Homework 1: Phys 7230 (Spring 2022)

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Due: January 24, 2022

***Problem* 1:** Consider two (otherwise) closed systems A and B at respective temperatures T_A and T_B in thermal contact with each other, so that heat Q can flow between them.

Using the 2nd law of thermodynamics – total entropy S of a closed system increases under equilibration – show that heat Q flows from the hotter to the colder system as A and B come to thermal equilibrium.

Hint: $TdS \geq Q$.

Solution.

First, for a closed system, the total energy $U = U_A + U_B$ is conserved. So $dU_B = -dU_A$. The total entropy is $S = S_A + S_B$. Since the volume and number of particles of each system is constant, we have

$$dS = \left(\frac{\partial S}{\partial U_A}\right)_{N_A,V_A} dU_A + \left(\frac{\partial S}{\partial U_B}\right) dU_B = \left(\frac{1}{T_A} - \frac{1}{T_B}\right) dU_A \geq 0 \tag{1.1}$$

The last inequality is the 2nd law of thermodynamics. Now, without loss of generality, assume $T_A > T_B$, then it must be the case that $dU_A < 0$. Since the change in energy is the heat exchanged ($dU_A = Q$), it follows that the hotter system (A) loses energy in the form of heat to the lower system (B). □

***Problem* 2** (Spin-1/2 paramagnet)**:** Consider a magnet of N noninteracting spin-1/2 magnetic moments in an external magnetic field $\mathbf{B}$, with Hamiltonian given by Zeeman energy,

$$\mathcal{H} = -\sum_{i=1}^{N} \boldsymbol{\mu}_i \cdot \mathbf{B} = -\sum_{i=1}^{N} \mu_B B \sigma_i \equiv -\sum_{i=1}^{N} h\sigma_i \tag{2.1}$$

where μ_B is Bohr magneton (carrying units of magnetic moment) and $\sigma_i = \pm 1$ labels the two Zeeman spin states of nth spin.

(a) For fixed dimensionless magnetization $M = N_\uparrow - N_\downarrow$, (i) what is the multiplicity $\Omega(M, N)$? (ii) What is the corresponding probability $P(M, N)$? Check that $\sum_{M=-N}^{N} P(M, N) = 1$.

(b) Compute the multiplicity $\Omega(E)$ for this system at a total energy E and sketch/plot it as a function of full range of accessible energies.

Hint: Note that the magnetization M is proportional to the energy E.

(c) Derive the relation between temperature $T(E)$ and energy E and plot $T(E)$.

(d) Calculate the (i) magnetization $m(T,B) = \mu_B \sum_{i=1}^{N} \sigma_i$, (ii) obtain its asymptotic forms in the classical $\mu_B B/k_B T \ll 1$ and quantum $\mu_B B/k_B T \gg 1$ limits and (iii) plot it as a function of T at a couple of fixed values of B and as a function of B at a couple of fixed values of T.

Hint: (i) Notice that magnetization is proportional to the energy E, (ii) Eliminate E in favor of T, (iii) Use lowest order Stirling approximation throughout to simplify the factorials in your expression.

(e) Compute the linear magnetic susceptibility $\chi(T,B) = \partial m/\partial B\big|_{B\to 0}$, show that it exhibits Curie form $\chi_{\text{Curie}} = a/T$, extracting the coefficient a.

(f) Compute the heat capacity (specific heat), $C_v(T) = T(\partial S/\partial T)_{V,N}$, extract its low and high temperatures asymptotics, and sketch it, noting its limiting forms and the crossover temperature.

Solution.

(a) Given $M = N_\uparrow - N_\downarrow$ and $N = N_\uparrow + N_\downarrow$, we can write

$$N_\uparrow = \frac{1}{2}(N+M) \qquad \text{and} \qquad N_\downarrow = \frac{1}{2}(N-M). \tag{2.2}$$

(i) The multiplicity of $\Omega(M,N)$ is just the combination of $N_\uparrow$ in N total magnets

$$\Omega(M,N) = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow! N_\downarrow!} = \frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!} \tag{2.3}$$

(ii) For N total magnets, each with 2 possible states, the sample size is 2^N. So the probability $P(M,N)$ is

$$P(M,N) = \frac{1}{2^N}\Omega(M,N) = \frac{1}{2^N}\frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!} \tag{2.4}$$

Using the binomial theorem,

$$2^N = \sum_{N_\uparrow=0}^{N} \binom{N}{N_\uparrow} = \sum_{N_\uparrow=0}^{N} \Omega(M,N), \tag{2.5}$$

