Homework 9: Phys 5210 (Fall 2021)

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***Problem* 1:** In atomic physics experiments such as the ones done in JILA, a cloud of interacting atoms is sometimes placed in a rotating external potential. It is convenient to study these atoms in a reference frame which rotates together with the potential where their potential energy does not depend on time. As we know the velocity of a particle at a position $\mathbf{r}$ in this rotating reference frame is related to the velocity of the same particle in the stationary reference frame by $\mathbf{v}_0 = \mathbf{v} + \mathbf{\Omega} \times \mathbf{r}$, where $\mathbf{\Omega}$ is the constant angular velocity of rotation, $\mathbf{v}$ is the velocity of the particle in the rotating reference frame and $\mathbf{v}_0$ is the velocity in the stationary reference frame. The Lagrange function in the rotating reference frame of a system of particles, each with mass $\mathbf{m}$, labelled by the index $j = 1, 2, \ldots, N$, is then given by

$$\mathcal{L} = \sum_{j=1}^{N} \frac{mv_{0j}^2}{2} - U(\mathbf{r}) = \sum_{j=1}^{N} \left(\frac{mv_j^2}{2} + m\mathbf{v}_j \cdot (\mathbf{\Omega} \times \mathbf{r}_j) + \frac{m(\mathbf{\Omega} \times \mathbf{r}_j)^2}{2} \right) - U(\mathbf{r}) \tag{1.1}$$

Find the Hamiltonian of this system in the rotating reference frame.

Solution.

Given the Lagrange function (1.1), the canonical momenta are

$$\mathbf{p}_j = \frac{\partial \mathcal{L}}{\partial \mathbf{v}_j} = m\mathbf{v}_j + m\big(\mathbf{\Omega} \times \mathbf{r}_j\big) \Rightarrow \mathbf{v}_j = \frac{\mathbf{p}_j}{m} - \mathbf{\Omega} \times \mathbf{r}_j \tag{1.2}$$

where $\mathbf{v}_j = \dot{\mathbf{r}}_j$. Then we can write the Hamiltonian as

$$\begin{aligned}
\mathcal{H} &= \sum_{j=1}^{N} \mathbf{p}_j \cdot \mathbf{v}_j - \mathcal{L} \\
&= \sum_{j=1}^{N} \Bigg[\frac{p_j^2}{m} - \mathbf{p}_j \cdot \big(\mathbf{\Omega} \times \mathbf{r}_j\big) \\
&\qquad - \frac{m}{2}\left(\frac{\mathbf{p}_j}{m} - \mathbf{\Omega} \times \mathbf{r}_j\right)^2 - \big(\mathbf{p}_j - m\mathbf{\Omega} \times \mathbf{r}_j\big) \cdot \big(\mathbf{\Omega} \times \mathbf{r}_j\big) - \frac{m}{2}\big(\mathbf{\Omega} \times \mathbf{r}_j\big)^2 \Bigg] + U(\mathbf{r}) \\
&= \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} - \mathbf{p}_j \cdot \big(\mathbf{\Omega} \times \mathbf{r}_j\big) \right] + U(\mathbf{r})
\end{aligned} \tag{1.3}$$

□

***Problem* 2:** Inspired by *Goldstein*, Chapter 8, Problem 35. Consider a system with this Lagrangian

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right)e^{2\gamma t} \tag{2.1}$$

where $\gamma > 0, \omega \neq 0, m > 0$.

(a) Write down the Euler-Lagrange equation and solve it with arbitrary initial conditions at $t = 0$.

(b) Find the Hamiltonian, write down the Hamilton equations of motion and verify that they are equivalent to the Euler-Lagrange equation.

(c) For arbitrary initial conditions such as in part (a), how do the position x, the momentum p, and the Hamiltonian $\mathcal{H}$ of this system behave as $t \to \infty$?

Solution.

