Homework 12: Phys 7310 (Fall 2021)

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December 3, 2021

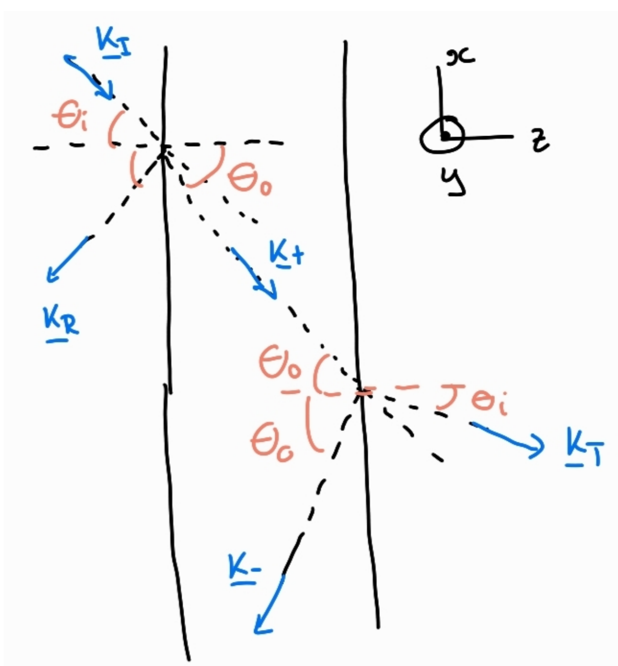
***Problem* 12.1** (Waves crossing a gap)**:** Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap ($n = 1$) of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i. For linear polarization *both* parallel to *and* perpendicular to the plane of incidence,

(a) calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power;

(b) for i greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap (only do perpendicular polarization).

Solution.

(a)



Given the geometry of the problem as sketched above, the incident I, reflected R, intermediate ($\pm$ in the gap), and transmitted T wavevectors can be written as

$$\mathbf{k}_I = k_1(-\sin\theta_i\hat{\mathbf{x}} + \cos\theta_i\hat{\mathbf{z}}) \tag{12.1.1a}$$
$$\mathbf{k}_R = k_1(-\sin\theta_i\hat{\mathbf{x}} - \cos\theta_i\hat{\mathbf{z}}) \tag{12.1.1b}$$
$$\mathbf{k}_+ = k_2(-\sin\theta_o\hat{\mathbf{x}} + \cos\theta_o\hat{\mathbf{z}}) \tag{12.1.1c}$$
$$\mathbf{k}_- = k_2(-\sin\theta_o\hat{\mathbf{x}} - \cos\theta_o\hat{\mathbf{z}}) \tag{12.1.1d}$$
$$\mathbf{k}_T = k_3(-\sin\theta_i\hat{\mathbf{x}} + \cos\theta_i\hat{\mathbf{z}}) \tag{12.1.1e}$$
$$\tag{12.1.1f}$$

The boundary conditions applied at the interface between two medium 1 and 2 are

$$(\epsilon_1\mathbf{E}_1 - \epsilon_2\mathbf{E}_2)\cdot\hat{\mathbf{z}} = 0 \tag{12.1.2a}$$
$$(\mathbf{B}_1 - \mathbf{B}_2)\cdot\hat{\mathbf{z}} = 0 \tag{12.1.2b}$$
$$(\mathbf{E}_1 - \mathbf{E}_2)\times\hat{\mathbf{z}} = 0 \tag{12.1.2c}$$
$$(\mathbf{B}_1 - \mathbf{B}_2)\times\hat{\mathbf{z}} = 0 \tag{12.1.2d}$$

Now, we consider two cases

Case 1: Parallel polarization

The electric field polarization is in the (xz) plane. So we can write from Faraday's law the electromagnetic fields for $z < 0$ as

$$\mathbf{E}_I = E_{0I}(\cos\theta_i\hat{\mathbf{x}} + \sin\theta_i\hat{\mathbf{z}})e^{i(\mathbf{k}_I\cdot\mathbf{x}-\omega t)} \tag{12.1.3a}$$
$$\mathbf{B}_I = \sqrt{\mu\epsilon}\,\hat{\mathbf{k}}_I\times\mathbf{E}_I = \sqrt{\mu\epsilon}E_{0I}e^{i(\mathbf{k}_I\cdot\mathbf{x}-\omega t)}\hat{\mathbf{y}} \tag{12.1.3b}$$
$$\mathbf{E}_R = E_{0R}(\cos\theta_i\hat{\mathbf{x}} - \sin\theta_i\hat{\mathbf{z}})e^{i(\mathbf{k}_R\cdot\mathbf{x}-\omega t)} \tag{12.1.3c}$$
$$\mathbf{B}_R = \sqrt{\mu\epsilon}\,\hat{\mathbf{k}}_R\times\mathbf{E}_R = -\sqrt{\mu\epsilon}E_{0R}e^{i(\mathbf{k}_R\cdot\mathbf{x}-\omega t)}\hat{\mathbf{y}} \tag{12.1.3d}$$

