Homework 4: Phys 7310 (Fall 2021)

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***Problem* 4.1** (Potential in a box)**:** A hollow cube has conducting walls defined by six planes $x = 0, y = 0, z = 0$ and $x = a, y = a, z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

(a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

(b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

(c) Find the surface charge density on the surface $z = a$.

Solution.

(a) First, we shift the coordinate in the z direction with a transformation $z \mapsto z - a/2$ so that the top wall is at $z = a/2$ and the bottom wall is at $z = -a/2$. Now, the boundary conditions in the xy plane remain the same as the case in Section 2.9 of Jackson. Thus, the potential may be written as

$$\Phi = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) Z_{nm}(z) \tag{4.1.1}$$

where $\alpha_n = n\pi/a$ and $\beta_m = m\pi/a$ for $n, m \in \mathbb{N}$. The function Z_{nm} satisfies the ordinary differential equation

$$\frac{d^2 Z_{nm}}{dz^2} = \gamma_{nm}^2 Z_{nm} \tag{4.1.2}$$

where $\gamma_{nm}^2 = \alpha_n^2 + \beta_m^2 = (\pi/a)\sqrt{n^2 + m^2}$. The general solution is

$$Z_{nm}(z) = A \sinh(\gamma_{nm} z) + B \cosh(\gamma_{nm} z) \tag{4.1.3}$$

The boundary conditions at the top and bottom walls require that $Z_{nm}(a/2) = Z_{nm}(-a/2)$. Thus, Z_{nm} cannot be odd, meaning A must be zero. Then we can write the potential as

$$\Phi = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \cosh(\gamma_{nm} z) \tag{4.1.4}$$

Now, at $z = a/2$,

$$\Phi(x, y, a/2) = V = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha n x) \sin(\beta_m y) \cosh(\gamma_{nm} z) \tag{4.1.5}$$

So we can find the coefficients A_{nm} from the orthogonality condition

$$\begin{aligned} A_{nm} &= \frac{4V}{a^2 \cosh\big(\gamma_{nm} a/2\big)} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx \int_0^a \sin\left(\frac{m\pi y}{a}\right) dy \\ &= \frac{4V}{\pi^2} \cosh\big(\gamma_{nm} a/2\big) \frac{\big[1 - (-1)^n\big]\big[1 - (-1)^m\big]}{nm} \end{aligned} \tag{4.1.6}$$

Finally, the potential is

$$\Phi = \frac{4V}{\pi^2} \sum_{n,m=1}^{\infty} \frac{[1 - (-1)^n][1 - (-1)^m]}{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \frac{\cosh(\gamma_{nm} z)}{\cosh\big(\gamma_{nm} a/2\big)} \tag{4.1.7}$$

(b) Using the following Python script, we calculate the potential Φ with the finite sum $\sum_{n=1}^{N} \sum_{m=1}^{M}$

```python
import numpy as np

V = 1
a = 1
x = a / 2
y = a / 2
z = 0

def Phi_nm(n, m):
    g = np.pi / a * np.sqrt(n ** 2 + m ** 2)
    return 4 * V / (np.pi ** 2) * (1 - (-1) ** n) * (1 - (-1) ** m) \
        / n / m * np.sin(n * np.pi * x / a) * np.sin(m * np.pi * y / a) \
        * np.cosh(g * z) / np.cosh(g * a / 2)

def Phi(N=1, M=1):
    y = 0
    for n in range(1, N + 1):
        for m in range(1, M + 1):
            y += Phi_nm(n, m)

    return y

for N in range(1, 7):
    for M in range(1, 7):
        print(f"(N,M)=({N},{M}): Phi={Phi(N,M):.5f}")
```

