Homework 3: Astr 5140 (Fall 2021)

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***Problem* 1** (Generalized Ohm's Law)**:** The MHD force equation is derived by a linear combination of the electron and ion fluid equations. Generalized Ohm's law is, simply stated, a different linear combination.

We begin by multiplying the ion force equation by m_e and the electron force equation by m_i, then subtract

$$m_e m_i n \frac{D\mathbf{u}_i}{Dt} = -m_e \boldsymbol{\nabla} P_i + n m_e e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - m_e m_i n \nu_{ie}(\mathbf{u}_i - \mathbf{u}_e) \tag{1.1a}$$

$$m_i m_e n \frac{D\mathbf{u}_e}{Dt} = -m_i \boldsymbol{\nabla} P_e - n m_i e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - m_e m_i n \nu_{ei}(\mathbf{u}_e - \mathbf{u}_i) \tag{1.1b}$$

We use quasi-neutrality and bundle the convective derivative into the symbol D.

$$\begin{aligned} m_e m_i n \left(\frac{D\mathbf{u}_i}{Dt} - \frac{D\mathbf{u}_e}{Dt} \right) &= (-m_e \boldsymbol{\nabla} P_i + m_i \boldsymbol{\nabla} P_e) + en\mathbf{E}(m_e + m_i) \\ &\quad + en(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} - m_e m_i n(\nu_{ei} + \nu_{ie})(\mathbf{u}_i - \mathbf{u}_e) \end{aligned} \tag{1.2}$$

(a) Divide each term by $en(m_i + m_e)$.

(b) Show that Term 1 can be re-written in the limit of $m_i \gg m_e$ as

$$\frac{m_e}{e^2} \frac{D(\mathbf{J}/n)}{Dt} \tag{1.3}$$

(c) Argue that since $m_i \gg m_e$ that, unless $P_i \gg P_e$ (very rare), Term 2 becomes

$$\frac{\boldsymbol{\nabla} P_e}{en} \tag{1.4}$$

(d) Term 3 is trivial. Term 4 is tricky as it must be broken into two parts. Add $\mathbf{u} = (m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_i + m_e)$, separate $\mathbf{u} \times \mathbf{B}$, then subtract $(m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_i + m_e)$. Show that the remaining four terms ($m_i \gg m_e$) can be approximated as

$$\frac{-\mathbf{J} \times \mathbf{B}}{en} \tag{1.5}$$

(e) Show that Term 5 can be written as $\mathbf{J}/\sigma$. Define σ.

In the end, you should arrive at

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{\mathbf{J} \times \mathbf{B}}{en} - \frac{\boldsymbol{\nabla} P_e}{en} + \frac{m_e}{ne^2}\frac{D\mathbf{J}}{Dt} + \text{small terms} \tag{1.6}$$

Note: If done exactly (full convective derivative and keeping small terms), n is not inside the derivative and furthermore, some of the "leftovers" from Term 1 cancel "leftovers" in Term 4.

Solution.

(a) Given the force equation with collisional terms,

$$n_s m_s \frac{D\mathbf{u}_s}{Dt} = -\boldsymbol{\nabla} P_s + n_s q_s(\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) + m_s n_s \mathbf{g} \tag{1.7}$$

we can write for ions and electrons (with collisional terms)

$$m_e m_i n \frac{D\mathbf{u}_i}{Dt} = -m_e \boldsymbol{\nabla} P_i + m_e n e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + m_e m_i n \mathbf{g} - m_e m_i n \nu_{ie}(\mathbf{u}_i - \mathbf{u}_e) \tag{1.8a}$$

$$m_i m_e n \frac{D\mathbf{u}_e}{Dt} = -m_i \boldsymbol{\nabla} P_e - m_i n e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + m_i m_e n \mathbf{g} - m_e m_i n \nu_{ei}(\mathbf{u}_e - \mathbf{u}_i) \tag{1.8b}$$

where we have assumed a quasi-neutral plasma ($n_i = n_e = n$). Subtracting (1.8a) with (1.8b), we get

$$\begin{aligned} m_i m_e n \frac{D}{Dt}(\mathbf{u}_i - \mathbf{u}_e) = {} & (-m_e \boldsymbol{\nabla} P_i + m_i \boldsymbol{\nabla} P_e) + ne(m_i + m_e)\mathbf{E} \\ & + ne(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} - m_i m_e n(\nu_{ei} + \nu_{ie})(\mathbf{u}_i - \mathbf{u}_e) \end{aligned} \tag{1.9}$$