Similarly, for $0 \le z \le d$,

$$\mathbf{E}_+ = E_{0+}(\cos\theta_o\hat{\mathbf{x}} + \sin\theta_o\hat{\mathbf{z}})e^{i(\mathbf{k}_+\cdot\mathbf{x}-\omega t)} \tag{12.1.4a}$$
$$\mathbf{B}_+ = \sqrt{\mu_0\epsilon_0}\,\hat{\mathbf{k}}_+\times\mathbf{E}_+ = \sqrt{\mu_0\epsilon_0}E_{0+}e^{i(\mathbf{k}_+\cdot\mathbf{x}-\omega t)}\hat{\mathbf{y}} \tag{12.1.4b}$$
$$\mathbf{E}_- = E_{0-}(\cos\theta_o\hat{\mathbf{x}} - \sin\theta_o\hat{\mathbf{z}})e^{i(\mathbf{k}_-\cdot\mathbf{x}-\omega t)} \tag{12.1.4c}$$
$$\mathbf{B}_- = \sqrt{\mu_0\epsilon_0}\,\hat{\mathbf{k}}_-\times\mathbf{E}_- = -\sqrt{\mu_0\epsilon_0}E_{0-}e^{i(\mathbf{k}_-\cdot\mathbf{x}-\omega t)}\hat{\mathbf{y}} \tag{12.1.4d}$$

and for $z > d$,

$$\mathbf{E}_T = E_{0T}(\cos\theta_i\hat{\mathbf{x}} + \sin\theta_i\hat{\mathbf{z}})e^{i(\mathbf{k}_T\cdot\mathbf{x}-\omega t)} \tag{12.1.5a}$$
$$\mathbf{B}_T = \sqrt{\mu\epsilon}\,\hat{\mathbf{k}}_T\times\mathbf{E}_T = \sqrt{\mu\epsilon}E_{0T}e^{i(\mathbf{k}_T\cdot\mathbf{x}-\omega t)}\hat{\mathbf{y}} \tag{12.1.5b}$$

Applying the boundary conditions at $z = 0$, we first get from (12.1.2a),

$$\epsilon\sin\theta_i(E_{0I} - E_{0R}) = \epsilon_0\sin\theta_0(E_{0+} - E_{0-}) \Rightarrow E_{0+} - E_{0-} = n(E_{0I} - E_{0R}) \tag{12.1.6}$$

and (12.1.2c),

$$\cos\theta_i(E_{0I}+E_{0R}) = \cos\theta_o(E_{0+}+E_{0-}) \Rightarrow E_{0+}+E_{0-} = \frac{\cos\theta_i}{\cos\theta_o}(E_{0I}+E_{0R}) \tag{12.1.7}$$

Similarly, at $z=d$, we attain from (12.1.2a),

$$\begin{aligned} \epsilon_0 \sin\theta_i \Big(E_{0+}e^{ik_2 d\cos\theta_o} - E_{0-}e^{-ik_2 d\cos\theta_o} \Big) &= \epsilon \sin\theta_i E_{0T} e^{ik_3 d\cos\theta_i} \\ \Rightarrow \qquad E_{0+}e^{ik_2 d\cos\theta_o} - E_{0-}e^{-ik_2 d\cos\theta_o} &= nE_{0T}e^{ik_3 d\cos\theta_i} \end{aligned} \tag{12.1.8}$$

and from (12.1.2c),

$$\begin{aligned} \cos\theta_o \Big(E_{0+}e^{ik_2 d\cos\theta_o} + E_{0-}e^{-ik_2 d\cos\theta_o} \Big) &= \cos\theta_i E_{0T} e^{ik_3 d\cos\theta_i} \\ \Rightarrow \qquad E_{0+}e^{ik_2 d\cos\theta_o} + E_{0-}e^{-ik_2 d\cos\theta_o} &= \frac{\cos\theta_i}{\cos\theta_o} E_{0T}e^{ik_3 d\cos\theta_i} \end{aligned} \tag{12.1.9}$$

Solving the system (12.1.6), (12.1.7), (12.1.8), (12.1.9) with Mathematica, we get

$$E_{0+} = E_{0T}\frac{\cos\theta_i + n\cos\theta_o}{2\cos\theta_o} e^{ik_3 d\cos\theta_i} e^{-ik_2 d\cos\theta_o} \tag{12.1.10a}$$

$$E_{0-} = E_{0T}\frac{\cos\theta_i - n\cos\theta_o}{2\cos\theta_o} e^{ik_3 d\cos\theta_i} e^{-ik_2 d\cos\theta_o} \tag{12.1.10b}$$

