Homework 5: Phys 7320 (Spring 2022)

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***Problem* 5.1** (Fresnel Diffraction)**:** A perfectly conducting flat screen occupies half of the xy plane (i.e., $x < 0$). A plane wave of intensity I_0 and wave number k is incident along the z axis from the region $z < 0$. Discuss the values of the diffracted fields in the plane parallel to the xy plane defined by $z = Z > 0$. Let the coordinates of the observation point be $(X, 0, Z)$.

(a) Show that, for the usual scalar Kirchhoff approximation and in the limit $Z \gg X$ and $\sqrt{kZ} \gg 1$, the diffracted field is

$$\psi(X, 0, Z, t) \approx I_0^{1/2} e^{ikZ - i\omega t} \left(\frac{1+i}{2i}\right) \sqrt{\frac{2}{\pi}} \int_{-\xi}^{\infty} e^{it^2} dt \tag{5.1.1}$$

where $\xi = (k/2Z)^{1/2} X$.

(b) Show that the intensity can be written

$$I = |\psi|^2 = \frac{I_0}{2}\left[\left(C(\xi) + \frac{1}{2}\right)^2 + \left(S(\xi) + \frac{1}{2}\right)^2\right] \tag{5.1.2}$$

where $C(\xi)$ and $S(\xi)$ are the so-called Fresnel integrals. Determine the asymptotic behavior of I for ξ large and positive (illuminated region) and ξ large and negative (shadow region). What is the value of I at $X = 0$? Make a sketch of I as a function of X for fixed Z.

Hint: Use the Dirichlet ("generalized") version of the Kirchhoff integral as in class or Jackson (10.85), with the subleading $+i/(kR)$ neglected, and expand

$$R = \left[(x - x') + y'^2 + z^2\right]^{1/2} \approx z\left[1 + \frac{(x - x')^2 + y'^2}{2z^2} + \dots\right]. \tag{5.1.3}$$

There are several definitions of Fresnel integrals with slightly different conventions. The Abramowitz and Stegun definition is

$$C(u) = \sqrt{\frac{2}{\pi}} \int_0^u dt \cos\left(t^2\right), \qquad S(u) = \sqrt{\frac{2}{\pi}} \int_0^u dt \sin\left(t^2\right). \tag{5.1.4}$$

Note that in the result Jackson quotes for part (a), t is used for two completely different things – in $e^{-i\omega t}$ it is the time, while in $e^{it^2}dt$ it is an integration variable related to x and x'. This is obviously terrible; the time dependence is just the usual harmonic dependence and plays no further role.

Solution.

(a) First, we expand the distance $R = \sqrt{(X-x')^2 + y'^2 + Z^2}$ as hinted.

$$R = Z\sqrt{1 + \frac{(X-x')^2 + y'^2}{Z^2}} \approx Z\left[1 + \frac{(X-x')^2 + y'^2}{2Z^2}\right]. \tag{5.1.5}$$

Then, as Z is large, $R \sim Z$ and $kR \ll 1$. The generalized Kirchhoff integral (10.85, Jackson) can be written as

$$\psi = \frac{kZ}{2\pi i}\int_{S_1} \frac{e^{ikR}}{R^2}\psi(\mathbf{x}')dx'dy', \tag{5.1.6}$$

where $S_1 = \big\{(x,y,0)|x,y \in \mathbb{R} \wedge x \geq 0\big\}$. On S_1, ψ assumes the form without the conducting screen $\psi = \psi_0 e^{-i\omega t}$ where $\psi_0^2 = I_0$. It follows that

