Homework 10: Phys 7320 (Spring 2022)

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***Problem* 10.1** (The electric field of a relativistic particle)**:** In class we showed that the Faraday tensor for a charged particle undergoing arbitrary motion is given by

$$F^{\mu\nu} = \frac{e}{U\cdot(x-r(\tau))}\frac{d}{d\tau}\left[\frac{(x-r(\tau))^\mu U^\nu - (x-r(\tau))^\nu U^\mu}{U\cdot(x-r(\tau))}\right]\Bigg|_{\tau\to\tau_0}, \tag{10.1.1}$$

where x^μ is the observation point $r^\mu(\tau)$ is the trajectory of the particle in spacetime, and U^μ is its 4-velocity (this is also Jackson (14.11), though he uses V^μ instead of U^μ). The denominators contain the 4-vector dot product $U\cdot(x-r)\equiv U_\mu(x-r)^\mu$. Let's use this to find the form of the electric field generated by a relativistic charged particle undergoing arbitrary motion.

(a) Take the derivative to show that

$$\frac{1}{e}F^{\mu\nu} = \frac{(x-r)^\mu\alpha^\nu - (x-r)^\nu\alpha^\mu}{(U\cdot(x-r))^2} + \frac{((x-r)^\mu U^\nu - (x-r)^\nu U^\mu)(c^2-\alpha\cdot(x-r))}{(U\cdot(x-r))^3}, \tag{10.1.2}$$

where $\alpha^\mu\equiv dU^\mu/d\tau$ is the 4-acceleration.

(b) We will find an electric field of the form

$$\mathbf{E} = \mathbf{E}_v + \mathbf{E}_a, \tag{10.1.3}$$

where $\mathbf{E}_v$ has no factors of the 3-acceleration $\mathbf{a}\equiv d\mathbf{u}/dt$, while $\mathbf{E}_a$ is linear in $\mathbf{a}$. First find $\mathbf{E}_v$; you can try to match to Jackson (14.14). The electric field components are embedded in the Faraday tensor as $E_i = F^{i0}$. Helpful formulas: the 4-vectors can be expressed in terms of their time component and spatial components as

$$\begin{aligned}(x-r)^\mu &= (R, R\hat{\mathbf{n}})\\ U^\mu &= (c\gamma, \gamma\mathbf{u})\\ \alpha^\mu &= (\gamma^4\boldsymbol{\beta}\cdot\mathbf{a}, \gamma^2\mathbf{a}+\gamma^4\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{a}))\end{aligned} \tag{10.1.4}$$

with $\boldsymbol{\beta}=\mathbf{u}/c$.

(c) Now find $\mathbf{E}_a$. To get the form with two cross-products given in Jackson (14.14), try using a vector calculus identity on Jackson's form.

Solution.

(a) First, we calculate

$$\frac{d}{d\tau}\big[U\cdot(x-r)\big]=\frac{dU_\mu}{d\tau}(x-r)^\mu+U_\mu\left(-\frac{dr}{d\tau}\right)^\mu=\alpha_\mu\cdot(x-r)^\mu-U_\mu U^\mu=\alpha\cdot(x-r)-c^2. \tag{10.1.5}$$

Then, by the product rule,

$$\begin{aligned}\frac{d}{d\tau}\left[\frac{(x-r)^\mu U^\nu-(x-r)^\nu U^\mu}{U\cdot(x-r)}\right]&=\frac{-U^\mu U^\nu+(x-r)^\mu\alpha^\nu+U^\nu U^\mu-(x-r)^\nu\alpha^\mu}{U\cdot(x-r)}\\&\quad-\frac{(x-r)^\mu U^\nu-(x-r)^\nu U^\mu}{[U\cdot(x-r)]^2}\frac{d}{d\tau}\big[U\cdot(x-r)\big]\\&=\frac{(x-r)^\mu\alpha^\nu-(x-r)^\nu\alpha^\mu}{U\cdot(x-r)}\\&\quad+\frac{[(x-r)^\mu U^\nu-(x-r)^\nu U^\mu][c^2-\alpha\cdot(x-r)]}{[U\cdot(x-r)]^2}\end{aligned} \tag{10.1.6}$$

Thus, it follows that

$$\frac{1}{e}F^{\mu\nu}=\frac{(x-r)^\mu\alpha^\nu-(x-r)^\nu\alpha^\mu}{[U\cdot(x-r)]^2}+\frac{[(x-r)^\mu U^\nu-(x-r)^\nu U^\mu][c^2-\alpha\cdot(x-r)]}{[U\cdot(x-r)]^3}. \tag{10.1.7}$$

(b) First, given the 4-vector components, we can write

$$U\cdot(x-r)=\gamma R(c-\mathbf{u}\cdot\hat{\mathbf{n}})=\gamma Rc(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}), \tag{10.1.8}$$

and

$$\alpha\cdot(x-r)=\gamma^4R(\boldsymbol{\beta}\cdot\mathbf{a})(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})-\gamma^2R\mathbf{a}\cdot\hat{\mathbf{n}}. \tag{10.1.9}$$