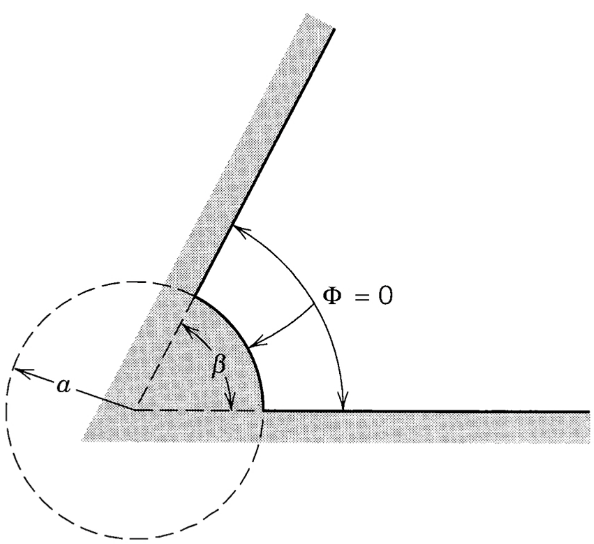
The average potential on the walls is $\langle\Phi\rangle = V/3$. In the script, with $V = 1$, we found that Φ reaches 0.333 when $(N, M) = (3, 5)$ or $(N, M) = (5, 3)$. So it takes 15 terms in total to reach an accuracy of at least 3 significant figures. Also, for large N and M, Φ converges to 1/3.

(c) At $z = a/2$, the normal vector $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, so

$$\begin{aligned}\sigma &= -\epsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=a/2} \\ &= -\frac{4\epsilon_0 V}{\pi a}\sum_{n,m=1}^{\infty}\frac{\sqrt{n^2+m^2}}{nm}\big[1-(-1)^n\big]\big[1-(-1)^m\big]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\tanh\big(\gamma_{nm}a/2\big)\end{aligned} \tag{4.1.8}$$

□

***Problem* 4.2** (Almost a corner)**:** The two-dimensional region, $\rho \geq a, 0 \leq \phi \leq \beta$, is bounded by conducting surfaces at $\phi = 0, \rho = a$, and $\phi = \beta$ held at zero potential, as indicated in the sketch. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.



(a) Write down a solution for the potential $\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite ρ.

(b) Keeping only the lowest nonvanishing terms, calculate the electric field components E_ρ and E_ϕ and also the surface charge densities $\sigma(\rho, 0), \sigma(\rho, \beta)$, and $\sigma(a, \phi)$ on the three boundary surfaces.

(c) Consider $\beta = \pi$ (a plane conductor with a half-cylinder of radius a on it). Show that far from the half-cylinder the lowest order terms of part (b) give a uniform electric field normal to the plane. Sketch the charge density on and in the neighborhood of the half-cylinder. For fixed electric field strength far from the plane, show that the total charge on the half-cylinder (actually charge per unit length in the z direction) is twice as large as would reside on a strip of width $2a$ in its absence. Show that the extra portion is drawn from regions of the plane nearby, so that the total charge on a strip of width large compared to a is the same whether the half-cylinder is there or not.

Solution.

(a) By separation of variable, we write $\Phi(\rho,\phi) = R(\rho)\Psi(\phi)$. Similar to Section 2.11 in Jackson, the functions R and Ψ satisfy the ordinary differential equations

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \nu^2 \qquad \text{and} \qquad \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -\nu^2 \tag{4.2.1}$$

If $\nu = 0$, then the solutions are

$$R_{\nu=0}(\rho) = A_0 + B_0 \ln\rho \qquad \text{and} \qquad \Psi_{\nu=0}(\phi) = C_0 + D_0\phi \tag{4.2.2}$$

However, the boundary conditions force these zeroth terms to vanish. At $\phi = 0$, $\Psi = C_0 = 0$ and $\phi = \beta \neq 0$, $\Psi = D_0\beta = 0$. So both C_0 and D_0 are zero. Now, for $\nu \neq 0$, the solutions are

$$R(\rho) = A\rho^{\nu} + b\rho^{-\nu} \tag{4.2.3a}$$
$$\Psi(\phi) = C\cos(\nu\phi) + D\sin(\nu\phi) \tag{4.2.3b}$$

At $r = a$, $R = Aa^{\nu} + Ba^{-\nu} = 0$, so $B = -Aa^{2\nu}$. At $\phi = 0$, $\Psi = C = 0$. At $\phi = \beta$, $\Psi = D\sin(\nu\beta) = 0$. Since the non-trivial solution has $D \neq 0$, $\sin(\nu\beta) = 0$ and thus $\nu = n\pi/\beta$ for $n \in \mathbb{N}$. Then we can write the general solution for finite ρ as

$$\Phi(\rho,\phi) = \sum_{n\in\mathbb{N}} A_n\left(\rho^{n\pi/\beta} - a^{2n\pi/\beta}\rho^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right) \tag{4.2.4}$$