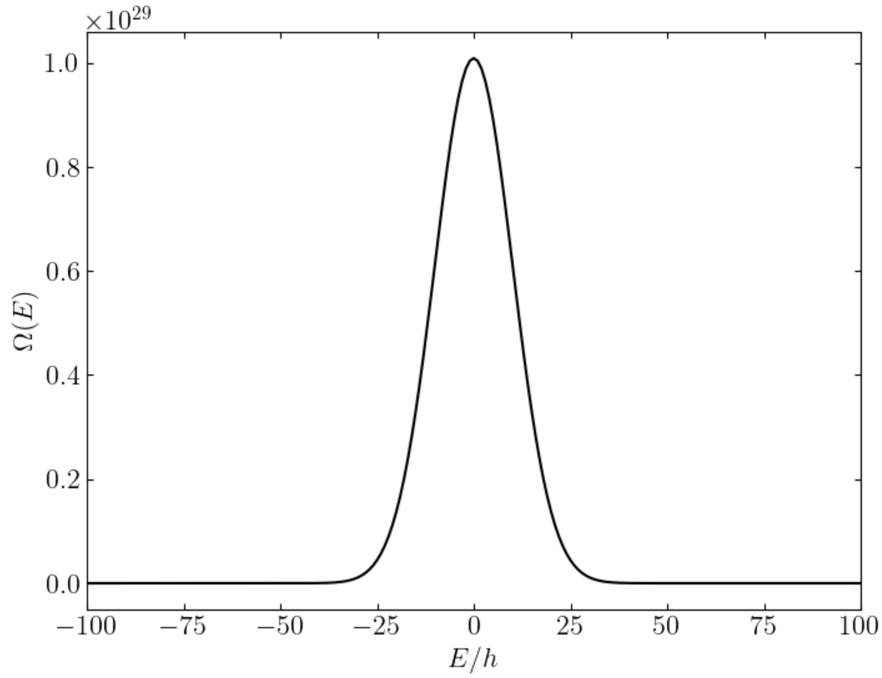
we can convert the summation back to M. For $0 \le N_\uparrow \le N$, $-N \le M \le N$. Thus,

$$1 = \frac{1}{2^N}\sum_{M=-N}^{N} \binom{N}{N_\uparrow} = \sum_{M=-N}^{N} P(M,N) \tag{2.6}$$

(b) By definition, $E = -hM$. So from (2.3),

$$\Omega(E) = \frac{N!}{\left(\frac{N-E/h}{2}\right)!\left(\frac{N+E/h}{2}\right)!} \tag{2.7}$$

for $-hN \le E \le hN$. A plot of $\Omega(E)$ is included below



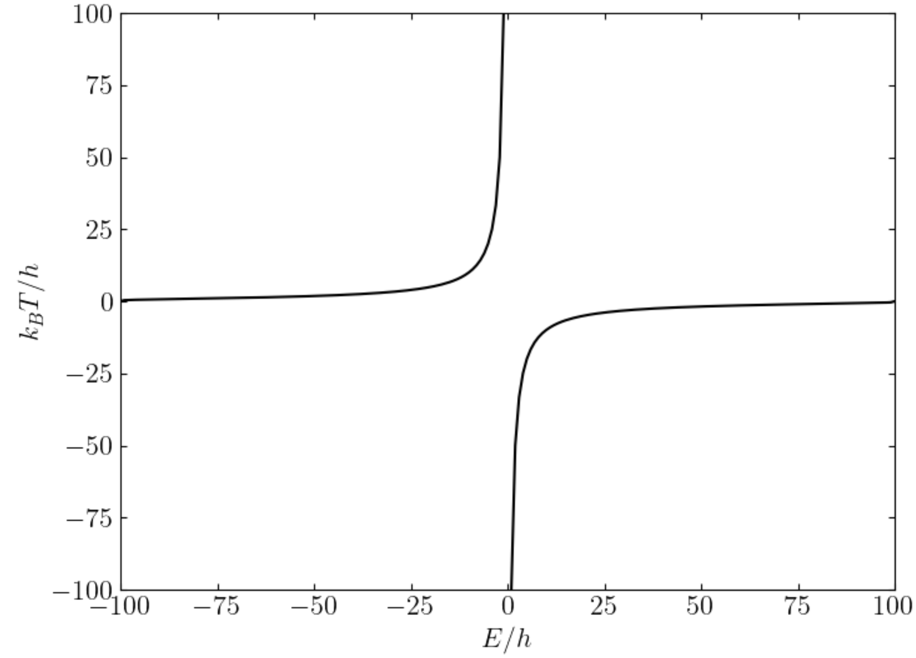
(c) By definition, the entropy is

$$
\begin{aligned}
S/k_B &= \ln \Omega \\
&\approx N \ln N - N - \left(\frac{N - E/h}{2}\right) \ln \left(\frac{N - E/h}{2}\right) + \frac{N - E/h}{2} \\
&\quad - \left(\frac{N + E/h}{2}\right) \ln \left(\frac{N + E/h}{2}\right) + \frac{N + E/h}{2} \\
&= N \ln N - \left(\frac{N - E/h}{2}\right) \ln \left(\frac{N - E/h}{2}\right) - \left(\frac{N + E/h}{2}\right) \ln \left(\frac{N + E/h}{2}\right)
\end{aligned}
\tag{2.8}
$$

by Stirling approximation. Then the temperature is

$$
\frac{1}{k_B T} = \frac{\partial (S/k_B)}{\partial E} = \frac{1}{2h} \ln \left(\frac{N - E/h}{N + E/h}\right) \Rightarrow \frac{k_B T}{h} = \frac{2}{\ln \left(\frac{N - E/h}{N + E/h}\right)}
\tag{2.9}
$$