Dividing each term by $en(m_i + m_e)$, we get

$$\begin{aligned} \frac{1}{e}\frac{m_i m_e}{m_i + m_e}\frac{D}{Dt}(\mathbf{u}_i - \mathbf{u}_e) = {} & \frac{1}{en(m_i + m_e)}(-m_e \boldsymbol{\nabla} P_i + m_i \boldsymbol{\nabla} P_e) + \mathbf{E} \\ & + \frac{m_e \mathbf{u}_i + m_i \mathbf{u}_e}{m_i + m_e} \times \mathbf{B} - \frac{1}{e}\frac{m_i m_e}{m_i + m_e}(\nu_{ei} + \nu_{ie})(\mathbf{u}_i - \mathbf{u}_e) \end{aligned} \tag{1.10}$$

(b) The current is $\mathbf{J} = en(\mathbf{u}_i - \mathbf{u}_e)$. Thus, we can rewrite the first term as

$$\frac{1}{e^2}\frac{m_i m_e}{m_i + m_e}\frac{D(\mathbf{J}/n)}{Dt} = \frac{1}{e^2}\frac{m_e}{1 + m_e/m_i}\frac{D(\mathbf{J}/n)}{Dt} \approx \frac{m_e}{e^2}\frac{D(\mathbf{J}/n)}{Dt} \tag{1.11}$$

where $m_i \gg m_e$. Expanding the convective derivative, we can write

$$\begin{aligned} \frac{D(\mathbf{J}/n)}{Dt} &= \frac{\partial(\mathbf{J}/n)}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\left(\frac{\mathbf{J}}{n}\right) \\ &= -\frac{\mathbf{J}}{n^2}\frac{\partial n}{\partial t} + \frac{1}{n}\frac{\partial \mathbf{J}}{\partial t} + \mathbf{u} \cdot \left[-\frac{\mathbf{J}}{n^2}\boldsymbol{\nabla} n + \frac{1}{n}\boldsymbol{\nabla}\mathbf{J}\right] \\ &= -\frac{\mathbf{J}}{n^2}\left[\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{u})\right] + \frac{1}{n}\frac{\partial \mathbf{J}}{\partial t} + \frac{1}{n}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{J} \\ &= \frac{1}{n}\frac{D\mathbf{J}}{Dt} \end{aligned} \tag{1.12}$$

(c) The second term can be rewritten as

$$\frac{1}{en}\left[-\frac{m_e/m_i}{1+m_e/m_i}\boldsymbol{\nabla}P_i+\frac{1}{1+m_e/m_i}\boldsymbol{\nabla}P_e\right]\approx\frac{\boldsymbol{\nabla}P_e}{en} \tag{1.13}$$

where the first term vanishes when $m_e/m_i \ll 1$ unless $P_i \gg P_e$.

(d) Adding and subtracting $\mathbf{u}$ to the velocity, the fourth term is

$$\begin{aligned}\frac{m_e\mathbf{u}_i+m_i\mathbf{u}_e}{m_i+m_e}\times\mathbf{B} &= \left[\mathbf{u}+\frac{m_e\mathbf{u}_i-m_i\mathbf{u}_e}{m_i+m_e}-\frac{m_i\mathbf{u}_i+m_e\mathbf{u}_e}{m_i+m_e}\right]\times\mathbf{B} \\ &= \left[\mathbf{u}-\frac{m_i-m_e}{m_i+m_e}(\mathbf{u}_i-\mathbf{u}_e)\right]\times\mathbf{B} \\ &\approx \mathbf{u}\times\mathbf{B}-(\mathbf{u}_i-\mathbf{u}_e)\times\mathbf{B} \\ &= \mathbf{u}\times\mathbf{B}-\frac{\mathbf{J}\times\mathbf{B}}{en}\end{aligned} \tag{1.14}$$

