Homework 3: Phys 7310 (Fall 2021)

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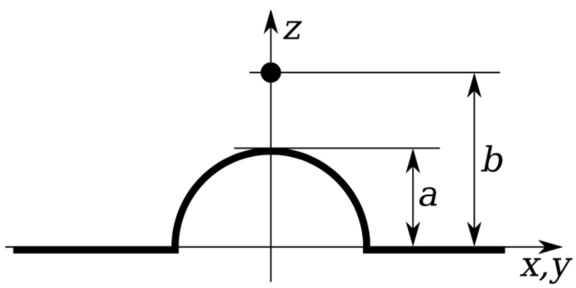
***Problem* 3.1** (A plate with a half-sphere)**:**
An infinite grounded conducting plate has a bulge in form of a haf-sphere with radius a. A



point charge q is put on the symmetry axis of the system at a distance $b > a$ from the center of the half-sphere (see figure). Using the method of image, calculate the potential Φ above the plate, the force on the point charge q, and the total charge induced on the half-sphere.

Solution.

This conductor has both the geometry of (i) the infinite plane conductor and (ii) the spherical conductor, already solved in Jackson and in class. The boundary condition requires that (I) the potential at the infinite plane ($x^2 + y^2 \geq a^2$) is zero and (II) the potential on the hemisphere ($z = \sqrt{a^2 - x^2 - y^2}$) is zero.

In (i), there needs an image charge $-q$ placed at $z = -b$ to satisfy (I). In (ii), an image charge $q' = -(a/b)q$ is placed at $z' = a^2/b$ to satisfy (II). However, now that there is a second image charge, (I) is no longer satisfied. As a remedy, there needs a third image charge $-q' = (a/b)q$ at $z = -z'$ so that (I) is satisfied. Note that by symmetry, (II) is also satisfied with this third image charge (imagine a situation where $-q$ at $z = -b$ is the real charge and we have to place a charge $q'' = -(a/b)(-q)$ at $z'' = -a^2/b$ for the potential at $|\mathbf{x}| = a$ to be zero. This applies for both hemispheres, so the existence of the third image charge does not affect the potential at the bulge).

Now, we can write the total potential as

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\Bigg\{\left[x^2+y^2+(z-b)^2\right]^{-1/2} - \left[x^2+y^2+(z+b)^2\right]^{-1/2} \\ + \frac{a}{b}\left[\left[x^2+y^2+\left(z+a^2/b\right)^2\right]^{-1/2} - \left[x^2+y^2+\left(z-a^2/b\right)^2\right]^{-1/2}\right]\Bigg\} \tag{3.1.1}$$

To verify that this potential is zero at the conductor, we consider two cases:

Case 1: $z = 0$ (on the plane conductor)

$$\begin{aligned}\Phi &\sim \left[x^2+y^2+b^2\right]^{-1/2} - \left[x^2+y^2+b^2\right]^{-1/2} \\ &\quad + \frac{a}{b}\left[x^2+y^2+a^4/b^2\right]^{-1/2} - \frac{a}{b}\left[x^2+y^2+a^4/b^2\right]^{-1/2} \\ &= 0\end{aligned} \tag{3.1.2}$$

Case 2: $z = \sqrt{a^2-x^2-y^2}$ (on the spherical bulge)

$$\begin{aligned}\Phi &\sim \left[a^2-z^2+(z-b)^2\right]^{-1/2} - \left[a^2-z^2+(z+b)^2\right]^{-1/2} \\ &\quad + \frac{a}{b}\left\{\left[a^2-z^2+(z+a^2/b)^2\right]^{-1/2} - \left[a^2-z^2+(z-a^2/b)^2\right]^{-1/2}\right\} \\ &= \left[a^2-2zb+b^2\right]^{-1/2} - \left[a^2+2zb+b^2\right]^{-1/2} \\ &\quad \left(\frac{b^2}{a^2}\right)^{-1/2}\left\{\left[a^2+2(a^2/b)z+a^4/b^2\right]^{-1/2} - \left[a^2-2(a^2/b)z+a^4/b^2\right]^{-1/2}\right\} \\ &= \left[a^2-2zb+b^2\right]^{-1/2} - \left[a^2+2zb+b^2\right]^{-1/2} + \left[b^2+2zb+a^2\right]^{-1/2} - \left[b^2-2zb+a^2\right]^{-1/2} \\ &= 0\end{aligned} \tag{3.1.3}$$

