1 < n \leq 2 \\ 0 & Otherwise \end{cases}$$

which is the pdf of $X + Y$.

Problem 3

We have $X + Y > 0$,We can know the mgf of X and Y is

$$\frac{1}{1 - t\lambda}$$

Since X and Y are independent, we have

$$M_{X+Y}(t) = \frac{1}{1 - t\lambda}\frac{1}{1 - t\lambda} = (\frac{1}{1 - t\lambda})^2$$

We can find that $M_{X+Y}(t)$ is the mgf of Gamma distribution with $\alpha = 2, \beta = \lambda$. Hence:

$$X + Y \sim Gamma(2, \lambda)$$

Hence, its pdf is:

$$f_{X+Y}(z) = \frac{1}{\Gamma(2)\lambda^2} z e^{-\frac{z}{\lambda}} = \frac{1}{\lambda^2} z e^{-\frac{z}{\lambda}}$$

Now consider $X - Y$, we have $X - Y \in R$:

$$M_{X-Y}(t) = E(e^{Xt}e^{Y(-t)}) = M_X(t)M_Y(-t) = \frac{1}{1 - t\lambda}\frac{1}{1 + t\lambda} = \frac{1}{1 - (t\lambda)^2}$$

Problem 3 continued on next page. . .

We find that this is the mgf of Double Exponential with $\mu = 0, \sigma = \lambda$. Hence:

$$X - Y \sim DoubleExponential(0, \lambda)$$

Its pdf is:

$$f_{X-Y}(z) = \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}$$

Problem 4

The set $\{X > 0, Y > 0\}$ is mapped into the set $\{0 < U < 1, V > 0\}$. And we have:

$$X = UV, Y = V - UV, |J| = V$$

Hence:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{(UV)^{\alpha_1 - 1}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \frac{(V(1-U))^{\alpha_2 - 1}}{\Gamma(\alpha_2)\beta^{\alpha_2}} e^{-\frac{V}{\beta}} V \\ &= \frac{1}{\Gamma(\alpha_1 + \alpha_2)\beta^{\alpha_1 + \alpha_2}} V^{\alpha_1 + \alpha_2 - 1} e^{\frac{-V}{\beta}} \times \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} U^{\alpha_1 - 1}(1-U)^{\alpha_2 - 1} \end{aligned}$$

We can see $f_{U,V}(u,v)$ can be factored into two densities. Hence U,V are independent. which done the job in Part b. And we can see that $V \sim Gamma(\alpha_1 + \alpha_2, \beta), U \sim beta(\alpha_1, \alpha_2)$. For cdf, we know:

$$F_U(u) = \frac{B(u; \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)}, F_V(v) = \frac{\gamma(\alpha_1 + \alpha_2, \frac{v}{\beta})}{\Gamma(\alpha_1 + \alpha_2)}$$

where B is the Beta fucntion, and:

$$B(u; \alpha_1, \alpha_2) = \int_0^u t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1} dt, \gamma(\alpha_1 + \alpha_2, \frac{v}{\beta}) = \int_0^{\frac{v}{\beta}} t^{\alpha_1 + \alpha_2 - 1} e^{-t} dt$$

With the independent, we can get:

$$F_{U,V}(u,v) = F_U(u)F_V(v) = \frac{B(u; \alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \frac{\gamma(\alpha_1 + \alpha_2, \frac{v}{\beta})}{\Gamma(\alpha_1 + \alpha_2)}$$

Problem 5

Part a

We have set:$\{X > 0, D > X\}$:

$$Y = X - D, |J| = 1$$

Hence:

$$f_{X,D}(x,d) = e^{-x} e^{-(x-d)} = e^{-(2x-d)}$$

From problem 3, we can know that X-Y is a double Double Exponential distribution, and its pdf is:

$$f_D(d) = \frac{1}{2} e^{-|d|}$$

Hence:

$$f_{X|D}(x|0) = \frac{f_{X,D}(x,0)}{f_D(0)} = 2e^{-2x}$$

Part b

We have set:$\{X > 0, Q > 0\}$.and:

$$Y = X/Q, |J| = \frac{X}{Q^2}$$

Hence:

$$f_{X,Q}(x,q) = \frac{x}{q^2}e^{-x}e^{-\frac{x}{q}} = \frac{x}{q^2}e^{-(\frac{x}{q}+x)}$$

