# Homework 6: Astr 5140 (Fall 2021)

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***Problem* 1** (Polarization drift)**:** In addition to $\mathbf{E} \times \mathbf{B}$ drift, there are of several high-order plasma drifts. The polarization drift is one such drift that comes from a time-varying electric field. Let $\mathbf{E}$ lie in the $y$ direction and $\mathbf{B}$ in the $z$ direction. We know that $v = \mathbf{E} \times \mathbf{B}/B^2$. In this problem we want to derive $\mathbf{v}_p$ the polarization drift in the $y$ direction.

(a) Show that $\mathbf{v}_p = (m/qB^2)d\mathbf{E}_y/dt$. Treat $d\mathbf{E}_y/dt$ as a constant. Hint: See Boyd & Sanderson 2.12 (pg. 37).

(b) How does the polarization drift differ from the $\mathbf{E} \times \mathbf{B}$ drift? Which has a faster polarization drift, ions or electrons? Why? What is the current associated with the polarization drift?

*Solution.*

(a) Given $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, the Lorentz force equation is

$$m\dot{\mathbf{v}} = m(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q(E\hat{\mathbf{y}} - v_x B\hat{\mathbf{y}} + v_y B\hat{\mathbf{x}}) \Rightarrow \begin{cases} \dot{v}_x &= \Omega_c v_y \\ \dot{v}_y &= qE/m - \Omega_c v_x \end{cases} \tag{1.1}$$

where $\Omega_c = qB/m$. Then $v_y$ follows the differential equation

$$\ddot{v}_y = \frac{q\dot{E}}{m} - \Omega_c^2 v_y = -\Omega_c^2\left(v_y - \frac{m}{qB^2}\dot{E}\right) \tag{1.2}$$

Let $\overline{v}_y = v_y - \frac{m}{qB^2}\dot{E}$, then since $\ddot{E} = 0$, $\overline{v}_y$ satisfies

$$\ddot{\overline{v}}_y = -\Omega_c^2 \overline{v}_y \tag{1.3}$$

A general solution to this is

$$\overline{v}_y = v_\perp \sin(\Omega t + \delta) \Rightarrow v_y = v_\perp \sin(\Omega t + \delta) + \frac{m}{qB^2}\frac{dE}{dt} \tag{1.4}$$

where $v_\perp$ and $\delta$ depend on initial conditions. Thus, there is a drift $v_p = (m/qB^2)\dot{E}_y$ in the $y$ direction beside the $\mathbf{E} \times \mathbf{B}$ drift in the $x$ direction

$$v_x = \frac{qE}{m\Omega_c} - \frac{\dot{v}_y}{\Omega_c} = -v_\perp \cos(\Omega_c t + \delta) + \frac{E}{B} \tag{1.5}$$

(b) While the $\mathbf{E} \times \mathbf{B}$ drift does not depend on the properties of the charged particle (mass and charge), the polarization drift does. Since $v_p \sim m$, ions drift faster than electrons. The current associated to this drift is

$$J_y = nqv_p = \frac{nm}{B^2}\frac{\partial E}{\partial t} = \frac{\rho}{B^2}\frac{\partial E}{\partial t} \tag{1.6}$$

□

***Problem* 2** (Inertial Alfvén wave)**:** In this problem, let $\mathbf{B}_0$ be in the $z$ direction and the motion, $\mathbf{u}_1$ and $\mathbf{B}_1$ in the $x$ direction. An Alfvén wave that is nearly perpendicular will have $k_y \gg k_z$ and will develop strong curents parallel to $\mathbf{B}_0$. Assume that the parallel current can be represented by electron motion

$$\frac{\partial J_z}{\partial t} = \frac{ne^2}{m_e}E_z \tag{2.1}$$

while the perpendicular current is from the polarization drift

$$J_y = \frac{\rho_0}{B_0^2}\frac{\partial E_y}{\partial t} \tag{2.2}$$

Derive the dispersion relation of the inertial Alfvén wave using the above combined with Maxwell's equations. Express $\omega$ in terms of $\mathbf{k}, V_A$, and the electron skin depth ($\lambda_e = c/\omega_{pe}$). Hint: One does not need the fluid equations once $\mathbf{J}$ is determined.