The electric field generated by the three image charges at $z = b$ is

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left\{\frac{a}{b}\left[\frac{1}{(b+a^2/b)^2} - \frac{1}{(b-a^2/b)^2}\right] - \frac{1}{4b^2}\right\}\hat{\mathbf{z}} \\ &= -\frac{q}{4\pi\epsilon_0}\left[\frac{4a^3b^3}{(b^4-a^4)^2} + \frac{1}{4b^2}\right]\hat{\mathbf{z}}\end{aligned} \tag{3.1.4}$$

Thus, the force exerted on the real charge q is

$$\mathbf{F} = q\mathbf{E} = -\frac{q^2}{4\pi\epsilon_0}\left[\frac{4a^3b^3}{(b^4-a^4)^2} + \frac{1}{4b^2}\right]\hat{\mathbf{z}} \tag{3.1.5}$$

Note that we can rewrite (3.1.1) in spherical coordinates

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}-b\hat{\mathbf{z}}|}-\frac{1}{|\mathbf{x}+b\hat{\mathbf{z}}|}+\frac{a}{b}\left(\frac{1}{\left|\mathbf{x}+(a^2/b)\hat{\mathbf{z}}\right|}-\frac{1}{\left|\mathbf{x}-(a^2/b)\hat{\mathbf{z}}\right|}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\Bigg\{\left(r^2+b^2-2br\cos\theta\right)^{-1/2}-(r^2+b^2+2br\cos\theta)^{-1/2} \\
&\qquad +\frac{a}{b}\left[(r^2+a^4/b^2+2(a^2/b)r\cos\theta)^{-1/2}-(r^2+a^4/b^2-2(a^2/b)r\cos\theta)^{-1/2}\right]\Bigg\}
\end{aligned}
\tag{3.1.6}
$$

where $|\mathbf{x}|=r$. With the help of Mathematica, we can find the directional derivative of Φ in the direction normal to the spherical bulge ($\partial/\partial n=\partial/\partial r$)

$$
\begin{aligned}
\frac{\partial\Phi}{\partial r} &= \frac{q}{4\pi\epsilon_0}\Bigg\{\frac{b\cos\theta-r}{(r^2+b^2-2br\cos\theta)^{3/2}}+\frac{b\cos\theta+r}{(r^2+b^2+2br\cos\theta)^{3/2}} \\
&\qquad +\frac{(b^2/a^2)r-b\cos\theta}{(a^2+(b^2/a^2)r^2-2br\cos\theta)^{3/2}}-\frac{(b^2/a^2)r+b\cos\theta}{(a^2+(b^2/a^2)r^2+2br\cos\theta)^{3/2}}\Bigg\}
\end{aligned}
\tag{3.1.7}
$$

The surface charge density at $r=a$ is thus

$$
\sigma=-\epsilon_0\left.\frac{\partial\Phi}{\partial r}\right|_{r=a}=-\frac{q}{4\pi}\frac{b^2-a^2}{a}\left[\frac{1}{(a^2+b^2-2ab\cos\theta)^{3/2}}-\frac{1}{(a^2+b^2+2ab\cos\theta)^{3/2}}\right] \tag{3.1.8}
$$

Then we can calculate the total induced charge on the bulge (where $da=a^2\sin\theta d\theta d\phi$)

$$
\begin{aligned}
Q_{\text{induced}} &= \int_{\text{bulge}}\sigma da \\
&= -\frac{q}{2}a(b^2-a^2)\int_0^{\pi/2}\left[\frac{1}{(a^2+b^2-2ab\cos\theta)}^{3/2}-\frac{1}{(a^2+b^2+2ab\cos\theta)}^{3/2}\right]\sin\theta d\theta \\
&= -\frac{q}{4}\frac{b^2-a^2}{b}\left[\int_{(a-b)^2}^{a^2+b^2}\frac{du_-}{u_-^{3/2}}+\int_{(a+b)^2}^{a^2+b^2}\frac{du_+}{u_+^{3/2}}\right] \qquad (u_\pm=a^2+b^2\pm 2ab\cos\theta) \\
&= \frac{q}{2}\frac{b^2-a^2}{b}\left[\frac{2}{\sqrt{a^2+b^2}}-\frac{1}{b-a}-\frac{1}{a+b}\right] \\
&= \frac{b^2-a^2}{b\sqrt{a^2+b^2}}q-q
\end{aligned}
\tag{3.1.9}
$$