From (12.1.6) and (12.1.7), we can write

$$\begin{aligned} 2E_{0I} &= \frac{1}{n}(E_{0+}-E_{0-}) + \frac{\cos\theta_o}{\cos\theta_i}(E_{0+}+E_{0-}) \\ &= \frac{\cos\theta_i + n\cos\theta_o}{n\cos\theta_i}E_{0+} - \frac{\cos\theta_i - n\cos\theta_o}{n\cos\theta_i}E_{0-} \\ \Rightarrow \quad 4n\cos\theta_i\cos\theta_o \frac{E_{0I}}{E_{0T}} &= e^{ik_3 d\cos\theta_i}\Big[(\cos\theta_i + n\cos\theta_o)^2 e^{-ik_2 d\cos\theta_o} - (\cos\theta_i - n\cos\theta_o)^2 e^{ik_2 d\cos\theta_o} \Big] \\ \Rightarrow \quad 4n\cos\theta_i\cos\theta_o \left| \frac{E_{0I}}{E_{0T}} \right| &= 4\Big| (\cos^2\theta_i + n^2\cos^2\theta_o)\sin\alpha + 2i\cos\theta_i\cos\theta_o\cos\alpha \Big| \\ & \qquad\qquad (\alpha = k_2 d\cos\theta_o) \\ \Rightarrow \quad n^2\cos^2\theta_i\cos^2\theta_o \frac{1}{T} &= \Big(\cos^2\theta_i + n^2\cos^2\theta_o\Big)^2 \sin^2\alpha + 4\cos^2\theta_i\cos^2\theta_o\cos^2\alpha \end{aligned} \tag{12.1.11}$$

Thus, the transmission coefficient is

$$T = \frac{n^2\cos^2\theta_i\cos^2\theta_o}{(\cos^2\theta_i + n^2\cos^2\theta_o)^2\sin^2\alpha + 4\cos^2\theta_i\cos^2\theta_o\cos^2\alpha} \tag{12.1.12}$$

where $\alpha = (\omega d/c)\cos\theta_o$ and $\cos\theta_o = \sqrt{1-n^2\sin^2\theta_i}$ from Snell's Law. The reflection coefficient is just trivially $R = 1-T$, by energy conservation.

Case 2: Perpendicular polarization

Similar to the first case, we can write the fields for $z < 0$ as

$$\mathbf{E}_I = E_{0I} e^{i(\mathbf{k}_I \cdot \mathbf{x} - \omega t)} \hat{\mathbf{y}} \tag{12.1.13a}$$
$$\mathbf{B}_I = \sqrt{\mu\epsilon} E_{0I} (-\cos\theta_i \hat{\mathbf{x}} - \sin\theta_i \hat{\mathbf{z}}) e^{i(\mathbf{k}_I \cdot \mathbf{x} - \omega t)} \tag{12.1.13b}$$
$$\mathbf{E}_R = E_{0R} e^{i(\mathbf{k}_R \cdot \mathbf{x} - \omega t)} \hat{\mathbf{y}} \tag{12.1.13c}$$
$$\mathbf{B}_R = \sqrt{\mu\epsilon} E_{0R} (\cos\theta_i \hat{\mathbf{x}} - \sin\theta_i \hat{\mathbf{z}}) e^{i(\mathbf{k}_R \cdot \mathbf{x} - \omega t)} \tag{12.1.13d}$$

For $0 \leq z \leq d$,

$$\mathbf{E}_+ = E_{0+} e^{i(\mathbf{k}_+ \cdot \mathbf{x} - \omega t)} \hat{\mathbf{y}} \tag{12.1.14a}$$
$$\mathbf{B}_+ = \sqrt{\mu_0\epsilon_0} E_{0+} (-\cos\theta_o \hat{\mathbf{x}} - \sin\theta_o \hat{\mathbf{z}}) e^{i(\mathbf{k}_+ \cdot \mathbf{x} - \omega t)} \tag{12.1.14b}$$
$$\mathbf{E}_- = E_{0-} e^{i(\mathbf{k}_- \cdot \mathbf{x} - \omega t)} \hat{\mathbf{y}} \tag{12.1.14c}$$
$$\mathbf{B}_- = \sqrt{\mu_0\epsilon_0} E_{0-} (\cos\theta_o \hat{\mathbf{x}} - \sin\theta_o \hat{\mathbf{z}}) e^{i(\mathbf{k}_- \cdot \mathbf{x} - \omega t)} \tag{12.1.14d}$$