A plot of the normalized temperature $k_B T/h$ is included below



(d) First, we can invert (2.9) to solve for E as below

$$E = -hN\frac{e^{2h/k_BT} - 1}{e^{2h/k_BT} + 1} = -hN\tanh\left(\frac{h}{k_BT}\right) = -hN\tanh\left(\frac{\mu_B B}{k_BT}\right) \tag{2.10}$$

But the energy E can be written in terms of m as $E = -mB$ since $m = \mu_B M$. So (i) the magnetization is

$$m = \mu_B N \tanh\left(\frac{\mu_B B}{k_BT}\right) \tag{2.11}$$

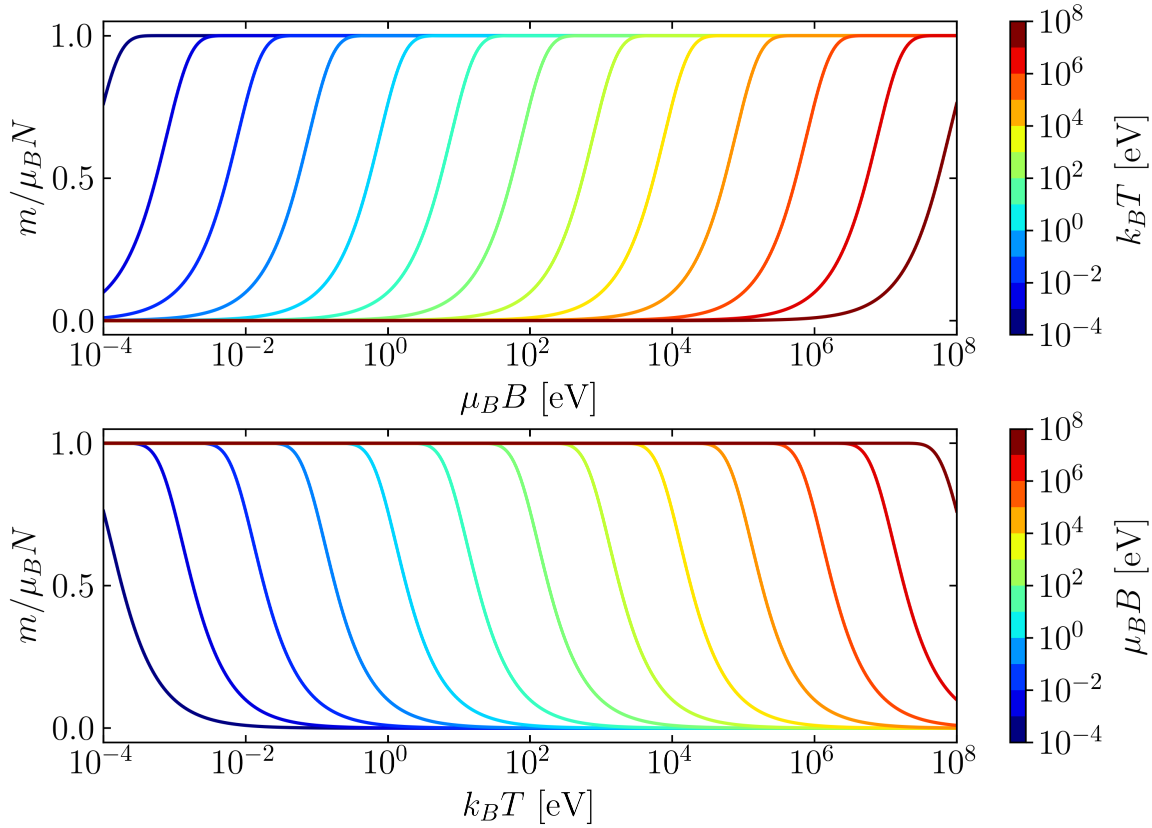
(ii) For $\mu_B B/k_BT \ll 1$, tanh is a linear function

$$m\left(\frac{\mu_B B}{k_BT}\right) = \frac{\mu_B^2 nB}{k_BT} \tag{2.12}$$

and for $\mu_B B/k_BT \gg 1$, $\tanh(x) \to 1$, so the magnetization is a constant

$$m\left(\frac{\mu_B B}{k_BT}\right) = \mu_B N \tag{2.13}$$

(iii) A plot of $m(T, B)/\mu_B N$ is included below



(e) From (2.11), we can calculate

$$\chi = \lim_{B\to 0} \frac{\partial m}{\partial B} = \frac{\mu^2 N}{k_B T} \lim_{B\to 0} \operatorname{sech}^2\left(\frac{\mu_B B}{k_B T}\right) = \frac{\mu^2 N}{k_B T} \tag{2.14}$$

So χ follows Curie's law with $a = \mu^2 N/k_B$.

