Homework 7: Phys 7320 (Spring 2022)

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***Problem* 7.1** (A spaceship accelerating)**:** In this problem we will study the spacetime trajectory of an object that is accelerating at a constant rate in its own (instantaneous) rest frame, and then use it to think about an alien visit.

Consider a trajectory of a spaceship moving through space and time given by

$$ct(\tau) = \frac{c^2}{a}\sinh\left(\frac{a\tau}{c}\right), \qquad x(\tau) = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], \tag{7.1.1}$$

where τ is the proper time of the spaceship, c is the speed of light, a is a constant that will turn out to be the acceleration of the ship in its own frame, and t and x are coordinates in an inertial reference frame. Consider only one-dimensional motion, so we can ignore y and z. We see that at $\tau = 0$ the ship starts out at the spacetime origin $x = t = 0$.

(a) Calculate the components of the four-velocity $U^\mu = dx^\mu/d\tau$ for the spaceship. Using $U^\mu = (c\gamma, \gamma u)$, identify γ and u as functions of τ. Then invert the relation $t(\tau)$ and find u s a function of t. Find and discuss the limits of $u(t)$ for small t and for $t \to \infty$.

(b) The ship is accelerating, so it does not define an inertial frame. However, at any moment in proper time there is an instantaneous rest frame moving with the same velocity as the ship at that moment. For the proper time τ_0, find the Lorentz transformation bringing the spaceship momentarily to rest, using the hyperbolic (rapidity) form.

(c) Find the components of the four-acceleration $\alpha^\mu = dU^\mu/d\tau$ in the inertial frame for general proper times, and then at $\tau = \tau_0$ use the Lorentz transformation you found in the last part to find the four-acceleration in the instantaneous rest frame, thus showing that a is indeed the magnitude of the acceleration experienced by the spaceship in its own frame.

(d) Aliens are traveling from Proxima Centauri to the Earth by uniformly accelerating their spaceship for half the trip, and then uniformly decelerating with the same magnitude acceleration for the second half of the trip. What is the elapsed time of the total one-way trip for the aliens on the spaceship, and for observers on the Earth? (For the sake of the problem, pretend Earth and Proxima are in the same stationary, inertial reference frame.) Use the interstellar distance of $d = 4.25$ light years and an acceleration of $8g$ (which our bodies would have a hard time with, but these are tough aliens).

Solution.

(a) By definition, we differentiate

$$U^0 = \frac{d(ct)}{d\tau} = c\cosh\left(\frac{a\tau}{c}\right), \qquad \text{and} \qquad U^1 = \frac{dx}{d\tau} = c\sinh\left(\frac{a\tau}{c}\right). \tag{7.1.2}$$

Then it follows that

$$\gamma = \cosh\left(\frac{a\tau}{c}\right), \qquad \text{and} \qquad u = c\tanh\left(\frac{a\tau}{c}\right) = c\frac{\sinh\left(a\tau/c\right)}{\sqrt{1+\sinh^2\left(a\tau/c\right)}}. \tag{7.1.3}$$