(b) The electric field is $\mathbf{E} = -\boldsymbol{\nabla}\Phi = -(\partial\Phi/\partial\rho)\hat{\boldsymbol{\rho}} - (1/\rho)(\partial\Phi/\partial\phi)\hat{\boldsymbol{\phi}}$. Thus, up to the first non-trivial term, the components are

$$E_\rho = -\frac{\pi A_1}{\beta}\left(\rho^{\pi/\beta-1} + a^{2\pi/\beta}\rho^{-\pi/\beta-1}\right)\sin\left(\frac{\pi\phi}{\beta}\right) \tag{4.2.5}$$

and

$$E_\phi = -\frac{\pi A_1}{\beta}\left(\rho^{\pi/\beta-1} - a^{2\pi/\beta}\rho^{-\pi/\beta-1}\right)\cos\left(\frac{\pi\phi}{\beta}\right) \tag{4.2.6}$$

Now, the surface-charge density is $\sigma = \epsilon_0\mathbf{E}\cdot\hat{\mathbf{n}}$ at a conductor. At $\phi = 0$, $\hat{\mathbf{n}} = \hat{\boldsymbol{\phi}}$. Thus,

$$\sigma(\rho,0) = \epsilon_0\, E_\phi\bigg|_{\phi=0} = -\frac{\pi\epsilon_0}{\beta}A_1\left(\rho^{\pi/\beta-1} - a^{2\pi/\beta}\rho^{-\pi/\beta-1}\right) \tag{4.2.7}$$

Similarly, at $\phi = \beta$, $\hat{\mathbf{n}} = -\hat{\boldsymbol{\phi}}$ and we can write

$$\sigma(\rho,\beta) = -\epsilon_0\, E_\phi\bigg|_{\phi=\beta} = -\frac{\pi\epsilon_0}{\beta}A_1\left(\rho^{\pi/\beta-1} - a^{2\pi/\beta}\rho^{-\pi/\beta-1}\right) \tag{4.2.8}$$

At $\rho = a$, $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$, so

$$\sigma(a, \phi) = \epsilon_0 E_\rho = -\frac{2\pi\epsilon_0}{\beta} A_1 a^{\pi/\beta - 1} \sin\left(\frac{\pi\phi}{\beta}\right) \tag{4.2.9}$$

(c) At $\beta = \pi$, the electric field components are

$$E_\rho = -A_1 \left[1 + \left(\frac{a}{\rho}\right)^2\right] \sin\phi \tag{4.2.10a}$$

$$E_\phi = -A_1 \left[1 - \left(\frac{a}{\rho}\right)^2\right] \cos\phi \tag{4.2.10b}$$

We can now write the electric field in Cartesian coordinates

$$\begin{aligned}
\mathbf{E} &= E_\rho \hat{\boldsymbol{\rho}} + E_\phi \hat{\boldsymbol{\phi}} \\
&= E_\rho(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}) + E_\phi(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) \\
&= \big(E_\rho \cos\phi - E_\phi \sin\phi\big)\hat{\mathbf{x}} + \big(E_\rho \sin\phi + E_\phi \cos\phi\big)\hat{\mathbf{y}} \\
&= -2A_1 \sin\phi\cos\phi\left(\frac{a}{\rho}\right)^2 \hat{\mathbf{x}} - A_1\left[1 + \left(\frac{a}{\rho}\right)^2 (\sin^2\phi - \cos^2\phi)\right]\hat{\mathbf{y}}
\end{aligned} \tag{4.2.11}$$

