Homework 1: Astr 5140 (Fall 2021)

Tien Vo

September 3, 2021

***Problem* 1** (Maxwellian distribution)**:** The most often used particle distribution in plasma physics is the drifting Maxwellian separate parallel and perpendicular temperature

$$f(\mathbf{v}) = Ae^{-\frac{m}{2}\left(\frac{(v_x-u_x)^2}{T_\perp}+\frac{(v_y-u_y)^2}{T_\perp}+\frac{(v_z-u_z)^2}{T_\parallel}\right)} \tag{1.1}$$

where $\mathbf{u}$ is the drift velocity (fluid velocity), $\mathbf{v}$ is the individual particle's velocity, and

$$A = n\left(\frac{m}{2\pi T_\perp}\right)\left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \tag{1.2}$$

(a) Show that

$$\int_{-\infty}^{\infty} f(\mathbf{v}) = n \tag{1.3}$$

Hint: Substitute the variable $\mathbf{w} = \mathbf{v} - \mathbf{u}$. Do the integration for one dimension, then deduce the result for the other two dimension.

(b) Show that:

$$\int_{-\infty}^{\infty} \mathbf{v} f(\mathbf{v}) = n\mathbf{u} \tag{1.4}$$

Hint: Use symmetry arguments (e.g. the odd functions integrate to 0) to avoid carrying out the integration. Be succint. Carry out the integration for one direction and deduce the result for the other directions.

(c) Show that:

$$\int_{-\infty}^{\infty} \mathbf{vv} f(\mathbf{v}) = n\mathbf{uu} + \frac{n\mathbf{T}}{m} \tag{1.5}$$

where $\mathbf{T} = \mathrm{diag}\big(T_\perp, T_\perp, T_\parallel\big)$. *Hint*: Solve one diagonal term, for example $v_x v_x$, and one off-diagonal term, for example $v_x v_y$. Deduce the results for the remaining terms. Again, use symmetry arguments where possible.

(d) Sketch f versus v_x by hand.

Solution.

(a) Note from (1.1) that $f(\mathbf{v}) = n\prod_{i\in\{x,y,z\}} g(v_i)$ where

$$g(v_i) = \sqrt{\frac{m}{2\pi T_i}}\exp\left[-\frac{m(v_i-u_i)^2}{2T_i}\right] \tag{1.6}$$

where $T_x = T_y = T_\perp$ and $T_z = T_\parallel$. Let $w_i = \sqrt{m/2T_i}(v_i - u_i)$, then

$$\int_{-\infty}^{\infty} g(v_i)dv_i = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-w_i^2}dw_i = 1 \tag{1.7}$$

It follows that

$$\int_{\mathbb{R}^3} f(\mathbf{v})d\mathbf{v} = n \prod_{i\in\{x,y,z\}} \int_{-\infty}^{\infty} g(v_i)dv_i = n \tag{1.8}$$

(b) Using (1.6) and the substitution $w_i = \sqrt{m/2T_i}(v_i - u_i)$, we can also calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} v_i g(v_i)dv_i &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2T_i}{m}}w_i + u_i\right)e^{-w_i^2}dw_i \\
&= \frac{1}{\sqrt{\pi}}\left[\sqrt{\frac{2T_i}{m}}\int_{-\infty}^{\infty} w_i e^{-w_i^2}dw_i + u_i\int_{-\infty}^{\infty} e^{-w_i^2}dw_i\right] \\
&= u_i
\end{aligned} \tag{1.9}$$

where the first term in the second equality is zero because w_i is odd, while $e^{-w_i^2}$ is even, and the domain $\mathbb{R}$ is even. Now, in the x direction of (1.4),

$$\begin{aligned}
\int_{\mathbb{R}^3} v_x f(\mathbf{v})d\mathbf{v} &= n \prod_{i\in\{x,y,z\}} \int_{-\infty}^{\infty} v_x g(v_i)dv_i \\
&= n\int_{-\infty}^{\infty} v_x g(v_x)dv_x \int_{-\infty}^{\infty} g(v_y)dv_y \int_{-\infty}^{\infty} g(v_z)dv_z \\
&= nu_x
\end{aligned} \tag{1.10}$$

The same must apply for y and z. So we can conclude that $\int_{\mathbb{R}^3}\mathbf{v}f(\mathbf{v})d\mathbf{v} = n\mathbf{u}$.

