Homework 7: Phys 7310 (Fall 2021)

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***Problem* 7.1** (Dielectric cylinder)**:** A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity. Determine the potential and electric field in the three regions, neglecting end effects.

Solution.

Since we are neglecting end effects on the long cylinder, we can treat this as a two-dimensional problem. From (2.71, Jackson), the general solution to the Laplace equation with polar symmetry is

$$\Phi(\rho,\phi) = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty} (a_n\rho^n + b_n\rho^{-n})(A_n\cos n\phi + B_n \sin n\phi) \tag{7.1.1}$$

Call I, II, III the regions with $\rho \leq a$, $a < \rho < b$, and $\rho \geq b$, respectively. Since the origin is contained in I, $B_0^{\mathrm{I}} = b_n^{\mathrm{I}} = 0$. Then we can write

$$\Phi_{\mathrm{I}} = A_0^{\mathrm{I}} + \sum_{n=1}^{\infty} \rho^n (A_n^{\mathrm{I}} \cos n\phi + B_n^{\mathrm{I}} \sin n\phi) \tag{7.1.2}$$

Now, at large $\rho \gg b$, the potential outside can be written as

$$\Phi_{\mathrm{III}} = A_0^{\mathrm{III}} + \sum_{n=1}^{\infty} (a_n^{\mathrm{III}}\rho^n + b_n^{\mathrm{III}}\rho^{-n})(A_n^{\mathrm{III}}\cos n\phi + B_n^{\mathrm{III}}\sin n\phi) = -E_0\rho\cos\phi \tag{7.1.3}$$

Thus, $A_0^{\mathrm{III}} = B_n^{\mathrm{III}} = 0$ and

$$\Phi_{\mathrm{III}} = \left(a_1^{\mathrm{III}}\rho + b_1^{\mathrm{III}}\rho^{-1}\right)\cos\phi \approx a_1^{\mathrm{III}}\rho\cos\phi = -E_0\rho\cos\phi \tag{7.1.4}$$

Then $a_1^{\mathrm{III}} = -E_0$ and we can write the potential outside the cylinder as

$$\Phi_{\mathrm{III}} = (-E_0\rho + b_1^{\mathrm{III}}\rho^{-1})\cos\phi \tag{7.1.5}$$

There is no restriction on the potential in II, except for that it connects the potential in I and III. Now, we impose the boundary condition that $E_\parallel$ is continuous at $\rho = a$.

$$\left.\frac{\partial \Phi_{\mathrm{I}}}{\partial \phi}\right|_{\rho=a} = \left.\frac{\partial \Phi_{\mathrm{II}}}{\partial \phi}\right|_{\rho=a}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} n a_n \Big(-A_n^{\mathrm{I}} \sin n\phi + B_n^{\mathrm{I}} \cos n\phi\Big) = \sum_{n=1}^{\infty} n \Big(a_n^{\mathrm{II}} a^n + b_n^{\mathrm{II}} a^{-n}\Big)\Big(-A_n^{\mathrm{II}} \sin n\phi + B_n^{\mathrm{II}} \cos n\phi\Big) \tag{7.1.6}$$

Thus, for $n \geq 1$,

$$A_n^{\mathrm{I}} = \Big(a_n^{\mathrm{II}} + b_n^{\mathrm{II}} a^{-2n}\Big) A_n^{\mathrm{II}} \qquad \text{and} \qquad B_n^{\mathrm{I}} = \Big(a_n^{\mathrm{II}} + b_n^{\mathrm{II}} a^{-2n}\Big) B_n^{\mathrm{II}} \tag{7.1.7}$$

Similarly, at $\rho = b$,

$$\left.\frac{\partial \Phi_{\mathrm{II}}}{\partial \phi}\right|_{\rho=b} = \left.\frac{\partial \Phi_{\mathrm{III}}}{\partial \phi}\right|_{\rho=b}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} n \Big(a_n^{\mathrm{II}} b^n + b_n^{\mathrm{II}} b^{-n}\Big)\Big(-A_n^{\mathrm{II}} \sin n\phi + B_n^{\mathrm{II}} \cos n\phi\Big) = -\Big(-E_0 b + b_1^{\mathrm{III}} b^{-1}\Big) \sin\phi \tag{7.1.8}$$

