Homework 11: Phys 7310 (Fall 2021)

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***Problem* 11.1** (Magnetic monopoles)**:** (a) Consider Maxwell's equations with free space permittivity ϵ_0 and permeability μ_0. Show that when $\rho = \mathbf{J} = 0$, these equations obey a *duality symmetry* under which

$$\mathbf{E} \to c\mathbf{B} \qquad \text{and} \qquad \mathbf{B} \to -\frac{\mathbf{E}}{c} \tag{11.1.1}$$

When ρ and $\mathbf{J}$ are non-zero, the duality symmetry doesn't exist. Show that the duality symmetry can be restored by adding magnetic charges with density ρ_m and current $\mathbf{J}_m$ to Maxwell's equations in suitable places, and also including an action of duality on the charges and currents which you should determine. Thus with magnetic charges added, electrodynamics is fully symmetric between electric and magnetic phenomena.

(b) Consider a vector potential in spherical coordinates

$$\mathbf{A} = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{11.1.2}$$

where g is a constant. Show that the associated magnetic field $\mathbf{B}$ is the field of a *magnetic monopole*, that is, a Coulomb magnetic field. What quantity plays the role of "magnetic charge"?

(c) If there is a magnetic Coulomb field, then the divergence of the magnetic field is not zero at the origin. However, we expect that any divergence of a curl of something well-defined is *always* zero. We should therefore ask what went wrong. What went wrong is that the vector potential above is not well-defined everywhere. Where besides the origin is the vector potential $\mathbf{A}$ singular? Be sure to treat apparently indeterminate quantities carefully. The locus of singularities is called the *Dirac string*.

If we have a magnetic monopole we can never get rid of the Dirac string, but gauge transformations can move it around. Define the new vector potential

$$\mathbf{A}' = \frac{g(-1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{11.1.3}$$

Find a gauge transformation relating $\mathbf{A}'$ to $\mathbf{A}$. Where is the Dirac string for $\mathbf{A}'$?

Other gauge transformations can move the string to other locations. Since the location of the Dirac string is arbitrary and it only shows up in $\mathbf{A}$, never in $\mathbf{B}$, it is not a physical object. (One may think of it intuitively as like the end of an infinitely thin, infinitely long solenoid, or an infinitely long chain of dipoles, either way unobservable; see Jackson figure 6.8). To be well-defined everywhere, we must use more than one vector potential, such as $\mathbf{A}$ and $\mathbf{A}'$, since each one is singular somewhere.

(d) In quantum mechanics, when one performs a gauge transformation on Φ and $\mathbf{A}$ as in Jackson (6.12) and (6.13), one must also perform a gauge transformation on the wavefunction $\psi(\mathbf{x},t)$. If the particle has charge q, this gauge transformation is

$$\psi(\mathbf{x},t) \to \psi(\mathbf{x},t)' = e^{iq\Lambda(\mathbf{x},t)/\hbar}\psi(\mathbf{x},t) \tag{11.1.4}$$

One may show that the Schrödinger equation then transforms in a well-defined way under gauge transformations.

Imagine such a particle of charge q is moving around in the presence of the magnetic monopole. If we use the two different vector potentials $\mathbf{A}$ and $\mathbf{A}'$, we must use two wavefunctions ψ and ψ' for the same particle. Assuming the wavefunction ψ is well-defined, the wavefunction ψ' will only be well-defined if $e^{iq\Lambda/\hbar}(\phi = 0) = e^{iq\Lambda/\hbar}(\phi = 2\pi)$, since $\phi = 0, 2\pi$ represent the same point. What constraint must we place on q and g to ensure ψ' is well-defined? This constraint is called *Dirac quantization condition.* If a magnetic monopole exists (none has ever been observed, but a wide class of so-called grand unified theories predict they might), what must we conclude about every electric charge in the universe?

Solution.

