Homework 3: Phys 5210 (Fall 2021)

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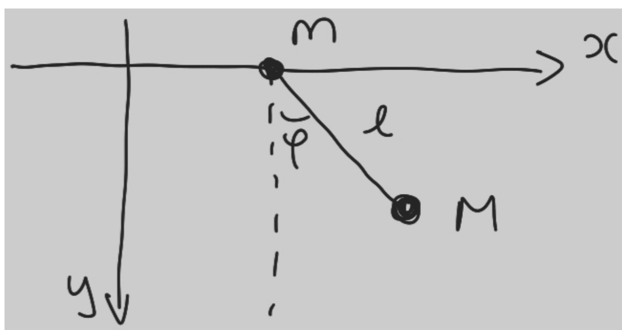
September 19, 2021

***Problem* 1:** A mass M is connected by a massless rod of length l to a bead of mass m which can slide along a horizontal railing without friction, as shown in the figure. The mass M and the bead undergo motion where their center of mass does not move in the horizontal direction, with gravity pulling on M downwards. Find the trajectory which the mass M follows in its motion.



Solution.

Define the coordinates as drawn in the figure. The position $\mathbf{r}_m = (x_m, y_m)$ of the bead and $\mathbf{r}_M = (x_M, y_M)$ of the mass M are as follows

$$\begin{aligned} x_m &= x & x_M &= x + l\sin\varphi \\ y_m &= 0 & y_M &= -l\cos\varphi \end{aligned} \tag{1.1}$$

By definition, the position of the center of mass is thus

$$\begin{aligned} \mathbf{r}_{COM} &= \frac{m}{m+M}\mathbf{r}_m + \frac{M}{m+M}\mathbf{r}_M \\ &= \left(x + \frac{M}{m+M}l\sin\varphi\right)\hat{\mathbf{x}} - \frac{M}{m+M}l\cos\varphi\hat{\mathbf{y}} \\ &= \left(x_M + \frac{m}{m+M}l\sin\varphi\right)\hat{\mathbf{x}} - \frac{M}{m+M}l\cos\varphi\hat{\mathbf{y}} \end{aligned} \tag{1.2}$$

Requiring that $\dot{x}_{COM} = 0$ leads to the differential equation

$$\frac{dx_M}{dt} = -\frac{m}{m+M}l\cos\varphi\frac{d\varphi}{dt} \tag{1.3}$$

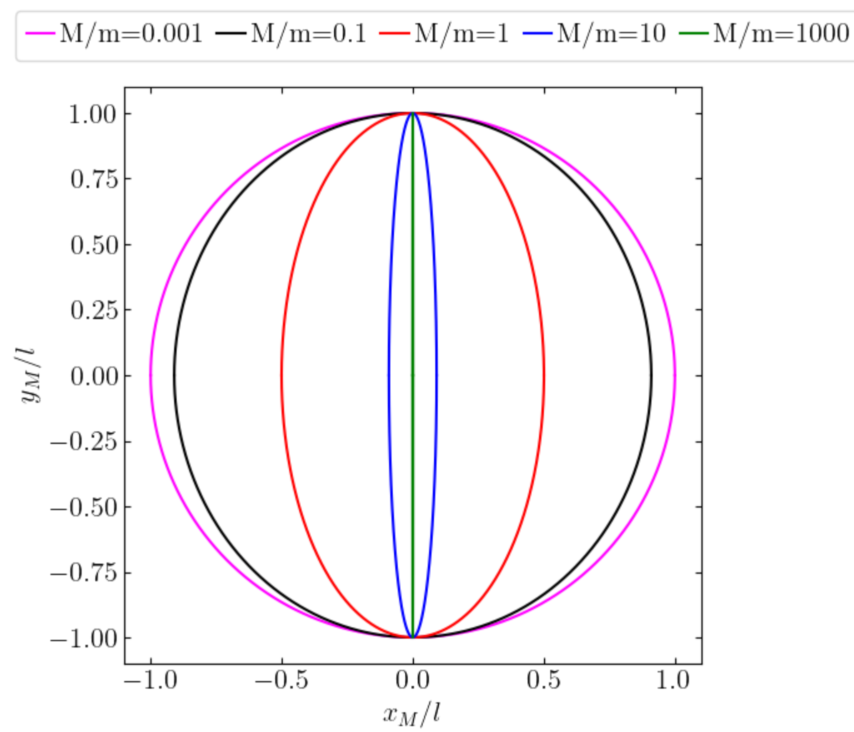
Integrating both sides, we can write $x_M = -ml\sin\varphi/(m+M)$. Now, we have

$$x_M^2 = \left(\frac{m}{m+M}\right)^2 l^2 \sin^2\varphi \qquad \text{and} \qquad y_M^2 = l^2\cos^2\varphi \tag{1.4}$$

Thus, the position of the mass M follows

$$\left(1+\frac{M}{m}\right)^2 x_M^2 + y_M^2 = l^2 \tag{1.5}$$

This is an elliptical trajectory. The dependence of its ellipticity on the mass ratio M/m is demonstrated in the figure below. If the bead is much more massive than the mass (magenta curve), then the bead is very inertial and will not accelerate too much. In this case, the mass has a circular trajectory, as expected of a single pendulum. The ellipticity increases as the bead becomes less massive compared to the mass. In the other limit where the mass is much more massive than the bead (green curve), then the center of mass is much closer to the mass. Since we require that the motion of the center of mass is only vertical, the mass should also roughly follow this trajectory. □



***Problem* 2:** Planets in their motion around the sun are supposed to follow elliptical trajectories. However, in practice their trajectories deviate from perfect ellipses. Some of that deviation can be explained if gravitational pull of other planets is taken into account. It has been known since the 19th century that the shift of the perihelion of Mercury (the point where it is the closest to the sun) cannot be explained merely by gravity of other planets.