(f) By the chain rule,

$$C_V = T\frac{\partial S}{\partial E}\frac{\partial E}{\partial T} = \frac{\partial E}{\partial T} = Nk_B\left(\frac{\mu_B B}{k_B T}\right)^2 \operatorname{sech}^2\left(\frac{\mu_B B}{k_B T}\right) \tag{2.15}$$

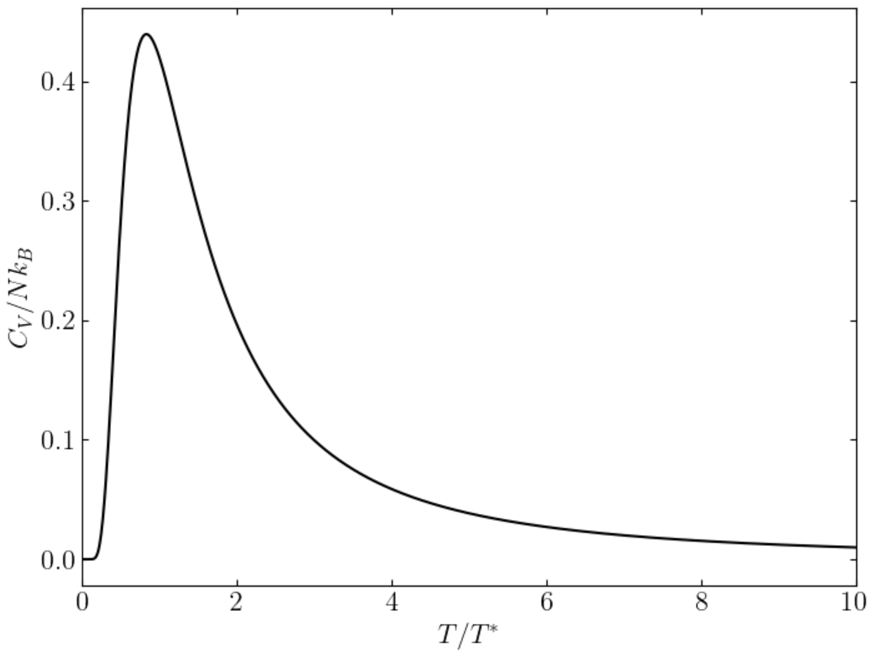
For $x = \mu_B B/k_B T \ll 1$ (high temperature), $\operatorname{sech}(x) \to 1$ and the heat capacity is

$$C_V\left(\frac{\mu_B B}{k_B T} \ll 1\right) \approx Nk_B\left(\frac{\mu_B B}{k_B T}\right)^2 \tag{2.16}$$

For $x \gg 1$ (low temperature), $\operatorname{sech}^2(x)$ dominates and so $C_V \to 0$. Letting the crossover temperature to be $T^* = \mu_B B/k_B$, then we can rewrite the heat capacity in normalized form

$$\frac{C_V}{Nk_B} = \left(\frac{T^*}{T}\right)^2 \operatorname{sech}^2\left(\frac{T^*}{T}\right) \tag{2.17}$$

A plot of (2.17) is included below



□

***Problem* 3** (Quamtum harmonic oscillators: Einstein solid)**:** Consider N decoupled 3D quantum harmonic oscillators as a model of atomic vibrations in a crystalline solid (Einstein phonons), described by the familiar quantum Hamiltonian

$$\hat{\mathcal{H}} = \sum_{i}^{N} \left[\frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{r}_i^2 - \frac{3}{2} \hbar \omega_0 \right] \tag{3.1}$$

where for convenience I defined $\hat{\mathcal{H}}$ with zero point energy subtracted off.

(a) Let's warm up on a single harmonic oscillator, computing its degeneracy $g(n) = \Omega(E = \hbar\omega_0 n)$ a fixed total energy $E \equiv \hbar\omega_0 n = \hbar\omega_0 \sum_{\alpha=1}^{d} n_\alpha$ (where $n_\alpha \in \mathbb{Z}$ are integer quantum numbers for $\alpha = x, y, \ldots$) for the cases of (i) 2D and (ii) 3D.

(b) Recalling the eigenvalues $E[\{n_\alpha\}] = \hbar\omega_0 \sum_{\alpha=1}^{3N} n_\alpha$ ($\alpha = x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ ranging from 1 to $3N$) for the harmonic oscillator Hamiltonian, compute the multiplicity

$$\Omega(E) = \sum_{\{n_\alpha\}} \delta_{E, E[\{n_\alpha\}]} \tag{3.2}$$

taking $E = \hbar\omega_0 n$ ($n \in \mathbb{Z}$).