*Solution.*

Given $\mathbf{B}_0 = B_0\hat{\mathbf{z}}, \mathbf{u}_1 = u_1\hat{\mathbf{x}}, \mathbf{B}_1 = B_1\hat{\mathbf{x}}$, and $\mathbf{k} = k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$, we can Fourier transform Ampere's Law and write

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{J} \Rightarrow i(k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}) \times (B_1\hat{\mathbf{x}}) = iB_1\big(-k_y\hat{\mathbf{z}} + k_z\hat{\mathbf{y}}\big) = \mu_0\mathbf{J} \tag{2.3}$$

Thus,

$$-ik_yB_1 = \mu_0J_z = i\frac{\omega_{pe}^2}{\omega c^2}E_z \qquad \text{and} \qquad ik_zB_1 = \mu_0J_y = -\frac{i\omega}{V_A^2}E_y \tag{2.4}$$

where we have also Fourier transformed (2.1) and (2.2). Similarly, from Faraday's Law,

$$(k_yE_z - k_zE_y) = -\frac{\omega c^2k_y^2}{\omega_{pe}^2}B_1 + \frac{k_z^2V_A^2}{\omega}B_1 = \omega B_1 \Rightarrow -k_y^2\lambda_e^2 + \frac{k_z^2V_A^2}{\omega^2} = 1 \tag{2.5}$$

Rewriting, we get the dispersion relation

$$\omega^2 = \frac{k_z^2V_A^2}{1 + k_y^2\lambda_e^2} \tag{2.6}$$

□

***Problem* 3** (High Mach shocks)**:** The universe is full of plasmas in motion that encounter obstacles such as stellar and planetary magnetospheres or other moving plasmas. If the motions exceed the Alfvén or sound speed, a shock can form. One possibility is that the flow perpendicular to the magnetic field. Mathematically, we model the shock with time-stationary "jump" conditions in a frame where the shock is stationary.

(a) Write down the Rankine-Hugoniot jump conditions applicable for a perpendicular shock.

(b)Using the compression ratio $r = \rho_2/\rho_1$, derive an expression for $P_2$ in terms of $r, \rho_1, u_1$, and $B_1$ by combining equations for continuity, magnetic field, and force.

(c) Repeat the above calculation using the energy equation instead of the force equation, that is, derive another expression for $P_2$ in terms of $r, \rho_1, u_1$, and $B_1$ based on the energy equation.

(d) With a good bit of algebra, one could solve for the compression ratio $r$ in terms of $P_1, \rho_1, u_1$, and $B_1$. Instead, let's introduce two new quantities. Write down expressions for the upstream Mach numbers in terms of $P_1, \rho_1, u_1$, and $B_1$.

(e) Derive approximations for expressions (a) and (b) assuming that $M_A \gg 1$ and $M_s \gg 1$.

(f) Demonstrate that, if $\gamma = 5/3$, the compression ratio for a high-Mach shock is 4.

*Solution.*

(a) The jump conditions are as follows

$$\rho_1 u_1 = \rho_2 u_2 \tag{3.1a}$$

$$u_1 B_1 = u_2 B_2 \tag{3.1b}$$

$$P_1 + \rho_1 u_1^2 + \frac{B_1^2}{2\mu_0} = P_2 + \rho_2 u_2^2 + \frac{B_2^2}{2\mu_0} \tag{3.1c}$$

$$u_1\left(\frac{1}{2}\rho_1 u_1^2 + \frac{\gamma}{\gamma-1}P_1 + \frac{B_1^2}{\mu_0}\right) = u_2\left(\frac{1}{2}\rho_2 u_2^2 + \frac{\gamma}{\gamma-1}P_2 + \frac{B_2^2}{\mu_0}\right) \tag{3.1d}$$

