Homework 6: Phys 7310 (Fall 2021)

Tien Vo

October 7, 2021

***Problem* 6.1** (The potential inside a cylinder)**:** (a) A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

(b) For the cylinder in the previous part, the cylindrical surface is made of two equal half-cylinders, one at potential V and the other at potential $-V$, so that

$$V(\phi, z) = \begin{cases} V, & \phi \in (-\pi/2, \pi/2) \\ -V, & \phi \in (\pi, 2, 3\pi/2) \end{cases} \tag{6.1.1}$$

Find the potential inside the cylinder.

Solution.

(a) By separation of variables, we write $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$ and plug back into Laplace equation

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{Q''}{Q} + \frac{Z''}{Z} = 0 \tag{6.1.2}$$

Because Z has to vanish at finite z, we can let $Z''(z) = -k^2 Z(z)$ for some constant k, the general solution to which is

$$Z(z) = a\cos(kz) + b\sin(kz) \tag{6.1.3}$$

The boundary conditions require that $Z(0) = a = 0$ and $Z(L) = b\sin(kL) = 0$. The latter makes k discrete $k_n = n\pi/L$ for some $n \in \{1, 2, 3, \ldots\}$. Now, our differential equation becomes

$$\rho^2\frac{R''}{R} + \rho\frac{R'}{R} - k^2\rho^2 + \frac{Q''}{Q} = 0 \tag{6.1.4}$$

Letting $Q''(\phi) = -\nu^2 Q(\phi)$, the solution for Q is $A\cos(m\phi) + B\sin(m\phi)$. Then the differential equation in ρ is

$$R''(\rho) + \frac{1}{\rho}R'(\rho) - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R(\rho) = 0 \tag{6.1.5}$$

Let $x = k\rho$, the solution for R is then

$$R(\rho) = CI_\nu(k\rho) + DK_\nu(k\rho) \tag{6.1.6}$$

for $\nu \in \mathbb{N}$. Since $K_\nu \to \infty$ for $\rho \to 0$, D has to be zero because the potential is finite in the cylinder. Combining the previous results, the general solution to the potential is

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m(k_n\rho)[A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)]\sin\left(\frac{n\pi z}{L}\right) \tag{6.1.7}$$

Now, applying the last boundary condition, we can write $\Phi(b, \phi, z) = V(\phi, z)$ and

$$V(\phi, z) = \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} I_{m'}(k_{n'}b)\Big[A_{n'm'}\cos\big(m'\phi\big) + B_{n'm'}\sin\big(m'\phi\big)\Big]\sin\left(\frac{n'\pi z}{L}\right) \tag{6.1.8}$$

Multiplying both sides with $2/\pi L\cos(m\phi)\sin\big(n\pi z/L\big)$ and integrating, we can write from the orthogonality condition

$$\frac{2}{\pi L}\int_0^L \int_0^{2\pi} V(\phi, z)\cos(m\phi)\sin\left(\frac{n\pi z}{L}\right)d\phi dz = \sum_{m'=0}^{\infty} \sum_{n'=1}^{\infty} I_{m'}(k_{n'}b)A_{n'm'}\delta_{mm'}\delta_{nn'} = I_m(k_nb)A_{nm} \tag{6.1.9}$$

Thus,

$$A_{nm} = \frac{2}{\pi L}\frac{1}{I_m(n\pi b/L)}\int_0^L \int_0^{2\pi} V(\phi, z)\cos(m\phi)\sin\left(\frac{n\pi z}{L}\right)d\phi dz \tag{6.1.10}$$

Similarly,

$$B_{nm} = \frac{2}{\pi L}\frac{1}{I_m(n\pi b/L)}\int_0^L \int_0^{2\pi} V(\phi, z)\sin(m\phi)\sin\left(\frac{n\pi z}{L}\right)d\phi dz \tag{6.1.11}$$

Then we can rewrite the solution (6.1.7) as

$$\Phi(\rho, \phi, z) = \frac{2}{\pi L}\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\big[a_{nm}\cos(m\phi) + b_{nm}\sin(m\phi)\big]\sin\left(\frac{n\pi z}{L}\right) \tag{6.1.12}$$

where

$$a_{nm} = \int_0^L \int_0^{2\pi} V(\phi', z')\cos\big(m\phi'\big)\sin\left(\frac{n\pi z'}{L}\right)d\phi' dz' \tag{6.1.13a}$$

$$b_{nm} = \int_0^L \int_0^{2\pi} V(\phi', z')\sin\big(m\phi'\big)\sin\left(\frac{n\pi z'}{L}\right)d\phi' dz' \tag{6.1.13b}$$

