Homework 13: Phys 7320 (Spring 2022)

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***Problem* 13.1** (Noether currents for rotations and boosts)**:** (0) An infinitesimal rotation or Lorentz boost can be written in the form

$$x'^{\mu} = x^{\mu} + L^{\mu}_{\nu} x^{\nu}, \tag{13.1.1}$$

where L^{μ}_{ν} is constant, small and obeys $L_{\mu\nu} = -L_{\nu\mu}$ (this is Jackson's (11.89) in tensor notation; L is proportional to either the rotation angle θ or the boost parameter ζ). Take a general Lagrangian density $\mathcal{L}$ with no particular form depending on some fields ϕ_i, and show that the Lagrangian density $\mathcal{L}(x)$ changes by a total derivative under these transformations. (Compare to what we did in class to derive the Noether currents for translations.) Then show the associated Noether current takes the form (up to a possible constant factor)

$$J^{\mu} = L_{\nu\rho} M^{\mu\nu\rho}, \tag{13.1.2}$$

with $M^{\mu\nu\rho}$ as in Jackson (12.109),

$$M^{\mu\nu\rho} = T^{\mu\nu} x^{\rho} - T^{\mu\rho} x^{\nu}. \tag{13.1.3}$$

We have now motivated the form of Jackson's (12.109) for a general Lagrangian. Now when you do Jackson's part (a) and (b), use the improved form of the electromagnetic energy-momentum tensor (12.113) in the expression for $M^{\mu\nu\rho}$ as in (12.117).

Consider the various conservation laws that are contained in the integral of $\partial_{\alpha} M^{\alpha\beta\gamma} = 0$ over all space, where $M^{\alpha\beta\gamma}$ is defined in (12.117).

(a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

(b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\mathbf{X}}{dt} = \frac{c^2 \mathbf{P}_{\text{em}}}{E_{\text{em}}}, \tag{13.1.4}$$

where $\mathbf{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\mathbf{X} \int u d^3x = \int \mathbf{x} u d^3x, \tag{13.1.5}$$

where u is the electromagnetic energy density and E_{em} and $\mathbf{P}_{\text{em}}$ re the total energy and momentum of the fields.

Solution.

(0) First, the Jacobian of the transformation (13.1.1) is

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} + L^{\mu}_{\nu}. \tag{13.1.6}$$

Expanding $\mathcal{L}(x')$ to first order in $\epsilon^{\mu} = L^{\mu}_{\nu}x^{\nu}$, we get

$$\mathcal{L}(x') = \mathcal{L}(x) + \epsilon^{\mu}\partial_{\mu}\mathcal{L}(x) + \mathcal{O}\big(\epsilon^2\big). \tag{13.1.7}$$

Writing $K^{\mu} = L^{\mu}_{\nu}x^{\nu}\mathcal{L}$ and using the product rule, we get

$$\partial_{\mu}K^{\mu} = L^{\mu}_{\nu}\delta^{\nu}_{\mu}\mathcal{L} + L^{\mu}_{\nu}x^{\nu}\partial_{\mu}\mathcal{L} = \delta\mathcal{L}, \tag{13.1.8}$$

where $L^{\mu}_{\nu}\delta^{\nu}_{\mu} = L^{\mu}_{\mu} = 0$, since L is antisymmetric. So the Lagrangian density $\mathcal{L}$ changes by a total derivative of K. Also, the scalar fields $\phi_i(x)$ transform as

$$\phi_i(x') = \phi_i(x) + \epsilon^{\mu}\partial_{\mu}\phi_i(x) + \mathcal{O}\big(\epsilon^2\big). \tag{13.1.9}$$

