Homework 11: Phys 7320 (Spring 2022)

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Due: April 13th, 2022

***Problem* 1** (Power distributions for simple motion)**:** Using the Liénard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge e, moving

(a) along the z axis with instantaneous position $z(t) = d\cos\omega_0 t$. Show how the time-averaged $dP/d\Omega$ and P match to results from earlier in the semester using the dipole moment,

(b) in a circle of radius R in the xy plane with constant angular frequency ω_0. Express Θ (the angle between the acceleration and the observation point) as a function of time and the spherical coordinates of the observer. For this part the time-averaged P should match results from earlier in the semester with the dipole moment. (The angular distribution in $dP/d\Omega$ is different from what you're used to; this is because the dipole is not pointing in the z-direction, but instead rotating in the xy-plane, and consequently the configuration corresponds to spherical harmonics with $l = 1, m = \pm 1$ instead of $l = 1, m = 0$; see table 9.1 in Jackson section 9.9.)

Sketch the angular distribution of the radiation and determine the total power radiated in each case. Also calculate the dipole moment as a function of time, and cast it as a complex moment $\mathbf{p}$ using the usual form $\mathbf{p}_{\text{real}} = \text{Re}\left(\mathbf{p}e^{-i\omega t}\right)$.

Solution.

(a) Given the position, we can calculate the instantaneous acceleration

$$a = \frac{d^2 z}{dt^2} = -\omega_0^2 z = -\omega_0^2 d\cos\omega_0 t. \tag{1.1}$$

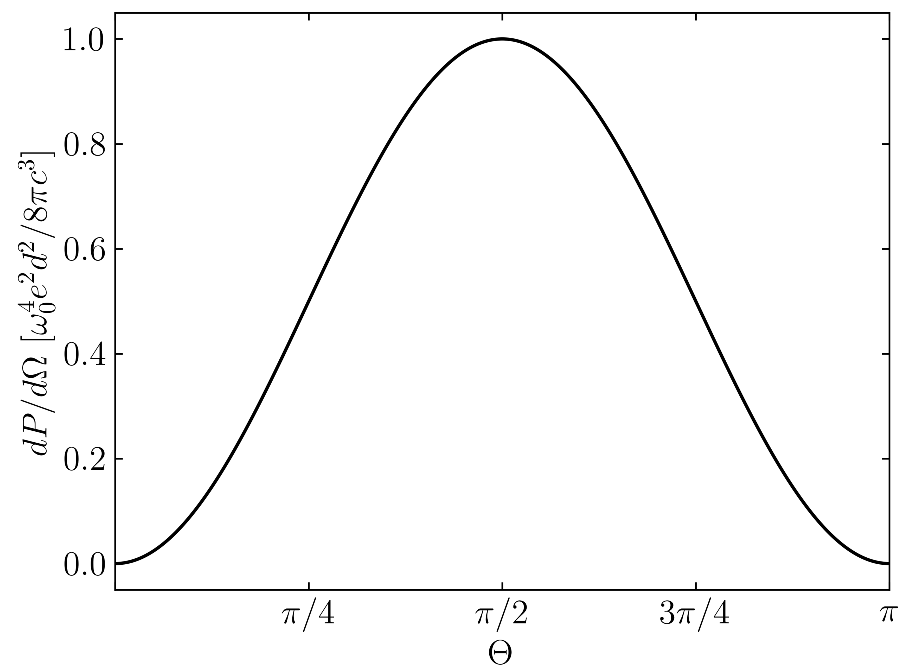
Then, from (14.21, Jackson), the radiation pattern is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} a^2 \sin^2\Theta = \frac{e^2\omega_0^4 d^2}{4\pi c^3}\sin^2\Theta\cos^2(\omega_0 t). \tag{1.2}$$

Averaged over time, this gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2\omega_0^4 d^2}{8\pi c^3}\sin^2\Theta = \frac{c}{8\pi}\frac{\omega_0^4}{c^4}(ed)^2\sin^2\Theta. \tag{1.3}$$

This is the same form as (9.23, Jackson) in Gaussian units. In analogy, the oscillating charge forms a dipole $\mathbf{p} = ede^{-i\omega t}\hat{\mathbf{z}}$ when averaged over time. The wavenumber is given by the frequency of oscillation $k = \omega_0/c$. A plot of $dP/d\Omega$ is given below



