Homework 8: Phys 7310 (Fall 2021)

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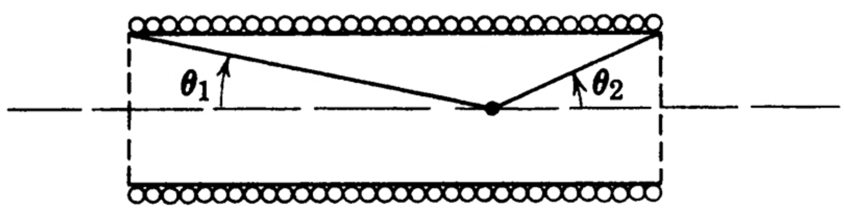
***Problem* 8.1** (Magnetic field in a solenoid)**:** (a) A right-circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is

$$B_z = \frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2) \tag{8.1.1}$$

where the angles are defined in the figure.



(b) Show that near the axis and near the (vertical) middle of the solenoid, the magnetic field is mainly parallel to the axis, but has a small radial component,

$$B_\rho \approx 24\mu_0 NI \frac{a^2 z\rho}{L^4} \tag{8.1.2}$$

with L the length and a the radius of the solenoid, and we take the vertical center of the solenoid to be at $z = 0$. You may assume the solenoid is long, $L \gg a$. Hint: write $\cos\theta_{1,2}$ in terms of L, a, and z and find an expression for B_z as a series. How can you relate B_z to B_ρ?

(c) Using similar methods, again assuming a long solenoid $L \gg a$, show that at the end of the solenoid, the magnetic field near the axis has components

$$B_z \approx \frac{\mu_0 NI}{2} \qquad \text{and} \qquad B_\rho \approx \pm\frac{\mu_0 NI}{4}\frac{\rho}{a} \tag{8.1.3}$$

where the two signs are for the two ends of the cylinder (you need only check one).

Solution.

(a) First we find the magnetic field due to a single loop of current I with radius a located at $z = z_0$. The only component is in the $\hat{\boldsymbol{\phi}}$ direction. So we can write the volume current density in cylindrical coordinates as

$$\mathbf{J} = I\delta(r' - a)\delta(z' - z_0)\hat{\boldsymbol{\phi}} \tag{8.1.4}$$

such that the current through a small Amperian area $da' = dr'dz'$ perpendicular to the loop is

$$\int da' J_\phi = I \int dr'dz'\delta(r'-a)\delta(z'-z_0) = I \tag{8.1.5}$$

Now, from Biot-Savart law, the magnetic field due to this current on the z axis is

$$\begin{aligned}
\mathbf{B}_0 &= \frac{\mu_0}{I}4\pi \int_0^\infty \int_0^{2\pi} \int_{-\infty}^{\infty} r'dr'd\phi'dz'\delta(r'-a)\delta(z'-z_0)\hat{\boldsymbol{\phi}} \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \\
&= \frac{\mu_0 I}{4\pi} \int_0^\infty \int_0^{2\pi} \int_{-\infty}^{\infty} r'dr'd\phi'dz'\delta(r'-a)\delta(z'-z_0) \frac{(z-z')\cos\phi'\hat{\mathbf{x}} + (z-z')\sin\phi'\hat{\mathbf{y}} + r'\hat{\mathbf{z}}}{[r'^2+(z-z')^2]^{3/2}} \\
&= \frac{\mu_0 I}{2} \int_0^\infty \int_{-\infty}^{\infty} \frac{r'^2\delta(r'-a)\delta(z'-z_0)dr'dz'}{[r'^2+(z-z')^2]^{3/2}}\hat{\mathbf{z}} \\
&= \frac{\mu_0 I}{2} \frac{a^2}{[a^2+(z-z_0)^2]^{3/2}}\hat{\mathbf{z}}
\end{aligned} \tag{8.1.6}$$

where the x and y components have dropped out in the azimuthal integration due to azimuthal symmetry on the z axis. Now, when $NL \to \infty$, the density of loops n' is high enough that it can be written in differentials as $dn' = Ndz'$. We can now consider the contribution due to multiple loops placed from $z_0 = 0 \to L$

$$\begin{aligned}
\mathbf{B} &= \int d\mathbf{B}_0 \\
&= \frac{\mu_0 NI}{2} a^2 \int_0^L \frac{dz_0}{[a^2+(z-z_0)^2]^{3/2}}\hat{\mathbf{z}} \\
&= \frac{\mu_0 NI}{2} \left[\frac{z}{\sqrt{a^2+z^2}} + \frac{L-z}{\sqrt{a^2+(L-z)^2}} \right] \hat{\mathbf{z}} \\
&= \frac{\mu_0 NI}{2} [\cos\theta_1 + \cos\theta_2]\hat{\mathbf{z}}
\end{aligned} \tag{8.1.7}$$

where $\theta_{1,2}$ are defined as drawn in the figure.