For $z > d$,

$$\mathbf{E}_T = E_{0T} e^{i(\mathbf{k}_T \cdot \mathbf{x} - \omega t)} \hat{\mathbf{y}} \tag{12.1.15a}$$
$$\mathbf{B}_T = \sqrt{\mu\epsilon} E_{0T} (-\cos\theta_i \hat{\mathbf{x}} - \sin\theta_i \hat{\mathbf{z}}) e^{i(\mathbf{k}_T \cdot \mathbf{x} - \omega t)} \tag{12.1.15b}$$

Applying the boundary conditions (12.1.2b) at $z = 0$, we get

$$\sqrt{\mu\epsilon} \sin\theta_i (-E_{0I} - E_{0R}) = \sqrt{\mu_0\epsilon_0} \sin\theta_o (-E_{0+} - E_{0-}) \Rightarrow E_{0I} + E_{0R} = E_{0+} + E_{0-} \tag{12.1.16}$$

and at $z = d$, we get

$$\sqrt{\mu_0\epsilon_0} \sin\theta_o (-E_{0+} e^{ik_2 d\cos\theta_o} - E_{0-} e^{-ik_2 d\cos\theta_o}) = \sqrt{\mu\epsilon} \sin\theta_i E_{0T} e^{ik_3 d\cos\theta_i} \tag{12.1.17}$$
$$\Rightarrow \qquad E_{0+} e^{ik_2 d\cos\theta_o} + E_{0-} e^{-ik_2 d\cos\theta_o} = E_{0T} e^{ik_3 d\cos\theta_i} \tag{12.1.18}$$

Applying (12.1.2d) at $z = 0$, we get

$$\sqrt{\mu\epsilon} \cos\theta_i (-E_{0I} + E_{0R}) = \sqrt{\mu_0\epsilon_0} \cos\theta_o (-E_{0+} + E_{0-}) \Rightarrow E_{0+} - E_{0-} = \frac{n\cos\theta_i}{\cos\theta_o} (E_{0I} - E_{0R}) \tag{12.1.19}$$

and at $z = d$, we get

$$\sqrt{\mu_0\epsilon_0} \cos\theta_o \left(-E_{0+} e^{ik_2 d\cos\theta_o} + E_{0-} e^{-ik_2 d\cos\theta_o} \right) = -\sqrt{\mu\epsilon} \cos\theta_i E_{0T} e^{ik_3 d\cos\theta_i}$$
$$\Rightarrow \qquad E_{0+} e^{ik_2 d\cos\theta_o} - E_{0-} e^{-ik_2 d\cos\theta_o} = \frac{n\cos\theta_i}{\cos\theta_o} E_{0T} e^{ik_3 d\cos\theta_i} \tag{12.1.20}$$

Using Mathematica to solve (12.1.16), (12.1.17), (12.1.19), and (12.1.20), we get

$$E_{0+} = \frac{1}{2} E_{0T} \left(1 + \frac{n\cos\theta_i}{\cos\theta_o} \right) e^{ik_3 d\cos\theta_i} e^{-ik_2 d\cos\theta_o} \tag{12.1.21a}$$
$$E_{0-} = \frac{1}{2} E_{0T} \left(1 - \frac{n\cos\theta_i}{\cos\theta_o} \right) e^{ik_3 d\cos\theta_i} e^{ik_2 \cos\theta_o} \tag{12.1.21b}$$

Then we can write from (12.1.16) and (12.1.19) that

$$
\begin{aligned}
2E_{0I} &= \frac{\cos\theta_o + n\cos\theta_i}{n\cos\theta_i}E_{0+} - \frac{\cos\theta_o - n\cos\theta_i}{n\cos\theta_i}E_{0-} \\
\Rightarrow \quad 4n\cos\theta_i\cos\theta_o\frac{E_{0I}}{E_{0T}} &= e^{ik_3 d\cos\theta_i}\left[(\cos\theta_o + n\cos\theta_i)^2 e^{-i\alpha} - (\cos\theta_o - n\cos\theta_i)^2 e^{i\alpha}\right] \\
\Rightarrow \quad 4n\cos\theta_i\cos\theta_o\left|\frac{E_{0I}}{E_{0T}}\right| &= \left|4n\cos\theta_i\cos\theta_o\cos\alpha - 2i(n^2\cos^2\theta_i + \cos^2\theta_o)\sin\alpha\right| \\
\Rightarrow \quad 4n^2\cos^2\theta_i\cos^2\theta_o\frac{1}{T} &= 4n^2\cos^2\theta_i\cos^2\theta_o\cos^2\alpha + (n^2\cos^2\theta_i + \cos^2\theta_o)^2\sin^2\alpha \quad (12.1.22)
\end{aligned}
$$

