Homework 2: Phys 7320 (Spring 2022)

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***Problem* 2.1** (A four-charge quadrupole)**:** A radiating quadrupole consists of a square of side a with charge $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of radiation?

Solution.

First, the monopole moment is just the total charge, which we know by the set up that $Q = 0$. Second, the four charge is in configuration such that they form two dipoles, each with $p = qa$ but with opposite signs. So the total dipole moment is also zero. Now, we calculate the quadrupole tensor

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{x})d^3x \tag{2.1.1}$$

The trajectories of the charges as they rotate in the (xy)-plane around the origin is

$$x_{1,n} = r\cos\left[\omega t + (n-1)\frac{\pi}{2}\right] \tag{2.1.2a}$$

$$x_{2,n} = r\sin\left[\omega t + (n-1)\frac{\pi}{2}\right] \tag{2.1.2b}$$

$$x_{3,n} = 0 \tag{2.1.2c}$$

for $n \in \{1, 2, 3, 4\}$ and $r = a/\sqrt{2}$ so that they make a square of sidelength a. Thus,

$$\begin{aligned}\rho(\mathbf{x}) &= q\big[\delta(\mathbf{x}-\mathbf{x}_1) - \delta(\mathbf{x}-\mathbf{x}_2) + \delta(\mathbf{x}-\mathbf{x}_3) - \delta(\mathbf{x}-\mathbf{x}_4)\big] \\ &= q\sum_{n=1}^{4}(-1)^{n-1}\delta(x_1 - x_{1,n})\delta(x_2 - x_{2,n})\delta(x_3 - x_{3,n})\end{aligned} \tag{2.1.3}$$

Then the tensor elements can be written as

$$Q_{ij} = q\sum_{n=1}^{4}(-1)^{n-1}\int(3x_i x_j - r^2\delta_{ij})\delta(x_1 - x_{1,n})\delta(x_2 - x_{2,n})\delta(x_3)dx_1dx_2dx_3 \tag{2.1.4}$$

First, because the quadrupole lies entirely in the (xy)-plane, $Q_{i3} = 0$ for all i. Plugging (2.1.4) into Mathematica, we can find the non-trivial elements and write

$$Q_{ij} = 6qr^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3qa^2 \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} e^{-2i\omega t} \tag{2.1.5}$$

By definition (9.43, Jackson), the vector

$$\begin{aligned}
\mathbf{Q}(\mathbf{n}) &= Q_{ij} n_j \\
&= \frac{3qa^2}{r} e^{-2i\omega t} \big[(x+iy)\hat{\mathbf{x}} + (ix - y)\hat{\mathbf{y}}\big] \\
&= \frac{3qa^2}{r} e^{-2i\omega t} (x+iy)(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \\
&= 3qa^2 e^{2i(\phi - \omega t)} \sin\theta \Big(\sin\theta \hat{\mathbf{r}} + \cos\theta \hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}\Big)
\end{aligned} \tag{2.1.6}$$

and we can then calculate from (9.44, Jackson)

$$\begin{aligned}
\mathbf{H} &= -\frac{ick^3}{24\pi} \frac{e^{ikr}}{r} \mathbf{n} \times \mathbf{Q}(\mathbf{n}) \\
&= -\frac{ick^3 qa^2}{8\pi} \frac{e^{i(kr - 2\omega t - 2\phi)}}{r} \sin\theta \Big(\cos\theta \hat{\boldsymbol{\phi}} - i\hat{\boldsymbol{\theta}}\Big)
\end{aligned} \tag{2.1.7}$$

and

$$\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n} = -\frac{iZ_0 ck^3 qa^2}{8\pi} \frac{e^{i(kr - 2\omega t - 2\phi)}}{r} \sin\theta \Big(\cos\theta \hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}\Big) \tag{2.1.8}$$

From (9.45, Jackson), the antenna pattern is then

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^6}{1152\pi^2} \Big| \big[\mathbf{n} \times \mathbf{Q}(\mathbf{n}) \times \mathbf{n}\big] \Big|^2 = \frac{Z_0 c^2 k^6 q^2 a^4}{128\pi^2} \big(1 - \cos^4\theta\big) \tag{2.1.9}$$

and the total radiated power is

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{Z_0 c^2 k^6 q^2 a^4}{64\pi} \int_0^\pi d\theta \sin\theta (1 - \cos^4\theta) = \frac{Z_0 c^2 k^6 q^2 a^4}{40\pi} \tag{2.1.10}$$

From the oscillatory term in (2.1.7) and (2.1.7), the frequency of radiation is 2ω.