So $\delta\phi_i = L^{\mu}_{\nu}x^{\nu}\partial_{\mu}\phi_i$. The current associated with these changes is defined as

$$\begin{aligned} J^{\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_i)}\delta\phi_i - K^{\mu} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_i)}L^{\nu}_{\rho}x^{\rho}\partial_{\nu}\phi_i - L^{\mu}_{\rho}x^{\rho}\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_i)}L_{\nu\rho}x^{\rho}\partial^{\nu}\phi_i - g^{\mu\nu}L_{\nu\rho}x^{\rho}\mathcal{L} \\ &= L_{\nu\rho}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi_i)}\partial^{\nu}\phi_i - g^{\mu\nu}\mathcal{L}\right]x^{\rho} \\ &= L_{\nu\rho}T^{\mu\nu}x^{\rho}. \end{aligned} \tag{13.1.10}$$

By a change of indices ($\nu \mapsto \rho$ and $\rho \mapsto \nu$), we can also write

$$L_{\nu\rho}T^{\mu\nu}x^{\rho} = L_{\rho\nu}T^{\mu\rho}x^{\nu} = -L_{\nu\rho}T^{\mu\rho}x^{\nu}. \tag{13.1.11}$$

Thus, the current can be written as

$$J^{\mu} = \frac{1}{2}L_{\nu\rho}(T^{\mu\nu}x^{\rho} - T^{\mu\rho}x^{\nu}) \sim L_{\nu\rho}M^{\mu\nu\rho}, \tag{13.1.12}$$

up to a factor of 1/2.

(a) First, by definition, we note that

$$\begin{aligned} M^{0ij} &= \Theta^{0i}x^j - \Theta^{0j}x^i \\ &= \frac{1}{4\pi}\Big[(\mathbf{E}\times\mathbf{B})^i x^j - (\mathbf{E}\times\mathbf{B})^j x^i\Big] \qquad \text{(from (12.114, Jackson))} \\ &= -\frac{1}{4\pi}\epsilon^{ijk}\big[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\big]^k. \end{aligned} \tag{13.1.13}$$

This is just proportional to the field angular momentum density. Thus,

$$\int d^3x M^{0ij} = -\frac{1}{4\pi}\epsilon^{ijk} \int d^3x \big[\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\big]^k = -c\epsilon^{ijk} L^k_{\text{field}}. \tag{13.1.14}$$

Then, since $\partial_\mu M^{\mu ij} = 0$, we can write

$$\int d^3x \partial_0 M^{0ij} = -c\epsilon^{ijk}\partial_0 L^k_{\text{field}} = \int d^3x \partial_k M^{kij} \Rightarrow \epsilon^{ijk}\frac{\partial L^k_{\text{field}}}{\partial t} = -\int d^3x \partial_k M^{kij}. \tag{13.1.15}$$

This is a sort of continuity equation for the field angular momentum, where the RHS (by Gauss divergence law) is a surface integral of the Maxwell stress tensor, according to (12.115, Jackson). Thus, this conservation law states that the change of the field angular momentum is balanced by the *flux* of electromagnetic force on a given surface.

(b) Setting $\beta = 0$, we consider

$$M^{00i} = \Theta^{00}x^i - \Theta^{0i}x^0 = \frac{E^2 + B^2}{8\pi}x^i - \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^i x^0 = 2ux^i - 2cg^i x^0. \tag{13.1.16}$$

Integrating, we get

$$\int d^3x M^{00i} = 2\int d^3x x^i u - 2cx^0 d^3x g^i = 2X^i E_{\text{em}} - 2cx^0 P^i_{\text{em}}, \tag{13.1.17}$$

where $E_{\text{em}} = \int d^3x u$ is the total field energy and $\mathbf{P}_{\text{em}} = \int d^3x \mathbf{g}$ is the total field momentum. If M^{00i} is a conserved charge, then

$$\frac{dX^i}{dt}E_{\text{em}} - c^2 P^i_{\text{em}} = 0 \Rightarrow \frac{d\mathbf{X}}{dt} = \frac{c^2\mathbf{P}_{\text{em}}}{E_{\text{em}}}, \tag{13.1.18}$$

where we have also assumed energy and momentum conservation $dE_{\text{em}} = 0$ and $d\mathbf{P}_{\text{em}}/dt = 0$. $\square$

***Problem* 13.2** (Proca equation.)**:** The Proca equation for a "massive" vector field $A_\mu(x)$ coupled to a current J^ν is

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \frac{4\pi}{c} J^\nu. \tag{13.2.1}$$

Due to the "mass" term $\mu^2 A^\nu$, this equation has no gauge invariance. The Proca equation is obeyed by massive spin-1 fields like the fields of the W-boson and Z-boson, and also describes the photon when it is superconducting.