We can also calculate the total radiated power

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{e^2\omega_0^4 d^2}{8\pi c^3} \int_{-1}^{1} d(\cos\Theta)\sin^2\Theta \int_0^{2\pi} d\phi = \frac{\omega_0^4 e^2 d^2}{3c^3}. \tag{1.4}$$

(b) The position of the charge can be written as

$$\mathbf{r}_c = R(\cos\omega_0 t\hat{\mathbf{x}} + \sin\omega_0 t\hat{\mathbf{y}}). \Rightarrow \dot{\mathbf{r}}_c = \omega_0 R(-\sin\omega_0 t\hat{\mathbf{x}} + \cos\omega_0 t\hat{\mathbf{y}}). \tag{1.5}$$

Then, given that $\boldsymbol{\omega} = \omega_0\hat{\mathbf{z}}$, the acceleration is

$$\mathbf{a} = \boldsymbol{\omega} \times \dot{\mathbf{r}}_c = -\omega_0^2 R(\cos\omega_0 t\hat{\mathbf{x}} + \sin\omega_0\hat{\mathbf{y}}), \tag{1.6}$$

which is, as expected from harmonic oscillation, $-\omega_0^2\mathbf{r}_c$. Also, $|\mathbf{a}|$ is constant since the charge is in circular orbit. It follows from (14.21, Jackson) that

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\mathbf{a}|^2\sin^2\Theta = \frac{\omega_0^4 e^2 R^2}{4\pi c^3}\sin^2\Theta. \tag{1.7}$$

The point of observation has the following coordinates

$$\mathbf{x} = r(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}). \tag{1.8}$$

By definition, taking the dot product between (1.6) and (1.8) yields

$$\cos\Theta = \frac{\mathbf{a}\cdot\mathbf{x}}{|\mathbf{a}||\mathbf{x}|} = \sin\theta\cos(\omega_0 t - \phi). \tag{1.9}$$

Thus, the averaged radiation pattern is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega_0^4 e^2 R^2}{4\pi c^3} \left[1 - \sin^2\theta \left\langle \cos^2(\omega_0 t - \phi) \right\rangle \right]$$
$$= \frac{\omega_0^4 e^2 R^2}{4\pi c^3} \left[1 - \frac{1}{2}\sin^2\theta \right], \tag{1.10}$$

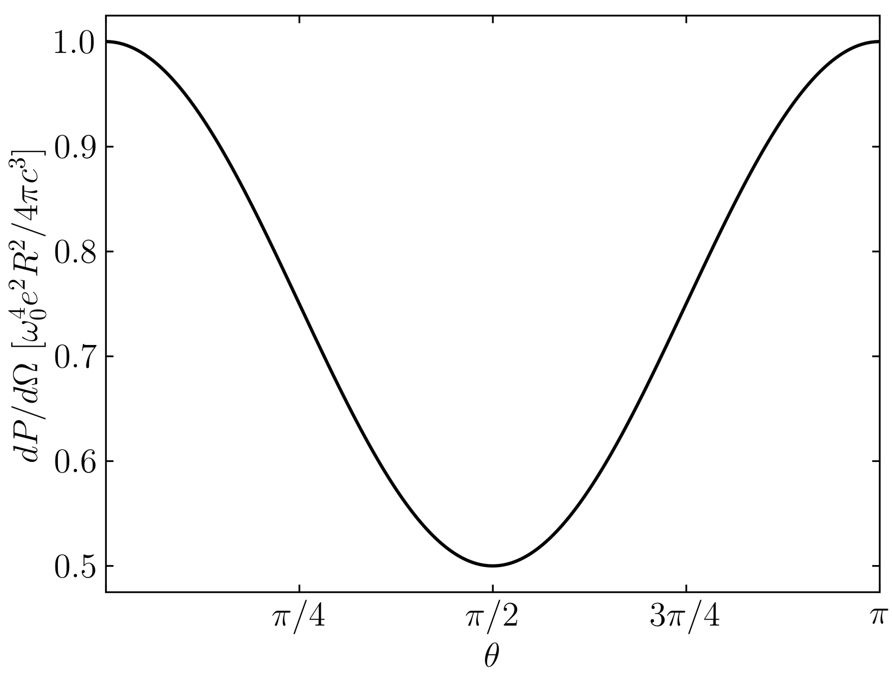
and the total radiated power is

$$P = \frac{\omega_0^4 e^2 R^2}{2c^3} \int_{-1}^{1} d(\cos\theta) \left[1 - \frac{1}{2}\sin^2\theta \right] = \frac{2\omega_0^4 e^2 R^2}{3c^3}. \tag{1.11}$$

This agrees with (9.24, Jackson) for a dipole

$$\mathbf{p} = eRe^{-i\omega t}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}). \tag{1.12}$$

A plot of (1.10) is included below.