(b) With the potential as in (6.1.1), the coefficients (6.1.13) for $m = 0$ become

$$a_{n0} = V\int_0^L \sin\left(\frac{n\pi z'}{L}\right)dz'\left[\int_{-\pi/2}^{\pi/2} d\phi' - \int_{\pi/2}^{3\pi/2} d\phi'\right] = 0$$

$$b_{n0} = 0 \tag{6.1.14a}$$

For $m \neq 0$,

$$a_{nm} = \frac{4VL}{mn\pi}\left[1-(-1)^n\right]\sin^3\left(\frac{m\pi}{2}\right) \tag{6.1.15a}$$

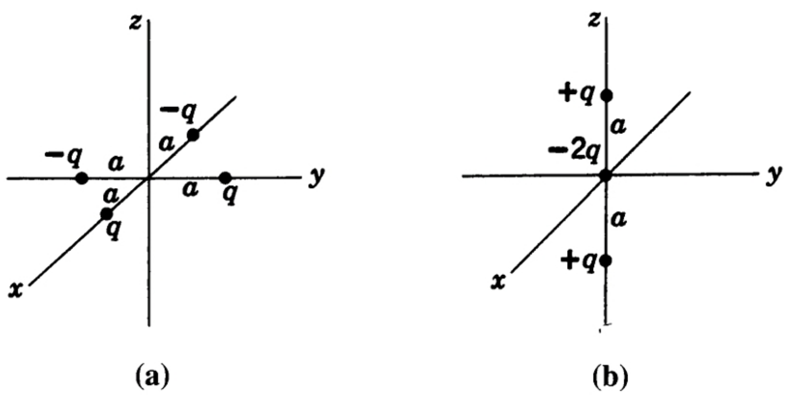
$$b_{nm} = -\frac{2VL}{mn\pi}\left[1-(-1)^n\right]\sin\left(\frac{m\pi}{2}\right)\sin\left(m\pi\right) = 0 \tag{6.1.15b}$$

From (6.1.15a), the only non-trivial terms have $n, m \in 2\mathbb{N}+1$ (odd). Then we can write the solution as

$$\Phi(\rho,\phi,z) = \frac{8V}{\pi^2}\sum_{n,m\in 2\mathbb{N}+1}\frac{\left[1-(-1)^n\right]}{mn}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\sin^3\left(\frac{m\pi}{2}\right)\cos(m\phi)\sin\left(\frac{n\pi z}{L}\right) \tag{6.1.16}$$

□

***Problem* 6.2** (Multiple moments)**:** Calculate the multipole moment q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.



Problem 4.1

(c) For the charge distribution of the second set b write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the xy plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for b in the xy plane. Plot it as a function of distance and compare with the result found in part c.

Divide out the asymptotic form in parts c and d to see the behavior at large distances more clearly.

Solution.

(a) The charge density in spherical coordinates can be written as

$$\rho(\mathbf{x}') = q\frac{\delta(r'-a)}{r'^2}\delta(\cos\theta')\left[\delta(\phi') + \delta\left(\phi' - \frac{\pi}{2}\right) - \delta(\phi'-\pi) - \delta\left(\phi' + \frac{\pi}{2}\right)\right] \tag{6.2.1}$$

Then the multipole moment is, by definition

$$q_{lm} = q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^\infty r'^l\delta(r'-a)dr'\int_{-1}^1 P_l^m(\cos\theta')\delta(\cos\theta')d(\cos\theta')$$
$$\times\int_0^{2\pi} e^{-im\phi'}\left[\delta(\phi')+\delta\left(\phi'-\frac{\pi}{2}\right)-\delta(\phi'-\pi)-\delta\left(\phi'+\frac{\pi}{2}\right)\right]d\phi'$$
$$= q\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}a^l P_l^m(0)\left[1+e^{-im\pi/2}-e^{-im\pi}-e^{im\pi/2}\right]$$
$$= qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m\left[1-(-1)^m-2i\sin\left(\frac{m\pi}{2}\right)\right]\left.\frac{d^m}{dx^m}P_l(x)\right|_{x=0} \tag{6.2.2}$$

Note from the square bracket that m has to be odd so that it does not vanish. Then $d^m P_l/dx^m$ is an odd polynomial if l is even. However, odd functions vanish at $x=0$. So l has to also be odd if m is odd. Thus, $l, m \in 2\mathbb{N}+1$. We can then write

$$q_{lm} = 2qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-i\sin\left(\frac{m\pi}{2}\right)\right] \tag{6.2.3}$$