(a) From the Euler-Lagrange equation for x,

$$\frac{d}{dt}\left[m\dot{x}e^{2\gamma t}\right] = -m\omega^2 x e^{2\gamma t} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \tag{2.2}$$

A guess to the solution of this differential equation is $x = Ce^{i\Omega t}$ for some Ω. Plugging back yields

$$-\Omega^2 + 2i\gamma\Omega + \omega^2 = 0 \Rightarrow \Omega_\pm = i\gamma \pm \sqrt{\omega^2 - \gamma^2} \tag{2.3}$$

Since there are two solutions to Ω, the general solution for is spanned by a linear combination

$$x(t) = C_1 e^{i\Omega_+ t} + C_2 e^{i\Omega_- t} = e^{-\gamma t}\left[C_1 e^{i\sqrt{\omega^2-\gamma^2}t} + C_2 e^{-i\sqrt{\omega^2-\gamma^2}t}\right] \tag{2.4}$$

where C_1, C_2 are some constants dependent on initial conditions. The physical solution is the real part of this

$$x(t) = e^{-\gamma t}\left[C_1 \cos\left(\sqrt{\omega^2-\gamma^2}t\right) + C_2 \sin\left(\sqrt{\omega^2-\gamma^2}t\right)\right] \tag{2.5}$$

Thus, given arbitrary conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, we can write the real constants C_1, C_2 as

$$C_1 = x_0 \qquad \text{and} \qquad C_2 = \frac{v_0 + \gamma x_0}{\sqrt{\omega^2 - \gamma^2}} \tag{2.6}$$

Then the final form of the solution, given initial conditions, is

$$x(t) = e^{-\gamma t}\left[x_0 \cos\left(\sqrt{\omega^2-\gamma^2}t\right) + \frac{v_0 + \gamma x_0}{\sqrt{\omega^2-\gamma^2}} \sin\left(\sqrt{\omega^2-\gamma^2}t\right)\right] \tag{2.7}$$

(b) From (2.1), the canonical momentum is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}e^{2\gamma t} \tag{2.8}$$

Then we can write the Hamiltonian as

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m}e^{-2\gamma t} - \frac{m}{2}\left(\frac{p}{m}e^{-2\gamma t}\right)^2 e^{2\gamma t} + \frac{1}{2}m\omega^2 x^2 e^{2\gamma t} = \frac{p^2}{2m}e^{-2\gamma t} + \frac{1}{2}m\omega^2 x^2 e^{2\gamma t} \tag{2.9}$$

The Hamiltonian equation of motion is thus

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}e^{-2\gamma t} \tag{2.10a}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x e^{2\gamma t} \tag{2.10b}$$

Taking the time derivative of (2.10a) one more time and substituting in (2.10b), we get

$$\ddot{x} = \frac{\dot{p}}{m}e^{-2\gamma t} - 2\gamma\frac{p}{m}e^{-2\gamma t} = -\omega^2 x - 2\gamma\dot{x} \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \tag{2.11}$$

This is the same differential equation obtained from the Euler-Lagrange equation. Thus, the Hamiltonian equation of motion is equivalent to that in the Lagrangian formulation.

(c) From (2.7), $x \to 0$ as $t \to \infty$. Also, the velocity is

$$\dot{x}(t) = -\gamma x(t) + e^{-\gamma t}\left[-x_0\sqrt{\omega^2 - \gamma^2}\cos\left(\sqrt{\omega^2 - \gamma^2}t\right) + (v_0 + \gamma x_0)\cos\left(\sqrt{\omega^2 - \gamma^2}t\right)\right] \tag{2.12}$$

Thus, the mechanical momentum $p_{\text{mech}} = m\dot{x} \to 0$ as $t \to \infty$. However, the canonical momentum $p = p_{\text{mech}}e^{2\gamma t} \sim e^{\gamma t} \to \infty$ as $t \to \infty$. Also, since $x(t) = e^{-\gamma t}f(t)$ and $p(t) = e^{\gamma t}g(t)$ where f, g are bounded functions, from (2.9), we can conclude that $\mathcal{H}$ is bounded. So $\mathcal{H}$ doesn't blow up where $t \to \infty$. $\square$