Then it folows that $B_n^{\mathrm{I}} = B_n^{\mathrm{II}} = 0$ due to symmetry and the only non-trivial term is $n = 1$ where

$$A_1^{\mathrm{II}} \Big(a_1^{\mathrm{II}} b + b_1^{\mathrm{II}} b^{-1}\Big) = -E_0 b + b_1^{\mathrm{III}} b^{-1} \tag{7.1.9}$$

To summarize, the potential everywhere is now

$$\Phi_{\mathrm{I}} = A_0^{\mathrm{I}} + \Big(a_1^{\mathrm{II}} + b_1^{\mathrm{II}} a^{-2}\Big) A_1^{\mathrm{II}} \rho \cos\phi \tag{7.1.10a}$$

$$\Phi_{\mathrm{II}} = A_0^{\mathrm{II}} + B_0^{\mathrm{II}} \ln\rho + \Big(a_1^{\mathrm{II}} + b_1^{\mathrm{II}} \rho^{-2}\Big) A_1^{\mathrm{II}} \rho \cos\phi \tag{7.1.10b}$$

$$\Phi_{\mathrm{III}} = \Big(-E_0 + b_1^{\mathrm{III}} \rho^{-2}\Big) \rho \cos\phi \tag{7.1.10c}$$

Now, imposing the condition that $D_\perp$ is continuous as a, we get

$$\epsilon_0 \left.\frac{\partial \Phi_{\mathrm{I}}}{\partial \rho}\right|_{\rho=a} = \epsilon \left.\frac{\partial \Phi_{\mathrm{II}}}{\partial \rho}\right|_{\rho=a}$$

$$\Rightarrow \qquad \epsilon_0 \Big(a_1^{\mathrm{II}} + b_1^{\mathrm{II}} a^{-2}\Big) A_1^{\mathrm{II}} \cos\phi = \epsilon \frac{B_0^{\mathrm{II}}}{a} + \epsilon \Big(a_1^{\mathrm{II}} - b_1^{\mathrm{II}} a^{-2}\Big) A_1^{\mathrm{II}} \cos\phi \tag{7.1.11}$$

By symmetry, $B_0^{\mathrm{II}} = 0$ and we can solve for b_1^{II} as

$$b_1^{\mathrm{II}} = \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1} a^2 a_1^{\mathrm{II}} \tag{7.1.12}$$

Also, because $B_0^{\mathrm{II}} = 0$, the continuity of Φ at $\rho = a$ requires that $A_0^{\mathrm{I}} = A_0^{\mathrm{II}} = 0$. The potential now looks like

$$\Phi_{\mathrm{I}} = \left(1 + \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\right) a_1^{\mathrm{II}} A_1^{\mathrm{II}} \rho \cos\phi \tag{7.1.13a}$$

$$\Phi_{\mathrm{II}} = \left(1 + \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{a^2}{\rho^2}\right) a_1^{\mathrm{II}} A_1^{\mathrm{II}} \rho \cos\phi \tag{7.1.13b}$$

$$\Phi_{\mathrm{III}} = \left(-E_0 + b_1^{\mathrm{III}} \rho^{-2}\right) \rho \cos\phi \tag{7.1.13c}$$

Finally, let $D_\perp$ be continuous at b, we get

$$\epsilon \frac{\partial \Phi_{\mathrm{II}}}{\partial \rho}\bigg|_{\rho=b} = \epsilon \frac{\partial \Phi_{\mathrm{III}}}{\partial \rho}\bigg|_{\rho=b}$$

$$\Rightarrow \qquad \epsilon \left(1 - \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{a^2}{b^2}\right) a_1^{\mathrm{II}} A_1^{\mathrm{II}} \cos\phi = \epsilon_0 \left(-E_0 - \frac{b_1^{\mathrm{III}}}{b^2}\right) \cos\phi$$

$$\Rightarrow \qquad \frac{\epsilon}{\epsilon_0} \left(1 - \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{a^2}{b^2}\right) a_1^{\mathrm{II}} A_1^{\mathrm{II}} = -E_0 - \frac{b_1^{\mathrm{III}}}{b^2} \tag{7.1.14}$$