Hint: Think about how to distribute n total quanta of excitations among $3N$ 1D oscillators, and use the lowest Stirling formula approximation for $N \gg 1$ and $n \gg 1$.

(c) Compute the entropy $S(E)$ and the corresponding $T(E)$, thereby extracting energy $E(T)$ as a function of temperature T, exploring its classical $\hbar\omega_0/k_BT \ll 1$ and quantum $\hbar\omega_0/k_BT \gg 1$ limiting functional forms. Plot $E(T)$, noting limiting forms.

(d) Compute heat capacity $C_v = T(\partial S/\partial T)_{V,N} = \partial E/\partial T$ and explore its classical (high T) and quantum (low T) limits, showing the expected equipartition $C_v = N_{\rm dof} k_B$ in the former and its breakdown in the latter limits. Plot $C_v(T)$, noting the crossover temperature.

(e) Consider a classical limit of the problem with small $\hbar\omega_0/k_BT$ such that $E[\{n_\alpha\}] = \sum_{\alpha=1}^{3N} \epsilon_\alpha$ and oscillator eigenvalues ϵ_α vary nearly continuously. Using this simplification compute

$$\Omega(E) = \Delta \prod_{\alpha=1}^{3N} \int \frac{d\epsilon_\alpha}{\hbar\omega_0} \delta(E - E[\{\epsilon_\alpha\}]) \tag{3.3}$$

as a multidimensional integral over ϵ_α.

Hint: It is helpful to use a result for a hypervolume of an N-dimensional space spanned by positive values of x_i coordinates, limited by a hyperplane $x_1 + x_2 + \ldots + x_N = R$,

$$V(R) = \int_{[\sum_{i=1}^N x_i] \leq R} dx_1 dx_2 \ldots dx_N = \int_0^R dr S(r) = R^N/N!, \tag{3.4}$$

where $S(R) = R^{N-1}/(N-1)!$ is the corresponding hyper-area at radius R needed for computation of the multiplicity $\Omega(E)$ and above integral is a constrained one indicated by a prime.

Solution.

(a) For the 2D case, there are n quanta to distribute into 2 partitions (n_x, n_y). The problem can be understood in terms of a sequence of 0's and 1's where 0's are the quanta and 1 is the separation between the two groups n_x and n_y. For example, if $n = 5$, 000100 is a possible sequence. So there are $n+1$ objects in total and the degeneracy is

$$g_{2D}(n) = \binom{n+1}{n} = n+1 \tag{3.5}$$

For the 3D case, the number of 1's is 2, to separate the 0's (quanta) into three groups n_x, n_y, n_z. So the degeneracy is

$$g_{3D}(n) = \binom{n+2}{n} = \frac{(n+1)(n+2)}{2} \tag{3.6}$$

(b) For the $3N$-dimensional case, there needs to be $3N-1$ 1's to seperate the 0's into $3N$ groups. So the multiplicity is

$$\Omega = \binom{n+3N-1}{n} = \frac{(n+3N-1)!}{n!(3N-1)!} \approx \frac{(n+3N)!}{n!(3N)!} \tag{3.7}$$

where we have assumed $n, N \gg 1$. Calculating $\ln\Omega$ and using the lowest order Stirling approximation, we get

$$\ln\Omega \approx (n+3N)\ln(n+3N) - n\ln n - 3N\ln 3N \tag{3.8}$$

So taking the exponent again gives us the multiplicity

$$\Omega(E) \approx \left(1 + \frac{3N\hbar\omega_0}{E}\right)^{E/\hbar\omega_0} \left(1 + \frac{E}{3N\hbar\omega_0}\right)^{3N} \tag{3.9}$$

where we have also written $n = E/\hbar\omega_0$.

(c) By definition, the entropy is

$$\begin{aligned} S/k_B = \ln\Omega &= \frac{E}{\hbar\omega_0}\ln\left(1 + \frac{3N\hbar\omega_0}{E}\right) + 3N\ln\left(1 + \frac{E}{3N\hbar\omega_0}\right) \\ &= 3N\left[\epsilon\ln\left(1 + \frac{1}{\epsilon}\right) + \ln\left(1 + \epsilon\right)\right] \end{aligned} \tag{3.10}$$