Up to $l=3$, the only non-trivial terms (with $m \geq 0$) are

$$q_{11} = qa\sqrt{\frac{3}{2\pi}}(-1+i), \qquad q_{31} = \frac{qa^3}{4}\sqrt{\frac{21}{\pi}}(1-i), \qquad \text{and} \qquad q_{33} = -\frac{qa^3}{4}\sqrt{\frac{35}{\pi}}(1+i) \tag{6.2.4}$$

(b) The charge density in spherical coordinates is

$$\rho(\mathbf{x}') = \frac{q}{2\pi}\frac{1}{r'^2}\left\{\delta\left(r'-a\right)\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]-\delta(r')\right\} \tag{6.2.5}$$

Then by definition, the multipole moment is

$$q_{lm} = \frac{q}{2\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^{2\pi} e^{-im\phi'}d\phi'\Bigg\{$$
$$\int_0^\infty dr' r'^l\delta(r'-a)\int_{-1}^1 d(\cos\theta')P_l^m(\cos\theta')\left[\delta(\cos\theta'-1)+\delta(\cos\theta'+1)\right]$$
$$-\int_0^\infty r'^l\delta(r')dr'\int_{-1}^1 d(\cos\theta')P_l^m(\cos\theta')\Bigg\}$$
$$\tag{6.2.6}$$

Note that the integration over ϕ' is only non-zero if $m=0$ because $e^{-im\phi'}$ is periodic in $[0,2\pi]$. Then (6.2.6) becomes

$$q_{lm} = q\sqrt{\frac{2l+1}{4\pi}}\left\{a^l\left[P_l(1)+P_l(-1)\right]-2\delta_{l0}\right\} \tag{6.2.7}$$

The only nontrivial term up to $l = 3$ is

$$q_{20} = qa^2\sqrt{\frac{5}{4\pi}} = \frac{1}{4}\sqrt{\frac{5}{\pi}}Q_{33} \Rightarrow Q_{33} = 4qa^2 \tag{6.2.8}$$

Now, $q_{22} = 0$ and $q_{2-2} = 0$ implies that

$$Q_{11} - Q_{22} = 2iQ_{12} \qquad \text{and} \qquad Q_{11} - Q_{22} = -2iQ_{12} \tag{6.2.9}$$

This means $Q_{12} = Q_{11} - Q_{22} = 0$ and $Q_{11} = Q_{22}$. Similarly, $q_{21} = q_{2-1} = 0$ implies that

$$Q_{13} = iQ_{23} \qquad \text{and} \qquad Q_{13} = -iQ_{23} \tag{6.2.10}$$

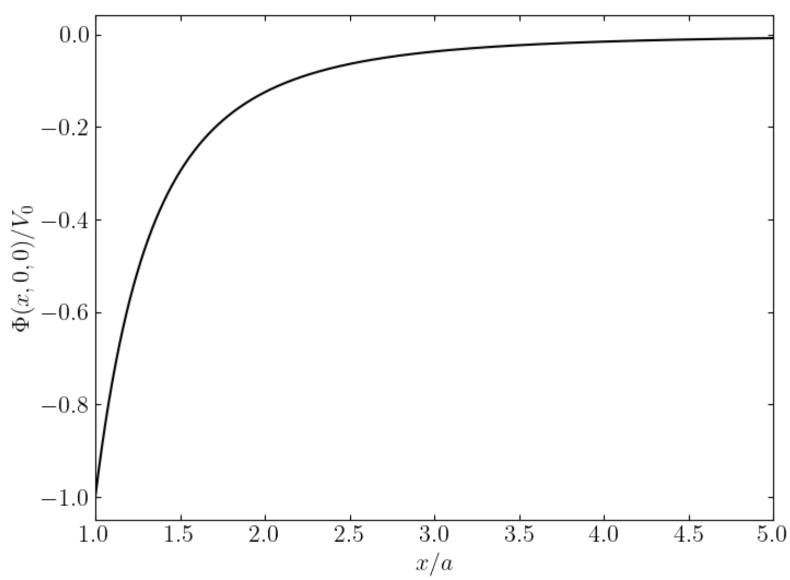
Then it must be that $Q_{13} = Q_{23} = 0$. Now, Q is traceless, so it must follow that

