Homework 10: Phys 7310 (Fall 2021)

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***Problem* 10.1** (Solutions for $\mathbf{E}$ and $\mathbf{B}$ in electrodynamics)**:** The potentials Φ and $\mathbf{A}$ are often easier to deal with, but it is possible to work with the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ directly. In particular, they also obey wave equations with velocity parameter c.

(a) Starting from Maxwell's equations in vacuum,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0,$$
$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{10.1.1}$$

show that $\mathbf{E}$ and $\mathbf{B}$ obey wave equations with sources as given in Jackon (6.49)–(6.50), and therefore have the solutions (6.51)–(6.52). (Thiscan also be obtained starting with the solutions for the potentials (6.48), but I want you to derive it without the potentials.)

(b) The solutions (6.51)–(6.52) are correct, but it is useful to recast them in a form that more obviously reduces to the static forms. Fill in the steps to derive (6.53) and (6.54), where we recall $\big[f(\mathbf{x}',t')\big]_{\text{ret}} \equiv f(\mathbf{x}', t - R/c)$.

(c) Use the results of part (b) to derive the solutions (6.55) and (6.56) for $\mathbf{E}$ and $\mathbf{B}$. These are general solutions for the electric and magnetic fields produced by fixed sources $\rho(\mathbf{x}',t')$ and $\mathbf{J}(\mathbf{x}',t')$ in the fully dynamic case. Show that in the static limit these reduce to Coulomb's Law (1.5) and the Biot-Savart Law (5.14).

(d) Comment on the R-dependence of the new terms that only appear in the time-dependent case; do these fall off faster or more slowly than the static fields? These are *radiation fields*.

Solution.

(a) Taking the curl of Ampere's Law and using the identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}$, we can write

$$\frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right) = -\mu_0\frac{\partial\mathbf{J}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$
$$\Rightarrow \qquad \nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\frac{1}{\epsilon_0}\left(-\boldsymbol{\nabla}\rho - \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\right) \tag{10.1.2}$$

Similarly, taking the curl of Faraday's Law, we get $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B} = -\nabla^2\mathbf{B}$ because $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Thus

$$-\nabla^2\mathbf{B} = \mu_0\boldsymbol{\nabla}\times\mathbf{J} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} \Rightarrow \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu_0\boldsymbol{\nabla}\times\mathbf{J} \tag{10.1.3}$$

(b) First, when taking the gradient of $[\rho]_{\rm ret} = [\rho]_{\rm ret}(\mathbf{x}', t - R/c)$, we have to use the chain rule

$$
\begin{aligned}
\boldsymbol{\nabla}'[\rho]_{\rm ret} &= \frac{\partial[\rho]_{\rm ret}}{\partial\mathbf{x}'} + \frac{\partial[\rho]_{\rm ret}}{\partial t'}\frac{\partial t'}{\partial\mathbf{x}'} \\
&= \big[\boldsymbol{\nabla}'\rho\big]_{\rm ret} + \left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}\boldsymbol{\nabla}'\big(t - R/c\big) \\
&= \big[\boldsymbol{\nabla}'\rho\big]_{\rm ret} - \frac{1}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}\boldsymbol{\nabla}'R \\
&= \big[\boldsymbol{\nabla}'\rho\big]_{\rm ret} + \frac{\hat{\boldsymbol{R}}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}
\end{aligned} \tag{10.1.4}
$$

Thus, we arrive at (6.53, Jackson)

$$
\big[\boldsymbol{\nabla}'\rho\big]_{\rm ret} = \boldsymbol{\nabla}'[\rho]_{\rm ret} - \frac{\hat{\boldsymbol{R}}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret} \tag{10.1.5}
$$