Then, by definition,

$$\begin{aligned}\frac{1}{e}\mathbf{E}&=\big[U\cdot(x-r)\big]^{-2}\left[\gamma^4R(\boldsymbol{\beta}\cdot\mathbf{a})(\hat{\mathbf{n}}-\boldsymbol{\beta})-\gamma^2R\mathbf{a}+\frac{(c\gamma R\hat{\mathbf{n}}-\gamma R\mathbf{u})(c^2-\alpha\cdot(x-r))}{U\cdot(x-r)}\right]\\&=\big[\gamma Rc(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})\big]^{-2}\left[\gamma^4R(\boldsymbol{\beta}\cdot\mathbf{a})(\hat{\mathbf{n}}-\boldsymbol{\beta})-\gamma^2R\mathbf{a}+\frac{(\hat{\mathbf{n}}-\boldsymbol{\beta})(c^2-\alpha\cdot(x-r))}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right]\\&=\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2R^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}+\big[\gamma Rc(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})\big]^{-2}\left[\frac{\gamma^2R\mathbf{a}\cdot\hat{\mathbf{n}}(\hat{\mathbf{n}}-\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}-\gamma^2R\mathbf{a}\right].\end{aligned} \tag{10.1.10}$$

Thus, in this form, we can write

$$\mathbf{E}_v=e\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2R^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}. \tag{10.1.11}$$

(c) The residue of (10.1.10) is

$$
\begin{aligned}
\mathbf{E}_a &= \frac{e}{Rc^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\left[\frac{(\mathbf{a}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}-\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}-\mathbf{a}\right] \\
&= \frac{e}{c}\frac{1}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\left[\frac{(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}-\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}-\dot{\boldsymbol{\beta}}\right] \\
&= \frac{e}{c}\frac{(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})(\hat{\mathbf{n}}-\boldsymbol{\beta})-\dot{\boldsymbol{\beta}}(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}. \qquad (10.1.12)
\end{aligned}
$$

However, from the vector algebraic identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$, the numerator is

$$
\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right] = (\hat{\mathbf{n}}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})-\dot{\boldsymbol{\beta}}[\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}-\boldsymbol{\beta})] = (\hat{\mathbf{n}}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})-\dot{\boldsymbol{\beta}}(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}). \qquad (10.1.13)
$$

So,

$$
\mathbf{E}_a = \frac{e}{c}\frac{\hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{R(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}. \qquad (10.1.14)
$$

□

***Problem* 10.2** (Two forms for the electric field of a charge in uniform motion)**:** A few weeks ago, we used a Lorentz transformation on the field of a stationary point charge (here called e) to derive an expression for the electric fied of the charge in uniform motion with velocity v (taken in the x-direction), as measured from a point with impact parameter b (in the y-direction)

$$\mathbf{E} = \frac{e\gamma\mathbf{r}}{\tilde{r}^3}, \tag{10.2.1}$$

where $\mathbf{r}$ points from the location of the charge *now* to the observation point,

$$\mathbf{r} = (vt)\hat{\mathbf{x}} + b\hat{\mathbf{y}}, \tag{10.2.2}$$

and $\tilde{r}$ looks like the radial variable but with an extra γ in the x-direction,

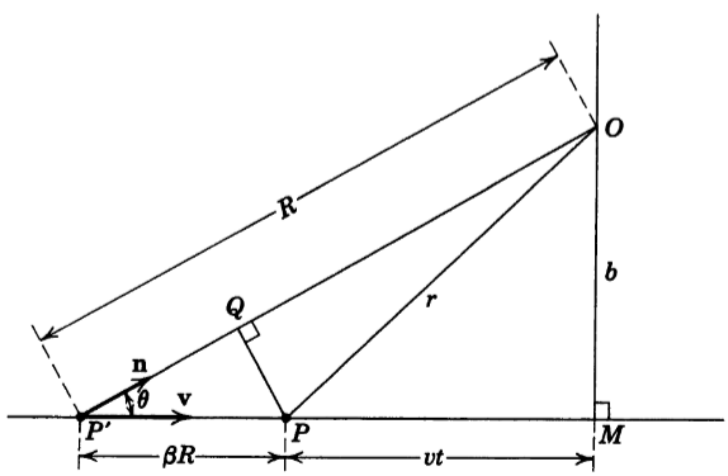
$$\tilde{r} \equiv \sqrt{b^2 + (\gamma vt)^2}. \tag{10.2.3}$$

On the other hand, in the previous problem we derived formulas for the electric field of a charged particle undergoing arbitrary motion. When the acceleration is zero, this becomes

$$\mathbf{E} = \mathbf{E}_v = \frac{e(\hat{\mathbf{n}} - \boldsymbol{\beta})}{\gamma^2(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3 R^2}, \tag{10.2.4}$$

where $\mathbf{R} = R\hat{\mathbf{n}}$ is the distance vector from the location of the particle *in the past*, at the retarded (delayed) time, to the observation point. It is not obvious that these forms of the electric field are the same, since one makes reference to the location of the particle now, and the other to the location of the particle in the past. We will show they are the same.