$$Q_{11} = Q_{22} = -\frac{1}{2}Q_{33} = -2qa^2 \tag{6.2.11}$$

(c) From (4.10, Jackson), we can write the potential as

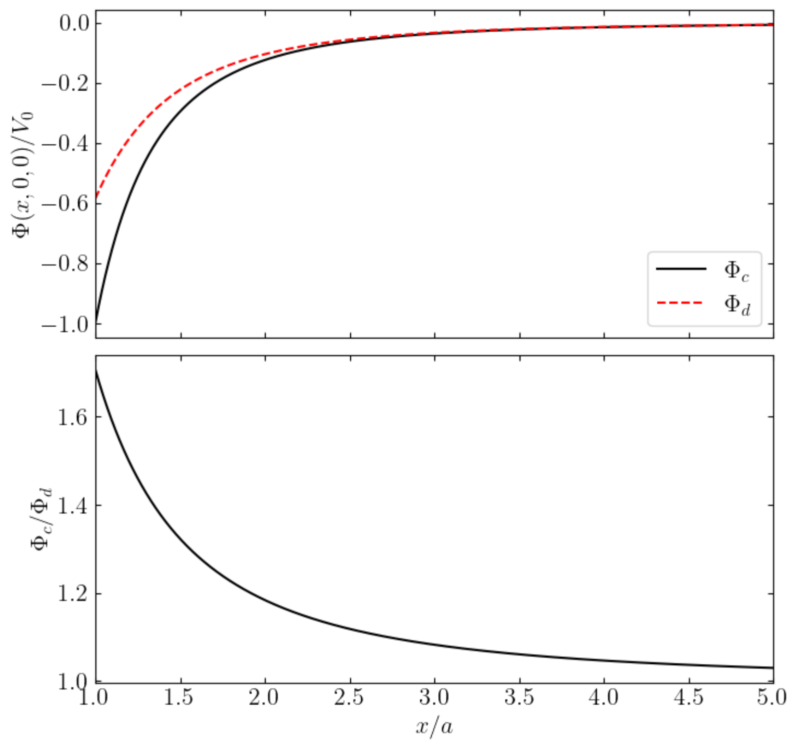
$$\begin{aligned}
\Phi &= \frac{1}{8\pi\epsilon_0}\left[Q_{11}\frac{x^2}{r^5} + Q_{22}\frac{y^2}{r^5} + Q_{33}\frac{z^2}{r^5}\right] \\
&= \frac{q}{4\pi\epsilon_0 a}\left[-\frac{(x/a)^2}{(r/a)^5} - \frac{(y/a)^2}{(r/a)^5} + 2\frac{(z/a)^2}{(r/a)^5}\right] \\
&= V_0\left[-\frac{(x/a)^2}{(r/a)^5} - \frac{(y/a)^2}{(r/a)^5} + 2\frac{(z/a)^2}{(r/a)^5}\right]
\end{aligned} \tag{6.2.12}$$

A plot of this potential in the xy plane for $y = 0$ is shown below.



(d) From Coulomb's Law, we can write the full potential as

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}-a\hat{\mathbf{z}}|}+\frac{1}{|\mathbf{x}+a\hat{\mathbf{z}}|}-\frac{2}{r}\right] = V_0\left[\frac{1}{|\mathbf{x}/a-\hat{\mathbf{z}}|}+\frac{1}{|\mathbf{x}/a+\hat{\mathbf{z}}|}-\frac{2}{r/a}\right] \tag{6.2.13}$$



As shown in the above figure, the two potentials agree at $x \gg a$. In the lower panel, the ratio between them converges to 1 at this limit. □

***Problem* 6.3** (Dipole as derivative of delta function)**:** A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\boldsymbol{\nabla}\delta(\mathbf{x}-\mathbf{x}_0) \tag{6.3.1}$$

Solution.

Note that the derivative of the delta function has the following property

$$\int f(\mathbf{x}')\boldsymbol{\nabla}'\delta(\mathbf{x}'-\mathbf{x}_0)d\mathbf{x}' = -\left.\boldsymbol{\nabla}'f(\mathbf{x}')\right|_{\mathbf{x}'=\mathbf{x}_0} \tag{6.3.2}$$

Using the density (6.3.1) to calculate the corresponding potential, we get

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{\mathbf{p}}{4\pi\epsilon_0} \cdot \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} \boldsymbol{\nabla}'\delta(\mathbf{x}'-\mathbf{x}_0) d\mathbf{x}' \\
&= \frac{1}{4\pi\epsilon_0}\mathbf{p} \cdot \boldsymbol{\nabla}'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\Bigg|_{\mathbf{x}'=\mathbf{x}_0} \\
&= \frac{1}{4\pi\epsilon_0}\mathbf{p} \cdot \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} \qquad (6.3.3)
\end{aligned}$$

From (4.10, Jackson), this is the potential due to a dipole moment $\mathbf{p}$ placed at $\mathbf{x}_0$. □

***Problem* 6.4** (Nucleus with a quadrupole moment)**:** A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $(\partial E_z/\partial z)_0$ along the z axis at the position of the nucleus.