(a) When $\rho = \mathbf{J} = 0$, Maxwell equations read

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{11.1.5}$$

Now, let $\mathbf{E}' = c\mathbf{B}$, $\mathbf{B}' = -\mathbf{E}/c$ and plug into the Maxwell equations. We get

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E}' &= c\boldsymbol{\nabla}\cdot\mathbf{B} = 0\\
\boldsymbol{\nabla}\cdot\mathbf{B}' &= -\frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E} = 0\\
\boldsymbol{\nabla}\times\mathbf{E}' &= c\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = -\frac{\partial\mathbf{B}'}{\partial t}\\
\boldsymbol{\nabla}\times\mathbf{B}' &= -\frac{1}{c}\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{c^2}\frac{\partial\mathbf{E}'}{\partial t}
\end{aligned} \tag{11.1.6}$$

Thus, the Maxwell equations are invariant under this transformation. Now, we rewrite Maxwell equations with the magnetic charge density ρ_m and current $\mathbf{J}_m$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = \mu_0\rho_m, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{11.1.7}$$

Then under the same transformation $\mathbf{E}' = c\mathbf{B}, \mathbf{B}' = -\mathbf{E}'/c$, they become

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E}' &= c\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{\rho_m/c}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \mathbf{B}' &= -\frac{1}{c}\boldsymbol{\nabla} \cdot \mathbf{E} = \mu_0(-c\rho) \\
\boldsymbol{\nabla} \times \mathbf{E}' &= c\boldsymbol{\nabla} \times \mathbf{B} = c\left(\mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) = -\mu_0(-c\mathbf{J}) - \frac{\partial \mathbf{B}'}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B}' &= -\frac{1}{c}\boldsymbol{\nabla} \times \mathbf{E} = \mu_0\frac{\mathbf{J}_m}{c} + \frac{1}{c^2}\frac{\partial \mathbf{E}'}{\partial t} \qquad (11.1.8)
\end{aligned}$$

Thus, this transformation turns $\rho, \rho_m, \mathbf{J}, \mathbf{J}_m \mapsto \rho', \rho'_m, \mathbf{J}', \mathbf{J}'_m$ where

$$\rho_m = c\rho', \qquad \rho = -\frac{1}{c}\rho'_m, \qquad \mathbf{J} = -\frac{1}{c}\mathbf{J}'_m, \qquad \mathbf{J}_m = c\mathbf{J}' \qquad (11.1.9)$$

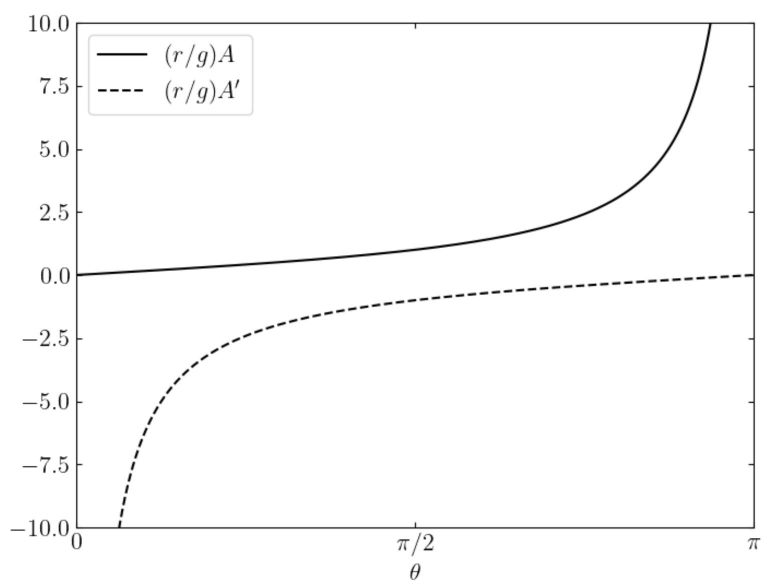
Formally, Maxwell equations are still invariant, so the *duality symmetry* is restored if we add $\rho_m, \mathbf{J}_m$ as above.

(b) The corresponding magnetic field $\mathbf{B}$ is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A\sin\theta)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA)\hat{\boldsymbol{\theta}} = \frac{g}{r^2}\hat{\mathbf{r}} \qquad (11.1.10)$$

Thus, this is a Coulomb field (radial and the magnitude follows an inverse square law). The constant g here plays the role of magnetic charge.

(c) Given $\mathbf{A}, \mathbf{A}'$ in (11.1.2) and (11.1.3), we plot the polar angle dependence in the figure below.