□

***Problem* 2.2** (An antenna: exact)**:** A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.

Solution.

(a) Given the current $I = I_0 \sin\big(2\pi z/d\big)e^{-i\omega t}$ where $2\pi/d = 2\pi/\lambda = k$, we can write the current density in the antenna as

$$\mathbf{J} = I_0 \sin(kz) e^{-i\omega t}\delta(x)\delta(y)\hat{\mathbf{z}} \tag{2.2.1}$$

Then from (9.8, Jackson), the vector potential is

$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{z}}\frac{\mu_0 I_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int d^3x' \sin\big(kz'\big)\delta(x')\delta(z')e^{-ik\mathbf{n}\cdot\mathbf{x}'} \\ &= \hat{\mathbf{z}}\frac{\mu_0 I_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int_{-\pi/k}^{\pi/k} dz' \sin\big(kz'\big)e^{-ikz'\cos\theta} \\ &= -i\hat{\mathbf{z}}\frac{\mu_0 I_0}{2\pi}\frac{e^{i(kr-\omega t)}}{kr}\frac{\sin(\pi u)}{1-u^2}\end{aligned} \tag{2.2.2}$$

where $u = \cos\theta$. Then the magnetic field is

$$\mathbf{H} = \frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A} = \frac{I_0}{2\pi}\frac{e^{i(kr-\omega t)}}{r}\frac{\sin\pi u}{1-u^2}\mathbf{n}\times\hat{\mathbf{z}} = \frac{I_0}{2\pi}\frac{e^{i(kr-\omega t)}}{r^2}\frac{\sin(\pi u)}{1-u^2}(y\hat{\mathbf{x}} - x\hat{\mathbf{y}}) \tag{2.2.3}$$

Now, we write $\mathbf{H} = H\mathbf{w}$ where $\mathbf{w} = (y\hat{\mathbf{x}} - x\hat{\mathbf{y}})$. By definition, the antenna pattern is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{2}\,\mathrm{Re}\left[r^2\mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}^*)\right] \\ &= \frac{Z_0}{2}\,\mathrm{Re}\Big\{\mathbf{x}\cdot\big[(\mathbf{H}\times\mathbf{x})\times\mathbf{H}^*\big]\Big\} \\ &= \frac{Z_0}{2}|H|^2\Big\{\mathbf{x}\cdot\big[(\mathbf{w}\times\mathbf{x})\times\mathbf{x}\big]\Big\} \\ &= \frac{Z_0}{2}|H|^2\big(x^2+y^2\big)r^2 \\ &= \frac{Z_0}{2}|H|^2 r^4(1-u^2) \\ &= \frac{Z_0 I_0^2}{8\pi^2}\frac{\sin^2(\pi u)}{1-u^2}\end{aligned} \tag{2.2.4}$$

where we have solved the term in the curly bracket with Mathematica. Fig 1 shows a plot of $dP/d\Omega$.

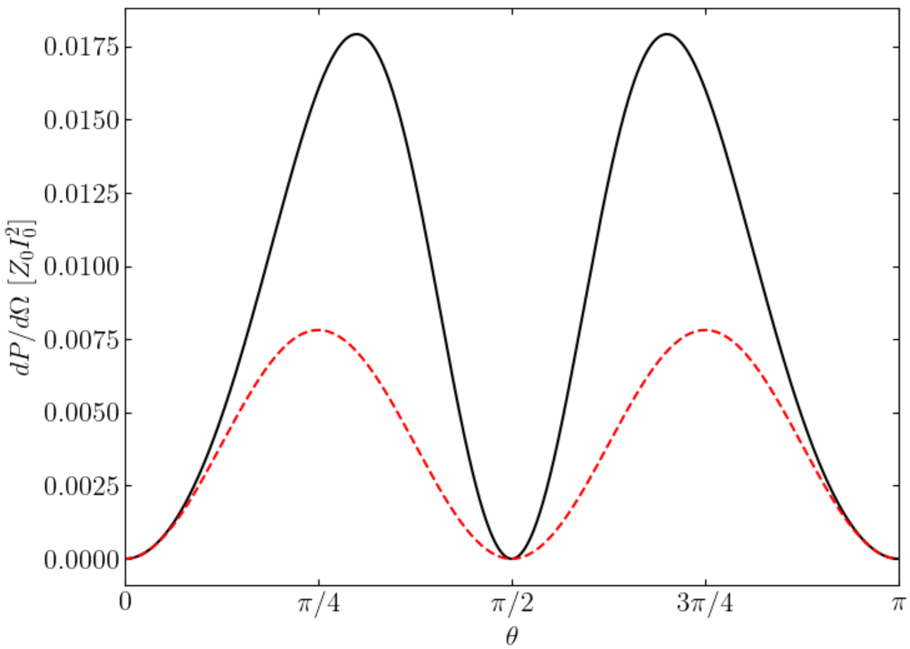

Figure 1: Angular distribution of the radiation pattern using exact derivation (solid black) and multipole expansion (dashed red).