Then the transmission coefficient is

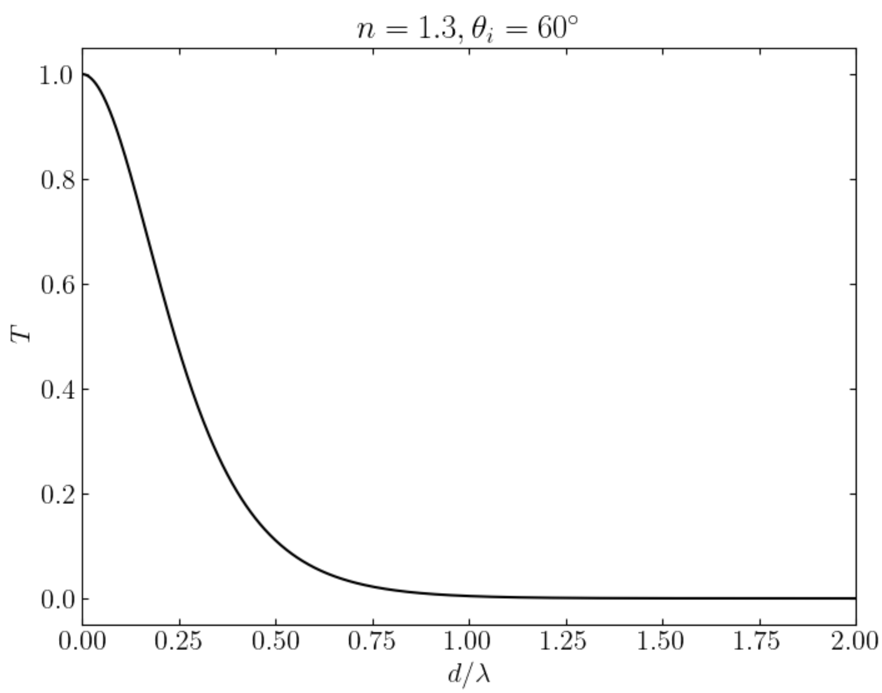
$$
T = \frac{4n^2\cos^2\theta_i\cos^2\theta_o}{4n^2\cos^2\theta_i\cos^2\theta_o\cos^2\alpha + (n^2\cos^2\theta_i + \cos^2\theta_o)^2\sin^2\alpha} \tag{12.1.23}
$$

and the reflection coefficient is simplify $R = 1 - T$, where $\cos\theta_o$ and α retain the same definitions as in the first case.

(b) If $\theta_i \geq \theta_c$ such that $n\sin\theta_c = 1$, then $1 < n^2\sin^2\theta_i$ and $\alpha = (\omega d/c)\sqrt{1 - n^2\sin^2\theta_i} = i(\omega d/c)\sqrt{n^2\sin^2\theta_i - 1} = i\psi$. It also follows that $\cos\alpha = \cosh\psi$ and $\sin\alpha = i\sinh\psi$. From (12.1.23), the transmission coefficient for angles above the critical angle θ_c is

$$
T = \frac{4n^2\cos^2\theta_i\cos^2\theta_o}{4n^2\cos^2\theta_i\cos^2\theta_o\cosh^2\psi - (n^2\cos^2\theta_i + \cos^2\theta_o)^2\sinh^2\psi} \tag{12.1.24}
$$

where $\psi = (\omega d/c)\sqrt{n^2\sin^2\theta_i - 1} = 2\pi\sqrt{n^2\sin^2\theta_i - 1}(d/\lambda)$. In the following, we plot the transmission coefficient for water ($n = 1.3$) and $\theta_i = 60°$



□

***Problem* 12.2** (Wave entering a conductor)**:** A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction $n(\omega)$ $[n^2(\omega) = \epsilon(\omega)/\epsilon_0]$.

(a) Show that the ratio of reflected power to incident power is

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2 \tag{12.2.1}$$

while the ratio of power transmitted into the medium to the incident power is

$$T = \frac{4\,\mathrm{Re}\,n(\omega)}{\left|1+n(\omega)\right|^2} \tag{12.2.2}$$

(b) Evaluate Re $\left[i\omega(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*)/2\right]$ as a function of (x, y, z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T. Compare the quantity mentioned to $T\mathbf{S}_i$ (the fraction of the incident flux that is transmitted) where $\mathbf{S}_i$ is the real, time-averaged Poynting flux $(1/2)\,\mathrm{Re}\,(\mathbf{E}\times\mathbf{H}^*)$.