Now, to derive (6.54, Jackson), we write the cross product in index notation

$$
\begin{aligned}
\boldsymbol{\nabla}' \times [\mathbf{J}]_{\rm ret} &= \epsilon_{ijk}\hat{\mathbf{e}}_i\partial_j J_k(\mathbf{x}', t - R/c) \\
&= \epsilon_{ijk}\hat{\mathbf{e}}_i\left[\partial_j J_k(\mathbf{x}', t') + \frac{\partial J_k}{\partial t'}\partial_j(t - R/c)\right] \\
&= \big[\boldsymbol{\nabla}'\mathbf{J}\big]_{\rm ret} + \epsilon_{ijk}\hat{\mathbf{e}}_i\partial_j t'\frac{\partial J_k}{\partial t'} \\
&= \big[\boldsymbol{\nabla}'\mathbf{J}\big]_{\rm ret} + \boldsymbol{\nabla}'(t - R/c) \times \left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret} \\
&= \big[\boldsymbol{\nabla}' \times \mathbf{J}\big]_{\rm ret} + \frac{\hat{\boldsymbol{R}}}{c} \times \left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}
\end{aligned} \tag{10.1.6}
$$

Thus,

$$
\big[\boldsymbol{\nabla}' \times \mathbf{J}\big]_{\rm ret} = \boldsymbol{\nabla}' \times [\mathbf{J}]_{\rm ret} + \frac{1}{c}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret} \times \hat{\boldsymbol{R}} \tag{10.1.7}
$$

(c) From (6.51, Jackson),

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{1}{R}\left[-\boldsymbol{\nabla}'\rho - \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret} \\
&= \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{1}{R}\left\{-\boldsymbol{\nabla}'[\rho]_{\rm ret} + \frac{\hat{\boldsymbol{R}}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret} - \frac{1}{c^2}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}\right\} \\
&= \frac{1}{4\pi\epsilon_0}\left\{-\int d^3x'\frac{1}{R}\boldsymbol{\nabla}'[\rho]_{\rm ret} + \int d^3x'\left\{\frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}\right\}\right\}
\end{aligned} \tag{10.1.8}
$$

Integrating the first term by parts, we get

$$\int d^3x' \frac{1}{R}\boldsymbol{\nabla}'[\rho]_{\text{ret}} = \int d^3x' \left[\boldsymbol{\nabla}'\left(\frac{[\rho]_{\text{ret}}}{R}\right) - [\rho]_{\text{ret}}\boldsymbol{\nabla}'\left(\frac{1}{R}\right)\right] = \oint \frac{[\rho]_{\text{ret}}}{R} d\mathbf{a}' - \int d^3x' [\rho]_{\text{ret}} \frac{\hat{\boldsymbol{R}}}{R^2} \tag{10.1.9}$$

The boundary term (surface integral) vanishes because $[\rho]_{\text{ret}}$ vanishes at ∞. So we can write the electric field as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{ \frac{[\rho]_{\text{ret}}}{R^2}\hat{\boldsymbol{R}} + \frac{\hat{\boldsymbol{R}}}{cR}\left[\frac{\partial \rho}{\partial t'}\right]_{\text{ret}} - \frac{1}{c^2 R}\left[\frac{\partial \mathbf{J}}{\partial t'}\right]_{\text{ret}} \right\} \tag{10.1.10}$$

Similarly, we can find the magnetic field by integrating by parts. From (6.52, Jackson),

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{R}\left[\boldsymbol{\nabla}' \times \mathbf{J}\right]_{\text{ret}} \\
&= \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{R}\left(\boldsymbol{\nabla}' \times [\mathbf{J}]_{\text{ret}} + \frac{1}{c}\left[\frac{\partial \mathbf{J}}{\partial t'}\right]_{\text{ret}} \times \hat{\boldsymbol{R}}\right) \\
&= \frac{\mu_0}{4\pi}\left\{\int d^3x' \left[\boldsymbol{\nabla}' \times \left(\frac{[\mathbf{J}]_{\text{ret}}}{R}\right) - \boldsymbol{\nabla}'\left(\frac{1}{R}\right) \times [\mathbf{J}]_{\text{ret}}\right] + \int d^3x' \left[\frac{\partial \mathbf{J}}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{\boldsymbol{R}}}{cR}\right\} \\
&= \frac{\mu_0}{4\pi}\int d^3x' \left\{-\frac{\hat{\boldsymbol{R}}}{R} \times [\mathbf{J}]_{\text{ret}} + \left[\frac{\partial \mathbf{J}}{\partial t'}\right]_{\text{ret}} \times \frac{\hat{\boldsymbol{R}}}{cR}\right\}
\end{aligned} \tag{10.1.11}$$

