Homework 9: Phys 7320 (Spring 2022)

Tien Vo

Due: March 30, 2022

***Problem* 9.1** (Relativistic particle in electric field)**:** A particle with mass m and charge e moves in a uniform, static, electric field $\mathbf{E}_0$.

(a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\mathbf{v}_0$ was perpendicular to the electric field.

(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

Solution.

(a) First, we write down the field-strength tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_0 & 0 & 0 \\ E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9.1.1}$$

Then the force equation is

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta = \frac{eE_0}{mc}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} U^0 \\ -U^1 \\ -U^2 \\ -U^3 \end{pmatrix} = \frac{eE_0}{mc}\begin{pmatrix} U^1 \\ U^0 \\ 0 \\ 0 \end{pmatrix}. \tag{9.1.2}$$

It then follows that

$$\frac{d^2U^0}{d\tau^2} = \left(\frac{eE_0}{mc}\right)^2 U^0, \qquad \frac{d^2U^1}{d\tau^2} = \left(\frac{eE_0}{mc}\right)^2 U^1, \qquad \text{and} \qquad \frac{dU^2}{d\tau} = \frac{dU^3}{d\tau} = 0. \tag{9.1.3}$$

The problem essentially becomes 2-dimensional, since the z dimension is ignorable. At $t = 0$, $U^2 = \gamma_0 v_0$ where $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$. Thus, $u_y = (\gamma_0/\gamma)v_0$. The general solution for the differential equations in U^0 and U^1 is

$$c\gamma = U^0 = A\sinh\left(\frac{eE_0\tau}{mc}\right) + B\cosh\left(\frac{eE_0\tau}{mc}\right) \tag{9.1.4a}$$

$$\gamma u_x = U^1 = \frac{mc}{eE_0}\frac{dU^0}{d\tau} = A\cosh\left(\frac{eE_0\tau}{mc}\right) + B\sinh\left(\frac{eE_0\tau}{mc}\right). \tag{9.1.4b}$$

Using initial conditions $U^0(\tau = 0) = c\gamma_0$ and $U^1(\tau = 0) = 0$, we can write

$$\gamma = \gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right) \tag{9.1.5a}$$

$$u_x = c\frac{\gamma_0}{\gamma} \sinh\left(\frac{eE_0\tau}{mc}\right). \tag{9.1.5b}$$

By simple integration, we get

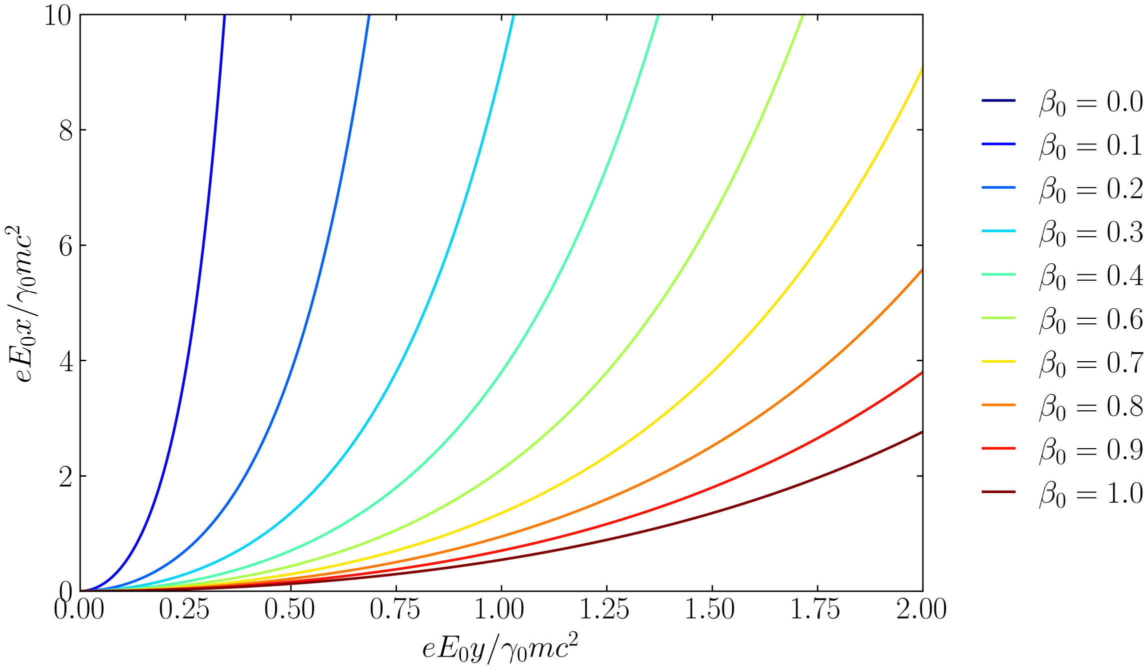
$$x = \int dt u_x = \int d\tau \gamma u_x = \frac{\gamma_0 mc^2}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right], \qquad \text{and} \qquad y = \int d\tau \gamma u_y = \gamma_0 v_0 \tau, \tag{9.1.6}$$

