Homework 3: Astr 5140 (Fall 2021)

Tien Vo

September 19, 2021

***Problem* 1** (Generalized Ohm's Law)**:** The MHD force equation is derived by a linear combination of the electron and ion fluid equations. Generalized Ohm's law is, simply stated, a different linear combination.

We begin by multiplying the ion force equation by m_e and the electron force equation by m_i, then subtract

$$m_e m_i n \frac{D\mathbf{u}_i}{Dt} = -m_e \boldsymbol{\nabla} P_i + n m_e e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - m_e m_i n \nu_{ie}(\mathbf{u}_i - \mathbf{u}_e) \tag{1.1a}$$

$$m_i m_e n \frac{D\mathbf{u}_e}{Dt} = -m_i \boldsymbol{\nabla} P_e - n m_i e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - m_e m_i n \nu_{ei}(\mathbf{u}_e - \mathbf{u}_i) \tag{1.1b}$$

We use quasi-neutrality and bundle the convective derivative into the symbol D.

$$\begin{aligned} m_e m_i n \left(\frac{D\mathbf{u}_i}{Dt} - \frac{D\mathbf{u}_e}{Dt} \right) &= (-m_e \boldsymbol{\nabla} P_i + m_i \boldsymbol{\nabla} P_e) + en\mathbf{E}(m_e + m_i) \\ &\quad + en(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} - m_e m_i n(\nu_{ei} + \nu_{ie})(\mathbf{u}_i - \mathbf{u}_e) \end{aligned} \tag{1.2}$$

(a) Divide each term by $en(m_i + m_e)$.

(b) Show that Term 1 can be re-written in the limit of $m_i \gg m_e$ as

$$\frac{m_e}{e^2} \frac{D(\mathbf{J}/n)}{Dt} \tag{1.3}$$

(c) Argue that since $m_i \gg m_e$ that, unless $P_i \gg P_e$ (very rare), Term 2 becomes

$$\frac{\boldsymbol{\nabla} P_e}{en} \tag{1.4}$$

(d) Term 3 is trivial. Term 4 is tricky as it must be broken into two parts. Add $\mathbf{u} = (m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_i + m_e)$, separate $\mathbf{u} \times \mathbf{B}$, then subtract $(m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_i + m_e)$. Show that the remaining four terms ($m_i \gg m_e$) can be approximated as

$$\frac{-\mathbf{J} \times \mathbf{B}}{en} \tag{1.5}$$

(e) Show that Term 5 can be written as $\mathbf{J}/\sigma$. Define σ.

In the end, you should arrive at

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{\mathbf{J} \times \mathbf{B}}{en} - \frac{\boldsymbol{\nabla} P_e}{en} + \frac{m_e}{ne^2}\frac{D\mathbf{J}}{Dt} + \text{small terms} \tag{1.6}$$

Note: If done exactly (full convective derivative and keeping small terms), n is not inside the derivative and furthermore, some of the "leftovers" from Term 1 cancel "leftovers" in Term 4.

Solution.

□

***Problem* 2** (Scale height in the solar corona)**:** The surface gravity of the Sun is $274\,\mathrm{m/s^2}$. Assume that the corona is entirely protons with $T_{\text{corona}} = 10^6\,^\circ\mathrm{K}$. Derive the isothermal scale height of the solar corona from the MHD equations assuming $\mathbf{g}$ is constant (use 1D). How does this value compare with the radius of the Sun (what % of R_{sun} is H_0)? Does your answer agree with hat is seen in UV or X-ray images?

Solution.

□

***Problem* 4** (Magnetic tension)**:** This problem is to develop an intuition for magnetic tension. Examine the diagram, which shows a field line if $\mathbf{B} = B_0[(z/z_0)\hat{\mathbf{x}} + \hat{\mathbf{z}}$. Examine the field line and convince yourself that the equation represents a curved magnetic field line. Derive the tension force at $z = 0$. What is the direction of the tension force? Argue that z_0 is the local radius of curvature.

Solution.

□