□

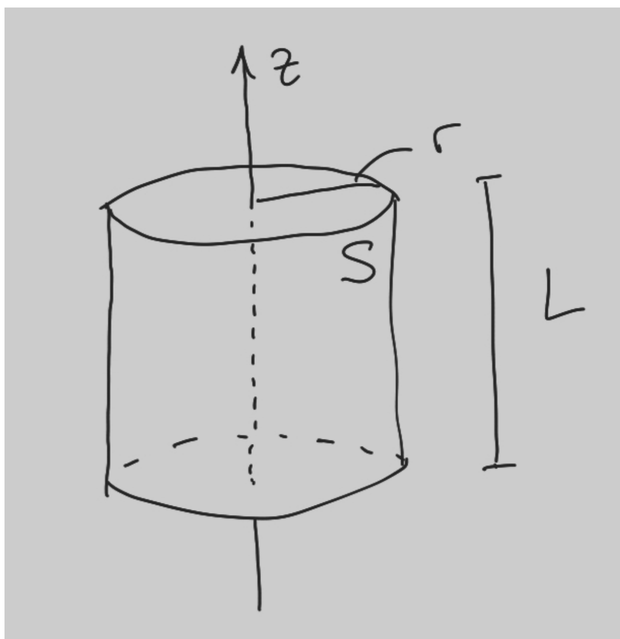
***Problem* 3.2** (A line charge)**:** In this problem, we consider an infinite line with a constant linear charge density λ (charge per unit length). Let the line charge sit at the origin $(x,y)=(0,0)$ in the xy-plane, and stretch infinitely in both directions along the z-axis.

Draw an appropriate Gaussian surface and find the electric field a distance r away from the line charge. Show that a potential for this electric field takes the form

$$\Phi(\mathbf{x}) = \alpha \ln \frac{r}{R} \tag{3.2.1}$$

where α is a constant you should determine in terms of λ and other things, and R is a *totally arbitrary* constant with units of length that we include to make the units in the log dimensionless; explain why R drops out of physically measurable quantities. Finally, if there is another line charge with linear charge density λ' a distance r away from the first charge, find the force per unit length experienced by this second line charge.

Solution.



Draw a cylindrical Gaussian surface S enclosing the line charge along a length L as above. By Gauss Law,

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} da = E 2\pi r L = \frac{\lambda L}{\epsilon_0} \tag{3.2.2}$$

Then by symmetry, we can write

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{r} \hat{\mathbf{r}} \tag{3.2.3}$$

By definition, the potential difference between $r = R$ and $r = r$ is then

$$\Phi(r) - \Phi(R) = -\int_R^r \mathbf{E} \cdot d\mathbf{l} = -\int_R^r E dr = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{r}{R} \tag{3.2.4}$$

So $\alpha = -\lambda/2\pi\epsilon_0$. By choosing $\Phi(R) = 0$, we arrive at an expression for the potential $\Phi(r)$. The choice of R is thus completely arbitrary. Also, we can only measure the potential difference, so R drops out of physically measurable quantities. Now, the line charge λ exerts a force on a charge $q' = \lambda' L$

$$\mathbf{F} = q'\mathbf{E} = -\frac{\lambda\lambda'}{2\pi\epsilon_0}\frac{L}{r}\hat{\mathbf{r}} \tag{3.2.5}$$

So the force per unit length is

$$\mathbf{f} = \frac{\mathbf{F}}{L} = -\frac{\lambda\lambda'}{2\pi\epsilon_0}\frac{1}{r}\hat{\mathbf{r}} \tag{3.2.6}$$

□

***Problem* 3.3** (Line charges and images)**:** A straight line charge with constant linear charge density λ is located perpendicular to the xy-plane in the first quadrant at (x_0, y_0). The intersecting planes $x = 0$, $y \geq 0$, and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x, y) = (\lambda/4\pi\epsilon_0)\ln\big(R^2/r^2\big)$ where $r^2 = (x-x_0)^2+(y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.

(b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1), (x_0 = 1, y_0 = 1), (x_0 = 1, y_0 = 2)$.

(c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi}\lambda\tan^{-1}\left(\frac{x_0}{y_0}\right) \tag{3.3.1}$$

What is the total charge on the plane $x = 0$?

(d) Show that far from the origin $[\rho \gg \rho_0$, where $\rho = \sqrt{x^2+y^2}$ and $\rho_0 = \sqrt{x_0^2+y_0^2}]$ the leading term in the potential is

$$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0y_0)(xy)}{\rho^4} \tag{3.3.2}$$

Interpret.