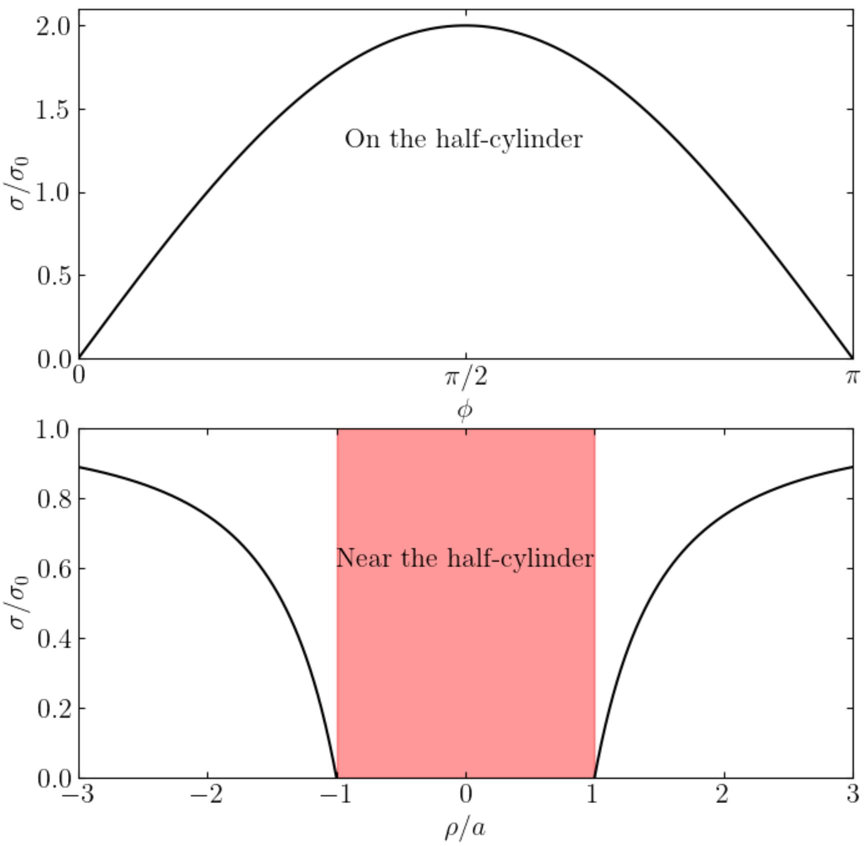
So for $\rho \to \infty$, $E_x \to 0$ and $E_y \to -A_1$. The electric field is thus uniform in the direction normal to the plane ($\hat{\mathbf{y}}$). The charge densities (4.2.7), (4.2.8), and (4.2.9) are now simplified to

$$\sigma(\rho, 0) = -\epsilon_0 A_1 \left[1 - \left(\frac{a}{\rho}\right)^2\right] \tag{4.2.12a}$$

$$\sigma(\rho, \beta) = -\epsilon_0 A_1 \left[1 - \left(\frac{a}{\rho}\right)^2\right] \tag{4.2.12b}$$

$$\sigma(a, \phi) = -2\epsilon_0 A_1 \sin\phi \tag{4.2.12c}$$

Note that $-\epsilon_0 A_1 = \epsilon_0 E_\infty = \sigma_0$ is the charge density of an infinite plane conductor. A plot of these charge densities is given below. The top panel shows the density on the half-cylindrical surface, while the bottom panel shows the density near the cylinder ($\rho \gtrsim a$ and $\rho \lesssim -a$).



From (4.2.12c), the total charge on the half-cylinder is

$$Q_{cyl} = \int \sigma da = 2a\sigma_0 \int_0^\pi \sin\phi d\phi \int_0^z dz' = 4a\sigma_0 z \tag{4.2.13}$$

Thus, the charge per unit length in z is

$$q_{cyl} = \frac{Q_{cyl}}{z} = 4a\sigma_0 \tag{4.2.14}$$

If there were no half-cylinder, the surface charge density would be σ_0. So the total charge per unit length z on the strip from $-a$ to a is

$$q_{strip} = \sigma_0 \int_{-a}^{a} dx = 2a\sigma_0 \tag{4.2.15}$$

Thus, $q_{cyl} = 2q_{strip}$. Now, we integrate for the charge near the half-cylinder. Because of the symmetry in (4.2.12a) and (4.2.12b), we know that the charge in the positive ρ region q_+ is the same as the charge in the negative ρ region q_-. Thus, we need only calculate q_+ from a to L where $L > a$

$$q_+ \approx \sigma_0 \int_a^L dx = \sigma_0(L-a) = q_- \tag{4.2.16}$$

The total charge including those on the half-cylinder and those near it is

$$q_{total} = q_{cyl} + 2q_+ = 2\sigma_0(L+a) = 2\sigma_0 L\left(1 + \frac{a}{L}\right) \tag{4.2.17}$$

If $L \gg a$, then $q_{total} = 2\sigma_0 L$. However, if there were no cylinder, the charge of the strip would be $q_{strip} = 2\sigma_0 L$, as indicated by (4.2.15). This means the extra charge calculated in (4.2.14) that leads to $q_{cyl} = 2q_{strip}$ is drawn from regions of the plane nearby. □

***Problem* 3.3** (A sphere with a hole in the top)**:** A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta) \tag{3.3.1}$$

where, for $l = 0, P_{l-1}(\cos\alpha) = -1$. What is the potential inside?