(e) Writing the last term in terms of the current, we get

$$-\frac{1}{e}\frac{m_i m_e}{m_i+m_e}(\nu_{ei}+\nu_{ie})(\mathbf{u}_i-\mathbf{u}_e)=-\frac{1}{e^2n}\frac{m_e}{1+m_e/m_i}(\nu_{ei}+\nu_{ie})\mathbf{J}=-\frac{m_e(\nu_{ei}+\nu_{ie})}{e^2n}\mathbf{J} \tag{1.15}$$

Defining $\sigma = e^2n/m_e(\nu_{ei}+\nu_{ie})$, it becomes $-\mathbf{J}/\sigma$.

Combining everything, we get

$$\frac{m_e}{ne^2}\frac{D\mathbf{J}}{Dt}=\frac{\boldsymbol{\nabla}P_e}{en}+\mathbf{E}+\mathbf{u}\times\mathbf{B}-\frac{\mathbf{J}\times\mathbf{B}}{en}-\frac{\mathbf{J}}{\sigma} \tag{1.16}$$

Rewriting this result yields the generalized Ohm's law

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}=\frac{\mathbf{J}}{\sigma}+\frac{\mathbf{J}\times\mathbf{B}}{en}-\frac{\boldsymbol{\nabla}P_e}{en}+\frac{m_e}{ne^2}\frac{D\mathbf{J}}{Dt} \tag{1.17}$$

□

***Problem* 2** (Scale height in the solar corona)**:** The surface gravity of the Sun is $274\,\mathrm{m/s^2}$. Assume that the corona is entirely protons with $T_{\text{corona}} = 10^6\,{}^\circ\mathrm{K}$. Derive the isothermal scale height of the solar corona from the MHD equations assuming $\mathbf{g}$ is constant (use 1D). How does this value compare with the radius of the Sun (what % of R_{sun} is H_0)? Does your answer agree with what is seen in UV or X-ray images?

Solution.

The 1D state equation is

$$\frac{\partial P}{\partial z}=\frac{\gamma T}{m}\frac{\partial \rho}{\partial z} \tag{2.1}$$

where $\gamma = 1$ in an isothermal plasma and $T = k_B T_{\text{corona}}$. From the force equation, we can also write

$$\frac{\partial P}{\partial z}=\rho g \tag{2.2}$$

Thus, we arrive at the following differential equation

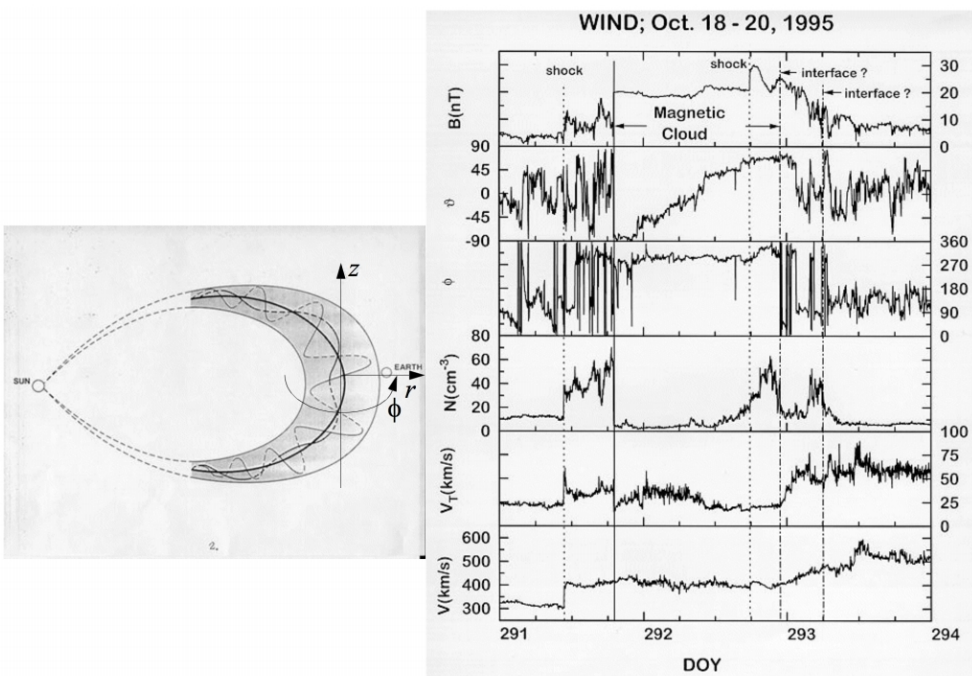
$$\frac{\partial n}{\partial z} = \frac{mg}{T} n \tag{2.3}$$

