Homework 5: Phys 5210 (Fall 2021)

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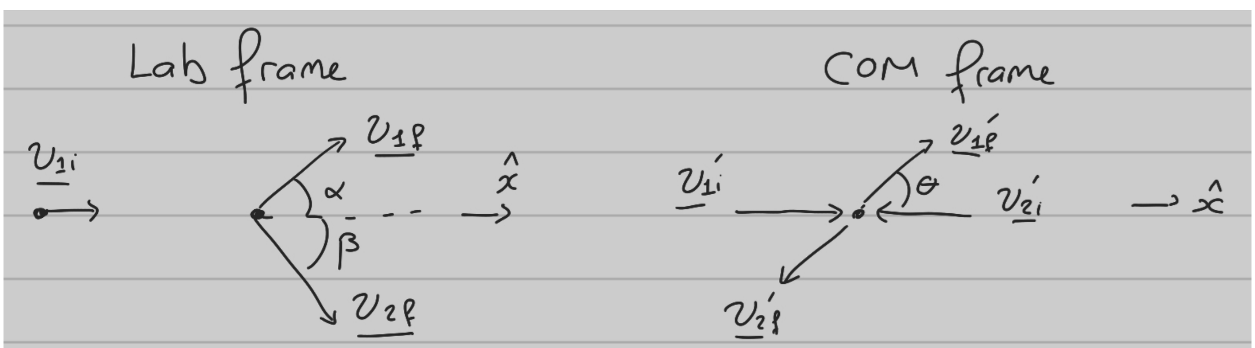
***Problem* 1:** A particle of mass m_1 moving with velocity $\mathbf{v}_1$ collides with a particle with mass m_2 which is stationary, that is $\mathbf{v}_2 = \mathbf{0}$. The process can be described by splitting the motion into the center of mass motion and relative motion. Center of mass continues to move with constant velocity throughout the collision. Relative motion can be described by means of a vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, which initially moves with the velocity $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1$. The collision process results in the relative velocity vector turning by a certain angle θ. Find the magnitudes of the final velocities of both particles, and the angles their final velocities form with $\mathbf{v}_1$.

Solution.



Let the un-primed coordinates be those of the lab frame, and the primed coordinates be those of the center-of-mass (COM) frame, as denoted above. We know that the initial velocity of the mass m_1 in the lab frame is $\mathbf{v}_{1i} = \mathbf{v}_1 = v_1\hat{\mathbf{x}}$ and the velocity of the COM is $\mathbf{V} = m_1/(m_1+m_2)\mathbf{v}_1 = V\hat{\mathbf{x}}$. By a transformation to the COM frame, the initial velocity of the first mass is also in the x direction

$$\mathbf{v}_{1i}' = \mathbf{v}_{1i} - \mathbf{V} = \frac{m_2}{m_1+m_2}v_1\hat{\mathbf{x}} \tag{1.1}$$

Now, in the COM frame, by momentum conservation, we can write

$$m_1\mathbf{v}_{1i}' + m_2\mathbf{v}_{2i}' = m_1\mathbf{v}_{1f}' + m_2\mathbf{v}_{2f}' = \mathbf{0} \tag{1.2}$$

This implies that

$$\mathbf{v}_{2i}' = -\frac{m_1}{m_2}\mathbf{v}_{1i}' \qquad \text{and} \qquad \mathbf{v}_{2f}' = -\frac{m_1}{m_2}\mathbf{v}_{1f}' \tag{1.3}$$

By energy conservation,

$$\begin{aligned}
m_1 v_{1i}'^2 + m_2 v_{2i}'^2 &= m_1 v_{1f}'^2 + m_2 v_{2f}'^2 \\
\Leftrightarrow \left(m_1 - \frac{m_1^2}{m_2} \right) v_{1i}'^2 &= \left(m_1 - \frac{m_1^2}{m_2} \right) v_{1f}'^2 \\
\Leftrightarrow \qquad\qquad v_{1i}' &= v_{1f}' \qquad (1.4)
\end{aligned}$$

It also follows from this that $v_{2i}' = v_{2f}'$. Then we can write the final velocities in the lab frame from (1.1) as

$$\mathbf{v}_{1f} = \mathbf{v}_{1f}' + \mathbf{V} = \mathbf{v}_{1i}' + \mathbf{V} = \big(V + v_{1i}' \cos\theta \big) \hat{\mathbf{x}} + v_{1i}' \sin\theta \hat{\mathbf{y}} \qquad (1.5)$$

and also from (1.3)

$$\mathbf{v}_{2f} = \mathbf{v}_{2f}' + \mathbf{V} = \mathbf{v}_{2i}' + \mathbf{V} = -\frac{m_1}{m_2} \mathbf{v}_{1f}' + \mathbf{V} = (V - \frac{m_1}{m_2} v_{1i}' \cos\theta) \hat{\mathbf{x}} - \frac{m_1}{m_2} v_{1i}' \sin\theta \hat{\mathbf{y}} \qquad (1.6)$$