Then $\mathbf{A}$ has a pole at $\theta = \pi$, beside at the origin and $\mathbf{A}'$ has a pole at $\theta = 0$. Now, we want to find Λ such that $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda$. Thus, it follows that

$$\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial\phi} = -\frac{2g}{r\sin\theta} \Rightarrow \Lambda = -2g\phi \tag{11.1.11}$$

(d) For ψ' to be well-defined, it must be that

$$1 = \exp\left(-\frac{4\pi i q g}{\hbar}\right) = \cos\left(\frac{4\pi q g}{\hbar}\right) - i\sin\left(\frac{4\pi q g}{\hbar}\right) \tag{11.1.12}$$

Thus, we must require $4\pi q g/\hbar = n2\pi$ for $n \in \mathbb{Z}$, or

$$\frac{qg}{\hbar} = \frac{n}{2} \tag{11.1.13}$$

If a magnetic monopole exists, then the electric charge must be quantized. □

***Problem* 11.2** (Polarizations of EM radiation)**:** Consider an electromagnetic wave in free space, where $\epsilon = \epsilon_0$ and $\mu = \mu_0$, and there are no charges or currents nearby. The wave has frequency ω. Let the direction of travel of the wave be $\hat{\mathbf{z}}$, so that Jackson's $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are just $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

(a) Assuming the (complex) electric field is given by Jackson (7.19) with $E_1 = E_2 = E$, find the *real* electric field. Determine its amplitude E_0 (this might not be the same as E!) and a unit polarization vector characterizing its direction. Find the real form of $\mathbf{B}$ as well, including magnitude and direction. Calculate the time-averaged energy density in two ways, one by finding the (real, time-dependent) energy density using the real fields (Jackson 6.106 in free space) and then time-averaging, and the other by using the complex time-averaged formula (the equation above Jackson (7.14)) and show your result agrees with Jackson (7.14) with $\epsilon \to \epsilon_0$. Is the direction of polarization changing with time? Is this wave linearly polarized, circularly polarized, or neither?

(b) Do all the same things and answer all the same questions for the case $E_1 = E, E_2 = iE$.

Solution.

(a) First, given $\mathbf{k} = k\hat{\mathbf{z}}$, we write the complex electric and magnetic fields

$$\mathbf{E} = E(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{i(kz-\omega t)}, \qquad \mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} = \frac{kE}{\omega}(\hat{\mathbf{y}} - \hat{\mathbf{x}})e^{i(kz-\omega t)} \tag{11.2.1}$$

The real electric and magnetic fields are thus

$$\mathbf{E}_R = E(\hat{\mathbf{x}} + \hat{\mathbf{y}})\cos{(kz-\omega t)}, \qquad \mathbf{B}_R = \frac{kE}{\omega}(\hat{\mathbf{y}} - \hat{\mathbf{x}})\cos{(kz-\omega t)} \tag{11.2.2}$$

The corresponding amplitudes are $E_0 = \sqrt{2}E$ and $B_0 = kE_0/\omega$. The unit polarization vector of the electric field is $\boldsymbol{\epsilon} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The energy density is, using (6.106, Jackson)

$$u = \frac{1}{2}\left(\epsilon_0 E_R^2 + \frac{B_R^2}{\mu_0}\right) = \epsilon_0 E^2\big(1+n^2\big)\cos^2(kz-\omega t) \Rightarrow \langle u\rangle = \frac{1}{2}\epsilon_0 E^2(1+n^2) \tag{11.2.3}$$

where $n = ck/\omega$ is the index of refraction. Now, the energy density using (7.14, Jackson) is

$$\langle u \rangle = \frac{1}{4}\left(2\epsilon_0 E^2 + \frac{2k^2E^2}{\omega^2\mu_0}\right) = \frac{1}{2}\epsilon_0 E^2\big(1+n^2\big) \tag{11.2.4}$$

These results agree. The polarization $\boldsymbol{\epsilon}$ does not change in time, and the wave is linearly polarized.

(b) The complex fields are

$$\mathbf{E} = E(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i(kz-\omega t)}, \qquad \mathbf{B} = \frac{kE}{\omega}(\hat{\mathbf{y}} - i\hat{\mathbf{x}})e^{i(kz-\omega t)} \tag{11.2.5}$$

Then the real fields are

$$\mathbf{E}_R = E\big[\cos(kz-\omega t)\hat{\mathbf{x}} - \sin(kz-\omega t)\hat{\mathbf{y}}\big], \qquad \mathbf{B}_R = \frac{kE}{\omega}\big[\sin(kz-\omega t)\hat{\mathbf{x}} + \cos(kz-\omega t)\hat{\mathbf{y}}\big] \tag{11.2.6}$$

The amplitudes are still $E_0 = \sqrt{2}E$ and $B_0 = kE_0/\omega$. The polarization of the electric field is $\boldsymbol{\epsilon} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$. In this case, it rotates clockwise in a plane perpendicular to $\mathbf{k}$ in time. Since the amplitude is the same, the energy density using (7.14, Jackson) should give the same result. Now, from the real fields,