There are two remaining unknowns, $a_1^{\mathrm{II}} A_1^{\mathrm{II}}$ and b_1^{III}. However, note that plugging (7.1.12) into (7.1.9) gives us another equation relating these two unknowns

$$\left(1 + \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1}\frac{a^2}{b^2}\right) a_1^{\mathrm{II}} A_1^{\mathrm{II}} = -E_0 + \frac{b_1^{\mathrm{III}}}{b^2} \tag{7.1.15}$$

Solving (7.1.14) and (7.1.15) using Mathematica, we get

$$a_1^{\mathrm{II}} A_1^{\mathrm{II}} = -E_0 b^2 \frac{2(\epsilon/\epsilon_0 + 1)}{b^2(\epsilon/\epsilon_0 + 1)^2 - a^2(\epsilon/\epsilon_0 - 1)^2} \tag{7.1.16}$$

and

$$b_1^{\mathrm{III}} = E_0 b^2 \frac{(b^2 - a^2)(\epsilon^2/\epsilon_0^2 - 1)}{b^2(\epsilon/\epsilon_0 + 1)^2 - a^2(\epsilon/\epsilon_0 - 1)^2} \tag{7.1.17}$$

Then we can write our final solution as

$$\Phi_{\mathrm{I}}(\rho, \phi) = -\frac{4b^2 \epsilon/\epsilon_0}{b^2(\epsilon/\epsilon_0 + 1)^2 - a^2(\epsilon/\epsilon_0 - 1)^2} E_0 \rho \cos\phi \tag{7.1.18a}$$

$$\Phi_{\mathrm{II}}(\rho, \phi) = -\frac{b^2}{\rho^2}\frac{2[(\rho^2 + a^2)\epsilon/\epsilon_0 + \rho^2 - a^2]}{b^2(\epsilon/\epsilon_0 + 1)^2 - a^2(\epsilon/\epsilon_0 - 1)^2} E_0 \rho \cos\phi \tag{7.1.18b}$$

$$\Phi_{\mathrm{III}}(\rho, \phi) = -\left[1 - \frac{b^2}{\rho^2}\frac{(b^2 - a^2)(\epsilon^2/\epsilon_0^2 - 1)}{b^2(\epsilon/\epsilon_0 + 1)^2 - a^2(\epsilon/\epsilon_0 - 1)^2}\right] E_0 \rho \cos\phi \tag{7.1.18c}$$

By definition, $\mathbf{E} = -\boldsymbol{\nabla}\Phi = -\big(\partial\Phi/\partial\rho\big)\hat{\boldsymbol{\rho}} - (1/\rho)\big(\partial\Phi/\partial\phi\big)\hat{\boldsymbol{\phi}}$. Thus, we can write

$$\mathbf{E}_{\mathrm{I}} = -\frac{4b^2\epsilon/\epsilon_0}{b^2(\epsilon/\epsilon_0+1)^2 - a^2(\epsilon/\epsilon_0-1)^2} E_0\Big(\cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}\Big) \tag{7.1.19}$$

Similarly,

$$\begin{aligned}\mathbf{E}_{\mathrm{II}} = \frac{b^2}{\rho^2}\frac{2E_0}{b^2(\epsilon/\epsilon_0+1)^2 - a^2(\epsilon/\epsilon_0-1)^2}\Bigg\{ \\ -\big[\rho^2(\epsilon/\epsilon_0+1) - a^2(\epsilon/\epsilon_0-1)\big]\cos\phi\hat{\boldsymbol{\rho}} + \big[\rho^2(\epsilon/\epsilon_0+1) + a^2(\epsilon/\epsilon_0-1)\big]\sin\phi\hat{\boldsymbol{\phi}}\Bigg\}\end{aligned} \tag{7.1.20}$$

