深蓝学院
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第二次作业思路讲解

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熟悉Eigen矩阵运算

QR分解：对矩阵列向量组α进行施密特正交化，得标准正交向量组β。

$$\begin{bmatrix}\beta_1 & \beta_2 & \cdots & \beta_n\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix}\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ 0 & k_{22} & \cdots & k_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & k_{nn}\end{bmatrix}$$

$\downarrow$ Q(正交矩阵)

$\downarrow$ 此矩阵的逆是 R(上三角矩阵)

Cholesky分解：分解为 $A = LL^T$

建议阅读：

https://zhuanlan.zhihu.com/p/84415000

https://zhuanlan.zhihu.com/p/84771043

旋转的表达

故 $|R|=1$

3. $q_1=\eta_1+x_1i+y_1j+z_1k$ $\quad q_2=\eta_2+x_2i+y_2j+z_2k$

$$q_1q_2=\eta_1\eta_2-(x_1x_2+y_1y_2+z_1z_2)$$
$$+(x_1\eta_2+\eta_1x_2+y_1z_2-y_2z_1)i$$
$$+(y_1\eta_2+\eta_1y_2+z_1x_2-x_1z_2)j$$
$$+(z_1\eta_2+\eta_1z_2+x_1y_2-y_1x_2)k$$

Graßmann积

令 $\varepsilon_1=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$, $\varepsilon_2=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$

$$q_1q_2=\begin{bmatrix}\eta_2\varepsilon_1+\eta_1\varepsilon_2+\varepsilon_1{}^\wedge\varepsilon_2, & \eta_1\eta_2-\varepsilon_1^T\cdot\varepsilon_2\end{bmatrix}$$

$$q_1^+q_2=\begin{bmatrix}\eta_1I+\varepsilon_1{}^\wedge & \varepsilon_1\\ -\varepsilon_1^T & \eta_1\end{bmatrix}\begin{bmatrix}\varepsilon_2\\ \eta_2\end{bmatrix}=\begin{bmatrix}\eta_1\varepsilon_2+\varepsilon_1{}^\wedge\varepsilon_2+\varepsilon_1\eta_2\\ -\varepsilon_1^T\varepsilon_2+\eta_1\eta_2\end{bmatrix}=q_1q_2$$

$$q_2^\oplus q_1=\begin{bmatrix}\eta_2I-\varepsilon_2{}^\wedge & \varepsilon_2\\ -\varepsilon_2^T & \eta_2\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \eta_1\end{bmatrix}=\begin{bmatrix}\eta_2\varepsilon_1-\varepsilon_2{}^\wedge\varepsilon_1+\eta_1\varepsilon_2\\ -\varepsilon_2^T\varepsilon_1+\eta_1\eta_2\end{bmatrix}$$

显然 $\varepsilon_1^T\varepsilon_2=\varepsilon_2^T\varepsilon_1$

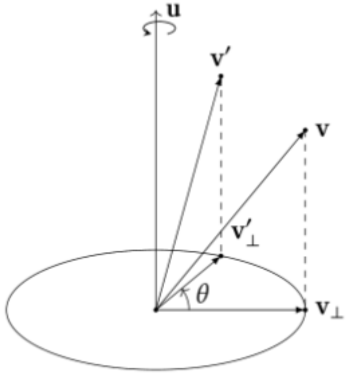
$$\varepsilon_1{}^\wedge\varepsilon_2=\begin{bmatrix}0&-z_1&y_1\\z_1&0&-x_1\\-y_1&x_1&0\end{bmatrix}\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}=\begin{bmatrix}y_1z_2-y_2z_1\\x_2z_1-x_1z_2\\x_1y_2-x_2y_1\end{bmatrix}$$
$$=-\varepsilon_2{}^\wedge\varepsilon_1$$

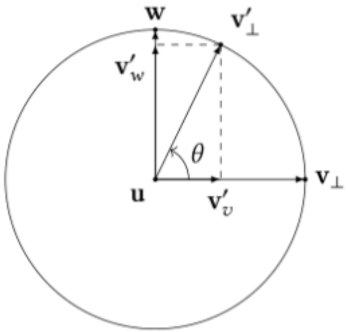
5 罗德里格斯公式的证明（2 分，约 1 小时）

罗德里格斯公式的证明

Introduce to quaternion中P11-16有详细的推导。

网盘链接：https://pan.baidu.com/s/1--QYjF2LL8cZNeLfkEo0Gw
提取码：jqm7





四元数运算性质的验证

$$q^{+} = \begin{bmatrix} \eta\mathbf{1} + \varepsilon^{\times} & \varepsilon \\ -\varepsilon^{\mathrm{T}} & \eta \end{bmatrix}, \quad q^{\oplus} = \begin{bmatrix} \eta\mathbf{1} - \varepsilon^{\times} & \varepsilon \\ -\varepsilon^{\mathrm{T}} & \eta \end{bmatrix},$$

相同，即取 ε 的反对称矩阵（它们都成叉积

四元数 q_1, q_2，四元数乘法可写成矩阵乘法

$$q_1 q_2 = q_1^{+} q_2$$

$$q_1 q_2 = q_2^{\oplus} q_1.$$

由提示 $p' = q^{+} q^{-1^{\oplus}} p$. $q^{-1} = \dfrac{q^{*}}{\|q\|^2}$. $q^{*} = [\eta, -\varepsilon]$

$$q^{+} = \begin{bmatrix} \eta I + \varepsilon^{\wedge} & \varepsilon \\ -\varepsilon^{T} & \eta \end{bmatrix}, \quad q^{\oplus} = \begin{bmatrix} \eta I - \varepsilon^{\wedge} & \varepsilon \\ -\varepsilon^{T} & \eta \end{bmatrix}$$

$$q^{+} q^{-1^{\oplus}} = \frac{1}{\|q\|^2} \begin{bmatrix} \eta I + \varepsilon^{\wedge} & \varepsilon \\ -\varepsilon^{T} & \eta \end{bmatrix} \begin{bmatrix} \eta I + \varepsilon^{\wedge} & -\varepsilon \\ \varepsilon^{T} & \eta \end{bmatrix} = \frac{1}{\|q\|^2} \begin{bmatrix} \eta^2 + 2\eta\varepsilon^{\wedge} + (\varepsilon^{\wedge})^2 + \varepsilon\varepsilon^{T} & 0 \\ 0^{T} & \varepsilon^{T}\varepsilon + \eta^2 \end{bmatrix}$$

故 $q^{+} q^{-1^{\oplus}} p = \dfrac{1}{\|q\|^2} \begin{bmatrix} \eta^2 + 2\eta\varepsilon^{\wedge} + (\varepsilon^{\wedge})^2 + \varepsilon\varepsilon^{T} & 0 \\ 0^{T} & \varepsilon^{T}\varepsilon + \eta^2 \end{bmatrix} \begin{bmatrix} \varepsilon_p \\ 0 \end{bmatrix}$ 的结果仍是一个虚四元数

$$R = \mathrm{Im}(q^{+} q^{-1^{\oplus}}) = [\eta^2 + 2\eta\varepsilon^{\wedge} + (\varepsilon^{\wedge})^2 + \varepsilon\varepsilon^{T}]$$

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感谢各位聆听！

Thanks for Listening