(b) By a translation $z \mapsto z + L/2$, we can rewrite (8.1.7) as

$$\begin{aligned}
B_z &= \frac{\mu_0 NI}{2} \left[\frac{L-2z}{4a^2+(L-2z)^2} + \frac{L+2z}{\sqrt{4a^2+(L+2z)^2}} \right] \\
&= \frac{\mu_0 NI}{2} \left[\frac{1-2z/L}{\sqrt{4(a/L)^2+(1-2z/L)^2}} + \frac{1+2z/L}{\sqrt{4(a/L)^2+(1+2z/L)^2}} \right]
\end{aligned} \tag{8.1.8}$$

Using Mathematica, we can expand B_z around small z/L and a/L

$$B_z \approx \mu_0 NI \left(1 - 2\frac{a^2}{L^2} \right) - 24\mu_0 NI \frac{z^2}{L^2}\frac{a^2}{L^2} \tag{8.1.9}$$

Now, since $\boldsymbol{\nabla}\cdot\mathbf{B}=0$, we can write the differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho B_\rho\big)+\frac{\partial B_z}{\partial z}=0 \tag{8.1.10}$$

Note that since $\partial B_z/\partial z\neq 0$, B_ρ has to be non-zero. The solution to (8.1.10) is, according to Mathematica,

$$B_\rho=-\frac{\rho}{2}\frac{\partial B_z}{\partial z}=24\mu_0 NI\frac{z\rho}{L^2}\frac{a^2}{L^2} \tag{8.1.11}$$

(c) Let us now rewriting (8.1.7) again without applying any translation

$$\begin{aligned}
B_z&=\frac{\mu_0 NI}{2}\left[\frac{1-z/L}{\sqrt{(a/L)^2+(1-z/L)^2}}+\frac{z/L}{\sqrt{(a/L)^2+(z/L)^2}}\right]\\
&\approx\frac{\mu_0 NI}{2}\left[1-\frac{1}{2}\frac{a^2}{L^2}+\frac{z}{a}\right]\\
&\approx\frac{\mu_0 NI}{2}
\end{aligned} \tag{8.1.12}$$

To find the small radial component, we have to retain the first-order term in z

$$B_\rho=-\frac{\rho}{2}\frac{\partial B_z}{\partial z}\approx-\frac{\mu_0 NI}{4}\frac{\rho}{a} \tag{8.1.13}$$

□

***Problem* 8.2** (Rotating sphere)**:** A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

Solution.

From Griffiths, we can write the volume current density from the surface current density $\mathbf{K}=\sigma\mathbf{v}=\sigma\omega a\sin\theta\hat{\boldsymbol{\phi}}$ as

$$\mathbf{J}=\mathbf{K}\delta(r'-a)=\sigma\omega a\sin\theta\delta(r'-a)\hat{\boldsymbol{\phi}} \tag{8.2.1}$$

Then the vector potential is

$$\begin{aligned}
\mathbf{A}&=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\
&=\mu_0\sigma\omega a\sum_{l,m}\frac{1}{2l+1}Y_{lm}(\Omega)\int d^3x'\frac{r_<^l}{r_>^{l+1}}\delta(r'-a)\sin\theta'(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}})Y_{lm}^*(\Omega')\\
&=\mu_0\sigma\omega a\sum_{l,m}\sqrt{\frac{1}{4\pi(2l+1)}\frac{(l-m)!}{(l+m)!}}Y_{lm}(\Omega)\int_0^\infty\frac{r'^2r_<^l}{r_>^{l+1}}\delta(r'-a)dr'\\
&\quad\times\int_0^\pi\sin^2\theta' P_l^m(\cos\theta')d\theta'\int_0^{2\pi}e^{-im\phi'}(-\sin\phi'\hat{\mathbf{x}}+\cos\phi'\hat{\mathbf{y}})d\phi'
\end{aligned} \tag{8.2.2}$$