Finally, the ρ component of $\mathbf{E}_{\mathrm{III}}$ is

$$\begin{aligned}E_\rho^{\mathrm{III}} = -&\frac{(\epsilon/\epsilon_0+1)[\rho^2(\epsilon/\epsilon_0+1) + b^2(\epsilon/\epsilon_0-1)] - a^2/b^2(\epsilon/\epsilon_0-1)[\rho^2(\epsilon/\epsilon_0-1) + b^2(\epsilon/\epsilon_0+1)]}{b^2(\epsilon/\epsilon_0+1)^2 - a^2(\epsilon/\epsilon_0-1)^2} \\ &\times\frac{b^2}{\rho^2}E_0\cos\phi\end{aligned} \tag{7.1.21}$$

and the ϕ component is

$$E_\phi^{\mathrm{III}} = \left[1 - \frac{b^2}{\rho^2}\frac{(b^2-a^2)(\epsilon^2/\epsilon_0^2-1)}{b^2(\epsilon/\epsilon_0+1)^2 - a^2(\epsilon/\epsilon_0-1)^2}\right]E_0\sin\phi \tag{7.1.22}$$

□

Problem 7.2 (Dielectric sphere)**:** A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a < d$) and dielectric constant ϵ/ϵ_0.

(a) Find the potential at all points in space as an expansion in spherical harmonics.

(c) Verify that, in the limit $\epsilon/\epsilon_0 \to \infty$, your result is the same as that for the conducting sphere.

Solution.

(a) Let the point charge q be placed at $\mathbf{x}' = d\hat{\mathbf{z}}$. Then the potential due to this point charge outside the sphere ($|\mathbf{x}| = r > a$) is

$$\Phi_q(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \tag{7.2.1}$$

where we have used the expansion (3.38, Jackson) with $r_< = \min(r,d), r_> = \max(r,d)$, and $\cos\theta = (\mathbf{x}/x)\boldsymbol{\cdot}\hat{\mathbf{z}}$. Now, without the presence of the point charge, the potential follows Laplace equation and has the following general solution

$$\Phi_{\text{dielectric}} = \begin{cases}\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & r \le a \\ \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) & r > a\end{cases} \tag{7.2.2}$$

because the dielectric has azimuthal symmetry. Now, by superposition principle, we can write the total potential everywhere as

$$\Phi_{r\leq a} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{7.2.3a}$$

$$\Phi_{r>a} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + B_l r^{-(l+1)}\right]P_l(\cos\theta) \tag{7.2.3b}$$

Note that we have also redefined $B_l \mapsto (q/4\pi\epsilon_0)B_l$ to simplify (7.2.3b). First, by continuity at $r = a$, we have

$$\sum_{l=0}^{\infty} A_l a^l P_l = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{a^l}{d^{l+1}} + \frac{B_l}{a^{l+1}}\right]P_l \Rightarrow A_l = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{d^{l+1}} + \frac{B_l}{a^{2l+1}}\right] \tag{7.2.4}$$

by orthogonality of P_l. Next, let $D_\perp$ be continuous at $r = a$, we can write

$$\epsilon\left.\frac{\partial\Phi_{r\leq a}}{\partial r}\right|_{r=a} = \epsilon_0\left.\frac{\partial\Phi_{r>a}}{\partial r}\right|_{r=a}$$

$$\Rightarrow \qquad \epsilon\sum_{l=1}^{\infty} lA_l a^{l-1}P_l = \frac{q}{4\pi}\left[-\frac{B_0}{a^2} + \sum_{l=1}^{\infty}\left[\frac{la^{l-1}}{d^{l+1}} - (l+1)\frac{B_l}{a^{l+2}}\right]P_l\right] \tag{7.2.5}$$

By orthogonality, B_0 and for $l \geq 1$,

$$A_l = \frac{q}{4\pi\epsilon}\left[\frac{1}{d^{l+1}} - \frac{l+1}{l}\frac{B_l}{a^{2l+1}}\right] \tag{7.2.6}$$

Solving (7.2.4) and (7.2.6) for A_l and B_l with $l \geq 1$ yields

$$A_l = \frac{q}{4\pi\epsilon d}\frac{2l+1}{l+(l+1)\epsilon_0/\epsilon}\frac{1}{d^{l+1}} \tag{7.2.7a}$$

$$B_l = \frac{l(\epsilon_0/\epsilon - 1)}{l+(l+1)\epsilon_0/\epsilon}\frac{a^{2l+1}}{d^{l+1}} \tag{7.2.7b}$$