(c) Again, using (1.6) and the substitution $w_i = \sqrt{m/2T_i}(v_i - u_i)$, we can calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} v_i^2 g(v_i)dv_i &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{2T_i}{m}w_i^2 + 2\sqrt{\frac{2T_i}{m}}u_i w_i + u_i^2\right)e^{-w_i^2}dw_i \\
&= \frac{1}{\sqrt{\pi}}\left[\frac{2T_i}{m}\int_{-\infty}^{\infty} w_i^2 e^{-w_i^2}dw_i + 2\sqrt{\frac{2T_i}{m}}u_i\int_{-\infty}^{\infty} w_i e^{-w_i^2}dw_i + u_i^2\int_{-\infty}^{\infty} e^{-w_i^2}dw_i\right] \\
&= \frac{T_i}{m} + u_i^2
\end{aligned} \tag{1.11}$$

where the second integral in the second equality is zero because w_i is odd and $e^{-w_i^2}$ is even over an even domain. Now, for $k \in \{x, y, z\}$, it follows that

$$\int_{\mathbb{R}^3} v_k^2 f(\mathbf{v})d\mathbf{v} = n\prod_{i\in\{x,y,z\}}\int_{-\infty}^{\infty} v_k^2 g(v_i)dv_i = n\left(\frac{T_k}{m} + u_k^2\right) \tag{1.12}$$

and similarly, for $j \neq k$,

$$\int_{\mathbb{R}^3} v_j v_k f(\mathbf{v}) d\mathbf{v} = n \prod_{i\in\{x,y,z\}} \int_{-\infty}^{\infty} v_j v_k g(v_i) dv_i = n u_j u_k \tag{1.13}$$

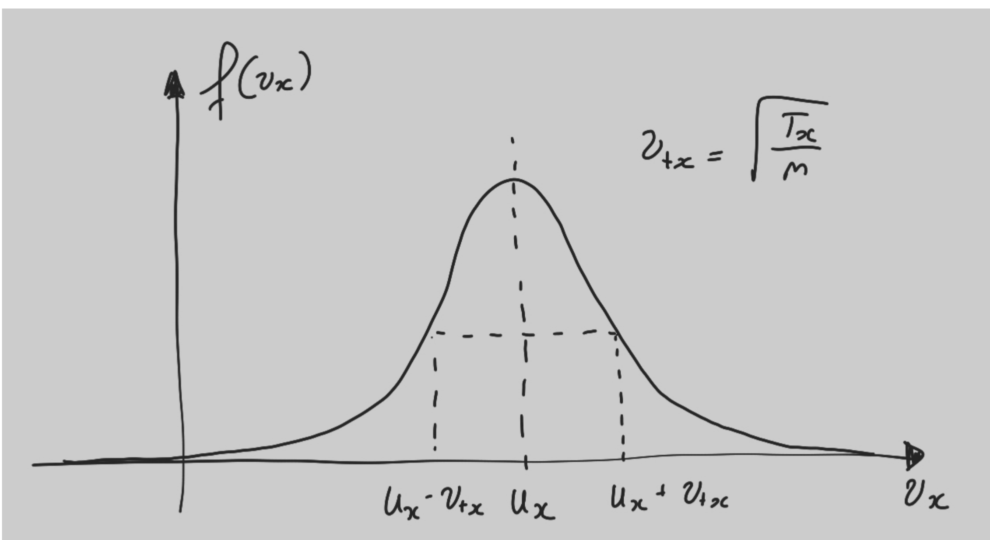
from (1.9). Note that these are general terms of the tensor

$$\int_{\mathbb{R}^3} \mathbf{v}\mathbf{v} f(\mathbf{v}) = \int_{\mathbb{R}^3} \begin{pmatrix} v_x^2 & v_x v_y & v_x v_z \\ v_y v_x & v_y^2 & v_y v_z \\ v_z v_x & v_z v_y & v_z^2 \end{pmatrix} f(\mathbf{v}) d\mathbf{v} \tag{1.14}$$

Thus, we can write explicitly from (1.12) and (1.13) that

$$\int_{\mathbb{R}^3} \mathbf{v}\mathbf{v} f(\mathbf{v}) d\mathbf{v} = n \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_y u_x & u_y^2 & u_y u_z \\ u_z u_x & u_z u_y & u_z^2 \end{pmatrix} + \frac{n}{m} \begin{pmatrix} T_\perp & 0 & 0 \\ 0 & T_\perp & 0 \\ 0 & 0 & T_\parallel \end{pmatrix} = n\mathbf{u}\mathbf{u} + \frac{n\mathbf{T}}{m} \tag{1.15}$$

(d) □



***Problem* 2** (Vlasov Equation)**:** As done in class, let particle n of a given species be defined as a Dirac delta function

$$\delta(\mathbf{X}_n(t) - \mathbf{x})\delta(\mathbf{V}_n(t) - \mathbf{v}) \tag{2.1}$$

where $\mathbf{X}_n(t)$ is the instantaneous position of a particle and $\mathbf{V}_n(t)$ is the instantaneous velocity of the particle. The distribution function for species s can be described as

$$F_s(x, v, t) = \sum_n \delta(\mathbf{x} - \mathbf{X}_n(t))\delta(\mathbf{v} - \mathbf{V}_n(t)) \tag{2.2}$$

Derive the "Vlasov Equation" for the unsmoothed distribution F using only electromagnetic force.