where $\rho = mn$. The solution is

$$n = n_0 e^{-mgz/T} = n_0 e^{-z/H_0} \tag{2.4}$$

where the scale height is $H_0 = T/mg \approx 3 \times 10^4$ km. This is $\approx 4\%$ of the Sun's radius. From EUVI images, the corona is usually 10s of the scale height wide. But the density profile follows roughly the exponential decay in (2.4). $\square$

***Problem* 3** (Force free flux rope)**:** Below are spacecraft observations of a force free flux rope passing by Earth after a coronal mass ejection. The data are (a) the total magnetic field, (b) the elevation angle from the solar ecliptic, (c) the azimuthal angle in the solar ecliptic (0° indicates anti-sunward), (d) the plasma density, (e) the thermal velocity, and (f) the solar wind speed.



Unlike the force free flux rope in an earlier problem in which α was constant, the observations indicate a steady rotation of the magnetic field (pay particular attention to the second panel – the elevation angle has a constant slope) that is best modeled by assuming cylindrical symmetry with the z axis aligned with the flux rope (see the left diagram) and the magnetic field obeys the relation: $B_\phi/B_z = \epsilon r$, where ϵ is a constant. Let the magnetic field at the center be $B_z = B_0$.

(a) Under the force free equations with α NOT constant, show that $\mathbf{\nabla} \times \mathbf{B} = \alpha \mathbf{B}$ can be expressed as

$$\frac{\partial B_z}{\partial r} = -\alpha B_\phi \qquad \text{and} \qquad \frac{1}{r}\frac{\partial (r B_\phi)}{\partial r} = \alpha B_z \tag{3.1}$$

Using the relationship $B_\phi/B_z = \epsilon r$, eliminate B_ϕ from the above equations.

(b) Eliminate α by combining the two equations to obtain one differential equation for B_z.

(c) Solve for B_z with manipulation followed by direct integration. Also derive B_ϕ and α.

(d) Plot B_z and B_ϕ a a function of r.

Solution.

(a) Assume cylindrical symmetry, so $\partial\mathbf{B}/\partial\phi = 0$. The curl of $\mathbf{B}$ in cylindrical coordinates is

$$\boldsymbol{\nabla}\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\hat{\mathbf{r}} - \frac{\partial B_z}{\partial r}\hat{\boldsymbol{\phi}} + \frac{1}{r}\frac{\partial(rB_\phi)}{\partial r}\hat{\mathbf{z}} \tag{3.2}$$

Now, the force free equation is $\boldsymbol{\nabla}\times\mathbf{B} = \alpha\mathbf{B} = \alpha(B_r\hat{\mathbf{r}} + B_\phi\hat{\boldsymbol{\phi}} + B_z\hat{\mathbf{z}})$. So we can write

$$\frac{\partial B_z}{\partial r} = -\alpha B_\phi \qquad \text{and} \qquad \frac{1}{r}\frac{\partial(rB_\phi)}{\partial r} = \alpha B_z \tag{3.3}$$

as desired. Since $B_\phi/B_z = \epsilon r$, we can eliminate B_ϕ from these differential equations

$$\frac{\partial B_z}{\partial r} = -\alpha\epsilon r B_z \qquad \text{and} \qquad \frac{\epsilon}{r}\frac{\partial(r^2B_z)}{\partial r} = \alpha B_z \tag{3.4}$$