(b) From the previous results, the total power is

$$P = \frac{Z_0 I_0^2}{4\pi} \int_0^1 du \frac{\sin^2(\pi u)}{1-u^2} \approx \frac{0.779}{4\pi} Z_0 I_0^2 \approx 0.062 Z_0 I_0^2 \tag{2.2.5}$$

The radiation resistance is

$$R_{\mathrm{rad}} = \frac{2P}{I_0^2} = \frac{0.779}{2\pi} Z_0 \approx 46.7\,\Omega \tag{2.2.6}$$

$\square$

***Problem* 2.3** (An antenna: multipole approximation)**:** Treat the linear antenna of Problem 9.16 by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) in the long wave-length approximation.

(b) Compare the shape of the angular distribution of radiated power for the lowest non-vanishing multipole with the exact distribution of Problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part (a). Compare with Problem 9.16b. Is there a paradox here?

Solution.

(a) From continuity equation, we can calculate the charge density using (2.2.1)

$$\rho = \frac{1}{i\omega}\frac{\partial J_z}{\partial z} = \frac{k}{i\omega} I_0 \cos(kz) e^{-i\omega t}\delta(x)\delta(y) \tag{2.3.1}$$

Then the electric dipole moment is

$$\mathbf{p} = \int d^3x' \mathbf{x}' \rho(\mathbf{x}') = \frac{k}{i\omega} I_0 e^{-i\omega t} \hat{\mathbf{z}} \int_{-d/2}^{d/2} z' \cos\big(kz'\big) dz' = \mathbf{0} \tag{2.3.2}$$

The magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2}\int d^3x' (\mathbf{x}' \times \mathbf{J}') = \frac{1}{2} I_0 e^{-i\omega t} \int d^3x' \sin(kz)\delta(x')\delta(y')(y'\hat{\mathbf{x}} - x'\hat{\mathbf{y}}) = \mathbf{0} \tag{2.3.3}$$

Finally, we calculate the quadrupole tensor

$$Q_{ij} = \frac{k}{i\omega} I_0 e^{-i\omega t} \int (3x_i x_j - r^2\delta_{ij}) \cos(kx_3)\delta(x_1)\delta(x_2) dx_1 dx_2 dx_3 \tag{2.3.4}$$

where we have switched to using (x_1, x_2, x_3) instead of (x, y, z) for convenience. Note that by symmetry (the antenna lies on the z-axis), all the off-diagonal terms vanish. Then we get

$$Q_{33} = \frac{k}{i\omega} I_0 e^{-i\omega t} \int_{-d/2}^{d/2} 2x_3^2 \cos(kx_3) dx_3 = i\frac{8\pi I_0}{\omega k^2} e^{-i\omega t} \tag{2.3.5}$$

and $Q_{11} = Q_{22} = -(1/2)Q_{33}$ so that Q_{ij} is traceless.

(b) From the previous results, the angular distribution is

$$\frac{dP}{d\Omega} = \frac{Z_0 c^2 k^6}{512\pi^2} |Q_{33}|^2 \sin^2\theta \cos^2\theta = \frac{Z_0 I_0^2}{32} \sin^2\theta\cos^2\theta \tag{2.3.6}$$

For a plot of this, see Fig 1. The angular distribution is still four-lobe, but the intensity is smaller and the locations of the maxima are shifted.

(c) Integrating the distribution from (b), we get

$$P = \frac{\pi}{16} Z_0 I_0^2 \int_0^\pi \sin^3\theta \cos^2\theta d\theta = \frac{\pi}{60} Z_0 I_0^2 \approx 0.052 Z_0 I_0^2 \tag{2.3.7}$$

and the radiation resistance is

$$R_{\text{rad}} = \frac{2P}{I_0^2} = \frac{\pi}{30} Z_0 = 39.45\,\Omega \tag{2.3.8}$$

These values are a bit lower than the results in Problem 2. But the total power is not too far from the exact solution. The discrepancy is only $\sim 16\%$. The rest might be due to higher order terms. $\square$