Note that (7.2.7b) is also zero when $l = 0$. So we can plug (7.2.7) into (7.2.3) for $l \geq 0$ to get the final solution

$$\Phi_{r\leq a}(r,\theta,\phi) = \frac{q}{4\pi\epsilon d}\sum_{l=0}^{\infty}\frac{2l+1}{l+(l+1)\epsilon_0/\epsilon}\frac{r^l}{d^l}P_l(\cos\theta) \tag{7.2.8a}$$

$$\Phi_{r>a}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + \frac{l(\epsilon_0/\epsilon - 1)}{l+(l+1)\epsilon_0/\epsilon}\frac{a^{2l+1}}{(rd)^{l+1}}\right]P_l(\cos\theta) \tag{7.2.8b}$$

(c) Rewriting (7.2.8a) in terms of ϵ/ϵ_0, we have

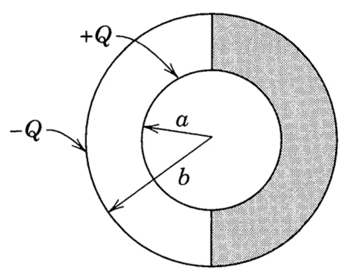
$$\Phi_{r\leq a} = \frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty}\frac{2l+1}{(\epsilon/\epsilon_0)l + l + 1}\frac{r^l}{d^l}P_l(\cos\theta) \approx \frac{q}{4\pi\epsilon_0 d} \tag{7.2.9}$$

for $\epsilon/\epsilon_0 \to \infty$. This is the potential inside a conducting sphere, since the electric field is zero inside. Now, for $r > a$, at the limit $\epsilon_0/\epsilon \to 0$,

$$\Phi_{r>a} \approx \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{a}{d} \frac{(a^2/d)^l}{r^{l+1}} \right] P_l(\cos\theta) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\mathbf{x} - d\hat{\mathbf{z}}|} - \frac{a/d}{|\mathbf{x} - (a^2/d)\hat{\mathbf{z}}|} \right] \quad (7.2.10)$$

This agrees with Section 2.1 in Jackson where we use an image charge $q' = -(a/d)q$ placed at $\mathbf{x}' = (a^2/d)\hat{\mathbf{z}}$ to find the total potential through the method of images. □

***Problem* 7.3** (Half a dielectric shell)**:** Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemi-spherical shell of dielectric (of dielectric constant ϵ/ϵ_0), as shown in the figure.



(a) Find the electric field everywhere between the sphere.
(b) Calculate the surface charge distribution on the inner sphere.
(c) Calculate the polarization charge density induced on the surface of the dielectric at $r = a$.

Solution.
(a) Let the system be positioned in a spherical coordinate system where the dielectric is located from $\theta = 0$ to $\theta = \pi/2$. Then it has azimuthal symmetry and the general solution for the potential with $0 \leq x = \cos\theta \leq 1$ is

$$\Phi_{x\geq 0} = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(x) \quad (7.3.1)$$

The potential is constant across the conductors. So we can evaluate (7.3.1) at $r = a$

$$A_0 + B_0 a^{-1} + \sum_{l=1}^{\infty} \left[A_l a^l + B_l a^{-(l+1)} \right] P_l(x) = V_a \quad (7.3.2)$$

where V_a is some constant potential that the sphere at $r = a$ is held at. By orthogonality, the coefficients for $l \geq 1$ must vanish and we have

$$B_l = A_l a^{2l+1} \quad (7.3.3)$$

Now, evaluating (7.3.1) at $r = b$, we can write

$$A_0 + B_0 b^{-1} + \sum_{l=1}^{\infty} A_l \Big(b^l + a^{2l+1} b^{-(l+1)} \Big) P_l(x) = V_b \tag{7.3.4}$$

where V_b is some constant potential at which the sphere at $r = b$ is held. By orthogonality, the $l \geq 1$ coefficients must vanish, so $A_l = B_l = 0$ for $l \geq 1$. The potential is thus