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{k^2E^2}{\omega^2\mu_0}\right) = \frac{1}{2}\epsilon_0 E^2(1+n^2) \Rightarrow \langle u \rangle = \frac{1}{2}\epsilon_0 E^2(1+n^2) \tag{11.2.7}$$

$\square$

***Problem* 11.3** (Reflection and transmission at a layered interface)**:** A plane wave is incident on a layered interface as shown in the figure. The indices of refraction of the three nonpermeable media are n_1, n_2, n_3. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.

(a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_1 = 1, n_2 = 2, n_3 = 3$; $n_1 = 3, n_2 = 2, n_3 = 1$; and $n_1 = 2, n_2 = 4, n_3 = 1$.

(b) The medium n_1 is part of an optical system (e.g., a lens); medium n_3 is air ($n_3 = 1$). It is desired to put an optical coating (medium n_2) on the surface so that there is no reflected wave for a frequency ω_0. What thickness d and index of refraction n_2 are necessary?

Solution.

(a) First, in medium 1, the fields are a superposition of an incident and reflected wave

$$\begin{aligned} \mathbf{E}_i &= E_i e^{i(k_1 z-\omega t)}\hat{\mathbf{x}} & \mathbf{B}_i &= \frac{E_i}{v_1}e^{i(k_1 z-\omega t)}\hat{\mathbf{y}} \\ \mathbf{E}_R &= E_R e^{i(-k_1 z-\omega t)}\hat{\mathbf{x}} & \mathbf{B}_R &= -\frac{E_R}{v_1}e^{i(-k_1 z-\omega t)}\hat{\mathbf{y}} \end{aligned} \tag{11.3.1}$$

where $v_1 = 1/\sqrt{\epsilon_1 \mu_1} = c/n_1 = \omega/k_1$. In medium 2, there are also backward and forward propagating waves

$$\mathbf{E}_+ = E_+ e^{i(k_2 z - \omega t)} \hat{\mathbf{x}} \qquad \mathbf{B}_+ = \frac{E_+}{v_2} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}$$
$$\mathbf{E}_- = E_- e^{i(-k_2 z - \omega t)} \hat{\mathbf{x}} \qquad \mathbf{B}_- = -\frac{E_-}{v_2} e^{i(-k_2 z - \omega t)} \hat{\mathbf{y}} \tag{11.3.2}$$

And in medium 3, there is only a transmitted wave since there is no surface to reflect off of

$$\mathbf{E}_T = E_T e^{i(k_3 z - \omega t)} \hat{\mathbf{x}} \qquad \mathbf{B}_T = \frac{E_T}{v_3} e^{i(k_3 z - \omega t)} \hat{\mathbf{y}} \tag{11.3.3}$$

We then apply the boundary conditions (7.37) for $z = 0$ and $z = d$ (assuming the 1–2 interface is at $z = 0$ and 2–3 interface is at $z = d$) of the parallel fields

$$E_i + E_R = E_+ + E_- \tag{11.3.4a}$$
$$E_+ e^{ik_2 d} + E_- e^{-ik_2 d} = E_T e^{ik_3 d} \tag{11.3.4b}$$
$$E_i - E_R = \frac{v_1}{v_2}(E_+ - E_-) \tag{11.3.4c}$$
$$E_+ e^{ik_2 d} - E_- e^{-ik_2 d} = \frac{v_2}{v_3} E_T e^{ik_3 d} \tag{11.3.4d}$$

where we have also let $\mu_1 = \mu_2 = \mu_3$. Since $v_j = c/n_j$, it is also possible to write $v_1/v_2 = n_2/n_1$ and $v_2/v_3 = n_3/n_2$. Solving this system with Mathematica yields

$$E_+ = \frac{1}{2}\left(1 + \frac{n_3}{n_2}\right) e^{i(k_3 - k_2)d} E_T \qquad \text{and} \qquad E_- = \frac{1}{2}\left(1 - \frac{n_3}{n_2}\right) e^{i(k_3 + k_2)d} E_T \tag{11.3.5}$$