(b) Find the magnitude and the direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

Solution.

(a) This problem has an azimuthal symmetry, so the potential can be generally written as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) \tag{3.3.2}$$

Inside the sphere, $r^{-(l+1)}$ blows up at $r = 0$, so $B_l = 0$ for all l. Similarly, outside the sphere, $A_l = 0$ because $r^l \to \infty$ at large radius. Then we have the solution inside and outside the sphere as

$$\Phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{3.3.3a}$$

$$\Phi_{out} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \tag{3.3.3b}$$

By continuity, $\Phi_{in}\big|_{r=R} = \Phi_{out}\big|_{r=R} \Rightarrow B_l = A_l R^{2l+1}$. Also, we can calculate the surface charge density at $r = R$ from the potential inside and outside the sphere

$$\begin{aligned}
\sigma &= -\epsilon_0 \left[\frac{\partial \Phi_{out}}{\partial r}\bigg|_{r=R} - \frac{\partial \Phi_{in}}{\partial r}\bigg|_{r=R} \right] \\
&= -\epsilon_0 \sum_{l=0}^{\infty} \left[-(l+1) B_l r^{-(l+2)} - l A_l R^{l-1} \right] P_l(\cos\theta) \\
&= \epsilon_0 \sum_{l=0}^{\infty} A_l (2l+1) R^{l-1} P_l(\cos\theta)
\end{aligned} \tag{3.3.4}$$

Now, we are also given the surface charge density

$$\sigma(\theta) = \begin{cases} \frac{Q}{4\pi R^2} & \theta > \alpha \\ 0 & \theta \leq \alpha \end{cases} \tag{3.3.5}$$

Thus, for $\theta > \alpha$ we can find the coefficients A_l by using the orthogonality condition

$$A_l(2l+1)R^{l-1} = \frac{2l+1}{2} \int_{\alpha}^{\pi} \sigma(\theta) P_l(\cos\theta) \sin\theta d\theta = \frac{Q}{4\pi\epsilon_0 R^2} \frac{2l+1}{2} \int_{\alpha}^{\pi} P_l(\cos\theta) \sin\theta d\theta \tag{3.3.6}$$

Thus,

$$A_l = \frac{Q}{8\pi\epsilon_0} R^{-(l+1)} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \tag{3.3.7}$$

Then we can write the solution for Φ as

$$\Phi_{in} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r^l}{R^{l+1}} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} P_l(\cos\theta) \tag{3.3.8a}$$

$$\Phi_{out} = \frac{Q}{8\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{R^l}{r^{l+1}} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} P_l(\cos\theta) \tag{3.3.8b}$$

Note that for $r \gg R$, the potential should be approximately that of a point charge

$$\Phi_{\infty} \approx \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int \sigma da = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \frac{Q}{2} \int_{\alpha}^{\pi} \sin\theta d\theta = \frac{Q(\cos\alpha + 1)}{8\pi\epsilon_0 r} \tag{3.3.9}$$

From (3.3.8b),

$$\Phi_{out} = \frac{Q}{8\pi\epsilon_0} \frac{1}{r} \left[\cos\alpha - P_{-1}(\cos\alpha) \right] + \mathcal{O}\left(r^{-2}\right) \approx \frac{Q(\cos\alpha - P_{-1}(\cos\alpha))}{8\pi\epsilon_0 r} \tag{3.3.10}$$

Comparing (3.3.9) and (3.3.10), we must then require that $P_{-1}(\cos\alpha) = -1$.

(b) The electric field is $\mathbf{E} = -\boldsymbol{\nabla}\Phi = -(\partial\Phi/\partial r)\hat{\mathbf{r}} - (1/r)(\partial\Phi/\partial\theta)\hat{\boldsymbol{\theta}}$. In the radial direction,

$$\begin{aligned}
E_r &= -\frac{Q}{8\pi\epsilon_0}\frac{\partial}{\partial r}\left[\frac{\cos\alpha+1}{R} + \sum_{l=1}^{\infty}\frac{r^l}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta)\right] \\
&= -\frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty} l\frac{r^{l-1}}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta) \\
&= -\frac{Q}{8\pi\epsilon_0}\left[-\frac{\sin^2\alpha}{2R^2}\cos\theta + \sum_{l=2}^{\infty} l\frac{r^{l-1}}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta)\right]
\end{aligned} \tag{3.3.11}$$