Solution.

(a) Place an "image" line charge λ_1 at $(x_0, -y_0)$, λ_2 at $(-x_0, -y_0)$, and λ_3 at $(-x_0, y_0)$. The total potential is then

$$\Phi(x, y) = -\frac{1}{4\pi\epsilon_0}\left[\lambda\ln\frac{(x-x_0)^2+(y-y_0)^2}{R^2} + \lambda_1\ln\frac{(x-x_0)^2+(y+y_0)^2}{R^2}\right.$$
$$\left. + \lambda_2\ln\frac{(x+x_0)^2+(y+y_0)^2}{R^2} + \lambda_3\ln\frac{(x+x_0)^2+(y-y_0)^2}{R^2}\right] \tag{3.3.3}$$

Evaluating at $x = 0$, we notice that if $\lambda_3 = -\lambda$, and $\lambda_1 = -\lambda_2$, then $\Phi(0, y) = 0$. Similarly, at $y = 0$, if $\lambda_1 = -\lambda$, then $\lambda_2 = \lambda$ and $\Phi(x, 0) = 0$. Then we can write

$$\Phi(x,y) = -\frac{\lambda}{4\pi\epsilon_0}\left[\ln\frac{(x-x_0)^2+(y-y_0)^2}{R^2} - \ln\frac{(x-x_0)^2+(y+y_0)^2}{R^2}\right.$$
$$\left. + \ln\frac{(x+x_0)^2+(y+y_0)^2}{R^2} - \ln\frac{(x+x_0)^2+(y-y_0)^2}{R^2}\right] \tag{3.3.4}$$

By definition, the electric field is $\mathbf{E}(x, y) = -\boldsymbol{\nabla}\Phi = -(\partial\Phi/\partial x)\hat{\mathbf{x}} - (\partial\Phi/\partial y)\hat{\mathbf{y}}$, where

$$E_x = \frac{\partial\Phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{x-x_0}{(x-x_0)^2+(y-y_0^2)} - \frac{x+x_0}{(x+x_0)^2+(y-y_0)^2}\right.$$
$$\left. - \frac{x-x_0}{(x-x_0)^2+(y+y_0)^2} + \frac{x+x_0}{(x+x_0)^2+(y+y_0)^2}\right] \tag{3.3.5a}$$

$$E_y = \frac{\partial\Phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{y-y_0}{(x-x_0)^2+(y-y_0^2)} - \frac{y-y_0}{(x+x_0)^2+(y-y_0)^2}\right.$$
$$\left. - \frac{y+y_0}{(x-x_0)^2+(y+y_0)^2} + \frac{y+y_0}{(x+x_0)^2+(y+y_0)^2}\right] \tag{3.3.5b}$$

where we have used Mathematica to skip the algebra. At $x = 0$, the tangent electric field is E_y,

$$E_y(0,y) \sim \frac{y-y_0}{x_0^2+(y-y_0)^2} - \frac{y-y_0}{x_0^2+(y-y_0)^2} - \frac{y+y_0}{x_0^2+(y+y_0)^2} + \frac{y+y_0}{x_0^2+(y+y_0)^2} = 0 \tag{3.3.6}$$

