Homework 9: Phys 7310 (Fall 2021)

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***Problem* 9.1** (Hard ferromagnetic cylinder)**:** A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform throughout its volume and parallel to its axis. Determine the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside. Use the magnetic scalar potential Φ_M. In addition, find H_z and B_z at $z \sim 0$ where 0 is the vertical middle of the cylinder, for the limits $a \ll L$ and $a \gg L$.

Solution.

From (5.100, Jackson),

$$\Phi_M(z) = \frac{1}{4\pi}\oint_S \frac{\hat{\mathbf{n}}' \cdot \mathbf{M}(\mathbf{x}')da'}{|z\hat{\mathbf{z}} - \mathbf{x}'|} \tag{9.1.1}$$

because $\boldsymbol{\nabla}' \cdot \mathbf{M} = 0$ inside the cylinder. The surface integral is only non-trivial on the caps $S_\pm$ at $z = \pm L/2$ because $\hat{\mathbf{n}}' = \pm\hat{\mathbf{z}}$. We can then evaluate for Φ_M

$$\begin{aligned}
\Phi_M(z) &= \frac{M_0}{4\pi}\left[\int_0^a \frac{\rho' d\rho'}{\sqrt{\rho'^2 + (z - L/2)^2}} - \int_0^a \frac{\rho' d\rho'}{\sqrt{\rho'^2 + (z + L/2)^2}}\right]\int_0^{2\pi} d\phi \\
&= \frac{M_0}{2}\left[\sqrt{\rho'^2 + (z - L/2)^2} - \sqrt{\rho'^2 + (z + L/2)^2}\right]\Bigg|_0^a \\
&= \frac{M_0}{2}\left[\sqrt{a^2 + \left(z - \frac{L}{2}\right)^2} - \sqrt{a^2 + \left(z + \frac{L}{2}\right)^2} + \left|z + \frac{L}{2}\right| - \left|z - \frac{L}{2}\right|\right]
\end{aligned} \tag{9.1.2}$$

Thus, the magnetic field H_z is

$$H_z = -\frac{\partial \Phi_m}{\partial z} = -\frac{M_0}{2}\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} - \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}} + \frac{z + L/2}{|z + L/2|} - \frac{z - L/2}{|z - L/2|}\right] \tag{9.1.3}$$

everywhere along the axis. The magnetic induction inside is

$$\begin{aligned}
B_{|z|\le L/2} &= \mu_0 H_z + \mu_0 M_0 \\
&= -\frac{\mu_0 M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\right] + \mu_0 M_0 \\
&= -\frac{\mu_0 M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right] \qquad (9.1.4)
\end{aligned}$$

and the magnetic induction outside is

$$\begin{aligned}
B_{|z|>L/2} &= \mu_0 H_z \\
&= -\frac{\mu_0 M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right] \qquad (9.1.5)
\end{aligned}$$

So the magnetic induction also has the same form everywhere along z.

From (9.1.3) and (9.1.5), at $z = 0$, the magnetic field and magnetic induction are

$$B_z = \mu_0 M_0 \frac{L}{\sqrt{4a^2+L^2}} \qquad (9.1.6a)$$

$$H_z = M_0\left[\frac{L}{\sqrt{4a^2+L^2}} - 1\right] \qquad (9.1.6b)$$

Then for $a \ll L$,

$$B_z = \mu_0 M_0 \frac{1}{\sqrt{1+4(a/L)^2}} \approx \mu_0 M_0\left[1 - 2\frac{a^2}{L^2}\right] \qquad (9.1.7a)$$

$$H_z = M_0\left[\frac{1}{\sqrt{1+4(a/L)^2}} - 1\right] \approx -2M_0\frac{a^2}{L^2} \qquad (9.1.7b)$$

Similarly, for $L \ll a$,

$$B_z = \mu_0 M_0 \frac{L/a}{\sqrt{4+(L/a)^2}} \approx \frac{\mu_0 M_0}{2}\frac{L}{a} \qquad (9.1.8a)$$

$$H_z = M_0\left[\frac{L/a}{\sqrt{4+(L/a)^2}} - 1\right] \approx -M_0\left[1 - \frac{L}{2a}\right] \qquad (9.1.8b)$$

□

***Problem* 9.2** (Self-inductance)**:** A circuit consists of a long thin conducting shell of radius a and a parallel return wire of radius b on axis inside. If the current is assumed distributed uniformly throughout the cross section of the wire, calculate the self-inductance per unit length. What is the self-inductance if the inner conductor is a thin hollow tube?

Solution.

Draw an Amperian circular loop perpendicular to the axis of the wire with a radius $r < b$. Then the enclosed current is

$$I_{\text{enc}} = \oint_S \mathbf{J} \cdot d\mathbf{a} = 4\pi r^2 J = I\frac{r^2}{b^2} \tag{9.2.1}$$

where I is the total current running through the wire. Then by Ampere's Law, the magnetic field is

$$\mathbf{H}_{r<b} = \frac{I_{\text{enc}}}{2\pi r}\hat{\boldsymbol{\phi}} = \frac{I}{2\pi b}\frac{r}{b}\hat{\boldsymbol{\phi}} \tag{9.2.2}$$

The magnetic induction is thus

$$\mathbf{B}_{r<b} = \mu\mathbf{H} = \frac{\mu I}{2\pi b}\frac{r}{b}\hat{\boldsymbol{\phi}} \tag{9.2.3}$$

For $b \leq r \leq a$, the enclosed current is I and

$$\mathbf{H}_{b\leq r\leq a} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}} \qquad \text{and} \qquad \mathbf{B}_{b\leq r\leq a} = \frac{\mu I}{2\pi r}\hat{\boldsymbol{\phi}} \tag{9.2.4}$$

By definition, the self-inductance is

$$L = \frac{1}{I^2}\int \frac{B^2}{\mu} d^3x = \frac{\mu}{2\pi}\left[\frac{1}{b^2}\int_0^b r\,dr + \int_b^a \frac{dr}{r}\right] = \frac{\mu}{2\pi}\left[\frac{1}{2} + \ln\frac{a}{b}\right] \tag{9.2.5}$$

If the wire is hollow, then there is no magnetic field inside the wire and the self-inductance is just

$$L_{\text{hollow}} = \frac{\mu}{2\pi}\ln\frac{a}{b} \tag{9.2.6}$$

$\square$