Adding (11.3.4a) and (11.3.4c), we can write

$$\begin{aligned} 2E_i &= \left(1 + \frac{n_2}{n_1}\right) E_+ + \left(1 - \frac{n_2}{n_1}\right) E_- \\ &= \frac{1}{2}\left[\left(1 + \frac{n_2}{n_1}\right)\left(1 + \frac{n_3}{n_2}\right) e^{i(k_3 - k_2)d} + \left(1 - \frac{n_2}{n_1}\right)\left(1 - \frac{n_3}{n_2}\right) e^{i(k_3 + k_2)d}\right] E_T \end{aligned} \tag{11.3.6}$$

Then it follows that

$$\frac{E_T^2}{E_i^2} = 4\left[\left(1 + \frac{n_3}{n_1}\right)^2 \cos^2 x + \left(\frac{n_2}{n_1} + \frac{n_3}{n_2}\right)^2 \sin^2 x\right]^{-1} \tag{11.3.7}$$

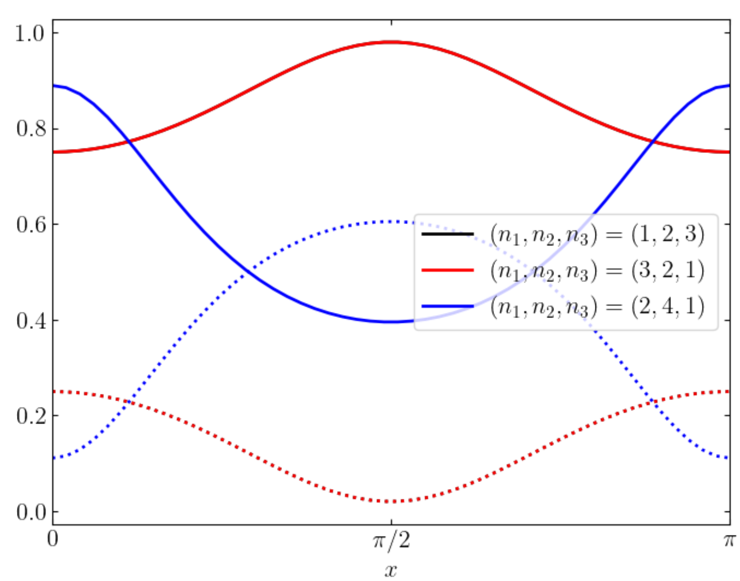
where $x = n_2 d\omega/c$. From (7.13, Jackson), the transmission coefficient is

$$\begin{aligned}
T &= \frac{S_T^2}{S_i^2} \\
&= \frac{\sqrt{\epsilon_3/\mu_3}E_T^2}{\sqrt{\epsilon_1/\mu_1}E_i^2} \\
&= \frac{n_3}{n_1}\frac{E_T^2}{E_i^2} \\
&= 4\frac{n_3}{n_1}\left[\left(1+\frac{n_3}{n_1}\right)^2 + \left(\frac{n_2^2}{n_1^2}+\frac{n_3^2}{n_2^2}-\frac{n_3^2}{n_1^2}-1\right)\sin^2 x\right]^{-1} \\
&= \frac{4n_1n_2^2n_3}{n_2^2(n_1+n_3)^2+[n_2^2(n_2^2-n_3^2)+n_1^2(n_3^2-n_2^2)]\sin^2 x} \qquad (11.3.8)
\end{aligned}$$

Then it follows that the reflection coefficient is

$$R = 1-T = 1-\frac{4n_1n_2^2n_3}{n_2^2(n_1+n_3)^2+[n_2^2(n_2^2-n_3^2)+n_1^2(n_3^2-n_2^2)]\sin^2 x} \qquad (11.3.9)$$

In the figure below, we show T (solid lines) and R (dotted lines) in three cases: $(n_1,n_2,n_3) \in \{(1,2,3),(3,2,1),(2,4,1)\}$. The first two cases (black and red) lay on top of each other.



(b) Given $n_3 = 1$, from (11.3.9), $R = 0$ when

$$n_2^2(1-n_1)^2+(n_2^2-1)(n_2^2-n_1^2)\sin^2 x = 0 \qquad \text{or} \qquad \sin(x) = \frac{n_2(n_1-1)}{\sqrt{(n_2^2-1)(n_1^2-n_2^2)}} \quad (11.3.10)$$

This requires that $n_1 > n_2$ and the thickness is determined by

$$d = \frac{c\omega_0}{n_2} \sin^{-1}\left[\frac{n_2(1-n_1)}{\sqrt{(n_2^2-1)(n_1^2-n_2^2)}}\right] \tag{11.3.11}$$

$\square$