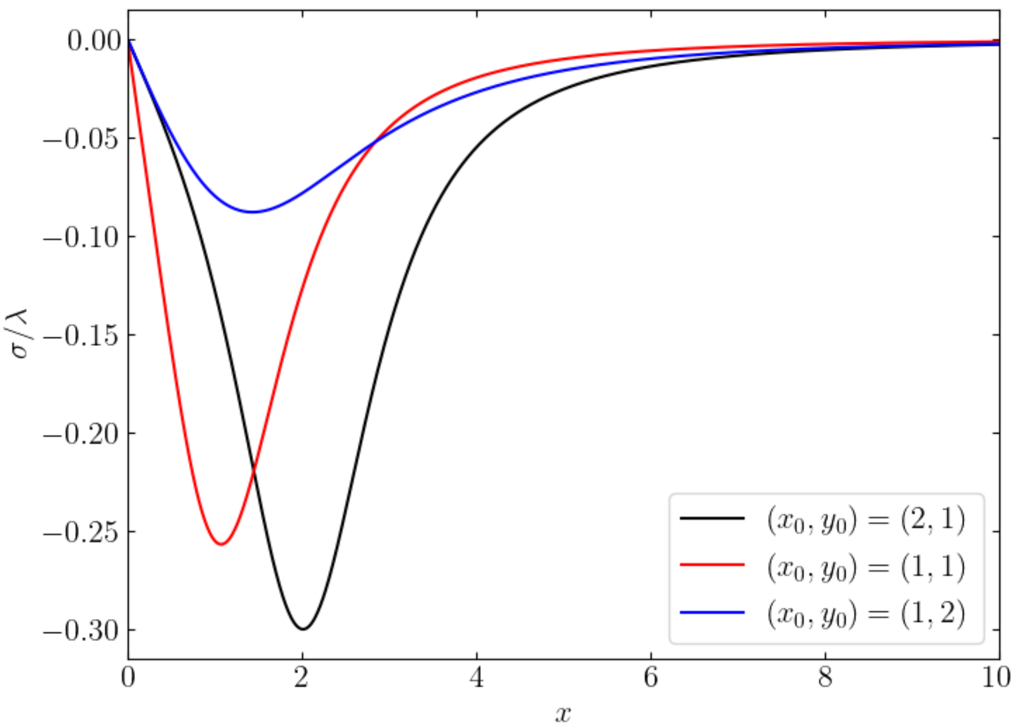
At $y = 0$, the tangent electric field is E_x,

$$E_x(x,0) \sim \frac{x-x_0}{(x-x_0)^2+y_0^2} - \frac{x+x_0}{(x+x_0)^2+y_0^2} - \frac{x-x_0}{(x-x_0)^2+y_0^2} + \frac{x+x_0}{(x+x_0)^2+y_0^2} = 0 \tag{3.3.7}$$

(b) Since $\sigma = -\epsilon_0\partial\Phi/\partial n = \epsilon_0 E_\perp$ near a conductor, we can write for the $y = 0$ conductor

$$\sigma = \epsilon_0\, E_y\Big|_{y=0} = \frac{\lambda}{\pi}\left[\frac{y_0}{(x+x_0)^2+y_0^2} - \frac{y_0}{(x-x_0)^2+y_0^2}\right] \tag{3.3.8}$$

The surface charge density for three cases of $(x_0, y_0) = (2, 1)$, $(x_0, y_0) = (1, 1)$, and $(x_0, y_0) = (1, 2)$ are plotted in black, red, and blue, respectively, below.



(c) At the plane $y = 0$, write an area element as $da = dxdz$. Thus, from part (b), the total charge per unit length in z is

$$\begin{aligned} Q_x &= \int_0^\infty \sigma dx \\ &= \frac{\lambda}{\pi}\left[\tan^{-1}\left(\frac{x+x_0}{y_0}\right) - \tan^{-1}\left(\frac{x-x_0}{y_0}\right)\right]\Bigg|_0^\infty \\ &= \frac{\lambda}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x_0}{y_0}\right) - \frac{\pi}{2} - \tan^{-1}\left(\frac{x_0}{y_0}\right)\right] \\ &= -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right) \end{aligned} \tag{3.3.9}$$

(d) From (3.3.4), we can perform a transformation to polar coordinate $(x, y) = \rho(\cos\theta, \sin\theta)$

and $(x_0, y_0) = \rho_0(\cos\theta_0, \sin\theta_0)$ and write

$$
\begin{aligned}
\Phi &= -\frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)}{\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta+\theta_0)}\right)+\ln\left(\frac{\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta-\theta_0)}{\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta+\theta_0)}\right)\right] \\
&= -\frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{1+\epsilon^2-2\epsilon\cos(\theta-\theta_0)}{1+\epsilon^2-2\epsilon\cos(\theta+\theta_0)}\right)+\ln\left(\frac{1+\epsilon^2+2\epsilon\cos(\theta-\theta_0)}{1+\epsilon^2+2\epsilon\cos(\theta+\theta_0)}\right)\right] \qquad (\epsilon\equiv\rho_0/\rho) \\
&\approx -\frac{\lambda}{4\pi\epsilon_0}\left[-4\sin(2\theta)\sin(2\theta_0)\epsilon^2+\mathcal{O}\big(\epsilon^3\big)\right] \\
&\approx \frac{4\lambda}{\pi\epsilon_0}\frac{(\rho\cos\theta)(\rho\sin\theta)(\rho_0\cos\theta_0)(\rho_0\sin\theta_0)}{\rho^4} \\
&= \frac{4\lambda}{\pi\epsilon_0}\frac{(xy)(x_0y_0)}{\rho^4}
\end{aligned}
\tag{3.3.10}
$$

where we have assumed $\rho \gg \rho_0$ in the third equality. This 2D potential is quadrupole, which is expected because there are two dipoles (formed by the 3 image charges and the real charge). □