(c) For a conductor, with $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, σ real, write out the results of parts (a) and (b) in the limit $\epsilon_0\omega \ll \sigma$. Express your answer in terms of $\delta = \sqrt{2/\mu\sigma\omega}$ as much as possible. Calculate $(1/2)\,\mathrm{Re}(\mathbf{J}^*\cdot\mathbf{E})$ and compare with the results of part (b). Do both enter the complex form of Poynting's theorem?

Solution.

(a) The reflection coefficient follows simply from (7.42, Jackson) that

$$R = \left|\frac{E_{0R}}{E_{0I}}\right|^2 = \left|\frac{1-n}{1+n}\right|^2 \tag{12.2.3}$$

since the derivations leading to that equation does not depend on the complexity of n. Now, we write $n = \mathrm{Re}(n) + i\,\mathrm{Im}(n)$ and calculate

$$\begin{aligned}
T &= 1 - R \\
&= 1 - \left(\frac{1-n}{1+n}\right)\left(\frac{1-n}{1+n}\right)^* \\
&= \frac{(1+\mathrm{Re}(n)+i\,\mathrm{Im}(n))(1+\mathrm{Re}(n)-i\,\mathrm{Im}(n)) - (1-\mathrm{Re}(n)-i\,\mathrm{Im}(n))(1-\mathrm{Re}(n)+i\,\mathrm{Im}(n))}{|1+n|^2} \\
&= \frac{4\,\mathrm{Re}(n)}{|1+n|^2}
\end{aligned} \tag{12.2.4}$$

where we have simplified the last step with Mathematica.

(b) First, the electric field of a wave propagating along the z direction is

$$\mathbf{E} = \mathbf{E}_0 e^{i(kz-\omega t)} = \mathbf{E}_0 e^{-\mathrm{Im}(k)z} e^{i(\mathrm{Re}(k)z-\omega t)} \tag{12.2.5}$$

where $\mathbf{E}_0$ is in the (xy) plane. It then folows that

$$|\mathbf{k}\times\mathbf{E}|^2 = |\mathbf{k}\times\mathbf{E}_0|e^{-2\,\mathrm{Im}(k)z} = |k|^2 E_0^2 e^{-2\,\mathrm{Im}(k)z} \tag{12.2.6}$$

So the rate of change in energy density is

$$\begin{aligned}
C &= \mathrm{Re}\,\frac{i\omega}{2}(\mathbf{E}\cdot\mathbf{D}^* - \mathbf{B}\cdot\mathbf{H}^*) \\
&= \mathrm{Re}\,\frac{i\omega}{2}\left(\epsilon^*|E|^2 - \frac{1}{\mu}|B|^2\right) \\
&= \mathrm{Re}\,\frac{i\omega}{2}\left[\epsilon^* E_0^2 e^{-2\,\mathrm{Im}(k)z} - \frac{1}{\mu\omega^2}|k|^2 E_0^2 e^{-2\,\mathrm{Im}(k)z}\right] \\
&= \mathrm{Re}\,\frac{i\omega}{2}\left(\frac{\epsilon^*}{\epsilon_0} - \frac{|k|^2}{\epsilon_0\mu\omega^2}\right)\epsilon_0 E_0^2 e^{-2\,\mathrm{Im}(k)z} \\
&= \mathrm{Re}\,\frac{i\omega}{2}\epsilon_0 E_0^2 e^{-2\,\mathrm{Im}(k)z}\left[(n^*)^2 - |n|^2\right] \\
&= \frac{\omega}{2}\epsilon_0 E_0^2 e^{-2\,\mathrm{Im}(k)z}\,\mathrm{Re}\left[2(\mathrm{Re}(n) - i\,\mathrm{Im}(n))\,\mathrm{Im}(n)\right] \\
&= \omega\epsilon_0 E_0^2 e^{-2\,\mathrm{Im}(k)z}\,\mathrm{Re}(n)\,\mathrm{Im}(n)
\end{aligned} \tag{12.2.7}$$

By definition, the incident Poynting flux is

$$\begin{aligned}
S_i &= \frac{1}{2}\left|\mathrm{Re}\left\{\mathbf{E}_i\times\mathbf{H}_i\right\}\right| \\
&= \frac{1}{2\mu_0}\left|\mathrm{Re}\left\{\mathbf{E}_i\times\mathbf{B}_i^*\right\}\right| \\
&= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left|\mathrm{Re}\left\{|E_i|^2\hat{\mathbf{k}}_i\right\}\right| \\
&= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|E_i|^2
\end{aligned} \tag{12.2.8}$$