(a) Show that the energy of quadrupole interaction is

$$W = -\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)_0 \qquad (6.4.1)$$

(b) If it is known that $Q = 2 \times 10^{-28}\,\mathrm{m}^2$ and that W/h is 10 MHz, where h is Planck's constant, calculate $(\partial E_z/\partial z)_0$ in units of $e/4\pi\epsilon_0 a_0^3$, where $a_0 = 4\pi\epsilon_0\hbar^2/me^2 = 0.529\times10^{-10}\,\mathrm{m}$ is the Bohr radius in hydrogen.

(c) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semimajor axis a and semiminor axis b. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is Ze. Given that Eu^{153} $(Z = 63)$ has a quadrupole moment $Q = 2.5 \times 10^{-28}\,\mathrm{m}^2$ and a mean radius

$$R = (a+b)/2 = 7 \times 10^{-15}\,\mathrm{m} \qquad (6.4.2)$$

determine the fractional difference in radius $(a-b)/R$.

Solution.

(a) From (4.24, Jackson), the energy is

$$\begin{aligned}
W &= -\frac{1}{6}\left(Q_{11}\frac{\partial E_x}{\partial x}\bigg|_0 + Q_{22}\frac{\partial E_y}{\partial y}\bigg|_0 + Q_{33}\frac{\partial E_z}{\partial z}\bigg|_0\right) \\
&= -\frac{eQ}{6}\left(-\frac{1}{2}\frac{\partial E_x}{\partial x}\bigg|_0 - \frac{1}{2}\frac{\partial E_y}{\partial y}\bigg|_0 + \frac{\partial E_z}{\partial z}\bigg|_0\right) \qquad (6.4.3)
\end{aligned}$$

where the first and second moment q and $\mathbf{p}$ are zero because this is a quadrupole, $Q_{11} = Q_{22} = -(1/2)Q_{33}$ as written in Jackson, Section 4.2, and $Q_{33} = eQ$. Now, at the origin, $\boldsymbol{\nabla}\cdot\mathbf{E}\big|_0 = 0$ because there's no charge density there. Thus, we can write

$$-\frac{1}{2}\left(\frac{\partial E_x}{\partial x}\bigg|_0 + \frac{\partial E_y}{\partial y}\bigg|_0\right) = \frac{1}{2}\frac{\partial E_z}{\partial z}\bigg|_0 \qquad (6.4.4)$$

Plugging this into (6.4.3) yields

$$W = -\frac{eQ}{6}\frac{3}{2}\left.\frac{\partial E_z}{\partial z}\right|_0 = -\frac{eQ}{4}\left.\frac{\partial E_z}{\partial z}\right|_0 \tag{6.4.5}$$

(b) Inverting (6.4.5), we can write

$$\left.\frac{\partial E_z}{\partial z}\right|_0 = -\frac{4W}{eQ} = -\frac{32\pi^2\epsilon_0\hbar(W/h)a_0^3}{e^2Q} \times \frac{e}{4\pi\epsilon_0 a_0^3} \approx -0.085\left[\frac{e}{4\pi\epsilon_0 a_0^3}\right] \tag{6.4.6}$$

(c) From the surface of the spheroid, we know that $|z| \leq a\sqrt{1 - r^2/b^2} = f(r)$. The volume of the spheroid is thus

$$V = \int_0^b \int_0^{2\pi} \int_{-f(r)}^{f(r)} r dr d\theta dz = \frac{4\pi}{3}ab^2 \tag{6.4.7}$$

Thus, the uniform charge density is

$$\rho = \frac{3Ze}{4\pi ab^2} \tag{6.4.8}$$

Then from (4.25, Jackson), we can calculate in cylindrical coordinates

$$\begin{aligned}
Q &= \frac{1}{e}\rho \int (2z^2 - r^2) r dr d\theta dz \\
&= \frac{3Z}{ab^2}\int_0^b r dr \int_0^{f(r)} (2z^2 - r^2) dz \\
&= \frac{2Z}{5}(a^2 - b^2) \\
&= \frac{252}{5}R(a - b)
\end{aligned} \tag{6.4.9}$$

Thus,

$$\frac{a-b}{R} = \frac{5}{252}\frac{Q}{R^2} \approx 0.1 \tag{6.4.10}$$

□