$$\Phi_{x\geq 0} = A_0 + B_0 r^{-1} \tag{7.3.5}$$

and we can write the electric field in the northen hemisphere as

$$\mathbf{E}_{x\geq 0} = -\boldsymbol{\nabla}\Phi_{x\geq 0} = -\frac{A}{r^2}\hat{\mathbf{r}} \tag{7.3.6}$$

where A is some constant. A similar argument applies for the southern hemisphere ($x < 0$) and we can write

$$\mathbf{E}_{x<0} = -\frac{B}{r^2}\hat{\mathbf{r}} \tag{7.3.7}$$

The electric field needs to be continuous at $x = 0$. Thus, $A = B$. Now, we can calculate the electric displacement

$$\mathbf{D} = \begin{cases} \epsilon \mathbf{E} & x \geq 0 \\ \epsilon_0 \mathbf{E} & x < 0 \end{cases} = \begin{cases} -\frac{\epsilon A}{r^2}\hat{\mathbf{r}} & x \geq 0 \\ -\frac{\epsilon_0 A}{r^2}\hat{\mathbf{r}} & x < 0 \end{cases} \tag{7.3.8}$$

Now, by Gauss Law, the charge enclosed in a spherical Gaussian surface S with $r \in (a, b)$ is

$$Q = \oint_S \mathbf{D}\cdot\hat{\mathbf{r}} da = 2\pi \int_{-1}^{1} D r^2 d(\cos\theta) = -2\pi A \left[\epsilon \int_0^1 d(\cos\theta) + \epsilon_0 \int_{-1}^{0} d(\cos\theta) \right] = -2\pi(\epsilon+\epsilon_0)A \tag{7.3.9}$$

Thus, we can invert for A and write the electric field everywhere between the spheres as

$$\mathbf{E} = \frac{Q}{2\pi(\epsilon+\epsilon_0)} \frac{1}{r^2}\hat{\mathbf{r}} \tag{7.3.10}$$

(b) From (7.3.10), the electric displacement is

$$\mathbf{D}_{x\geq 0} = \frac{Q}{2\pi} \frac{\epsilon}{\epsilon+\epsilon_0} \frac{1}{r^2}\hat{\mathbf{r}} \qquad \text{and} \qquad \mathbf{D}_{x<0} = \frac{Q}{2\pi} \frac{\epsilon_0}{\epsilon+\epsilon_0} \frac{1}{r^2}\hat{\mathbf{r}} \tag{7.3.11}$$

From (4.40, Jackson), choose $\hat{\mathbf{n}}_{21} = \hat{\mathbf{r}}$. So $\mathbf{D}_1 = \mathbf{0}$ and $\mathbf{D}_2 = \mathbf{D}$. The free surface charge density on the sphere at $r = a$ is thus

$$\rho_{x\geq 0} = \mathbf{D}_{x\geq 0}\cdot\hat{\mathbf{r}}\Big|_{r=a} = \frac{Q}{2\pi} \frac{\epsilon}{\epsilon+\epsilon_0} \frac{1}{a^2} \qquad \text{and} \qquad \rho_{x<0} = \mathbf{D}_{x<0}\cdot\hat{\mathbf{r}}\Big|_{r=a} = \frac{Q}{2\pi} \frac{\epsilon_0}{\epsilon+\epsilon_0} \frac{1}{a^2} \tag{7.3.12}$$

(c) From (4.36, Jackson) and (4.38, Jackson), the polarization is

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} = \frac{Q}{2\pi} \frac{\epsilon-\epsilon_0}{\epsilon+\epsilon_0} \frac{1}{r^2}\hat{\mathbf{r}} \tag{7.3.13}$$

Now, at $r = a$, the normal vector to the dielectric is $\hat{\mathbf{n}}_{21} = -\hat{\mathbf{r}}$. Then the bound surface charge density on the dielectric is

$$\sigma_{\text{pol}} = \mathbf{P}_1 \cdot \hat{\mathbf{n}}_{21} = - \mathbf{P}_{x \geq 0} \cdot \hat{\mathbf{r}} \Big|_{r=a} = \frac{Q}{2\pi} \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} \frac{1}{a^2} \tag{7.3.14}$$

□