To do this, use Jackson's figure 14.2, reproduced here:



Here O is the location of the observer, P is the location of the charged particle now, P' is the location at the retarded time, and Q is a point on OP' where a line from P meets OP' at a right angle. θ is the angle OP' makes with the x axis. From the setup, it is evident that $PM = vt, OM = b$, and $OP' = R$ by definition; make sure you understand these assignments. (Technically since the figure shows the particle on the negative x-axis, it is at

$t < 0$ and we should write $PM = v|t|$, but t will be squared in the final result so we can pretend it's positive.)

(a) Use the definition of the retarded time to explain why the segment $P'P$ has length βR. Explain geometrically why it then follows that

$$R(\hat{\mathbf{n}} - \boldsymbol{\beta}) = \mathbf{r}, \tag{10.2.5}$$

where $\mathbf{r}$ is the vector from the location of the particle *now* to the observation point (the segment PO). This shows that the expression for $\mathbf{E}_v$ does indeed point from the location of the particle now.

(b) Show that the lengths of the following segments take the values

$$\begin{aligned} P'Q &= R\boldsymbol{\beta} \cdot \hat{\mathbf{n}} \\ OQ &= R(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}) \\ PQ &= \beta b. \end{aligned} \tag{10.2.6}$$

For the last equality, use the fact that θ is in more than one right triangle.

(c) The segment PO with length r is the hypotenuse of two different right triangles. Use the Pythagorean theorem for both triangles and equate the values of r^2 to show that

$$R^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2 = \frac{\tilde{r}^2}{\gamma^2}. \tag{10.2.7}$$

(d) Combine the results from part (a) and part (c) to show that the two expressions for the electric field given above are in fact equal.

Solution.

(a) By definition, the retarded time is

$$t_{\text{ret}} = t + \frac{R}{c}, \tag{10.2.8}$$

where we have let $-t \to t$ to match the coordinates in the figure. It then follows that

$$vt_{\text{ret}} = vt + \beta R \tag{10.2.9}$$

is the distance the charge needs to traverse to reach $x = 0$, as observed by O. Thus, considering that vt is the distance the charge actually traverse, it follows that the PP' segment needs to be βR. By vector addition, we can then write $R\hat{\mathbf{n}} = \beta R\hat{\boldsymbol{\beta}} + \mathbf{r}$. Thus,

$$\mathbf{r} = R\hat{\mathbf{n}} - R\boldsymbol{\beta}\hat{\boldsymbol{\beta}} = R(\hat{\mathbf{n}} - \boldsymbol{\beta}). \tag{10.2.10}$$

(b) First, by Pythagorean theorem, we can write

$$(PQ)^2 = (\beta R)^2 - (P'Q)^2 = r^2 - (R - P'Q)^2 = r^2 - R^2 + 2R(P'Q) - (P'Q)^2. \tag{10.2.11}$$

Thus, we can solve for

$$P'Q = \frac{(1+\beta^2)R^2 - r^2}{2R} = \frac{(1+\beta^2)R^2 - (1+\beta^2 - 2\boldsymbol{\beta} \cdot \hat{\mathbf{n}})R^2}{2R} = R(\boldsymbol{\beta} \cdot \hat{\mathbf{n}}), \tag{10.2.12}$$

where we have calculated r from (10.2.10). It then follows immediately that

$$OQ = R - P'Q = R(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}). \tag{10.2.13}$$

Finally,

$$PQ = \sqrt{\beta^2 R^2 - (P'Q)^2} = R\sqrt{\beta^2 - (\boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2} = \beta R\sqrt{1 - \cos^2\theta} = \beta R \sin\theta = \beta R \frac{b}{R} = \beta b. \tag{10.2.14}$$

(c) By the Pythagorean theorem,

$$r^2 = b^2 + (vt)^2 = (OQ)^2 + (PQ)^2 = R^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2 + \beta^2 b^2. \tag{10.2.15}$$

Thus,

$$R^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2 = b^2(1 - \beta^2) + (vt)^2 = \frac{\tilde{r}^2}{\gamma^2}, \tag{10.2.16}$$

where $\tilde{r}^2 = b^2 + \gamma^2(vt)^2$ by definition.

(d) Then, from (10.2.4),

$$\mathbf{E} = \frac{e\mathbf{r}}{\gamma^2 R^3 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3} = \frac{e\mathbf{r}}{\gamma^2 \tilde{r}^3/\gamma^3} = \frac{\gamma e\mathbf{r}}{\tilde{r}^3}. \tag{10.2.17}$$

So the two expressions for the electric field are equal. □