***Problem* 3.4** (Green's function in Cartesian coordinates)**:** (a) Show that the Green function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \le x \le 1, 0 \le y \le 1$, has an expansion

$$G(x, y; x', y') = 2\sum_{n=1}^{\infty} g_n(y, y')\sin(n\pi x)\sin\big(n\pi x'\big) \tag{3.4.1}$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2}-n^2\pi^2\right)g_n(y, y') = -4\pi\delta(y'-y) \qquad \text{and} \qquad g_n(y,0)=g_n(y,1)=0 \tag{3.4.2}$$

(b) Taking for $g_n(y, y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x', y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin\big(n\pi x'\big)\sinh(n\pi y_<)\sinh\big[n\pi(1-y_>)\big] \tag{3.4.3}$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.

Solution.

(a) First, we expand $G(x, y; x', y')$ in a Fourier series with the orthonormal basis $\mathcal{B} = \left\{\sqrt{2}\sin(n\pi x)\right\}_{n=1}^{\infty}$ on the domain $x \in [0, 1]$

$$G(x, y; x', y') = \sum_{n,n'\in\mathbb{N}} 2g_{n,n'}(y, y')\sin(n\pi x)\sin\big(n'\pi x'\big) \tag{3.4.4}$$

Taking the Laplacian $\nabla^{2\prime} G = (\partial^2/\partial x'^2 + \partial^2/\partial y'^2)G$, we get from the property of Green's functions

$$\begin{aligned}\sum_{m,m'\in\mathbb{N}} &2\sin(m\pi x)\sin\!\left(m'\pi x'\right)\left(\frac{\partial^2}{\partial y'^2} - m'^2\pi^2\right)g_{m,m'}(y,y') \\ &= -4\pi\delta(x-x')\delta(y-y') \\ &= -4\pi\delta(y-y')\sum_{m\in\mathbb{N}} 2\sin(m\pi x)\sin\!\left(m\pi x'\right)\end{aligned} \tag{3.4.5}$$

where the last equality comes from completeness. Now, note that for our choice of basis $\mathcal{B}$,

$$\left\langle \sqrt{2}\sin(m\pi x)\middle|\sqrt{2}\sin(n\pi x)\right\rangle = \int_0^1 2\sin(m\pi x)\sin(n\pi x)dx = \delta_{m,n} \tag{3.4.6}$$

Thus, taking the inner product of both sides of (3.4.5) with $\sin(n\pi x)$, we get

$$\left\langle \text{LHS}\middle|\sqrt{2}\sin(n\pi x)\right\rangle = \sum_{m'\in\mathbb{N}} \sqrt{2}\sin\!\left(m'\pi x'\right)\left(\frac{\partial^2}{\partial y'^2} - m'^2\pi^2\right)g_{n,m'}(y,y') \tag{3.4.7}$$

and

$$\left\langle \text{RHS}\middle|\sqrt{2}\sin(n\pi x)\right\rangle = -4\pi\delta(y-y')\sqrt{2}\sin\!\left(n\pi x'\right) \tag{3.4.8}$$

Taking the inner product of both sides (3.4.7) and (3.4.8) again with $\sqrt{2}\sin(n\pi x')$, we get

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right)g_n(y,y') = -4\pi\delta(y-y') \tag{3.4.9}$$

Thus, g_n has to satisfy (3.4.9). Then the Green function can be written as

$$G(x,y;x',y') = 2\sum_{n\in\mathbb{N}} g_n(y,y')\sin(n\pi x)\sin\!\left(n\pi x'\right) \tag{3.4.10}$$

This obviously vanishes at $x' = 0$ and $x' = 1$ because $\sin(0) = \sin(n\pi) = 0$ for $n \in \mathbb{N}$. But we also have to require that $g_n(y,0) = g_n(y,1) = 0$ so that G vanishes at $y' = 0$ and $y' = 1$.

