Homework 8: Phys 5210 (Fall 2021)

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***Problem* 1:** Ever since the 16th century, we know that Earth orbits the Sun and not the other way around. However, we are modern scientists and understand that there's nothing wrong with looking at cosmos from the point of view of the Earth reference frame. In that reference frame the Sun rotates around the Earth, making a full rotation in close to 24 hours. The Sun is massive and its distance to the Earth is large, thus the centripetal force which makes it move this way must be very large. Identify the force which makes the Sun move in this way and show that it provides correct acceleration to the Sun in the reference frame which rotates with the Earth. Take into account that the vector connecting Earth with the Sun forms an angle $\theta < \pi/2$ with the Earth's angular velocity $\mathbf{\Omega}$. Neglect Earth's orbiting the Sun and the Sun's movement relative to the center of the galaxy. In other words, you can assume for the purpose of this problem that both the Earth and the Sun are stationary in space relative to distant galaxies, and that the Earth rotates about its axis with a constant angular velocity.

Solution.

The rotation of the Earth about its axis is $\mathbf{\Omega} = \Omega(\cos\theta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}})$ where $T = 2\pi/\Omega \approx 24\,\text{hr}$. In the inertial frame, let the Earth be at the origin and the Sun at $\mathbf{r} = r\hat{\mathbf{y}}$ where $r = 1\,\text{AU}$. Then the force on the Sun seen by an observer on Earth is

$$\mathbf{F} = -2M\mathbf{\Omega} \times \mathbf{v} - M\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = M\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \tag{1.1}$$

where we have written $\mathbf{v} = -\mathbf{\Omega}\times\mathbf{r}$. Also, by Newton's 2nd law, $\mathbf{F} = M\mathbf{a}$. So the acceleration of the Sun in the non-inertial frame rotating with the Earth is

$$\mathbf{a} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \Omega(\cos\theta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}}) \times (-\Omega r\hat{\mathbf{x}}) = \Omega^2 r\sin\theta(\cos\theta\hat{\mathbf{z}} - \sin\theta\hat{\mathbf{y}}) \tag{1.2}$$

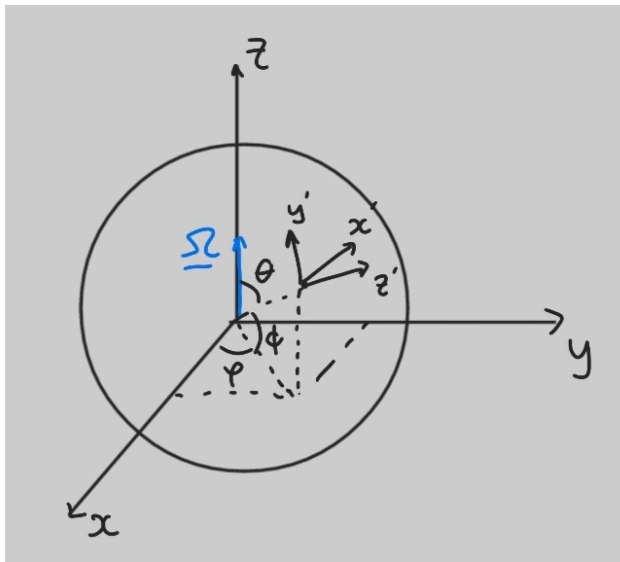
So the magnitude of this acceleration is

$$a = \Omega^2 r\sin\theta \approx 726\,\text{ms}^{-2} \tag{1.3}$$

where $\theta = 66.6^\circ$ from online sources. Note that this is also the centripetal acceleration if the Sun rotates around the Earth in circular orbit with radius $r_\perp = r\sin\theta$ and angular velocity Ω. So the force (1.1) gives the correct acceleration of the Sun. □

***Problem* 2:** Solve for the motion of the Foucault's pendulum. It is a pendulum attached to a very high ceiling of a building located at a latitude ϕ, swinging back and forth along the floor with frequency ω. For the purpose of calculating its kinetic energy, neglect the vertical motion of the pendulum, taking its movement as two dimensional. How long does it take for the plane of oscillations to do one full rotation?

Solution.



Define a coordinate system as above where $\hat{\mathbf{r}} = \hat{\mathbf{z}}', \hat{\boldsymbol{\theta}} = \hat{\mathbf{y}}'$, and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{x}}'$. Then the Earth's rotation is $\mathbf{\Omega} = \Omega(\cos\theta\hat{\mathbf{z}}' - \sin\theta\hat{\mathbf{y}}')$. The velocity of the pendulum on the surface of the Earth at $\theta = \pi/2 - \phi$, neglecting the $\hat{\mathbf{z}}'$ direction, is $\mathbf{v}' = \dot{\mathbf{x}}' = \dot{x}'\hat{\mathbf{x}}' + \dot{y}'\hat{\mathbf{y}}'$. Now, the force on a pendulum with frequency ω in the non-inertial reference frame of the Earth is

$$\begin{aligned}
& \mathbf{F} = m\mathbf{a}' = -m\omega^2\mathbf{x}' - 2m\mathbf{\Omega}\times\mathbf{v}' \\
\Rightarrow \quad & \mathbf{a}' = -\omega^2\mathbf{x}' - 2\Omega\big(\cos\theta\hat{\mathbf{z}}' - \sin\theta\hat{\mathbf{y}}'\big)\times\big(\dot{x}'\hat{\mathbf{x}}' + \dot{y}'\hat{\mathbf{y}}'\big) \\
\Leftrightarrow \quad & \mathbf{a}' = -\omega^2\mathbf{x}' - 2\Omega\cos\theta\big(\dot{x}'\hat{\mathbf{y}}' - \dot{y}'\hat{\mathbf{x}}'\big) \qquad (2.1)
\end{aligned}$$

where the $\hat{\mathbf{z}}'$ component is ignored. Writing $\cos\theta = \sin\phi$, we then have the following system of differential equations

$$\ddot{x} = -\omega^2 x + 2\Omega\sin\phi\dot{y} \qquad (2.2a)$$

$$\ddot{y} = -\omega^2 y - 2\Omega\sin\phi\dot{x} \qquad (2.2b)$$

where we have dropped the primes to simplify the expressions. Letting $z = x + iy$, then the system above can be simplified into a single differential equation

$$\ddot{z} = \ddot{x} + i\ddot{y} = -\omega^2(x+iy) - 2i\Omega\sin\phi(\dot{x} + i\dot{y}) \Rightarrow \ddot{z} + \omega^2 z + 2i\Omega\sin\phi\dot{z} = 0 \qquad (2.3)$$

Solving this with Mathematica yields the solution

$$z(t) = e^{-i\Omega\sin\phi t}\left[A\sin\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right) + B\cos\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right)\right] \qquad (2.4)$$

where A, B are constants dependent on initial conditions. We can also write explicitly

$$x(t) = \text{Re}\{z\} = \cos(\Omega \sin\phi t)\left[A\sin\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right) + B\cos\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right)\right] \tag{2.5a}$$

$$y(t) = \text{Im}\{z\} = -\sin(\Omega \sin\phi t)\left[A\sin\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right) + B\cos\left(\sqrt{\omega^2 + \Omega^2\sin^2\phi t}\right)\right] \tag{2.5b}$$

However, from (2.4), it is more apparent that the amplitude of the sinuisoidal solution (in the square bracket) is rotated in the $x'y'$ plane clockwise with a frequency $\Omega\sin\phi$. Thus, the time it takes for the plane of oscillations to do one full rotation is

$$T = \frac{2\pi}{\Omega\sin\phi} \tag{2.6}$$

□