$$\begin{aligned}
\psi &\approx \sqrt{I_0}\frac{k}{2\pi i Z}e^{i(kZ-\omega t)}\int_{-\infty}^{\infty} dy' e^{(1/2)i(k/Z)y'^2}\int_0^{\infty} dx' e^{ik(X-x')^2/2Z} \\
&= \sqrt{I_0}\frac{1}{2\pi i}\frac{k}{Z}e^{i(kZ-\omega t)}\sqrt{\frac{2\pi}{-ik/Z}}\int_{X\sqrt{k/2Z}}^{-\infty}\left(-\sqrt{\frac{2Z}{k}}\right)du e^{iu^2} \qquad (u^2 = k(X-x')^2/2Z) \\
&= \sqrt{I_0}\frac{1}{\sqrt{\pi}}\frac{1}{i\sqrt{-i}}e^{i(kZ-\omega t)}\int_{-\xi}^{\infty} du e^{iu^2} \qquad (\xi = X\sqrt{k/2Z}) \\
&= \sqrt{I_0}e^{i(kZ-\omega t)}\frac{1+i}{2i}\sqrt{\frac{2}{\pi}}\int_{-\xi}^{\infty} du e^{iu^2}
\end{aligned} \tag{5.1.7}$$

where $1/i\sqrt{-i} = (1/\sqrt{2})(1-i)$. This is the desired result.

(b) First, note that

$$\lim_{u\to\infty} C(u) = \lim_{u\to\infty} S(u) = \frac{1}{2}, \tag{5.1.8}$$

from the Abramowitz & Stegun definition. Then, we can rewrite our result for ψ in (a) as

$$\begin{aligned}
\psi &= I_0^{1/2}e^{i(kZ-\omega t)}\frac{1+i}{2i}\sqrt{\frac{2}{\pi}}\left(\int_0^{\xi} du e^{iu^2} + \int_0^{\infty} du e^{iu^2}\right) \\
&= I_0^{1/2}e^{i(kZ-\omega t)}\frac{1+i}{2i}\big[C(\xi) + iS(\xi) + C_\infty + iS_\infty\big] \\
&= I_0^{1/2}e^{i(kZ-\omega t)}\frac{1}{2}(1-i)\left[\left(C(\xi) + \frac{1}{2}\right) + i\left(S(\xi) + \frac{1}{2}\right)\right].
\end{aligned} \tag{5.1.9}$$

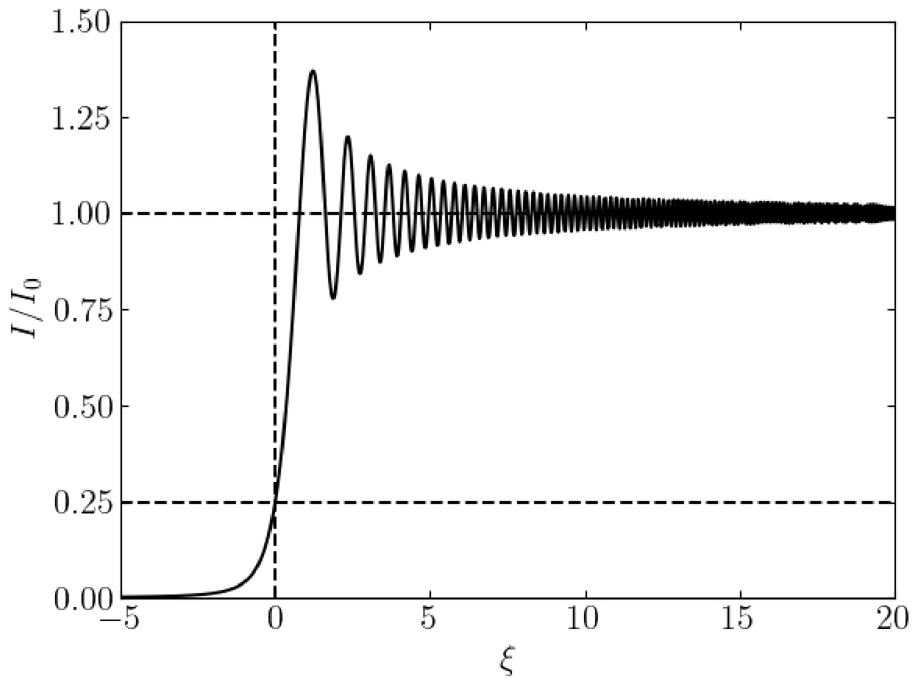
Then, it is trivial to calculate

$$I = |\psi|^2 = \frac{I_0}{2}\left[\left(C(\xi) + \frac{1}{2}\right)^2 + \left(S(\xi) + \frac{1}{2}\right)^2\right]. \tag{5.1.10}$$