Solution.

Given the distribution function (2.2), we can differentiate with the chain rule

$$\begin{aligned}\frac{\partial F_s}{\partial t} &= \sum_n \left[\frac{\partial \delta(\mathbf{x}-\mathbf{X}_n)}{\partial \mathbf{X}_n}\cdot\frac{\partial \mathbf{X}_n}{\partial t}\delta(\mathbf{v}-\mathbf{V}_n) + \delta(\mathbf{x}-\mathbf{X}_n)\frac{\partial \delta(\mathbf{v}-\mathbf{V}_n)}{\partial \mathbf{V}_n}\cdot\frac{\partial \mathbf{V}_n}{\partial t}\right] \\ &= -\sum_n \left\{\mathbf{V}_n\cdot\left[\frac{\partial \delta(\mathbf{x}-\mathbf{X}_n)}{\partial \mathbf{x}}\delta(\mathbf{v}-\mathbf{V}_n)\right] + \frac{\partial \mathbf{V}_n}{\partial t}\cdot\left[\delta(\mathbf{x}-\mathbf{X}_n)\frac{\partial \delta(\mathbf{v}-\mathbf{V}_n)}{\partial \mathbf{v}}\right]\right\} \\ &= -\sum_n \left\{\left[\mathbf{V}_n\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{\partial \mathbf{V}_n}{\partial t}\cdot\frac{\partial}{\partial \mathbf{v}}\right]\left[\delta(\mathbf{x}-\mathbf{X}_n)\delta(\mathbf{v}-\mathbf{V}_n)\right]\right\} \qquad (2.3)\end{aligned}$$

where the velocity $\mathbf{V}_n = \partial \mathbf{X}_n/\partial t$ by definition. $\partial \mathbf{V}_n/\partial t$ follows the equation of motion

$$\frac{\partial \mathbf{V}_n}{\partial t} = \frac{q}{m}(\mathbf{E} + \mathbf{V}_n \times \mathbf{B}) \qquad (2.4)$$

Now, recall a property of the delta function for some arbitrary function f

$$\sum_n f(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}_n) = \sum_n f(\mathbf{X}_n)\delta(\mathbf{x}-\mathbf{X}_n) \qquad (2.5)$$

Then we can rewrite (2.3) as

$$\begin{aligned}\frac{\partial F_s}{\partial t} &= -\left[\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial \mathbf{v}}\right]\sum_n \delta(\mathbf{x}-\mathbf{X}_n)\delta(\mathbf{v}-\mathbf{V}_n) \\ &= -\mathbf{v}\cdot\frac{\partial F_s}{\partial \mathbf{x}} - \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial F_s}{\partial \mathbf{v}} \qquad (2.6)\end{aligned}$$

Re-arranging, we arrive at the Vlasov equation

$$\frac{\partial F_s}{\partial t} + \mathbf{v}\cdot\frac{\partial F_s}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial F_s}{\partial \mathbf{v}} = 0 \qquad (2.7)$$

□

Problem 3 (Math Review)**:** We will use vector notation including cross products, curl, and divergences quite often in this course so it is useful to be able to manipulate them.

(a) Show that: $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

(b) Show that: $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$

Hint: One method is to use the Levi-Civita symbol, ξ_{ijk}, where

$$\xi_{ijk} = \begin{cases} 1 & (i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1)\} \\ -1 & (i,j,k) \in \{(3,2,1),(1,3,2),(2,3,1)\} \end{cases} \qquad (3.1)$$

Solution.

(a) Starting from the LHS, we can show that

$$\begin{aligned}
\text{LHS} &= (\partial_l \hat{\mathbf{e}}_l) \times \big(\xi_{ijk}\hat{\mathbf{e}}_i \partial_j A_k\big) \\
&= \xi_{ijk}\partial_l \partial_j A_k \hat{\mathbf{e}}_l \times \hat{\mathbf{e}}_i \\
&= \xi_{ijk}\xi_{lin}\partial_l \partial_j A_k \hat{\mathbf{e}}_n \\
&= \xi_{jki}\xi_{nli}\partial_l \partial_j A_k \hat{\mathbf{e}}_n \\
&= (\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn})\partial_l \partial_j A_k \hat{\mathbf{e}}_n \\
&= (\partial_j \hat{\mathbf{e}}_j)(\partial_k A_k) - \partial_j^2 (A_k \hat{\mathbf{e}}_k) \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned} \tag{3.2}$$