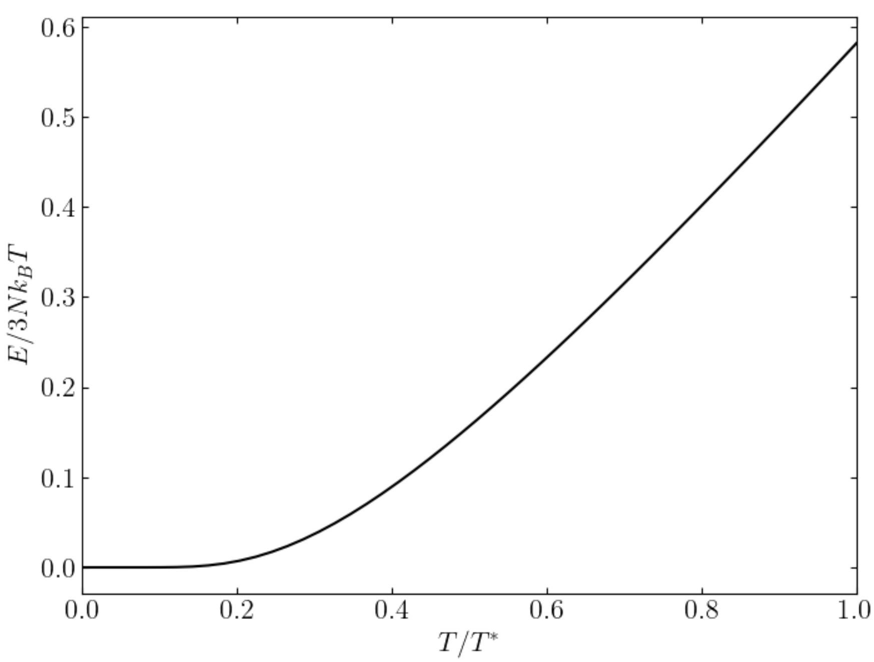
where $\epsilon \equiv E/3N\hbar\omega_0$. The temperature is then

$$\frac{3N\hbar\omega_0}{k_B T} = \frac{\partial(S/k_B)}{\partial\epsilon} = 3N\ln\left(1 + \frac{1}{\epsilon}\right) \Rightarrow T(E) = \frac{\hbar\omega_0}{k_B\ln\left(1 + 3N\hbar\omega_0/E\right)} \tag{3.11}$$

Inverting this result, we get

$$E(T) = \frac{3N\hbar\omega_0}{e^{\hbar\omega_0/k_B T} - 1} \tag{3.12}$$

For the classical limit ($x = \hbar\omega_0/k_B T \ll 1$), $e^x - 1 \approx x$ and $E(T) = 3Nk_BT$. For the quantum limit ($x \gg 1$), $e^x \to \infty$ and $E \sim e^{-x} \to 0$. Letting $T^* = \hbar\omega_0/k_B$, a plot of $E(T)$ is included below.



At high temperature (classical limit), E grows linearly in temperature and at low temperature (quantum limit), $E \to 0$, as expected.

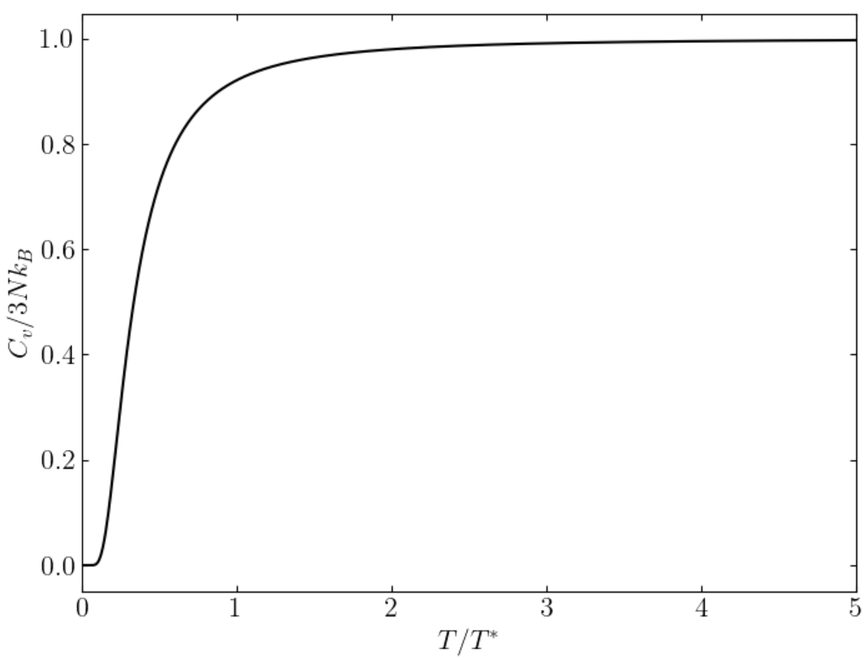
(d) By definition, the heat capacity is

$$C_v = \frac{\partial E}{\partial T} = 3Nk_B \left(\frac{T^*}{T}\right)^2 \frac{e^{T^*/T}}{(e^{T^*/T} - 1)^2} \tag{3.13}$$

For the quantum limit, C_v grows as $\sim (T^*/T)^2 e^{-T^*/T} \sim 0$ because of the exponential term. At large T (classical limit), we can set $x = T^*/T$ and write C_v as

$$C_v = 3Nk_B x^2 \frac{e^x}{(e^x - 1)^2} \approx 3Nk_B x^2 \frac{1+x}{x^2} \approx 3Nk_B \tag{3.14}$$

when $x \to 0$. This is the expected equipartition where $N_{\text{dof}} = 3N$. A plot of C_v is included below.