(a) Assuming the parameter $\mu \neq 0$ and the current is conserved, show that the Proca equation implies the vanishing 4-divergence

$$\partial_\mu A^\mu = 0. \tag{13.2.2}$$

Note this is not a gauge condition (there is no gauge invariance for massive A_μ) but a consequence of the Proca equation. Then show the Proca equation becomes

$$(\partial_\mu \partial^\mu + \mu^2) A^\nu = \frac{4\pi}{c} J^\nu, \tag{13.2.3}$$

which is a Klein-Gordon equation for each component A^ν with mass parameter μ, sourced by the current J^ν.

(b) Consider Proca waves in a region with no current, $J^\nu = 0$. Use a plane wave ansatz of the form

$$A_\mu = \epsilon_\mu e^{-ik\cdot x}, \tag{13.2.4}$$

where $k \cdot x$ is a 4-vector dot product involving a 4-wavevector $k^\mu = (\omega/c, \mathbf{k})$, and ϵ_μ is a constant polarization vector. Using the results of part (a), first find the relationship between ω and $\mathbf{k}$, showing that k^μ is a timelike wavevector, not null, and so the waves travel at less than the speed of light.

Then find a constraint relating ϵ_μ and k^μ. Since there is no gauge invariance, this is the only constraint obeyed by ϵ_μ. Since k^μ is timelike, you may choose a frame where $k^\mu = (\omega/c, \mathbf{0})$; what directions may ϵ_μ point in this frame? We see there are three possible polarizations for the (massive) Proca wave, unlike the ordinary (massless) Maxwell electromagnetic wave which has two.

(c) Consider a time-independent delta-function charge at the origin $\rho(x) = q\delta^3(\mathbf{x})$, and show the Proca equation is solved by the Yukawa potential,

$$\Phi(x) = q\frac{e^{-\mu r}}{r}, \tag{13.2.5}$$

with $\mathbf{A} = \mathbf{0}$. Thus the force fied for a massive vector field is exponentially suppressed, explaining why we don't see long-range forces from the W-boson and Z-boson.

Hint: Recall that the 3D Laplacian acting on $1/r$ gives a delta function, as is needed for the case of the usual Coulomb potential (which is the $\mu \to 0$ limit of the Yukawa potential). The other pieces can be treated by looking at the Laplacian in spherical coordinates.

Solution.

(a) By definition, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, so

$$\partial_{\mu\nu} F^{\mu\nu} = \partial_{\mu\nu}\partial^\mu A^\nu - \partial_{\mu\nu}\partial^\nu A^\mu = \partial_\mu\partial^\mu(\partial_\nu A^\nu) - \partial_\nu\partial^\nu(\partial_\mu A^\mu) = \Box(\partial_\nu A^\nu - \partial_\mu A^\mu) = 0, \tag{13.2.6}$$

where $\Box = \partial_\mu\partial^\mu$ is the 4-Laplacian. Thus, the derivative of the Proca equation becomes

$$\partial_{\mu\nu} F^{\mu\nu} + \mu^2 \partial_\nu A^\nu = \mu^2 \partial_\nu A^\nu = \frac{4\pi}{c}\partial_\nu J^\nu = 0, \tag{13.2.7}$$

since the current J^μ is conserved. This implies that $\partial_\nu A^\nu = 0$ if $\mu \neq 0$. Then, expanding the Proca equation, we can write