Thus, $E_r(r=0) = (Q/16\pi\epsilon_0 R^2)\sin^2\alpha\cos\theta$. In the polar direction,

$$\begin{aligned}
E_\theta &= -\frac{Q}{8\pi\epsilon_0 r}\frac{\partial}{\partial\theta}\left[\frac{\cos\alpha+1}{R} + \sum_{l=1}^{\infty}\frac{r^l}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta)\right] \\
&= -\frac{Q}{8\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{r^{l-1}}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\frac{l\cos\theta P_l(\cos\theta)-lP_{l-1}(\cos\theta)}{\sin\theta} \\
&\qquad\qquad \text{(from (3.29, Jackson))} \\
&= -\frac{Q}{8\pi\epsilon_0}\frac{\sin^2\alpha}{2R^2}\sin\theta - \frac{Q}{8\pi\epsilon_0}\sum_{l=2}^{\infty}\frac{r^{l-1}}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\frac{l\cos\theta P_l(\cos\theta)-lP_{l-1}(\cos\theta)}{\sin\theta}
\end{aligned} \tag{3.3.12}$$

Thus, $E_\theta(r=0) = -(Q/16\pi\epsilon_0 R^2)\sin^2\alpha\sin\theta$. Putting these results together, we can write the electric field as

$$\mathbf{E}(r=0) = \frac{Q}{16\pi\epsilon_0}\frac{\sin^2\alpha}{R^2}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) = \frac{Q}{16\pi\epsilon_0}\frac{\sin^2\alpha}{R^2}\hat{\mathbf{z}} \tag{3.3.13}$$

(c) First, from (3.3.13), whenever $\alpha \ll 1$, $\sin\alpha \approx \alpha$ and we can write

$$\mathbf{E} = \frac{Q}{16\pi\epsilon_0}\frac{\alpha^2}{R^2}\hat{\mathbf{z}} \tag{3.3.14}$$

So $\mathbf{E} \to \mathbf{0}$ as $\alpha \to 0$. This is the case of a closed conductor, and we know that the electric field in a hollow conductor must be zero. Now, in the other limit, let $\beta = \pi - \alpha$ and consider $\beta \ll 1$. Since $\sin\alpha = \sin(\pi-\beta) = \sin\beta$, $\mathbf{E} \to \mathbf{0}$ when $\beta \to 0$. Now, this makes sense because the total charge on the sphere is $Q(\cos\alpha+1)/2 = Q(1-\cos\beta)/2 \to 0$ when $\beta \to 0$.

Now, the potential inside the sphere is

$$
\begin{aligned}
\Phi_{in} &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta) \\
&= \frac{Q}{8\pi\epsilon_0}\left[\frac{\cos\alpha+1}{R}+\sum_{l=1}^{\infty}\frac{r^l}{R^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta)\right] \\
&= \frac{Q}{8\pi\epsilon_0}\left[\frac{\cos\alpha+1}{R}+\mathcal{O}\big(\alpha^2\big)\right] \qquad (3.3.15)
\end{aligned}
$$

As $\alpha \to 0$, to first order, Φ_{in} approaches the potential of the sphere at $r = R$. We know that $\mathbf{E} = \mathbf{0}$ in this case, so Φ_{in} is a constant. Because it has to satisfy the continuity with the outside potential, this is the expected result. In the other limit $\beta = \pi - \alpha \to 0$, $\cos\alpha = -\cos\beta$, so $\Phi_{in} \to 0$ for the same reason as the vanishing of the electric field (total charge $\to 0$).

Similarly, the potential outside the sphere is

$$
\begin{aligned}
\Phi_{out} &= \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta) \\
&= \frac{Q}{8\pi\epsilon_0}\left[\frac{\cos\alpha+1}{r}+\mathcal{O}\big(\alpha^2\big)\right] \qquad (3.3.16)
\end{aligned}
$$

As $\alpha \to 0$, this approachs the potential of a conducting sphere with charge Q

$$\Phi_{out} \to \frac{Q}{4\pi\epsilon_0 r} \qquad (3.3.17)$$

as expected. In the opposite limit, $\beta \to 0$ and $\Phi_{out} \to 0$, because the total charge decreases as in the other cases. □

***Problem* 4.4** (Conducting disc)**:** A thin, flat, conducting, circular disc of radius R is located in the xy plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2-\rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

(a) show that for $r > R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta) \qquad (4.4.1)$$

(b) find the potential for $r < R$.
(c) What is the capacitance of the disc?