(b) First, we write

$$g_n(y,y') = \begin{cases} A_n\sinh(n\pi y') + B_n\cosh(n\pi y') & y' < y \\ C_n\sinh(n\pi y') + D_n\cosh(n\pi y') & y' > y \end{cases} \tag{3.4.11}$$

For $y \in (0,1)$, $g_n(y,0) = B_n$. Thus we must require that $B_n = 0$ due to boundary conditions. Similarly, $g_n(y,1) = C_n\sinh(n\pi) + D_n\cosh(n\pi) = 0$. Thus we can write

$$g_n(y,y') = \begin{cases} A_n\sinh(n\pi y') & y' < y \\ C_n\left[\sinh(n\pi y') - \tanh(n\pi)\cosh(n\pi y')\right] & y' > y \end{cases} \tag{3.4.12}$$

By continuity at $y' = y$, we must require that $\lim_{y' \to y^-} g_n = \lim_{y' \to y^+} g_n$, which leads to

$$A_n = C_n \left[1 - \frac{\tanh(n\pi)}{\tanh(n\pi y)} \right] \tag{3.4.13}$$

Now, from the condition (3.4.9), we can integrate both sides with respect to y' in a small region $(y - \epsilon, y + \epsilon)$ around y and get

$$\left. \frac{\partial g_n}{\partial y'} \right|_{y-\epsilon}^{y+\epsilon} = n^2 \pi^2 \int_{y-\epsilon}^{y+\epsilon} g_n(y, y') dy' - 4\pi \tag{3.4.14}$$

The integral $I = \int_{y-\epsilon}^{y+\epsilon} g_n dy'$ in the RHS is

$$\begin{aligned} I &= A_n \int_{y-\epsilon}^{y} \sinh\big(n\pi y'\big) dy' + C_n \int_{y}^{y+\epsilon} \Big[\sinh\big(n\pi y'\big) - \tanh(n\pi) \cosh\big(n\pi y'\big) \Big] dy' \\ &= \frac{A_n}{n\pi} \big[\cosh(n\pi y) - \cosh[n\pi(y - \epsilon)] \big] \\ &\quad + \frac{C_n \tanh(n\pi)}{n\pi} \big[\cosh[n\pi(y + \epsilon)] - \cosh(n\pi y) + \sinh(n\pi y) - \sinh[n\pi(y + \epsilon)] \big] \end{aligned} \tag{3.4.15}$$

As $\epsilon \to 0$, $I \to 0$. So the jump condition is

$$\begin{aligned} -4\pi = \left. \frac{\partial g_n}{\partial y'} \right|_{y'=y+\epsilon} - \left. \frac{\partial g_n}{\partial y'} \right|_{y'=y-\epsilon} &= C_n n\pi \big[\cosh[n\pi(y + \epsilon)] - \tanh(n\pi) \sinh[n\pi(y + \epsilon)] \big] \\ &\quad - A_n n\pi \cosh[n\pi(y - \epsilon)] \end{aligned} \tag{3.4.16}$$

As $\epsilon \to 0$, we can use (3.4.13) and write from the jump condition

$$C_n = -\frac{4}{n} \frac{\sinh(n\pi y) \cosh(n\pi)}{\sinh(n\pi)} \tag{3.4.17}$$

It also follows that

$$A_n = -\frac{4}{n} \frac{\sinh(n\pi y)}{\tanh(n\pi)} \left[1 - \frac{\tanh(n\pi)}{\tanh(n\pi y)} \right] = -\frac{4}{n} \frac{\sinh[n\pi(y - 1)]}{\sinh(n\pi)} \tag{3.4.18}$$

Plugging these back into (3.4.12), we get

$$\begin{aligned} g_n &= \begin{cases} \frac{4}{n \sinh(n\pi)} \sinh(n\pi y') \sinh[n\pi(1 - y)] & y' < y \\ \frac{4}{n \sinh(n\pi)} \sinh(n\pi y) \sinh[n\pi(1 - y')] & y' > y \end{cases} \\ &= \frac{4}{n \sinh(n\pi)} \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)] \end{aligned} \tag{3.4.19}$$

where $y_< = \min(y, y')$ and $y_> = \max(y, y')$. Then the Green function (3.4.10) can be written as

$$G(x, y; x', y') = 8 \sum_{n \in \mathbb{N}} \frac{1}{n \sinh(n\pi)} \sinh(n\pi y_<) \sinh[n\pi(1 - y_>)] \sin(n\pi x) \sin\big(n\pi x'\big) \tag{3.4.20}$$

as desired. □