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***Problem* 2** (Radiation from motion in a magnetic field.)**:** A particle of mass m, charge q, moves in a plane perpendicular to a uniform, static magnetic induction B.

(a) Calculate the total energy radiated per unit time, expressing it in terms of the constants already defined and the ratio γ of the particle's total energy to its rest energy.

(b) If at time $t = 0$ the particle has a total energy $E_0 = \gamma_0 mc^2$, show that it will have energy $E = \gamma mc^2 < E_0$ at a time t, where

$$t \approx \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right), \tag{2.1}$$

provided $\gamma \gg 1$.

(c) If the particle is initially nonrelativistic and has a *kinetic* energy T_0 at $t = 0$, what is its kinetic energy at time t?

Solution.

(a) The relativistic motion of a charged particle in a uniform magnetic field is given in Jackson, Section 12.2

$$\mathbf{v} = \omega R(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{-i\omega t}, \tag{2.2}$$

where $\omega = qB/\gamma mc$ is the relativistic gyrofrequency, R is the gyroradius, and we have set $v_\parallel = 0$ since there is no acceleration in the parallel direction. Since $\mathbf{v}$ is perpendicular to $\boldsymbol{\omega}$, we have

$$\dot{\beta} = |\boldsymbol{\beta} \times \boldsymbol{\omega}| = \beta\omega, \qquad \text{and} \qquad \left|\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}\right| = \left|\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \boldsymbol{\omega})\right| = -\beta^2\omega. \tag{2.3}$$

From (14.26, Jackson), the total radiated power is

$$\begin{aligned} P &= \frac{2q^2}{3c}\gamma^6\left[\dot{\beta}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2\right] \\ &= \frac{2q^2}{3c}\gamma^6\left[\beta^2\omega^2 - \beta^4\omega^2\right] \\ &= \frac{2q^2}{3c}\gamma^4\beta^2\omega^2 \\ &= \frac{2q^4B^2}{3m^2c^3}\beta^2\gamma^2 \\ &= \frac{2q^4B^2}{3m^2c^3}(\gamma^2 - 1). \end{aligned} \tag{2.4}$$

(b) The radiated power is $P = -(\partial E/\partial t) = -mc^2(\partial\gamma/\partial t)$. Using the previous result, the time t is

$$t = \int_0^t dt' = -mc^2\int_{\gamma_0}^{\gamma}\frac{d\gamma}{P} = \frac{3m^3c^5}{2q^4B^2}\int_{\gamma_0}^{\gamma}\frac{d\gamma}{1-\gamma^2}. \tag{2.5}$$

But since $\gamma \gg 1$, we can ignore unity in the demoninator and write

$$t \approx \frac{3m^3c^5}{2q^4B^2}\int_{\gamma_0}^{\gamma}\frac{d\gamma}{-\gamma^2} = \frac{3m^3c^5}{2q^4B^2}\left(\frac{1}{\gamma} - \frac{1}{\gamma_0}\right). \tag{2.6}$$

(c) In the non-relativistic regime, the gyrofrequency $\omega_0 = qB/m$ is constant, and we can utilize the result from Problem 1(b). Again assuming $T_\parallel = 0$ since the constant translation contributes nothing to the radiation, the kinetic energy is $T = (1/2)mv_\perp^2 = (1/2)m\omega_0^2 R^2$, and we can write the total radiated power as

$$P = -\frac{\partial T}{\partial t} = \frac{4\omega_0^2 e^2}{3mc^2} T \Rightarrow T(t) = T_0 e^{-4\omega_0^2 e^2 t/3mc^3}. \tag{2.7}$$

So the kinetic energy is reduced exponentially with the rate $\Gamma = 4\omega_0^2 e^2/3mc^3$.

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