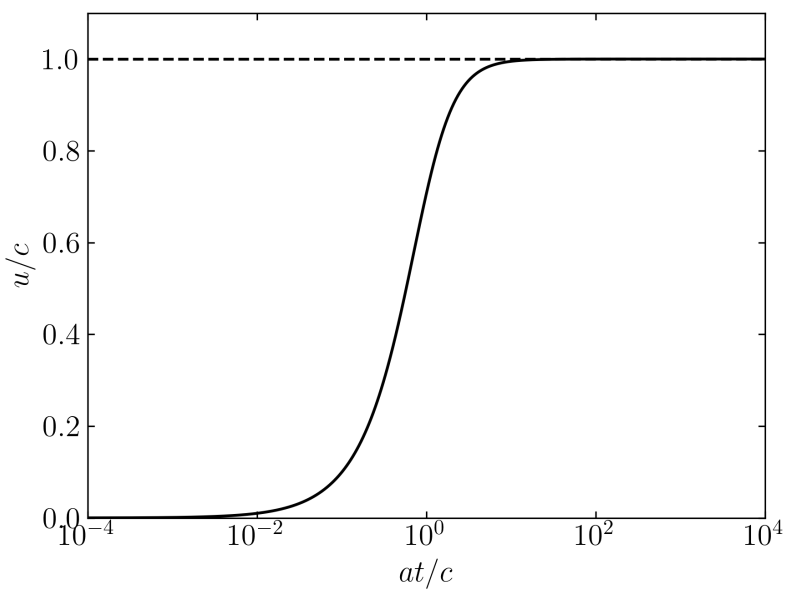
Inverting $t(\tau)$, we get

$$\tau(t) = \frac{c}{a}\sinh^{-1}\left(\frac{at}{c}\right). \tag{7.1.4}$$

Plugging this into the previous result, we obtain

$$u(t) = \frac{at}{\sqrt{1+a^2t^2/c^2}}. \tag{7.1.5}$$

Below we plot $u(t)$



Thus, $u = 0$ at small t and approaches c asymtotically at ∞. This means the spaceship accelerates from rest and reaches monotonically increasing velocity, but never past the speed of light.

(b) Note that in hyperbolic form, $\zeta_0 = a\tau_0/c$ is the rapidity. Thus, the Lorentz transformation

$$L(\tau_0) = \begin{pmatrix} \cosh\zeta_0 & -\sinh\zeta_0 \\ -\sinh\zeta_0 & \cosh\zeta_0 \end{pmatrix}, \tag{7.1.6}$$

brings the spaceship momentarily to rest.

(c) By definition,

$$\alpha^0 = a\sinh\left(\frac{a\tau}{c}\right), \qquad \text{and} \qquad \alpha^1 = a\cosh\left(\frac{a\tau}{c}\right). \tag{7.1.7}$$

Under the Lorentz transformation, α^μ transforms as

$$\alpha'^\mu = a \begin{pmatrix} \cosh\zeta_0 & -\sinh\zeta_0 \\ -\sinh\zeta_0 & \cosh\zeta_0 \end{pmatrix} \begin{pmatrix} \sinh\zeta_0 \\ \cosh\zeta_0 \end{pmatrix} = a\delta_{\mu,1} \tag{7.1.8}$$

Thus, the acceleration is truly a.

(d) In the Earth's (lab's) inertial frame, the accelerating/decelerating proper time is $\tau_\pm$, determined by

$$\frac{d}{2} = \pm\frac{c^2}{8g}\left[\cosh\left(\frac{8g\tau_\pm}{c}\right) - 1\right]. \tag{7.1.9}$$

Inverting this, we can write

$$\sinh\left(\frac{8g\tau_\pm}{c}\right) = \frac{4dg}{c^2}\sqrt{1 \pm \frac{c^2}{2dg}}. \tag{7.1.10}$$

Plugging this back into $t(\tau)$,

$$t_{\text{total}} = t(\tau_+) + t(\tau_-) = \frac{4d}{c}\left[\sqrt{1+\frac{c^2}{2dg}} - \sqrt{1-\frac{c^2}{2dg}}\right] \approx 1.94\,\text{yr}. \tag{7.1.11}$$

The total time measured by the aliens is

$$\tau_{\text{total}} = \tau_+ + \tau_- = \frac{c}{8g}\left[\sinh^{-1}\left(\frac{4dg}{c^2}\sqrt{1+\frac{c^2}{2dg}}\right) + \sinh^{-1}\left(\frac{4dg}{c^2}\sqrt{1-\frac{c^2}{2dg}}\right)\right] \approx 0.86\,\text{yr}, \tag{7.1.12}$$

which is smaller than the inertial frame measurement as expected, since the Proxima-Earth distance is Lorentz contracted in the spaceship frame. □

***Problem* 7.2** (Equal mass elastic scattering)**:** Two particles a and b, both with mass m, scatter elastically; label the outgoing particles, also of mass m, as c and d. Elastic scattering means the *total* 4-momentum before and after the collision is conserved. In the center-of-mass frame, the scattering angle is θ'. Show that the scattering angle θ in a frame where one incident particle is at rest and the other has energy E (the "lab frame") is given by

$$\cos^2\theta = \frac{\cos^2\theta'/2}{1 - \frac{E - mc^2}{E + mc^2}\sin^2\theta'/2}. \tag{7.2.1}$$

Comment on the nonrelativistic ($v \ll c$, $E \approx mc^2$) and ultrarelativistic ($v \approx c, E \gg mc^2$) limits. To make the algebra less painful, work in units where $c = 1$, and restore it by dimensional analysis at the end.