(b) Similarly, from the LHS

$$\begin{aligned}
\text{LHS} &= (\partial_l \hat{\mathbf{e}}_l) \cdot \big(\xi_{ijk}\hat{\mathbf{e}}_i A_j B_k\big) \\
&= \xi_{ijk}\partial_i (A_j B_k) \\
&= \xi_{ijk}\big(B_k \partial_i A_j + A_j \partial_i B_k\big) \\
&= \xi_{kij} B_k \partial_i A_j - \xi_{jik} A_j \partial_i B_k \\
&= \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})
\end{aligned} \tag{3.3}$$

□

***Problem* 4** (Quasi-neutral Plasma)**:** Calculate the condition of the ratio $\Delta N_c/N$ for gravity to dominate over the electromagnetic force on a proton near a star. N is the total number of protons in the star and ΔN_c is the number of unbalanced charges. Show that

$$\frac{\Delta N_c}{N} \ll 8 \times 10^{-37} \tag{4.1}$$

Solution.

The gravitational force exerted on a proton with mass m by a cluster of N protons in a star with mass $M = Nm$ at a distance r (assuming $r \gg \overline{d}$ where $\overline{d}$ is the mean distance among protons in the star) is

$$F_G = \frac{GNm^2}{r^2} \tag{4.2}$$

With a number ΔN_c of unbalanced charge, the electrostatic force on the proton at a distance r is

$$F_E = \frac{1}{4\pi\epsilon_0}\frac{\Delta N_c e^2}{r^2} \tag{4.3}$$

The attractive gravitational force (4.2) dominates the repulsive electrostatic force (4.3) when $F_G \gg F_E$, or in other words,

$$\frac{\Delta N_c}{N} \ll \frac{4\pi\epsilon_0 G m^2}{e^2} \approx 8 \times 10^{-37} \tag{4.4}$$

□

***Problem* 5** (Debye Shielding)**:** (a) Consider a conducting sphere of radius a and charge Q that is immersed in a collisionless, Maxwellian plasma that has density n_0, $T_i = 0$, but finite T_e. Let $\mathbf{B} = \mathbf{0}$. Solve the time-independent electron momentum equation

$$eEn_e + \gamma T_e \frac{\partial n_e}{\partial r} = 0 \tag{5.1}$$

to show that the isothermal equilibrium ($\gamma = 1$) electron density can be expressed as

$$n_e = n_0 e^{e\phi/T_e}, \qquad r > a \tag{5.2}$$

(b) Let the potential at the sphere be ϕ_0. In the limit $\phi_0 \ll T_e$, derive the potential ϕ as a function of r in spherical coordinates. Express your answer in terms of r, a, ϕ_0, and λ_D.

Solution.
(a) Assume the electron density is given by (5.2), then

$$\frac{\partial n_e}{\partial r} = \frac{e}{T_e}\frac{\partial \phi}{\partial r} n_e = -\frac{eE}{T_e} n_e \tag{5.3}$$

where $E = -\partial\phi/\partial r$ when $\mathbf{B} = \mathbf{0}$. This is nothing but (5.1) in isothermal equilibrium ($\gamma = 1$).
(b) When $\phi_0 \ll T_e$, we can write $n_e \approx n_0(1 + e\phi/T_e)$ and Poisson equation becomes

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right) = -\frac{e(n_i - n_e)}{\epsilon_0} \approx -\frac{en_0}{\epsilon_0}\left[1 - \left(1 + \frac{e\phi}{T_e}\right)\right] = \frac{n_0 e^2}{\epsilon_0 T_e}\phi = \frac{\phi}{\lambda_D^2} \tag{5.4}$$

where λ_D is the Debye length. A guess to the solution for this is

$$\phi = C\frac{e^{-kr}}{r} \tag{5.5}$$

Differentiating, we can write

$$\nabla^2\phi = -\frac{C}{r^2}\frac{\partial}{\partial r}\left[(kr + 1)e^{-kr}\right] = Ck^2\frac{e^{-kr}}{r} = k^2\phi \tag{5.6}$$

From (5.4) and (5.6), $k = 1/\lambda_D$. Now, $\phi = \phi_0$ when $r = a$. Thus,

$$C\frac{e^{-a/\lambda_D}}{a} = \phi_0 \Rightarrow C = a\phi_0 e^{a/\lambda_D} \tag{5.7}$$

So the full solution to the Poisson equation is

$$\phi(r) = a\phi_0 e^{(a-r)/\lambda_D} \tag{5.8}$$

□