From (5.1.8), $\lim_{\xi\to\infty} I = I_0$. However, at the opposite limit,

$$\lim_{u\to-\infty} C(u) = \lim_{u\to-\infty} S(u) = -\frac{1}{2}. \tag{5.1.11}$$

Thus, $\lim_{\xi\to-\infty} I = 0$. At $X = \xi = 0$, it is clear from the bounds of the Fresnel integrals that $C(0) = S(0) = 0$. So $I(\xi = 0) = I_0/4$. A plot of I is shown below where the limiting behavior and the value of I at $\xi = 0$ are confirmed.



$\square$

***Problem* 5.2** (Diffraction through a circular hole)**:** A linearly polarized plane wave of amplitude E_0 and wave number k is incident on a circular opening of radius a in an otherwise perfectly conducting flat screen. The incident wave vector makes an angle α with the normal to the screen. The polarization vector is perpendicular to the plane of incidence.

(a) Calculate the diffracted fields and the power per unit solid angle transmitted through the opening, using the vector Smythe-Kirchhoff formula (10.101) with the assumption that the tangential electric field in the opening is the unperturbed incident field.

(b) Compare your result in part (a) with the standard scalar Kirchhoff approximation and with the result in Section 10.9 for the polarization vector in the plane of incidence.

Hint: In (a), work in the Fraunhofer limit. In class (also Jackson pp. 491–492), we discussed the same case except the initial polarization was in the plane of incidence; now on this homework problem the initial polarization is perpendicular to the plane of incidence, $\boldsymbol{\epsilon}_0 = \hat{\mathbf{y}}$. Go through the steps and fill out details. For part (b), again use the Dirichet Kirchhoff formula with the subleading $+i/(kR)$ dropped.

Solution.

(a) First, the incident electric field can be written as

$$\mathbf{E}_i = E_0\hat{\mathbf{y}}e^{ik(\sin\alpha x+\cos\alpha z)}. \tag{5.2.1}$$

So, with a normal $\hat{\mathbf{n}} = \hat{\mathbf{z}}$,

$$\hat{\mathbf{n}}\cdot\mathbf{E}(\mathbf{x}') = -E_0\hat{\mathbf{x}}e^{ik\sin\alpha x'} = -E_0\hat{\mathbf{x}}e^{ik\rho\sin\alpha\cos\beta}, \tag{5.2.2}$$

where $\mathbf{x}' \in S_1$ and $x' = \rho\cos\beta$. Now, plugging into (10.109, Jackson), we get

$$\mathbf{E}(\mathbf{x}) = \frac{-ie^{ikr}}{2\pi r}E_0(\mathbf{k}\times\hat{\mathbf{x}})\int_{S_1}\rho d\rho d\beta e^{ik\rho\sin\alpha\cos\beta}e^{-ik\rho(\sin\theta\cos(\phi-\beta))}. \tag{5.2.3}$$

Performing a change of variable $\xi = (\sin^2\theta+\sin^2\alpha-2\sin\theta\sin\alpha\cos\phi)^{1/2}$, we can calculate

$$\mathbf{E}(\mathbf{x}) = -\frac{ie^{ikr}E_0}{2\pi r}(\mathbf{k}\times\hat{\mathbf{x}})\int_0^a\rho d\rho 2\pi J_0(k\rho\xi) = -ie^{ikr}E_0\frac{a}{r}\left(\hat{\mathbf{k}}\times\hat{\mathbf{x}}\right)\frac{J_1(ka\xi)}{\xi}, \tag{5.2.4}$$