Solution.

(a) Given a surface charge density $\sigma = \alpha/\sqrt{R^2-\rho^2}$ where ρ is the radius in cylindrical coordinates and α is some constant, we can integrate to find the charge of an infinitesimally thin ring

$$q_{ring} = \int_0^{2\pi} \sigma \rho d\rho d\phi = 2\pi\alpha \frac{\rho d\rho}{\sqrt{R^2 - \rho^2}} \tag{4.4.2}$$

In Section 3.3 of Jackson, the potential contribution (at $\mathbf{x} = r\hat{\mathbf{z}}$ with $r < R$) of a thin ring placed at $b = 0$ is

$$d\Phi = \frac{\alpha}{2\epsilon_0} \frac{\rho d\rho}{\sqrt{R^2 - \rho^2}} \sum_{l=0}^{\infty} \frac{r^l}{\rho^{l+1}} P_l(0) \tag{4.4.3}$$

Then we can integrate for the total potential

$$\Phi = \frac{\alpha}{2\epsilon_0} \sum_{l=0}^{\infty} \left(\int_0^R \frac{d\rho}{\sqrt{R^2 - \rho^2 \rho^l}} \right) r^l P_l(0) = \frac{\alpha}{2\epsilon_0} \left[\int_0^R \frac{d\rho}{\sqrt{R^2 - \rho^2}} + C(r) \right] \tag{4.4.4}$$

where $C(r) = C_1 r + C_2 r^2 + \ldots$ is a polynomial in r. At $r = 0$, $C(r) = 0$ and

$$\Phi = \frac{\pi\alpha}{4\epsilon_0} = V \Rightarrow \alpha = \frac{4\epsilon_0 V}{\pi} \tag{4.4.5}$$

for some fixed value V.

For $r > R$, the potential contribution by a thin ring is

$$d\Phi = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{\rho^{l+1} d\rho}{\sqrt{R^2 - \rho^2}} \frac{P_l(0)}{r^{l+1}} = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{\rho^{2l+1} d\rho}{\sqrt{R^2 - \rho^2}} \frac{P_{2l}(0)}{r^{2l+1}} \tag{4.4.6}$$

where we have used the fact that $P_l(0)$ is only non-zero for some even l. Then, we can integrate for the total potential

$$\begin{aligned} \Phi &= \frac{2V}{\pi} \sum_{l=0}^{\infty} \left(\int_0^R \frac{\rho^{2l+1} d\rho}{\sqrt{R^2 - \rho^2}} \right) \frac{P_{2l}(0)}{r^{2l+1}} \\ &= \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{\Gamma(l+1)}{\Gamma(l+3/2)} \left(\frac{R}{r} \right)^{2l+1} P_{2l}(0) \end{aligned} \tag{4.4.7}$$

where $\Gamma(l) = (l-1)!$ is the Gamma function. Now, we also know that

$$P_{2l}(0) = \frac{(-1)^l (2l)!}{2^{2l} (l!)^2} \tag{4.4.8}$$

Plugging this into (4.4.7) results in

$$\begin{aligned} \Phi &= \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{\sqrt{\pi}}{2} \frac{l!}{(l+1/2)!} \frac{(-1)^l (2l)!}{2^{2l} (l!)^2} \\ &= \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{(-1)^l (2l)!}{(2l+1)!} \left(\frac{R}{r} \right)^{2l} \\ &= \frac{2V}{\pi} \frac{R}{r} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r} \right)^{2l} \end{aligned} \tag{4.4.9}$$

where we have used $l!(l+1/2)! = \sqrt{\pi}2^{-1-2l}(2l+1)!$ in the second equality. (4.4.9) is the solution on the z axis. So for every point in space,

$$\Phi(r,\theta) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{2l}\frac{(-1)^l}{2l+1}P_{2l}(\cos\theta) \tag{4.4.10}$$

due to azimuthal symmetry.

(c) The total charge on the disc is

$$Q = \int_0^R \sigma\rho d\rho \int_0^{2\pi} d\phi = 2\pi\alpha \int_0^R \frac{\rho d\rho}{\sqrt{R^2-\rho^2}} = 8\epsilon_0 VR \tag{4.4.11}$$

Thus, the capacitance is $C = Q/V = 8\epsilon_0 R$. □