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu + \mu^2 A^\nu = \left(\partial_\mu\partial^\mu + \mu^2\right)A^\nu = \frac{4\pi}{c}J^\nu. \tag{13.2.8}$$

(b) From the previous result, assuming $J^\mu = 0$,

$$\left(\partial_\mu\partial^\mu + \mu^2\right)A^\nu = \epsilon^\nu\left(-k_\mu k^\mu + \mu^2\right)e^{-ik\cdot x} = 0. \tag{13.2.9}$$

Since $\epsilon^\nu \neq 0$, it follows that

$$k_\mu k^\mu = \left(\frac{\omega}{c}\right)^2 - k^2 = \mu^2 > 0. \tag{13.2.10}$$

So k^μ is timelike. Also, the dispersion relation is

$$\omega^2 = c^2k^2 + c^2\mu^2. \tag{13.2.11}$$

The phase velocity is

$$\frac{\omega}{k} = c\sqrt{1 + \frac{\mu^2}{k^2}}, \tag{13.2.12}$$

and the group velocity is

$$\frac{d\omega}{dk} = \frac{c^2}{\omega/k} = \frac{c}{\sqrt{1+\mu^2/k^2}}. \tag{13.2.13}$$

Since the denominator is always larger than unity because $\mu > 0$, $d\omega/dk$ is less than the speed of light. Also, since $\partial_\mu A^\mu = 0$, it follows that

$$\partial_\mu A^\mu = -ik_\mu \epsilon^\mu e^{-ik_\nu x^\nu} = 0 \Rightarrow k_\mu \epsilon^\mu = 0. \tag{13.2.14}$$

Without loss of generality, assume $\mathbf{k} = k\hat{\mathbf{x}}$ and let the Lorentz boost be along $\hat{\mathbf{x}}$

$$\Lambda = \begin{pmatrix} \frac{\omega^2}{c^2\mu^2} & -\frac{k\omega}{c\mu^2} & 0 & 0 \\ -\frac{k\omega}{c\mu^2} & \frac{\omega^2}{c^2\mu^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{13.2.15}$$

Then it follows that

$$k'^{\mu} = \Lambda^{\mu}_{\nu} k^{\nu} = \begin{pmatrix} \omega/c \\ \mathbf{0} \end{pmatrix}. \tag{13.2.16}$$

From the above constraint that $k_\mu \epsilon^\mu = 0$. The transformed ϵ'^μ must also be perpendicular with k'^μ. So it can point in any of the three space-like directions, resulting in 3 possible polarizations.

(c) We expand the LHS of the Proca equation for $A^0 = \Phi$, since $A^i = 0$.

$$\begin{aligned}
\text{LHS} &= (-\boldsymbol{\nabla}^2 + \mu^2)\Phi \\
&= \mu^2 \Phi - q\left[\frac{1}{r}\boldsymbol{\nabla}^2\left(e^{-\mu r}\right) + e^{-\mu r}\boldsymbol{\nabla}^2\left(\frac{1}{r}\right) + 2\boldsymbol{\nabla}\left(e^{-\mu r}\right)\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right)\right] \\
&= \mu^2 \Phi - q\left[\frac{\mu^2}{r}e^{-\mu r} - \frac{2\mu}{r}e^{-\mu r} - 4\pi\delta(\mathbf{x})e^{-\mu r} + 2\frac{\mu}{r^2}e^{-\mu r}\right] \\
&= 4\pi q \delta^3(\mathbf{x}) \\
&= \text{RHS},
\end{aligned} \tag{13.2.17}$$

where we have used the familiar results

$$\boldsymbol{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{x}), \qquad \text{and} \qquad \boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{\mathbf{r}}}{r^2}, \tag{13.2.18}$$

and also, we have recognized that $J^0 = c\rho(\mathbf{x}) = cq\delta^3(\mathbf{x})$. Thus, the Yukawa potential solves the Proca equation. $\square$