(b) From (3.1a) and (3.1b), the compression ratio is

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{B_2}{B_1} \tag{3.2}$$

Then from the force equation (3.1c),

$$\begin{aligned}
P_2 &= P_1 + \rho_1 u_1^2 + \frac{B_1^2}{2\mu_0} - \rho_2 u_2^2 - \frac{B_2^2}{2\mu_0} \\
&= P_1 + \rho_1 u_1^2\left(1 - \frac{\rho_2}{\rho_1}\frac{u_2^2}{u_1^2}\right) + \frac{B_1^2}{2\mu_0}\left(1 - \frac{B_2^2{}^2}{B_1}\right) \\
&= P_1 + \rho_1 u_1^2\left(1 - \frac{1}{r}\right) + \frac{B_1^2}{2\mu_0}\left(1 - r^2\right) \qquad (3.3)
\end{aligned}$$

(c) Similary, from (3.1d),

$$
\begin{aligned}
P_2 &= \frac{\gamma-1}{\gamma}\left[\frac{u_1}{u_2}\left(\frac{1}{2}\rho_1 u_1^2 + \frac{\gamma}{\gamma-1}P_1 + \frac{B_1^2}{\mu_0}\right) - \frac{1}{2}\rho_2 u_2^2 - \frac{B_2^2}{\mu_0}\right] \\
&= P_1 + \frac{\gamma-1}{\gamma}\left[\frac{1}{2}\rho_1 u_1^2\left(\frac{u_1}{u_2} - \frac{\rho_2}{\rho_1}\frac{u_2^2}{u_1^2}\right) + \frac{B_1^2}{\mu_0}\left(\frac{u_1}{u_2} - \frac{B_2^2}{B_1^2}\right)\right] \\
&= P_1 + \frac{\gamma-1}{\gamma}\left[\frac{1}{2}\rho_1 u_1^2\left(r - \frac{1}{r}\right) + \frac{B_1^2}{\mu_0}\left(r - r^2\right)\right] \qquad (3.4)
\end{aligned}
$$

(d) By definition,

$$M_A^2 + \frac{u_1^2}{V_A^2} = \frac{(1/2)\rho_1 u_1^2}{B_1^2/2\mu_0} \qquad \text{and} \qquad M_s^2 = \frac{u_1^2}{\gamma T_1/m} = \frac{1}{\gamma}\frac{\rho_1^2 u_1^2}{P_1} \qquad (3.5)$$

(e) At high $M_A$ and $M_s$, $\rho_1 u_1^2 \gg B_1^2/2\mu_0$ and $\rho_1 u_1^2 \gg P_1$. Thus, from (3.3) and (3.4),

$$P_2 \approx \rho_1 u_1^2\left(1 - \frac{1}{r}\right) \approx \frac{\gamma-1}{\gamma}\frac{1}{2}\rho_1 u_1^2(r - \frac{1}{r}) \Rightarrow \frac{\gamma-1}{\gamma} = 2\frac{1-1/r}{r-1/r} \Leftrightarrow \frac{\gamma-1}{2\gamma}\left(r^2-1\right) = r-1 \qquad (3.6)$$

(f) When $\gamma = 5/3$, (3.6) becomes

$$\frac{1}{5}r^2 - r + \frac{4}{5} = 0 \Rightarrow r = 1 \text{ or } r = 4 \qquad (3.7)$$

The first solution is trivial. So for a high Mach shock, $r = 4$. □

***Problem* 4** (Harris Current Sheet - Review)**:** Assume a sheet of current in which $J$ flows in the $y$ direction. There is no external magnetic field. Let $P = 0$ and $B_z = \mp B_0$ at $x = \pm\infty$. One way to find a physical solution is to set $\mathbf{J}$ to be proportional to $P$. Derive a solution for $\mathbf{B}, \mathbf{J}$, and $P$ by assuming $J_y = 2P/(LB_0)$. $L$ is a characteristic length. Plot $B_z, P$, and $J_y$ as a function of $x$.