such that $x(t = 0) = y(t = 0) = 0$.

(b) Using the previous results, we can write

$$\frac{eE_0 x}{\gamma_0 mc^2} = \cosh\left(\frac{eE_0 y}{\gamma_0 v_0 mc}\right) - 1. \tag{9.1.7}$$

Below, we plot this trajectory for a few different values of $\beta_0 = v_0/c$.



Although the acceleration is along the electric field ($\hat{\mathbf{x}}$), u_y changes as $1/\gamma$ due to relativity, leading to a linear growth in y in proper time. However, x grows exponentially. So x increases faster than y for a particle starting from rest ($u_{x0} = u_{y0} = 0$). For a relativistic particle moving perpendicular to the field, this is the same, however, it takes longer to steer it to point from $\hat{\mathbf{y}}$ to $\hat{\mathbf{x}}$. Here, short time means $\overline{y} = eE_0y/\gamma_0mc^2 \ll \beta_0$, while long time means $\overline{y} \gg \beta_0$. As soon as $\overline{y}$ grows past β_0, we observe significant change in x. $\square$

***Problem* 9.2** (Electric and magnetic fields at an angle)**:** Static, uniform electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, make an angle of θ with respect to each other.

(a) Show that a Lorentz transformation can be made to go to a frame where $\mathbf{E}'$ and $\mathbf{B}'$ are parallel, and that the boost velocity satisfies the equation

$$\frac{\boldsymbol{\beta}}{1+\beta^2} = \frac{\mathbf{E}\times\mathbf{B}}{E^2+B^2}. \tag{9.2.1}$$

Hint: You can start with the trial solution $\boldsymbol{\beta} = \lambda(\mathbf{E}\times\mathbf{B})$ and then show there is a λ that works.

(b) For $\mathbf{E}$ and $\mathbf{B}$ parallel, show that with appropriate constants of integration, etc., the parametric solution can be written

$$x = AR\sin\phi, \qquad y = AR\cos\phi, \qquad z = \frac{R}{\rho}\sqrt{1+A^2}\cosh(\rho\phi), \qquad ct = \frac{R}{\rho}\sqrt{1+A^2}\sinh(\rho\phi), \tag{9.2.2}$$

where $R = (mc^2/eB), \rho = E/B$, A is an arbitrary constant, and ϕ is the parameter (actually c/R times the proper time).

Solution.

(a) Let $\boldsymbol{\beta} = \lambda(\mathbf{E}\times\mathbf{B})$, as hinted. Then the transformed fields are

$$\mathbf{E}' = \gamma\big[(1-\lambda B^2)\mathbf{E} + \lambda(\mathbf{E}\cdot\mathbf{B})\mathbf{B}\big], \tag{9.2.3}$$

and

$$\mathbf{B}' = \gamma\big[(1-\lambda E^2)\mathbf{B} + \lambda(\mathbf{E}\cdot\mathbf{B})\mathbf{E}\big]. \tag{9.2.4}$$