(b) Combining the equations in (3.4), we have

$$\frac{\partial B_z}{\partial r} = -\epsilon^2\frac{\partial(r^2B_z)}{\partial r} = -\epsilon^2\left(2rB_z + r^2\frac{\partial B_z}{\partial r}\right) \Rightarrow \left(1+\epsilon^2r^2\right)\frac{\partial B_z}{\partial r} = -2\epsilon^2 rB_z \tag{3.5}$$

(c) The differential equation in (3.5) is separable into

$$\frac{dB_z}{B_z} = -\frac{2\epsilon^2 r dr}{1+\epsilon^2r^2} \tag{3.6}$$

Integrating both sides, we get

$$\begin{aligned} \ln B_z &= -\int\frac{2\epsilon^2 r dr}{1+\epsilon^2r^2} \\ &= -\frac{du}{u} && (u\equiv 1+\epsilon^2r^2) \\ &= -\ln\left[B_0(1+\epsilon^2r^2)\right] && (3.7) \end{aligned}$$

So the solution is

$$B_z = \frac{B_0}{1+\epsilon^2r^2} \tag{3.8}$$

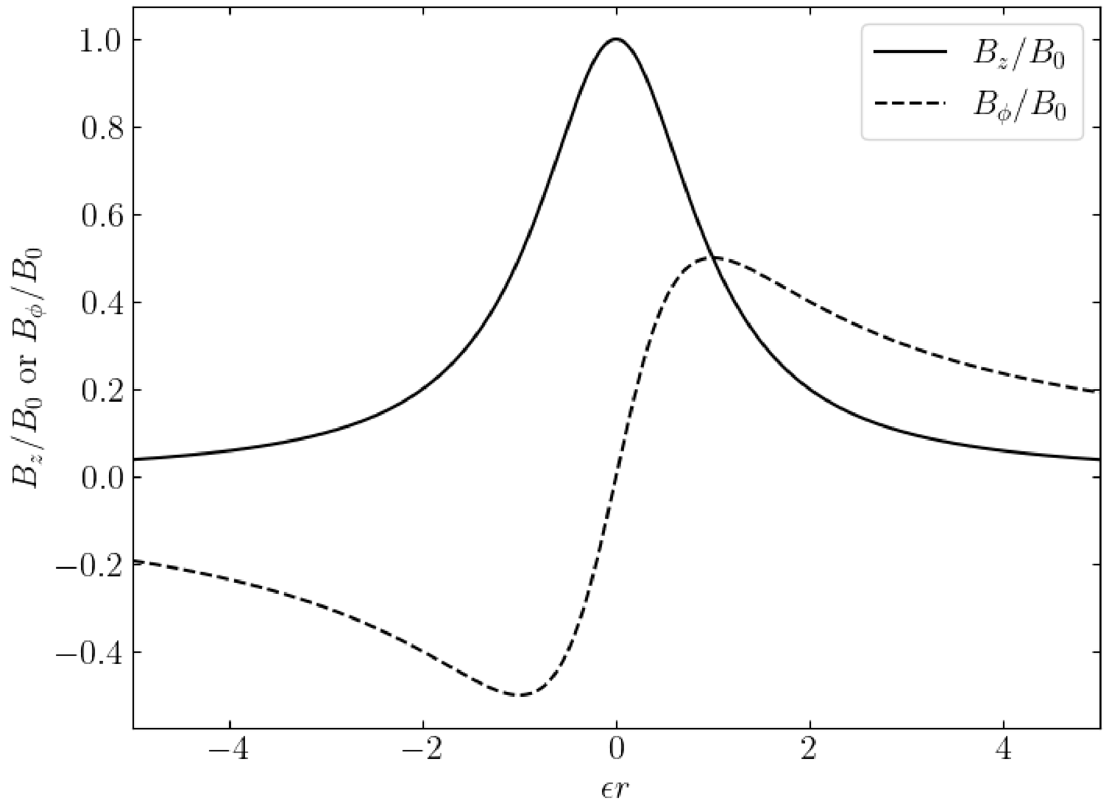
Also, by the relation $B_\phi/B_z = \epsilon r$, we can write