□

***Problem* 10.2** (Causality in Coulomb gauge)**:** In the Coulomb gauge, $\Phi(\mathbf{x}, t)$ is instantaneous (that is, it is determined by the behavior of $\rho(\mathbf{x}, t)$ at the same time), but $\mathbf{A}(\mathbf{x}, t)$ is causal (it is the solution of the wave equation). Causality of $\mathbf{B}$ follows from causality of $\mathbf{A}$ (you do not need to show this). Starting from equations for Φ and $\mathbf{A}$ in Coulomb gauge, show that $\mathbf{E}$ is also causal. What is the source of $\mathbf{E}$? (Hint: it should be the same as what you got in the previous problem.)

Solution.

Starting from the LHS of the wave equation for $\mathbf{A}$ (6.24, Jackson) in Coulomb gauge, we can use the definition of $\mathbf{E}$ and write

$$\begin{aligned}
\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= \nabla^2(-\boldsymbol{\nabla}\Phi - \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(-\boldsymbol{\nabla}\Phi - \mathbf{E}) \\
&= -\boldsymbol{\nabla}\left(\frac{\rho}{\epsilon_0}\right) - \nabla^2 \mathbf{E} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}
\end{aligned} \tag{10.2.1}$$

Equating this to the RHS of (6.24, Jackson), we can write

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t} \tag{10.2.2}$$

So $\mathbf{E}$ also follows the wave equation of the general form (6.32, Jackson), meaning it is causal. The RHS determines the source of $\mathbf{E}$ where the source distribution $f(\mathbf{x}, t)$ is

$$f(\mathbf{x}, t) = -\frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\rho - \frac{\mu_0}{4\pi}\frac{\partial \mathbf{J}}{\partial t} \tag{10.2.3}$$

□

***Problem* 10.3** (A conducting shell and the stress tensor)**:** A (perfectly) conducting spherical shell of radius a is placed in a uniform electric field $\mathbf{E}_0$. Find the force tending to separate the two halves of the sphere across a diametrical plane perpendicular to $\mathbf{E}_0$ in two ways:
(a) Using the stress tensor.
(b) By integrating the appropriate projection of $\sigma^2/2\epsilon_0$ over a hemisphere.
You may use old results for a conducting sphere in a uniform electric field.

Solution.
(a) From (2.15, Jackson), the electric field strength on the surface of the conductor is $|\mathbf{E}| = 3E_0 \cos\theta$. So by definition, the stress tensor is

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta - \frac{1}{2}E^2\delta_{\alpha\beta}\right] = \frac{1}{2}\epsilon_0 E^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{9}{2}\epsilon_0 E_0^2 \cos^2\theta \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.3.1}$$

Then by definition, the force on one hemisphere of the shell is

$$\begin{aligned} \mathbf{F} &= \int \mathbf{T}\cdot\hat{\mathbf{r}}da \\ &= \frac{9}{2}\epsilon_0 E_0^2 \int \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \cos^2\theta a^2 \sin\theta d\theta d\phi \\ &= \frac{9}{2}\epsilon_0 E_0^2 a^2 \int_0^{\pi/2} \sin\theta\cos^3\theta d\theta \int_0^{2\pi} d\phi\hat{\mathbf{z}} \\ &= \frac{9}{4}\pi a^2 \epsilon_0 E_0^2 \hat{\mathbf{z}} \end{aligned} \tag{10.3.2}$$

(b) By symmetry, the force has to be in the z direction. So using (2.15, Jackson),

$$F_z = \int \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}da = \frac{9}{2}\epsilon_0 E_0^2 a^2 \int_0^{\pi/2} \sin\theta\cos^3\theta d\theta \int_0^{2\pi} d\phi = \frac{9}{4}\pi a^2\epsilon_0 E_0^2 \tag{10.3.3}$$

This is the same result as (a). □