*Solution.*

The force equation is

$$0 = -\boldsymbol{\nabla} P - \boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right) \Rightarrow P + \frac{B^2}{2\mu_0} = \frac{B_0^2}{2\mu_0} \qquad (4.1)$$

where the RHS is given from boundary conditions. From Faraday's Law,

$$-\frac{\partial B_z}{\partial x} = \mu_0 J = \frac{B_0^2 - B^2}{LB_0} \qquad (4.2)$$

We can solve this differential equation by separation of variables

$$\int \frac{dx}{L} = -B_0 \int \frac{dB_z}{B_0^2 - B^2} \Rightarrow \frac{x}{L} = -\tanh^{-1}\left(\frac{B}{B_0}\right) \qquad (4.3)$$

Then the magnetic field is

$$B = -B_0 \tanh\left(\frac{x}{L}\right) \tag{4.4}$$

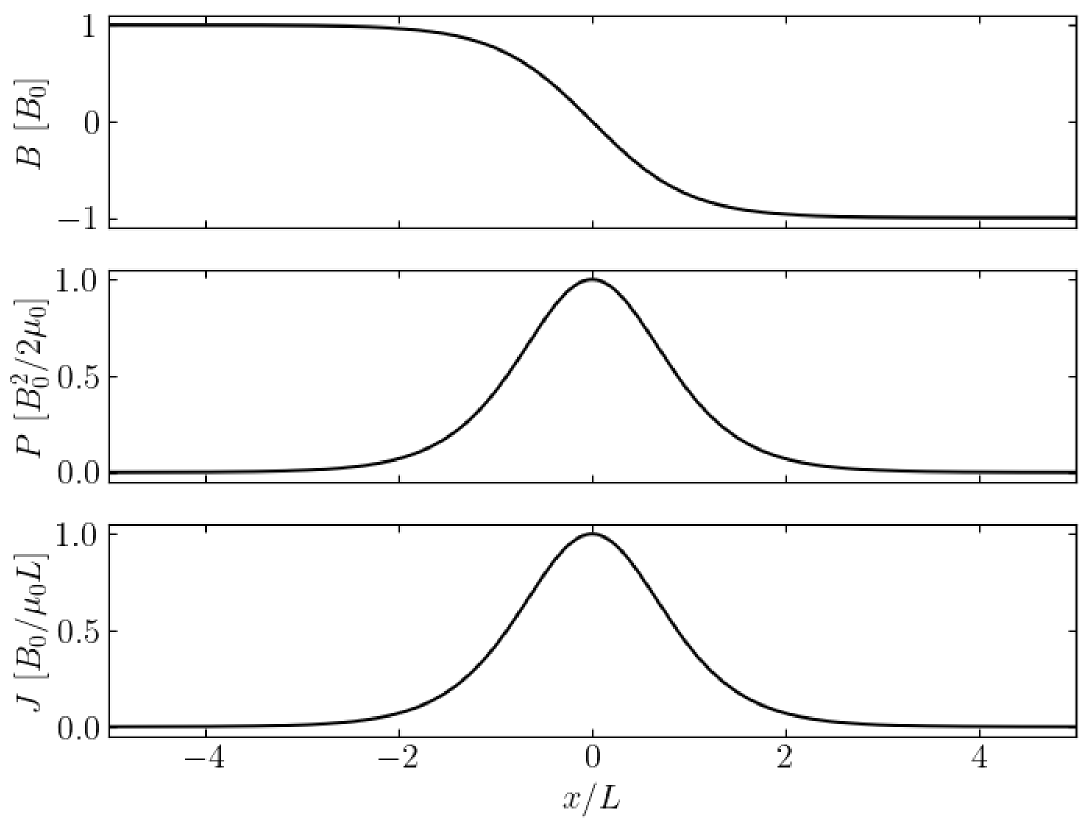
Note that $B \to \mp B_0$ as $x \to \pm\infty$. This would create a current in the $+y$ direction through the right hand rule. From (4.1),

$$P = \frac{1}{2\mu_0}(B_0^2 - B^2) = \frac{B_0^2}{2\mu_0}\,\mathrm{sech}^2\left(\frac{x}{L}\right) \tag{4.5}$$

and by assumption,

$$J = \frac{B_0}{\mu_0 L}\,\mathrm{sech}^2\left(\frac{x}{L}\right) \tag{4.6}$$

Their plots are as below.

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