The azimuthal integration vanishes for $m^2 \neq 1$. Thus, the vector potential now becomes

$$\mathbf{A} = \mu_0\sigma\omega a \sum_{l=1}^{\infty} \frac{1}{\sqrt{4\pi(2l+1)}} R_l \pi \sqrt{\frac{(l-1)!}{(l+1)!}} \int_0^{\pi} \sin^2\theta' P_l^1(\cos\theta') d\theta' \Big[Y_{l,1}(i\hat{\mathbf{x}} + \hat{\mathbf{y}}) + Y_{l,1}^*(-i\hat{\mathbf{x}} + \hat{\mathbf{y}}) \Big] \tag{8.2.3}$$

where R_l is the radial integration. Note also that

$$\int_0^{\pi} \sin^2\theta' P_l^1(\cos\theta') d\theta' = \int_{-1}^{1} \sqrt{1-x^2} P_l^1(x) dx = -\int_{-1}^{1} (1-x^2) P_l'(x) dx = -\frac{4}{3} \tag{8.2.4}$$

only for $l = 1$. Then the vector potential simplifies to

$$\begin{aligned}
\mathbf{A} &= -\frac{4}{3}\sqrt{\frac{\pi}{24}} \mu_0\sigma\omega a R_1 \Big[i(Y_{1,1} - Y_{1,1}^*)\hat{\mathbf{x}} + (Y_{1,1} + Y_{1,1}^*)\hat{\mathbf{y}} \Big] \\
&= \frac{1}{3}\mu_0\sigma\omega a \sin\theta R_1(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}) \\
&= \frac{1}{3}\mu_0\sigma\omega a \sin\theta R_1 \hat{\boldsymbol{\phi}}
\end{aligned} \tag{8.2.5}$$

The radial integral evaluates to

$$R_1(r' > r) = r\int_0^{\infty} \delta(r'-a) dr' = r \tag{8.2.6a}$$

$$R_1(r' < r) = \int_0^{\infty} \frac{r'^3}{r^2} \delta(r'-a) dr' = \frac{a^3}{r^2} \tag{8.2.6b}$$

Then we can write the final solution as

$$\mathbf{A}_{r'<r} = \frac{\mu_0\sigma\omega}{3} ar\sin\theta\hat{\boldsymbol{\phi}} \tag{8.2.7a}$$

$$\mathbf{A}_{r'>r} = \frac{\mu_0\sigma\omega}{3}\frac{a^4}{r^2}\sin\theta\hat{\boldsymbol{\phi}} \tag{8.2.7b}$$

Then since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, it follows that

$$\mathbf{B}_{r'<r} = \frac{2}{3}\mu_0\sigma\omega a\Big(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\Big) \tag{8.2.8a}$$

$$\mathbf{B}_{r'>r} = \frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^3}\Big(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\Big) \tag{8.2.8b}$$

□

***Problem* 8.3** (Image current)**:** A current distribution $\mathbf{J}(\mathbf{r})$ exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ_r and filling the half-space, $z < 0$.

(a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability μ_r by an image current distribution, $\mathbf{J}^*$, with components

$$\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \qquad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \qquad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z) \tag{8.3.1}$$

(b) Show that for $z < 0$ the magnetic induction appears to be due to a current distribution $[2\mu_r/(\mu_r + 1)]\mathbf{J}$ in a medium of unit relative permeability.

Solution.

(a,b) Suppose that the magnetic field in both half-infinite regions is given by the indicated currents. Then we can write

$$\mathbf{B}_{z\geq 0} = \frac{\mu_0}{4\pi} \int d^3x' \big[\mathbf{J}(\mathbf{x}') + \mathbf{J}^*(\mathbf{x}')\big] \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{8.3.2a}$$

$$\mathbf{B}_{z<0} = \frac{\mu_0}{4\pi} \frac{2\mu_r}{\mu_r + 1} \int d^3x' \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \tag{8.3.2b}$$

where the position vector can be splitted into two terms $\mathbf{x} = \mathbf{x}_\perp + z\hat{\mathbf{z}}$ where $\mathbf{x}_\perp = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ and

$$\mathbf{J}^*(\mathbf{x}_\perp, z) = \frac{\mu_r - 1}{\mu_r + 1} \big[\mathbf{J}_\perp(\mathbf{x}_\perp, -z) - J_z(\mathbf{x}_\perp, -z)\hat{\mathbf{z}}\big] \tag{8.3.3}$$

Now, we need only show that the magnetic field (8.3.2) follows $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and Ampere's Law, or

$$\lim_{z\to 0^-} \mathbf{B} \cdot \hat{\mathbf{z}} = \lim_{z\to 0^+} \mathbf{B} \cdot \hat{\mathbf{z}} \qquad \text{and} \qquad \lim_{z\to 0^-} \mathbf{B} \times \hat{\mathbf{z}} = \mu_r \lim_{z\to 0^+} \mathbf{B} \times \hat{\mathbf{z}} \tag{8.3.4}$$

because we let the normal vector to the boundary between the two material be in the $\hat{\mathbf{z}}$ direction.