Then we can calculate their magnitudes

$$\begin{aligned}
v_{1f} &= \big(v_{1i}'^2 + V^2 + 2 v_{1i}' V \cos\theta \big)^{1/2} \\
&= \left[\frac{m_2^2}{(m_1+m_2)^2} + \frac{m_1^2}{(m_1+m_2)^2} + 2\frac{m_1 m_2}{(m_1+m_2)^2} \cos\theta \right]^{1/2} v_1 \\
&= \frac{\sqrt{m_1^2 + m_2^2 + 2 m_1 m_2 \cos\theta}}{m_1 + m_2} v_1 \qquad (1.7)
\end{aligned}$$

and

$$\begin{aligned}
v_{2f} &= \left(\frac{m_1^2}{m_2^2} v_{1i}'^2 + V^2 - 2\frac{m_1}{m_2} v_{1i}' V \cos\theta \right)^{1/2} \\
&= \left[\frac{m_1^2}{(m_1+m_2)^2} + \frac{m_1^2}{(m_1+m_2)^2} - 2\frac{m_1^2}{(m_1+m_2)^2} \cos\theta \right]^{1/2} v_1 \\
&= \frac{m_1}{m_1 + m_2} \sqrt{2(1 - \cos\theta)} v_1 \qquad (1.8)
\end{aligned}$$

Also, the angle α, β between $\mathbf{v}_{1f}, \mathbf{v}_{2f}$ and $\mathbf{v}_{1i}$ are

$$\alpha = \tan^{-1} \frac{v_{1f,y}}{v_{1f,x}} = \tan^{-1} \left[\frac{\sin\theta}{\cos\theta + V/v_{1i}'} \right] = \tan^{-1} \left[\frac{\sin\theta}{\cos\theta + m_1/m_2} \right] \qquad (1.9)$$

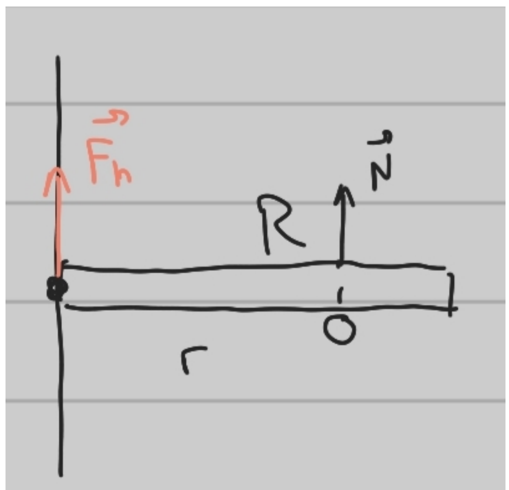
and

$$\beta = \tan^{-1} \frac{v_{2f,y}}{v_{2f,x}} = \tan^{-1} \left[\frac{\sin\theta}{\cos\theta - (m_2/m_1)(V/v_{1i}')} \right] = \tan^{-1} \left[\frac{\sin\theta}{\cos\theta - 1} \right] \qquad (1.10)$$

□

***Problem* 2:** A doorstop protects the wall from being slammed by a door when it is pushed open. How far from the hinges should the doorstop be placed so that there would be no force on the hinges as the door stops thus protecting the hinges from breaking? Think of the door as a very thin rectangle of a uniform density attached to the hinges at one end. To avoid thinking of the door "falling over" the doorstop, think of the doorstop as a vertical pole as tall as the door itseld, and of the door as hitting this pole at the same instant in time along its entire length.

Solution.



By Newton's 2nd Law, the total force on the door with mass M (at the center of mass) is the sum of the force on the hinge ($\mathbf{F}_h$) and the normal force due to the doorstop $\mathbf{N}$

$$F_h + N = Ma = \frac{MR}{2}\alpha \tag{2.1}$$

where we have related the angular acceleration at the center of mass of the door ($R/2$ from the hinge) to the linear acceleration $a = \alpha R/2$. Now, only the normal force due to the doorstop causes a torque on the door

$$Nr = I\alpha \tag{2.2}$$

where $I = (1/3)MR^2$ is the moment of inertia of the door (thin rectangle of width R and uniformly distributed mass) and r is the distance of the doorstop from the hinge. Substituting (2.1) into (2.2) and setting the force on the hinge $F_h = 0$ results in

$$\frac{MRr}{2}\alpha = I\alpha \Rightarrow \frac{MRr}{2} = \frac{1}{3}MR^2 \tag{2.3}$$

Solving for r, we get $r = (2/3)R$. □