Hint: Consider 4-vector dot products $p_a \cdot p_b, p_b \cdot p_c$, and $p_a \cdot p_c$. These are 4-scalars and therefore the same in all frames, that is $p_a \cdot p_b = p'_a \cdot p'_b$ and so on. You will also want to recall the relativistic relations between 3-momentum, mass and energy. Use the $p_a \cdot p_b$ equation to solve for E' in terms of E and m, and the $p_b \cdot p_c$ equation to solve for E_c in terms of E, m and θ'; also show the useful relation

$$E_c - m = \frac{E - m}{2}(1 + \cos\theta'). \tag{7.2.2}$$

Then use $p_a \cdot p_c$ to find an expression for $\cos\theta$ that you can turn into the answer needed.

Solution.

First, with $p_a = \begin{pmatrix} E & \mathbf{p} \end{pmatrix}^T$ and $p_b = \begin{pmatrix} m & \mathbf{0} \end{pmatrix}^T$, where the 3-vectors are boldened, we can calculate $p_a \cdot p_b = mE$. By Lorentz invariance,

$$mE = p'_a \cdot p'_b = \begin{pmatrix} E' \\ \mathbf{p}' \end{pmatrix} \cdot \begin{pmatrix} E' \\ -\mathbf{p}' \end{pmatrix} = E'^2 + p'^2 = 2E'^2 - m^2 \Rightarrow E'^2 = \frac{m(E+m)}{2}. \tag{7.2.3}$$

Now, $p_b \cdot p_c = \begin{pmatrix} m & \mathbf{0} \end{pmatrix}^T \cdot \begin{pmatrix} E_c & \mathbf{p}_c \end{pmatrix}^T = mE_c$, but also,

$$mE_c = p'_b \cdot p'_c = \begin{pmatrix} E' \\ -\mathbf{p}' \end{pmatrix} \cdot \begin{pmatrix} E' \\ \mathbf{p}'_c \end{pmatrix} = E'^2 + p'p'_c\cos\theta'. \tag{7.2.4}$$

Now, note that the magnitude of the 4-vectors p'_b and p'_c is $E'^2 - p'^2 = E'^2 - p'^2_c = m^2$, since the mass are the same. So we can rewrite

$$mE_c = E'^2 + \left(E'^2 - m^2\right)\cos\theta' = m\frac{E+m}{2} + m\frac{E-m}{2}\cos\theta'. \tag{7.2.5}$$

Thus,

$$E_c - m = (E - m)\cos^2 x, \tag{7.2.6}$$

where $x = \theta'/2$. Finally, we can calculate $p_a \cdot p_c = EE_c - pp_c\cos\theta = E'^2 - p'p'_c\cos\theta' = p'_a \cdot p'_c$. Plugging in previous results,

$$p\frac{p_c}{E_c}\cos\theta = (E+m)\left(1 - \frac{m}{E_c}\right) = (E+m)\frac{(E-m)}{E_c}\cos^2 x = \frac{p^2}{E_c}\cos^2 x. \tag{7.2.7}$$

This leads to

$$\begin{aligned}
\cos^2\theta &= \frac{p^2}{p_c^2}\cos^4 x \\
&= \frac{(E^2-m^2)\cos^2 x}{2m(E-m)+(E-m)^2(1-\sin^2 x)} \\
&= \frac{(E^2-m^2)\cos^2 x}{E^2-m^2-(E-m)^2\sin^2 x} \\
&= \frac{\cos^2 x}{1-\frac{(E-m)^2}{(E-m)(E+m)}\sin^2 x} \\
&= \frac{\cos^2(\theta'/2)}{1-\frac{E-m}{E+m}\sin^2(\theta'/2)} \\
&= \frac{\cos^2(\theta'/2)}{1-\frac{E-mc^2}{E+mc^2}\sin^2(\theta'/2)},
\end{aligned} \tag{7.2.8}$$

where we have corrected for $c=1$ in the beginning. $\square$