First, let us consider the former condition

$$\begin{aligned}
\lim_{z\to 0^+} \mathbf{B} \cdot \hat{\mathbf{z}} = \frac{\mu_0}{4\pi} \Bigg\{ &\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3} \big[(y - y')J_x(\mathbf{x}'_\perp, z') - (x - x')J_y(\mathbf{x}'_\perp, z')\big] \\
&+ \int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3} \Big[(y - y')J^*_x(\mathbf{x}'_\perp, z') - (x - x')J^*_y(\mathbf{x}'_\perp, z')\Big] \Bigg\} \\
= \frac{\mu_0}{4\pi} \Bigg\{ &\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3} \big[(y - y')J_x(\mathbf{x}'_\perp, z') - (x - x')J_y(\mathbf{x}'_\perp, z')\big] \\
&+ \frac{\mu_r - 1}{\mu_r + 1} \int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3} \big[(y - y')J_x(\mathbf{x}'_\perp, -z') - (x - x')J_y(\mathbf{x}'_\perp, -z')\big] \Bigg\}
\end{aligned} \tag{8.3.5}$$

Let $\overline{z}' = -z'$ in the second integration. Then $\int_{-\infty}^{\infty} dz' = -\int_{\infty}^{-\infty} d\overline{z}' = \int_{-\infty}^{\infty} d\overline{z}'$. Also, $|\mathbf{x}_\perp - \mathbf{x}'| = \sqrt{(x - x')^2 + (y - y')^2 + \overline{z}'^2}$. So the first terms in the integral are not affected and we can write

$$\begin{aligned}
\lim_{z\to 0^+} \mathbf{B}\cdot\hat{\mathbf{z}} &= \frac{\mu_0}{4\pi}\Bigg\{\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[(y-y')J_x(\mathbf{x}'_\perp, z') - (x-x')J_y(\mathbf{x}'_\perp, z')\big] \\
&\qquad + \frac{\mu_r - 1}{\mu_r + 1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[(y-y')J_x(\mathbf{x}'_\perp, z') - (x-x')J_y(\mathbf{x}'_\perp, z')\big]\Bigg\} \\
&= \frac{\mu_0}{4\pi}\frac{2\mu_r}{\mu_r+1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[(y-y')J_x(\mathbf{x}'_\perp, z') - (x-x')J_y(\mathbf{x}'_\perp, z')\big] \\
&= \frac{\mu_0}{4\pi}\frac{2\mu_r}{\mu_r+1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[\mathbf{J}(\mathbf{x}') \times (\mathbf{x}_\perp - \mathbf{x}')\big]\cdot\hat{\mathbf{z}} \\
&= \lim_{z\to 0^-} \mathbf{B}\cdot\hat{\mathbf{z}}
\end{aligned} \tag{8.3.6}$$

So the first condition is satisfied. Now, consider the second one

$$\begin{aligned}
\mu_r \lim_{z\to 0^+} \mathbf{B}\times\hat{\mathbf{z}} &= \frac{\mu_0\mu_r}{4\pi}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[(\mathbf{J} + \mathbf{J}^*) \times (\mathbf{x}_\perp - \mathbf{x}')\big]\times\hat{\mathbf{z}} \\
&= \frac{\mu_0\mu_r}{4\pi}\Bigg\{\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\Big[[(x-x')J_z + z'J_x]\hat{\mathbf{x}} + [(y-y')J_z + z'J_y]\hat{\mathbf{y}}\Big] \\
&\qquad + \frac{\mu_r - 1}{\mu_r + 1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\Big[[-(x-x')J_z + z'J_x]\hat{\mathbf{x}} + [-(y-y')J_z + z'J_y]\hat{\mathbf{y}}\Big]\Bigg\}
\end{aligned} \tag{8.3.7}$$

By letting $z' \to -z'$ in the second integral, we get

$$\begin{aligned}
\mu_r \lim_{z\to 0^+} \mathbf{B}\times\hat{\mathbf{z}} &= \frac{\mu_0}{4\pi}\frac{2\mu_r}{\mu_r+1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\Big[[(x-x')J_z + z'J_x]\hat{\mathbf{x}} + [(y-y')J_z + z'J_y]\hat{\mathbf{y}}\Big] \\
&= \frac{\mu_0}{4\pi}\frac{2\mu_r}{\mu_r+1}\int \frac{d^3x'}{|\mathbf{x}_\perp - \mathbf{x}'|^3}\big[\mathbf{J} \times (\mathbf{x}_\perp - \mathbf{x}')\big]\times\hat{\mathbf{z}} \\
&= \lim_{z\to 0^-} \mathbf{B}\times\hat{\mathbf{z}}
\end{aligned} \tag{8.3.8}$$

Thus, (8.3.2) follows all boundary conditions. $\square$