Homework 4: Phys 7320 (Spring 2022)

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***Problem* 4.1** (Critical opalescence)**:** In the scattering of light by a gas very near the critical point the scattered light is observed to be "whiter" (i.e., its spectrum is less predominantly peaked toward the blue) than far from the critical point. Show that this can be understood by the fact that the volumes of the density fluctuations become large enough that Rayleigh's law fails to hold. In particular, consider the lowest order approximation to the scattering by a unifrom dielectric sphere of radius a whose dielectric constant ϵ_r differs only slightly from unity.

Show that for $ka \gg 1$, the differential cross section is sharply peaked in the forward direction and the total scattering cross section is approximately

$$\sigma \approx \frac{\pi}{2}(ka)^2 |\epsilon_r - 1|^2 a^2 \tag{4.1.1}$$

with a k^2, rather than k^4, dependence on frequency.

Hint: We are now in the small-wavelength limit, so we cannot assume the scattering is just due to dipoles. Instead, use the Born approximation. You may assume the incident radiation is unpolarized and may average over polarizations in the usual way. The integral for the total cross-section peaks strongly in the forward direction, so you can replace $qa \approx ka\theta$, $d\cos\theta = \theta d\theta$, and take the range of θ from 0 to infinity. You'll get an integral

$$\sigma \approx 2\pi |\epsilon_r - 1|^2 k^2 a^4 \int_0^\infty x dx \frac{j_1(x)^2}{x^2} \tag{4.1.2}$$

At that point Bessel function identities near Jackson (9.90) can be used to solve the integral if you are adventurous, or you may plug it into Mathematica. Notice how the Rayleigh k^4 is softened by the extended source to $\sigma \sim k^2$.

Solution.

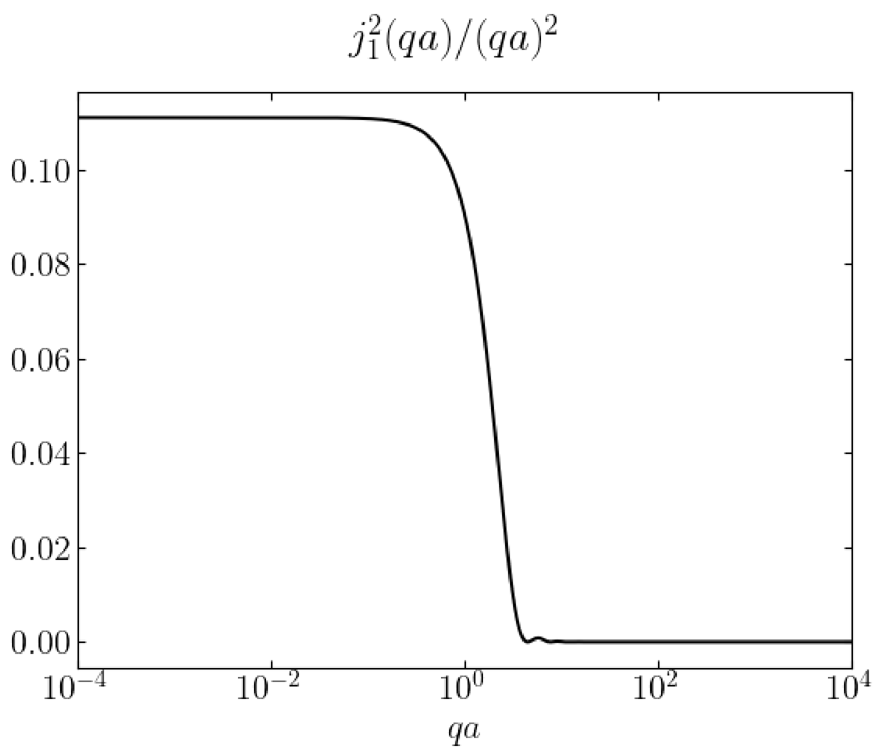
From the equation above (10.32, Jackson), we can calculate the differential cross section

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{|\boldsymbol{\epsilon}^* \cdot \mathbf{A}_{\rm sc}|^2}{D_0^2} \\
&= k^4|\epsilon_r - 1|^2|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \left[\frac{\sin(qa) - qa\cos(qa)}{q^3}\right]^2 \\
&= k^4a^6|\epsilon_r - 1|^2|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \left[\frac{\sin(qa) - qa\cos(qa)}{(qa)^3}\right]^2 \\
&= k^4a^6|\epsilon_r - 1|^2\frac{j_1^2(qa)}{(qa)^2}|\boldsymbol{\epsilon}^* \cdot \boldsymbol{\epsilon}_0|^2 \qquad (4.1.3)
\end{aligned}$$

where we have written $\delta\epsilon/\epsilon_0 = \epsilon_r - 1$. Averaging over all polarizations, we get

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle_{\rm pol} = \frac{k^4a^6}{2}|\epsilon_r - 1|^2\frac{j_1^2(qa)}{(qa)^2}(1 + \cos^2\theta) \qquad (4.1.4)$$

A plot of $j_1^2(qa)/(qa)^2$ is plotted below.



This indicates a sharp peak for $qa \lesssim 1$. Thus, we can write $qa \approx ka\theta$ and integrate for the total cross section

$$\begin{aligned}
\sigma &= \pi k^4 a^6 |\epsilon_r - 1|^2 \int_{-1}^{1} d(\cos\theta) \frac{j_1^2(ka\theta)}{(ka\theta)^2} \left(1 + \cos^2\theta\right) \\
&\approx \pi k^4 a^6 |\epsilon_r - 1|^2 \int_0^\infty \theta d\theta \frac{j_1^2(ka\theta)}{(ka\theta)^2} (1 + \cos^2\theta) \\
&= \pi k^2 a^4 |\epsilon_r - 1|^2 \int_0^\infty x dx \frac{j_1^2(x)}{x^2} \left[1 + \cos^2\left(\frac{x}{ka}\right)\right] \\
&\approx 2\pi k^2 a^4 |\epsilon_r - 1|^2 \int_0^\infty dx \frac{j_1^2(x)}{x} \\
&\approx \frac{\pi}{2} k^2 a^4 |\epsilon_r - 1|^2
\end{aligned} \tag{4.1.5}$$

where we have let $x/ka \to 0$ since $ka \gg 1$ and numerically integrated the integral in the final equality to be approximately $\pi/4$. $\square$