We can also calculate the transmitted Poynting flux (energy flux) from the rate of change in energy density by integrating over the half-infinite medium

$$\begin{aligned}
S_t &= \int_0^\infty dz C \\
&= \int_0^\infty dz\omega\epsilon_0 E_0^2\,\mathrm{Re}(n)\,\mathrm{Im}(n)e^{-2\omega\,\mathrm{Im}(n)z/c} \\
&= \frac{c}{2\omega\,\mathrm{Im}(n)}\omega\epsilon_0 E_0^2\,\mathrm{Re}(n)\,\mathrm{Im}(n) \\
&= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|E_t|^2\,\mathrm{Re}(n)
\end{aligned} \tag{12.2.9}$$

Then the ratio between these fluxes is exactly the same as the transmission coefficient

$$\frac{S_t}{S_i} = \mathrm{Re}(n)\left|\frac{E_t}{E_i}\right|^2 = T \tag{12.2.10}$$

Thus, $S_t = TS_i$. The rate of change in energy density accounts for the transmitted energy flux.

(c) First, $n^2 = 1 + i(\sigma/\omega\epsilon_0) = 1 + i/x \approx i/x = (1/x)e^{i\pi/2}$ if $x \ll 1$. Then we can write

$$n = \frac{1}{\sqrt{x}} e^{i\pi/4} \tag{12.2.11}$$

From part (a), the reflection coefficient is

$$\begin{aligned} R &= \left| \frac{1-n}{1+n} \right|^2 \\ &= \left| 1 + \frac{2}{\sqrt{x}e^{3i\pi/4} - 1} \right|^2 \\ &\approx \left| -1 - 2\sqrt{x}e^{3i\pi/4} \right|^2 \\ &= \left| -1 - \sqrt{2x}(-1+i) \right|^2 \\ &= (\sqrt{2x} - 1)^2 + 2x \\ &\approx 1 - 2\sqrt{2x} \\ &= 1 - 2\frac{\delta\omega}{c} \end{aligned} \tag{12.2.12}$$

where $\delta = \sqrt{2/\mu\sigma\omega}$. Similarly, the transmission coefficient is

$$\begin{aligned} T &= \frac{4\,\mathrm{Re}(n)}{|1+n|^2} \\ &= \frac{4/\sqrt{2x}}{1 + (1/\sqrt{x})e^{i\pi/4}}^{2} \\ &= \frac{4/\sqrt{2x}}{(1 + 1/\sqrt{2x})^2 + 1/2x} \\ &\approx 2\sqrt{2x} \\ &= 2\frac{\delta\omega}{c} \end{aligned} \tag{12.2.13}$$

which guarantees that $R + T = 1$. From part (b), the rate of change in energy density is

$$\begin{aligned} C &= \omega\epsilon_0 E_0^2 e^{-2\omega\,\mathrm{Im}(n)z/c}\,\mathrm{Re}(n)\,\mathrm{Im}(n) \\ &= \omega\epsilon_0 E_0^2 e^{-2\omega z/c\sqrt{2x}} \frac{1}{2x} \\ &= \frac{1}{2}\sigma E_0^2 e^{-2z/\delta} \\ &= \frac{1}{\mu\omega\delta^2} e^{-2z/\delta} E_0^2 \end{aligned} \tag{12.2.14}$$

Now, we calculate with the generalized Ohm's Law, $\mathbf{J} = \sigma \mathbf{E}$, to confirm that

$$\frac{1}{2} \operatorname{Re}\{\mathbf{J}^* \cdot \mathbf{E}\} = \frac{\sigma}{2} \operatorname{Re}\{\mathbf{E}^* \cdot \mathbf{E}\} = \frac{\sigma}{2} E_0^2 e^{-2\operatorname{Im}(k)z} = C \tag{12.2.15}$$

Since these are the same, the energy transmitted must be lost to Joule heating. □

***Problem* 12.3** (Wave propagation through a stack of layers)**:** A monochromatic plane wave of frequency ω is incident normally on a stack of layers of various thicknesses t_j and lossless indices of refraction n_j. Inside the stack, the wave has both forward and backward moving components. The change in the wave through any interface and also from one side of a layer to the other can be described by means of 2×2 transfer matrices. If the electric field is written as

$$E = E_+ e^{ikx} + E_- e^{-ikx} \tag{12.3.1}$$

in each layer, the transfer matrix equation $E' = TE$ is explicitly

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} \tag{12.3.2}$$

(a) Show that the transfer matrix for propagation inside, but across, a layer of index of refraction n_j and thickness t_j is

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} = I \cos\left(k_j t_j\right) + i\sigma_3 \sin\left(k_j t_j\right) \tag{12.3.3}$$

where $k_j = n_j \omega / c$, I is the unit matrix, and σ_k are the Pauli spin matrices of quantum mechanics. Show that the inverse matrix is T^*.