Since $\mathbf{E}'\times\mathbf{B}' = \mathbf{0}$, λ has to obey the following relation

$$\lambda^2(\mathbf{E}\cdot\mathbf{B})^2 = (1-\lambda B^2)(1-\lambda E^2) \Rightarrow \lambda^2E^2B^2\sin^2\theta - \lambda(E^2+B^2) + 1 = 0. \tag{9.2.5}$$

However, note that $|\boldsymbol{\beta}|^2 = \lambda^2E^2B^2\sin^2\theta$. Plugging this into the above equation yields

$$\lambda = \frac{1+\beta^2}{E^2+B^2}, \tag{9.2.6}$$

as desired. Thus, the boost velocity is given by (9.2.1).

(b) Now, dropping the prime, in the transformed frame, the field tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix}. \tag{9.2.7}$$

Then, the force equation is

$$\frac{dU^\alpha}{d\tau} = \frac{c}{R}\begin{pmatrix} 0 & 0 & 0 & -\rho \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \rho & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} U^0 \\ -U^1 \\ -U^2 \\ -U^3 \end{pmatrix} = \frac{c}{R}\begin{pmatrix} \rho U^3 \\ U^2 \\ -U^1 \\ \rho U^0 \end{pmatrix}. \tag{9.2.8}$$

Thus, the separate components of U^α follows the following differential equations

$$\frac{d^2U^0}{d\tau^2} = \frac{c^2\rho^2}{R^2}U^0 \qquad \frac{d^2U^1}{d\tau^2} = -\frac{c^2}{R^2}U^1 \tag{9.2.9}$$

$$\frac{d^2U^3}{d\tau^2} = \frac{c^2\rho^2}{R^2}U^3 \qquad \frac{d^2U^2}{d\tau^2} = -\frac{c^2}{R^2}U^2. \tag{9.2.10}$$

The general solutions to which are

$$U^0 = A\sinh(\rho\phi) + B\cosh(\rho\phi) \tag{9.2.11a}$$
$$U^1 = C\sin\phi + D\cos\phi \tag{9.2.11b}$$
$$U^2 = C\cos\phi - D\sin\phi \tag{9.2.11c}$$
$$U^3 = A\cosh(\rho\phi) + B\sinh(\rho\phi), \tag{9.2.11d}$$

where A, B, C, D are constants and $\phi = c\tau/R$. First, recall that $U^0 = c\gamma = cdt/d\tau$, we can integrate to get

$$ct = \int d\tau U^0 = \frac{R}{c\rho}\big[A\cosh(\rho\phi) + B\sinh(\rho\phi)\big]. \tag{9.2.12}$$

Forcing $t(\phi = 0) = 0$, $A = 0$ and we can rewrite $B \mapsto c\sqrt{1+A^2}$ so that

$$ct = \frac{R}{\rho}\sqrt{1+A^2}\sinh(\rho\phi), \tag{9.2.13}$$

for some arbitrary constant A. Then it follows that $\gamma = \sqrt{1+A^2}\cosh(\rho\phi)$ and $\gamma u_z = c\sqrt{1+A^2}\sinh(\rho\phi)$. So

$$z = \int d\tau\gamma u_z = \frac{R}{\rho}\sqrt{1+A^2}\cosh(\rho\phi). \tag{9.2.14}$$

Now, letting $C = 0$ so that the initial conditions match the desired ones. Then, note that by definition,

$$\gamma^2 = \sqrt{1 + \frac{(U^1)^2 + (U^2)^2}{c^2} + \frac{(U^3)^2}{c^2}} = 1 + \frac{D^2}{c^2} + (1+A^2)\sinh^2(\rho\phi) = (1+A^2)\cosh^2(\rho\phi). \tag{9.2.15}$$

We can then solve for $D = cA$. The x and y positions thus follow

$$x = \int d\tau U^1 = AR\sin\phi, \qquad \text{and} \qquad y = \int d\tau U^2 = AR\cos\phi. \tag{9.2.16}$$

Note that to get these solutions, we had to assume $t(\tau = 0) = 0$ and $C = 0$. So these aren't general solutions. □