where $\hat{\mathbf{k}}\times\hat{\mathbf{x}} = -\sin\theta\sin\phi\hat{\mathbf{z}}+\cos\theta\hat{\mathbf{z}}$. It then follows that

$$|\mathbf{E}|^2 = \left(\frac{a}{r}\right)^2E_0^2\left(\cos^2\theta+\sin^2\theta\sin^2\phi\right)\left|\frac{J_1(ka\xi)}{\xi}\right|^2, \tag{5.2.5}$$

and from (9.21, Jackson),

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{r^2}{2Z_0}|\mathbf{E}|^2\\ &= \frac{P_i}{\pi\cos\alpha}(\cos^2\theta+\sin^2\theta\sin^2\phi)\left|\frac{J_1(ka\xi)}{\xi}\right|^2\\ &= P_i\frac{(ka)^2}{4\pi\cos\alpha}(\cos^2\theta+\sin^2\theta\sin^2\phi)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2\end{aligned} \tag{5.2.6}$$

where $P_i = (E_0^2/2Z_0)\pi a^2 \cos\alpha$ is defined in (10.115).

(b) First, we note that at $\alpha = 0$, $\xi = \sin\theta$ and (5.2.6) reduces to

$$\frac{dP}{d\Omega} = \frac{P_i}{\pi}\left|\frac{J_1(ka\sin\theta)}{\tan\theta}\right|^2 \approx \frac{P_i}{\pi}\left|\frac{J_1(ka\sin\theta)}{\sin\theta}\right|^2. \tag{5.2.7}$$

since θ is restricted to a very small forward angle ($\theta \ll 1$). (10.114) would also reduce to the same form since the $\cos^2\phi\sin^2\theta$ term would vanish. Now, we derive the radiation pattern with scalar diffraction theory. Recall from (5.1.6),

$$E = \frac{kz}{2\pi i}\int_{S_1}\frac{e^{ikR}}{R^2}\psi(\mathbf{x}')\rho d\rho d\beta \tag{5.2.8}$$

where S_1 is the circular hole. ψ on S_1 is

$$\psi(\mathbf{x}') = E_0 e^{i(\mathbf{k}_0\cdot\mathbf{x}' - \omega t)} E_0 e^{-i\omega t} e^{ik\sin\alpha x'} = E_0 e^{-i\omega t} e^{ik\rho\sin\alpha\cos\beta}, \tag{5.2.9}$$

while $kR \approx kr - \mathbf{k}\cdot\mathbf{x}' = kr - k\rho\sin\theta\cos(\phi-\beta)$. Plugging back into (5.2.8), we get

$$E = \frac{kz}{2\pi i}E_0\frac{e^{i(kr-\omega t)}}{r^2}\int_0^a \rho d\rho\int_0^{2\pi} d\beta e^{ik\rho(\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta))}. \tag{5.2.10}$$

We have solved this integration before. The result is

$$\begin{aligned} E &= \frac{kz}{i}E_0 e^{i(kr-\omega t)}\frac{a^2}{r^2}\frac{J_1(ka\xi)}{ka\xi} \\ &= -ie^{i(kr-\omega t)}E_0\frac{a}{r}\cos\theta\frac{J_1(ka\xi)}{\xi} \end{aligned} \tag{5.2.11}$$

Then, the radiation pattern is

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|E|^2 = \frac{a^2E_0^2}{2Z_0}\cos^2\theta\left|\frac{J_1(ka\xi)}{\xi}\right|^2 = \frac{P_i}{\pi\cos\alpha}\cos^2\theta\left|\frac{J_1(ka\xi)}{\xi}\right|^2 \tag{5.2.12}$$

Once again, at $\alpha = 0$ and $\theta \ll 1$, this result reduces to (5.2.7). So all scalar and vector approximations reduce to the common expression (10.120, Jackson), as expected. Regarding their similarities, we observe that they all scale as $\left|J_1/\xi\right|^2$. $\square$