(b) Show that the transfer matrix to cross an interface from $n_1 (x < x_0)$ to $n_2 (x > x_0)$ is

$$T_{\text{interface}}(2, 1) = \frac{1}{2} \begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix} = I \frac{n+1}{2} - \sigma_1 \frac{n-1}{2} \tag{12.3.4}$$

where $n = n_1 / n_2$.

(c) Show that for a complete stack, the incident, reflected, and transmitted waves are related by

$$E_{\text{trans}} = \frac{\det(T)}{t_{22}} E_{\text{inc}}, \qquad E_{\text{refl}} = -\frac{t_{21}}{t_{22}} E_{\text{inc}} \tag{12.3.5}$$

where t_{ij} are the elements of T, the product of the forward-going transfer matrices, including from the material filling space on the incident side into the first layer and from the last layer into the medium filling the space on the transmitted side.

Solution.

(a) Given the electric field in the layer n_j

$$E(x) = E_+ e^{ik_j x} + E_- e^{-ik_j x} \tag{12.3.6}$$

By a translation $x \mapsto x + t_j$, the electric field becomes

$$E(x+t_j) = E_+ e^{ik_j(x+t_j)} + E_- e^{-ik_j(x+t_j)} = E_+ e^{ik_j t_j} e^{ik_j x} + E_- e^{-ik_j t_j} e^{-ik_j x} \tag{12.3.7}$$

Thus, the transferred amplitude of the forward and backward electric field is $E'_\pm$ where

$$E'_\pm = E_\pm e^{\pm ik_j t_j} \tag{12.3.8}$$

or

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} \tag{12.3.9}$$

Thus,

$$\begin{aligned} T_{\text{layer}} &= \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} \\ &= \begin{pmatrix} \cos\big(k_j t_j\big) + i\sin\big(k_j t_j\big) & 0 \\ 0 & \cos\big(k_j t_j\big) - i\sin\big(k_j t_j\big) \end{pmatrix} \\ &= I\cos\big(k_j t_j\big) + i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\big(k_j t_j\big) \\ &= I\cos\big(k_j t_j\big) + i\sigma_3\sin\big(k_j t_j\big) \end{aligned} \tag{12.3.10}$$

Since T_{layer} is diagonal, the inverse of T is

$$T^{-1} = \begin{pmatrix} 1/e^{ik_j t_j} & 0 \\ 0 & 1/e^{-ik_j t_j} \end{pmatrix} = \begin{pmatrix} e^{-ik_j t_j} & 0 \\ 0 & e^{ik_j t_j} \end{pmatrix} = T^* \tag{12.3.11}$$

(b) At the boundary, the electric field and magnetic field are continuous. Thus, it follows that

$$E_+ + E_- = E'_+ + E'_- \qquad \text{and} \qquad n_1(E_+ - E_-) = n_2(E'_+ - E'_-) \tag{12.3.12}$$

Solving this system of equations, we can write

$$E'_+ = \frac{1}{2}\big[(1+n)E_+ + (1-n)E_-\big] \tag{12.3.13a}$$

$$E'_- = \frac{1}{2}\big[(1-n)E_+ + (1+n)E_-\big] \tag{12.3.13b}$$

or,

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+n & 1-n \\ 1-n & 1+n \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} \tag{12.3.14}$$

So the transfer matrix is

$$\begin{aligned}
T_{\text{interface}} &= \frac{1}{2}\begin{pmatrix} 1+n & 1-n \\ 1-n & 1+n \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+n & 0 \\ 0 & 1+n \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1-n \\ 1-n & 0 \end{pmatrix} \\
&= I\frac{n+1}{2} + \sigma_1\frac{1-n}{2}
\end{aligned} \tag{12.3.15}$$

(c) At the last layer, there is no backward-propagating wave, so the transfer equation is

$$\begin{pmatrix} E_{0T} \\ 0 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} E_{0I} \\ E_{0R} \end{pmatrix} \tag{12.3.16}$$

which equates to

$$E_{0T} = t_{11}E_{0I} + t_{12}E_{0R} \qquad \text{and} \qquad t_{21}E_{0I} + t_{22}E_{0R} \tag{12.3.17}$$

From the latter, we can write

$$\frac{E_{0R}}{E_{0I}} = -\frac{t_{21}}{t_{22}} \tag{12.3.18}$$

and from the latter,

$$\frac{E_{0T}}{E_{0I}} = t_{11} + t_{12}\frac{E_{0R}}{E_{0I}} = \frac{t_{11}t_{22} - t_{12}t_{21}}{t_{22}} = \frac{\det(T)}{t_{22}} \tag{12.3.19}$$

□