We have:

$$f_Q(1) = \int_0^\infty xe^{-2x}dx = \frac{1}{4}$$

Hence:

$$f_{X|Q}(x|1) = 4xe^{-2x}$$

Problem 6

Part a

We have:

$$E(Y) = E(E(Y|X)) = E(X) = \frac{1}{2}$$

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)) = E(X^2) + Var(X) = \frac{5}{12}$$

$$Cov(X,Y) = E(XY) - \frac{1}{4} = E(E(XY|X)) - \frac{1}{4} = E(XE(Y|X)) - \frac{1}{4} = E(X^2) - \frac{1}{4} = \frac{1}{12}$$

Part b

Proof. We can know that:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = \frac{1}{\sqrt{2\pi}x}e^{-\frac{(y-x)^2}{2x^2}}$$

Let $U = \frac{Y}{X}$, we have set $\{U \in R, X \in (0,1)\}$. And we have:

$$Y = UX, |J| = X$$

Hence:

$$f_{U,X}(u,x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(ux-x)^2}{2x^2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}+u-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(u-1)^2}{2}} \times 1$$

which can be factored into two densities. which $U \sim Norm(1,1), X \sim Uni(0,1)$. Hence $\frac{Y}{X}$ and X are independent. □

Problem 7

We can know that:

$$E(Y) = E(E(Y|X)) = 10 - E(X), E(X) = 7 - \frac{E(Y)}{4}$$

Solve it to get:

$$E(Y) = 4, E(X) = 6$$

Also, we have:

$$E(XY) = E(X(10-X)) = 60 - E(X^2) = E(Y(7 - \frac{Y}{4})) = 28 - \frac{1}{4}E(Y^2)$$

We can get:

$$\frac{1}{4}E(Y^2) = E(X^2) - 32$$

We have:

$$Corr(X,Y) = \frac{E(XY) - 24}{\sqrt{E(X^2) - 6}\sqrt{E(Y^2) - 16}} = \frac{36 - E(X^2)}{\sqrt{E(X^2) - 36}\sqrt{4E(X^2) - 144}} = -\frac{E(X^2) - 36}{2\sqrt{E(X^2) - 36}\sqrt{E(X^2) - 36}} = -\frac{1}{2}$$

Problem 8

We have:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sqrt{12}} e^{-\frac{(y-2x+3)^2}{24}}$$

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{2\pi\sqrt{12}} e^{-(\frac{(y-2x+3)^2}{24}+\frac{x^2}{2})}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \frac{1}{2\pi\sqrt{12}} \int_{-\infty}^{\infty} e^{-\frac{y^2+6y+9-\frac{(y^2+6y+9)}{4}}{24}} e^{-\frac{(4x-\frac{3+y}{2})^2}{24}} dx$$

We let $t = 4x - \frac{3+y}{2}$, we have $dt = 4dx$. Hence, we get:

$$f_Y(y) = \frac{1}{2\pi\sqrt{12}} e^{-\frac{(y+3)^2}{32}} \frac{1}{4} \int_{-\infty}^{\infty} e^{-\frac{t^2}{24}} dt$$

The integration part is the unnormlized pdf of $N(0,12)$, And it should equal to $\sqrt{2\pi}\sqrt{12}$. Hence we get:

$$f_Y(y) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{(y+3)^2}{32}}$$

We can see from the pdf, Y is a normal distribution. we have:

$$E(Y) = -3, Var(Y) = 12 + 4Var(x) = 16$$

Thus,We have:

$$E(XY) = 2E(X^2) = 2$$

$$Corr(X,Y) = \frac{2}{4} = \frac{1}{2}$$