Homework 1: ASTR 5140 (Fall 2021)

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Problem 1. Maxwellian distribution. The most often used particle distribution in plasma physics is the drifting Maxwellian separate parallel and perpendicular temperature

$$f(\mathbf{v}) = Ae^{-\frac{m}{2}\left(\frac{(v_x-u_x)^2}{T_\perp}+\frac{(v_y-u_y)^2}{T_\perp}+\frac{(v_z-u_z)^2}{T_\parallel}\right)} \tag{1}$$

where $\mathbf{u}$ is the drift velocity (fluid velocity), $\mathbf{v}$ is the individual particle's velocity, and

$$A = n\left(\frac{m}{2\pi T_\perp}\right)\left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \tag{2}$$

(a) Show that

$$\int_{-\infty}^{\infty} f(\mathbf{v}) = n \tag{3}$$

Hint: Substitute the variable $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Do the integration for one dimension, then deduce the result for the other two dimension.

(b) Show that:

$$\int_{-\infty}^{\infty} \mathbf{v} f(\mathbf{v}) = n\mathbf{u} \tag{4}$$

Hint: Use symmetry arguments (e.g. the odd functions integrate to 0) to avoid carrying out the integration. Be succint. Carry out the integration for one direction and deduce the result for the other directions.

(c) Show that:

$$\int_{-\infty}^{\infty} \mathbf{v}\mathbf{v} f(\mathbf{v}) = n\mathbf{u}\mathbf{u} + \frac{n\mathbf{T}}{m} \tag{5}$$

where $\mathbf{T} = \mathrm{diag}\big(T_\perp, T_\perp, T_\parallel\big)$. *Hint*: Solve one diagonal term, for example $v_x v_x$, and one off-diagonal term, for example $v_x v_y$. Deduce the results for the remaining terms. Again, use symmetry arguments where possible.

(d) Sketch f versus v_x by hand.

Problem 2. Vlasov Equation. As done in class, let particle n of a given species be defined as a Dirac delta function

$$\delta(\mathbf{X}_n(t) - \mathbf{x})\delta(\mathbf{V}_n(t) - \mathbf{v}) \tag{6}$$

where $\mathbf{X}_n(t)$ is the instantaneous position of a particle and $\mathbf{V}_n(t)$ is the instantaneous velocity of the particle. The distribution function for species S can be described as

$$F_S(x, v, t) = \sum_n \delta(\mathbf{x} - \mathbf{X}_n(t))\delta(\mathbf{v} - \mathbf{V}_n(t)) \tag{7}$$

Derive the "Vlasov Equation" for the unsmoothed distribution F using only electromagnetic force.

Problem 3. Math Review. We will use vector notation including cross products, curl, and divergences quite often in this course so it is useful to be able to manipulate them.

(a) Show that: $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

(b) Show that: $\boldsymbol{\nabla} \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$

Hint: One method is to use the Levi-Civita symbol, ξ_{ijk}, where

$$\xi_{ijk} = \begin{cases} 1 & (i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1)\} \\ -1 & (i,j,k) \in \{(3,2,1),(1,3,2),(2,3,1)\} \end{cases} \tag{8}$$

Problem 4. Quasi-neutral Plasma. Calculate the condition of the ratio $\Delta N_c/N$ for gravity to dominate over the electromagnetic force on a proton near a star. N is the total number of protons in the star and ΔN_c is the number of unbalanced charges. Show that

$$\frac{\Delta N_c}{N} \ll 8 \times 10^{-37} \tag{9}$$

Problem 5. Debye Shielding. (a) Consider a conducting sphere of radius a and charge Q that is immersed in a collisionless, Maxwellian plasma that has density n_0, $T_i = 0$, but finite T_e. Let $\mathbf{B} = 0$. Solve the time-independent electron momentum equation

$$eEn_e + \gamma T_e \frac{\partial n_e}{\partial r} = 0 \tag{10}$$

to show that the isothermal equilibrium ($\gamma = 1$) electron density can be expressed as

$$n_e = n_0 e^{e\phi/T_e}, \qquad r > a \tag{11}$$

(b) Let the potential at the sphere be ϕ_0. In the limit $\phi_0 \ll T_e$, derive the potential ϕ as a function of r in spherical coordinates. Express your answer in terms of r, a, ϕ_0, and λ_D.