We now know that this shift is due to the effects of general relativity. If we did not know about it, we could perhaps hypothesize the existence of a correction to the Newtonian gravity which perhaps could be written as

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2} \tag{2.1}$$

where $\alpha = GMm$, M is the solar mass and m is the mass of a planet, and β is some constant. Find the shift of the point closest to the sun after the planet completes one full revolution about the Sun, assuming that $\beta/R \ll \alpha$ where R is the typical distance between the planet and the sun, by Taylor expanding in powers of β and keeping the lowest term.

Hint: In this kind of problems, we have to deal with integrals of the (rough) form

$$\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{E - U(r) - l^2/2\mu r^2}} \tag{2.2}$$

Direct Taylor expansion of this in powers of β is problematic because it leads to integrals of the form

$$\int_{r_{\min}}^{r_{\max}} \frac{\beta dr}{r^2(E + \alpha/r - l^2/2\mu r^2)^{3/2}} \tag{2.3}$$

This integral would be divergent as r approaches the limits of integration where the expression in the round bracket vanishes. Instead, rewrite this integral as

$$-\frac{2\mu}{l}\frac{\partial}{\partial l}\int_{r_{\min}}^{r_{\max}} dr r^2 \sqrt{E - U(r) - \frac{l^2}{2\mu r^2}} \tag{2.4}$$

and only then expand in powers of β.

Solution.

The energy of a two-body orbit system is

$$E = \frac{\mu \dot{r}^2}{2} + U(r) + \frac{l^2}{2\mu r^2} \tag{2.5}$$

where $\mu = mM/(m+M)$ is the reduced mass of the system, and r is the relative distance between the two bodies. As discussed in class, the angular momentum $l = \mu\dot{\varphi}r^2$ is conserved. From (2.5), we can write the differential equation

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)^{1/2} \tag{2.6}$$

By using $dt = d\varphi\mu r^2/l$, we can write

$$\begin{aligned}
d\varphi &= \frac{l}{\sqrt{2\mu}} r^{-2}\left(E - \frac{l^2}{2\mu r^2} - U(r)\right)^{-1/2} dr \\
&= -\sqrt{2\mu}\frac{\partial}{\partial l}\left(E - \frac{l^2}{2\mu r^2} - U(r)\right)^{1/2} dr \qquad \text{(from (2.4))} \\
&\approx -\sqrt{2\mu}\frac{\partial}{\partial l}\left[\sqrt{E - \frac{l^2}{2\mu r^2} + \frac{\alpha}{r}} - \frac{\beta}{2r^2}\left(E - \frac{l^2}{2\mu r^2} + \frac{\alpha}{r}\right)^{-1/2}\right] dr \\
&= \left\{ r^{-2}\left(-\frac{1}{r^2} + \frac{2\mu\alpha}{l^2}\frac{1}{r} + \frac{2\mu E}{l^2}\right)^{-1/2} + \mu\beta\frac{\partial}{\partial l}\left[\frac{1}{l} r^{-2}\left(-\frac{1}{r^2} + \frac{2\mu\alpha}{l^2}\frac{1}{r} + \frac{2\mu E}{l^2}\right)^{-1/2}\right]\right\} dr
\end{aligned} \tag{2.7}$$

Integrating both sides, we get

$$\varphi = I + \mu\beta\frac{\partial}{\partial l}\left(\frac{I}{l}\right) \tag{2.8}$$

where

$$I = \int r^{-2}\left(-\frac{1}{r^2} + \frac{2\mu\alpha}{l^2}\frac{1}{r} + \frac{2\mu E}{l^2}\right) dr \tag{2.9}$$

We have solved this integral in class. By a change of variables $r = 1/x$, we can write

$$I = -\int dx\left(-x^2 + \frac{2\mu\alpha}{l^2}x + \frac{2\mu E}{l^2}\right)^{-1/2} = -\int dx\big[-(x-x_+)(x-x_-)\big]^{-1/2} \tag{2.10}$$

where

$$x_\pm = \frac{\mu\alpha}{l^2} \pm \sqrt{\frac{\mu^2\alpha^2}{l^4} + \frac{2\mu E}{l^2}} \tag{2.11}$$

By another change of variables $x = (x_+ + x_-)/2 + (x_+ - x_-)y/2$, the integral I turns into

$$I = -\int \frac{dy}{\sqrt{1-y^2}} = -\sin^{-1}(y) \tag{2.12}$$

Since $x = 1/r$, x_+ correponds to the point at the closest approach and x_- corresponds to that at the furthest approach. From (2.8), integrating I from x_+ to x_- results in the angular difference of half of the system orbit, which leads to $I = \pi$. Here, note that I describes the angular evolution of the unperturbed system ($\beta = 0$). Without the perturbation, a full revolution of this orbital system corresponds to an angular sweep of $\varphi = 0 \to 2\pi$.

The second term in (2.8) describes the first order perturbation of this angular evolution when $\beta \neq 0$. Thus, the phase shift of the point of the closest approach in half an orbital period is

$$\Delta\varphi_{\text{half}} = \varphi - \pi = \mu\beta\pi\frac{\partial}{\partial l}\left(\frac{1}{l}\right) = -\pi\frac{\mu\beta}{l^2} \tag{2.13}$$

It then follows from symmetry of the orbital trajectory that the shift in one period is $\Delta\varphi_{\text{full period}} = -2\pi(\mu\beta/l^2)$. $\square$