The formal break occurs at $T = T^*$ as expected, where $T^* = \hbar\omega_0/k_B$ is the crossover temperature. Also, C_v is asymptotically constant as $T \to \infty$ as predicted in (3.14) and 0 as $T \to 0$.

(e) First, we can isolate the integration

$$\Omega(E) = \frac{\Delta}{(\hbar\omega_0)^{3N}} \prod_{\alpha=1}^{3N} \int d\epsilon_\alpha \delta(E - E[\{\epsilon_\alpha\}]) = \frac{\Delta}{(\hbar\omega_0)^{3N}} I_{3N} \tag{3.15}$$

I_{3N} is just the $3N-1$-dimensional hyper-area restricted by the constraint $E = \sum_{\alpha=1}^{3N} \epsilon_\alpha$. We can prove this by induction. First consider the 2D case where $E = \epsilon_1 + \epsilon_2$ and

$$I_2 = \int_0^\infty d\epsilon_1 \int_0^\infty d\epsilon_2 \delta(E - \epsilon_1 - \epsilon_2) \tag{3.16}$$

Let $u = E - \epsilon_1$, then $du = -d\epsilon_1$ and

$$I_2 = -\int_E^{-\infty} du \int_0^\infty d\epsilon_2 \delta(u - \epsilon_2) = \int_0^E du \int_0^\infty d\epsilon_2 \delta(u - \epsilon_2) = \int_0^E du = E = S_2(E) \tag{3.17}$$

where S_n is the given hyper-area in the problem. Now, to prove the $3N$-dimensional case, suppose the $3N-1$-dimensional case is true. Then,

$$\begin{aligned}
I_{3N} &= \int_0^\infty d\epsilon_{3N} \prod_{\alpha=1}^{3N-1} \int_0^\infty d\epsilon_\alpha \delta\left(E - \epsilon_{3N} - \sum_{\beta=1}^{3N-1} \epsilon_\beta\right) \\
&= \int_0^E du \prod_{\alpha=1}^{3N-1} \int_0^\infty d\epsilon_\alpha \delta\left(u - \sum_{\beta=1}^{3N-1} \epsilon_\beta\right) \qquad (u = E - \epsilon_{3N}) \\
&= \int_0^E du S_{3N-1}(u) \\
&= \frac{1}{(3N-2)!} \int_0^E u^{3N-2} du \\
&= \frac{E^{3N-1}}{(3N-1)!} \\
&= S_{3N}(E)
\end{aligned} \tag{3.18}$$

Finally, we can use this result to write

$$\Omega(E) = \frac{\Delta}{(\hbar\omega_0)^{3N}} S_{3N}(E) = \frac{1}{(3N-1)!} \frac{\Delta}{E} \left(\frac{E}{\hbar\omega_0}\right)^{3N} \tag{3.19}$$

But $(3N-1)! \approx (3N)! \approx (3N)^{3N}$ for $N \gg 1$. So

$$\Omega(E) \approx \frac{\Delta}{E} \left(\frac{E}{3N\hbar\omega_0}\right)^{3N} \tag{3.20}$$

□

***Problem* 4** (Boltzmann gas)**:** Consider N identical noninteracting particles in a 3D box of linear size L, described by a Hamiltonian

$$\mathcal{H}(\{\mathbf{p}_i\}) = \sum_i^N \frac{p_i^2}{2m} \tag{4.1}$$

(a) By integrating over $6N$ dimensional phase space, compute the multiplicity (number of microstates) in a shell of width Δ around energy E,

$$\Omega(E) = \frac{\Delta}{N!}\prod_i^N \left[\int \frac{d\mathbf{r}_i d\mathbf{p}_i}{(2\pi\hbar)^3}\right]\delta(E - \mathcal{H}(\{\mathbf{p}_i\})] \tag{4.2}$$

where $1/N!$ is the Gibbs "fudge" factor to crudely (we will see later why this fix fails at low temperatures; also see below) account for the identity of these classical particles, and we used the fact that 1 state corresponds to phase space area $dxdp = 2\pi\hbar$ to normalize the integration measure.

Hint: Use the expression for a surface area of a d-dimensional unit hypersphere, $S_d = 2\pi^{d/2}/\Gamma(d/2)$ (with $S_2 = 2\pi, S_3 = 4\pi, \ldots$).

(b) Compute the corresponding (i) entropy $S(E,V,N) = k_B \ln \Omega$, and (ii) find the temperature T_c at which the entropy becomes negative, i.e., unphysical.

(c) Using the expression for $S(E,V,N)$ found above, calculate the pressure $P(V,N) = T(\partial S/\partial V)_{E,N}$ for the Boltzmann gas, and show that it leads to the familiar ideal gas law, $PV = Nk_BT$.

(d) Compute the corresponding chemical potential $\mu = -T(\partial S/\partial N)_{E,V}$, expressing in terms of the thermal deBroglie wavelength $\lambda_{dB}(T) = h/\sqrt{2\pi m k_B T}$, T and density n.