$$B_\phi = \frac{\epsilon r}{1+\epsilon^2r^2}B_0 \tag{3.9}$$

Plugging this back into (3.4),

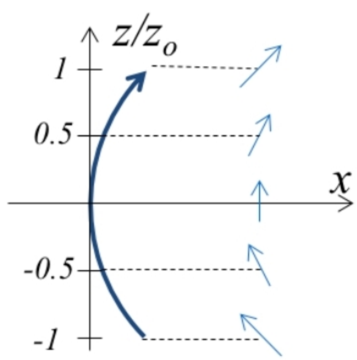
$$\frac{\partial B_z}{\partial r} = -\frac{2\epsilon^2 r}{(1+\epsilon^2r^2)^2}B_0 = -\frac{\alpha\epsilon r}{1+\epsilon^2r^2}B_0 \Rightarrow \alpha = \frac{2\epsilon}{1+\epsilon^2r^2} \tag{3.10}$$

(d)



□

Problem **4** (Magnetic tension)**:** This problem is to develop an intuition for magnetic tension. Examine the diagram, which shows a field line if $\mathbf{B} = B_0[(z/z_0)\hat{\mathbf{x}} + \hat{\mathbf{z}}]$. Examine the field line and convince yourself that the equation represents a curved magnetic field line. Derive the tension force at $z = 0$. What is the direction of the tension force? Argue that z_0 is the local radius of curvature.



Solution.

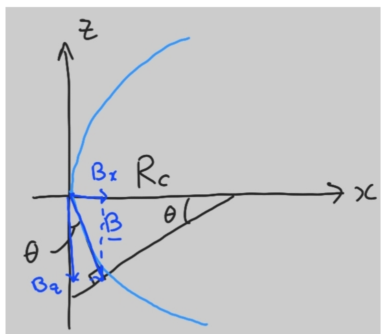
Given the magnetic field

$$\mathbf{B} = B_0\left(\frac{z}{z_0}\hat{\mathbf{x}} + \hat{\mathbf{z}}\right) \tag{4.1}$$

the tension force is, by definition

$$\frac{1}{\mu_0}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} = \frac{1}{\mu_0}\left[B_x\frac{\partial}{\partial x} + B_z\frac{\partial}{\partial z}\right]\mathbf{B} = \frac{1}{\mu_0}B_0\frac{B_0}{z_0}\hat{\mathbf{x}} = \frac{B_0^2}{\mu_0}\frac{1}{z_0}\hat{\mathbf{x}} \tag{4.2}$$

This tension force is constant and is in the x direction.