Solution.

(a) First, we can simplify

$$\begin{aligned}
\Omega(E) &= \frac{\Delta}{E}\frac{V^N}{N!}\frac{1}{h^{3N}}\prod_i^N \int d\mathbf{p}_i \delta\left(1 - \sum_j^N \frac{p_j^2}{2mE}\right) \\
&= \frac{\Delta}{E}\frac{V^N}{N!}\left(\frac{2mE}{h^2}\right)^{3N/2}\prod_i^N \int d\overline{\mathbf{p}}_i \delta\left(1 - \sum_j^N \overline{p}_j^2\right) \qquad (\overline{p}_j^2 = p_j^2/2mE)
\end{aligned}$$

The integration is now just the surface area of a $3N$-dimensional unit hypersphere, since it is constrained to the surface $1 = \sum_j^N \overline{p}_j^2$. So we can write

$$\begin{aligned}
\Omega(E) &= \frac{\Delta}{E}\frac{V^N}{N!}\left(\frac{2mE}{h^2}\right)^{3N/2}\frac{2\pi^{3N/2}}{\Gamma(3N/2)} \\
&\approx \frac{\Delta}{E}\frac{V^N}{\sqrt{2\pi N}e^{-N}N^N}\left(\frac{2\pi mE}{h^2}\right)^{3N/2}\frac{2}{\sqrt{3\pi N}e^{-3N/2}(3N/2)^{3N/2}} \\
&= \frac{2\Delta}{\sqrt{6}\pi N E}e^{5N/2}\frac{V^N}{N^N}\left(\frac{4\pi mE}{3Nh^2}\right)^{3N/2}
\end{aligned} \tag{4.3}$$

where we have written $\Gamma(3N/2) = (3N/2 - 1)! \approx (3N/2)!$ and used Stirling approximation on the factorials.

(b) By definition, the (i) entropy is

$$S/k_B = \ln\Omega = \ln\left(\frac{2\Delta}{\sqrt{6}\pi NE}\right) + N\left\{\frac{5}{2} + \ln\left[\frac{V}{N}\left(\frac{4\pi mE}{3Nh^2}\right)^{3/2}\right]\right\} \tag{4.4}$$

the first term is small compared to N and we retrieve the Sackur-Tetrode equation in the last two terms. Also, the temperature is

$$\frac{1}{k_BT} = \frac{\partial(S/k_B)}{\partial E} = \frac{3}{2}\frac{N}{E} \Rightarrow \frac{E}{N} = \frac{3}{2}k_BT \tag{4.5}$$

Now, from (4.4), $S < 0$ only when

$$\frac{V}{N}\left(\frac{4\pi mE}{3Nh^2}\right)^{3/2} = \frac{V}{N}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2} < e^{-5/2} \tag{4.6}$$

(ii) Solving this inequality, we get

$$T_c < \frac{h^2}{2\pi mk_B}\left(\frac{N}{V}\right)^{2/3} e^{-5/3} \tag{4.7}$$

(c) Differentiating S, we get

$$\frac{P}{T} = \frac{\partial S}{\partial V} = \frac{Nk_B}{V} \Rightarrow PV = Nk_BT \tag{4.8}$$

This is the ideal gas law.

(d) Differentiating S, we get

$$\mu = -k_BT\ln\left[\frac{8}{3\sqrt{3}}\frac{V}{N}\left(\frac{\pi mE}{h^2N}\right)^{3/2}\right] = -k_BT\ln\left[\frac{1}{n}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}\right] = k_BT\ln\left(n\lambda_{dB}^3\right) \tag{4.9}$$

□

***Problem* 5** (Classical harmonic oscillators: Einstein solid)**:** Consider N decouped 3D classical harmonic oscillators, described by the familiar Hamiltonian

$$\mathcal{H} = \sum_i^N\left[\frac{p_i^2}{2m} + \frac{1}{2}m\omega_0^2r_i^2\right] \tag{5.1}$$

as the classical version of the problem above, with $\mathbf{r}_i, \mathbf{p}_i$ the classical phase space coordinates. Let's repeat the analysis utilizing and generalizing the analysis for the Boltzmann gas, above.

(a) By integrating over $6N$ dimensional phase space, compute the multiplicity (number of microstates) in a shell of width Δ around energy E,

$$\Omega(E,N) = \Delta\prod_i^N\left[\int\frac{d\mathbf{r}_i d\mathbf{p}_i}{(2\pi\hbar)^3}\right]\delta[E - \mathcal{H}(\{\mathbf{p}_i\})], \tag{5.2}$$

where there is no $1/N!$ Gibbs "fudge" factor. Why?

(b) Compare your classical result with the high T classical limit ($\hbar\omega_0/k_BT$ is the relevant dimensionless parameter) of the quantum treatment of the system in problem 3(c,d,e) above.

(c) What is the relation of this classical analysis to the classical limit analysis in problem 3e?

Solution.

(a) $\square$