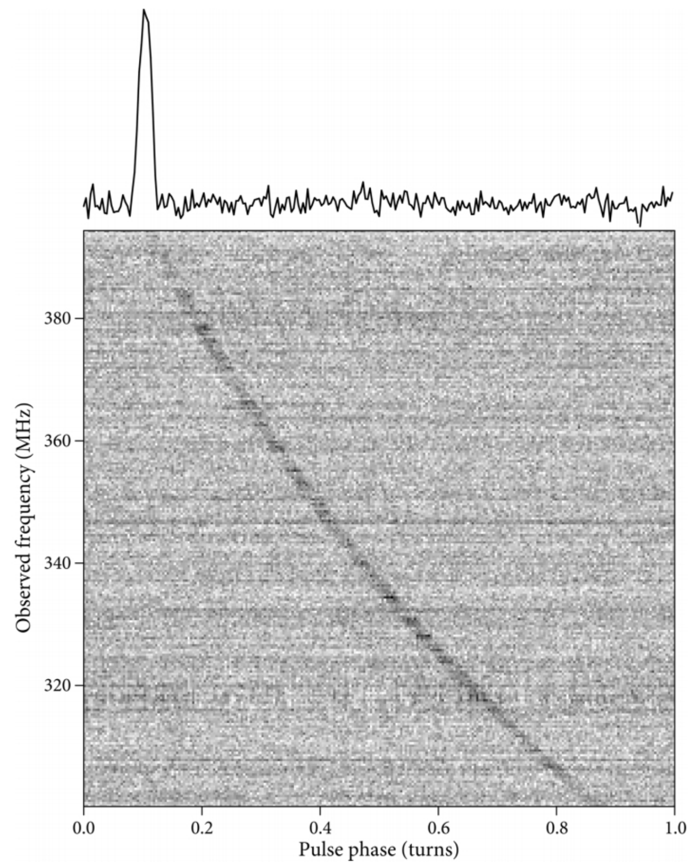


Draw a point equidistant (for $z/z_0 \ll 1$) to the magnetic field (blue) on the x axis at R_c as above. If $\sin\theta \ll 1$, we can write by geometry that

$$\tan\theta \approx \frac{B_x}{B_z} \approx \frac{z}{R_c} \tag{4.3}$$

But $B_x/B_z = z/z_0$ from (4.1). So $z/R_c = z/z_0 \Rightarrow z_0 = R_c$. z_0 is thus the local radius of the curvature. $\square$

***Problem* 5** (Pulsars)**:** Radio signals that we receive from space not only give us information about the source but can tell us about the interstellar medium. For example, impulsive signals can show strong frequency dispersion as diagramed below.



(a) We showed that the solution of a transverse light wave becomes $\omega^2 = c^2k^2 + \omega_{pe}^2$. Derive the group velocity $d\omega/dk$ of the light wave.

(b) Assuming that $\omega \gg \omega_{pe}$, show that the "delay" (from light-speed) it takes to travel a distance L is, to lowest order, is directly proportional to n_eL, the total electron content (column electron density). Derive an expression for the time delay in terms of ω and n_eL.

(c) Describe (in words is OK) how one can determine the average density of the interstellar medium from a pulsar frequency dispersion.

Solution.

(a) Taking the implicit differentiation of the dispersion relation, we get

$$2\omega d\omega = 2c^2kdk \Rightarrow v_g = \frac{d\omega}{dk} = \frac{c}{\omega/k} = c\left(1 + \frac{\omega_{pe}^2}{c^2k^2}\right)^{-1/2} \tag{5.1}$$

where v_g is the group velocity of the light wave.

(b) If $\omega \gg \omega_{pe}$, then from the dispersion relation,

$$1 = \frac{c^2k^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2} \approx \frac{c^2k^2}{\omega^2} \tag{5.2}$$

We can then approximate the group velocity as

$$v_g \approx c\left(1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\right) \tag{5.3}$$

The travel time is then

$$\tau = \frac{L}{v_g} \approx \frac{L}{c}\left(1 + \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\right) \tag{5.4}$$

So the delay is

$$\tau_{\text{delay}} = \tau - \tau_{\omega\to\infty} = \frac{L}{2c}\frac{\omega_{pe}^2}{\omega^2} = \frac{e^2}{2c\epsilon_0 m_e}\frac{\langle n_e\rangle L}{\omega^2} \tag{5.5}$$

where $\langle n_e\rangle L = \int_0^L n(s)ds$ is the total electron content.

(c) From (5.5), we can write

$$\langle n_e\rangle = \frac{2\epsilon_0 m_e \omega^2}{e^2}\frac{c\tau_{\text{delay}}}{L} \tag{5.6}$$

The average electron density can thus be determined from the delay time, the interstellar distance L, and the frequency ω through the above expression. One can approximate where $\tau_{\omega\to\infty}$ is in the graph and pick a point of finite frequency on the drifted power curve in the spectrogram to calculate the delay time. □