Homework 13: Phys 7310 (Fall 2021)

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December 8, 2021

***Problem* 13.1** (A wave packet)**:** An approximately monochromatic plane wave packet in one dimension has the instantaneous form, $u(x,0) = f(x)e^{ik_0x}$, with $f(x) = Ne^{-\alpha|x|/2}$ the modulation envelope. Calculate the wave-number spectrum $\left|A(k)\right|^2$ of the packet, sketch $\left|u(x,0)\right|^2$ and $\left|A(k)\right|^2$, evaluate explicitly the rms deviations from the means Δx and Δk (defined in terms of the intensity $\left|u(x,0)\right|^2$ and $\left|A(k)\right|^2$), and test inequality (7.82).

Solution.

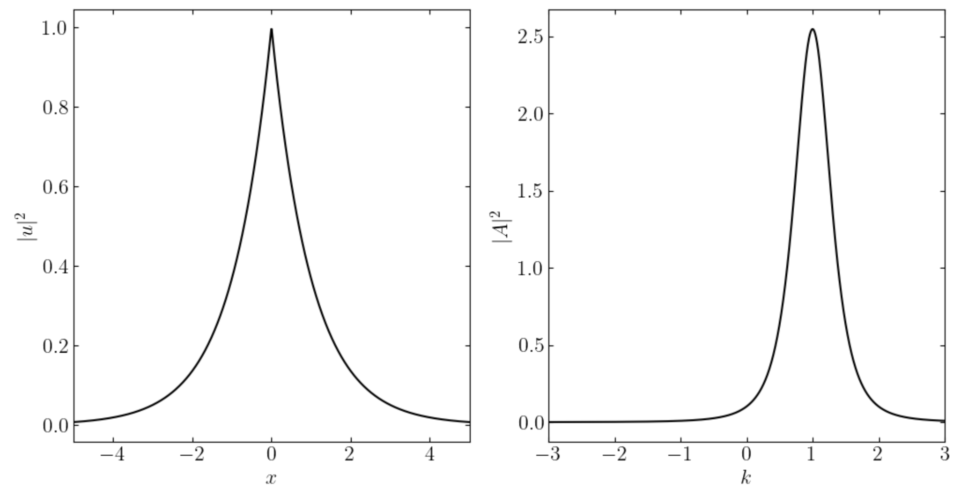
By definition,

$$\begin{aligned}
A(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-ikx}\left[u(x,0) + \frac{i}{\omega}\frac{\partial u}{\partial t}(x,0)\right] \\
&= \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-ikx}e^{-\alpha|x|/2}e^{ik_0x} \\
&= \frac{N}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} dx e^{i(k_0-k)x}e^{\alpha x/2} + \int_{0}^{\infty} dx e^{i(k_0-k)x}e^{-\alpha x/2}\right] \\
&= \frac{N}{\sqrt{2\pi}}\left[\frac{2i}{2(k-k_0)+i\alpha} + \frac{2}{2i(k-k_0)+\alpha}\right] \\
&= \frac{N}{\sqrt{2\pi}}\frac{4\alpha}{4(k-k_0)^2+\alpha^2} \qquad (13.1.1)
\end{aligned}$$

Then it follows that

$$\left|A(k)\right|^2 = \frac{8}{\pi}\frac{N^2\alpha^2}{[4(k-k_0)^2+\alpha^2]^2} \qquad (13.1.2)$$

For $N = \alpha = k_0 = 1$, we plot $|u|^2$ and $|A|^2$ as follows



For Δx, the kernel is already symmetric about $x = 0$. So by definition,

$$\left|x^2\right| = N^2 \int_{-\infty}^{\infty} dx x^2 e^{-\alpha|x|} = N^2 \left[\int_{-\infty}^{0} dx x^2 e^{\alpha x} + \int_{0}^{\infty} dx x^2 e^{-\alpha x}\right] = \frac{4N^2}{\alpha^3} \tag{13.1.3}$$

Similarly, we calculate

$$\mathcal{N}_x = \int_{-\infty}^{\infty} dx \left|u(x,0)\right|^2 = \int_{-\infty}^{0} dx e^{\alpha x} + \int_{0}^{\infty} dx e^{-\alpha x} = \frac{2N^2}{\alpha} \tag{13.1.4}$$

Then it follows that $\Delta x = \sqrt{|x^2|/\mathcal{N}_x} = \sqrt{2}/\alpha$. Now, since the kernel for k ($|A|^2$) is not symmetric, we can shift it by $k \mapsto k + k_0$ such that

$$\left|A(k)\right|^2 = \frac{8}{\pi} \frac{N^2 \alpha^2}{(4k^2 + \alpha^2)^2} \tag{13.1.5}$$

Then

$$\mathcal{N}_k = \frac{8N^2\alpha^2}{\pi} \int_{-\infty}^{\infty} \frac{dk}{(4k^2 + \alpha^2)^2} = \frac{2N^2}{\alpha} \tag{13.1.6}$$

and

$$\left\langle k^2 \right\rangle = \frac{8N^2\alpha^2}{\pi} \int_{-\infty}^{\infty} \frac{k^2 dk}{(4k^2 + \alpha^2)^2} = \frac{N^2 \alpha}{2} \tag{13.1.7}$$

Thus, $\Delta k = \sqrt{\langle k^2 \rangle / \mathcal{N}_k} = \alpha/2$. Then

$$\Delta x \Delta k = \frac{1}{\sqrt{2}} \approx 0.71 > \frac{1}{2} \tag{13.1.8}$$

So the inequality (7.82) is satisfied. $\square$

***Problem* 13.2** (A triangular waveguide)**:** A waveguide is constructed so that the cross section of the guide forms a right triangle with side of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \epsilon_r = 1$. Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

Solution.

First, the TE solution for a square waveguide from (8.42, Jackson) is

$$\psi = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) \tag{13.2.1}$$

which already satisfies the boundary conditions that $\partial\psi/\partial n = 0$ at $y = 0$ and $x = a$. Now, if the diagonal of the triangular waveguide is defined by the line $y = x$, then it must follow also that

$$\left.\frac{\partial\psi}{\partial n}\right|_{y=x} = \frac{1}{\sqrt{2}}(-\hat{\mathbf{x}}+\hat{\mathbf{y}})\cdot\boldsymbol{\nabla}\psi\Big|_{y=x} = \frac{1}{\sqrt{2}}\left(\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial x}\right)\Bigg|_{y=x} = 0 \tag{13.2.2}$$

or equivalently, that

$$\left.\frac{\partial\psi}{\partial x}\right|_{y=x} = \left.\frac{\partial\psi}{\partial y}\right|_{y=x} \tag{13.2.3}$$

This means the function ψ has to be symmetric under the transformation $x \mapsto y$ and $y \mapsto x$. However, (13.2.1) is not symmetric. But we can rewrite ψ as a linear combination

$$\psi = \frac{H_0}{\sqrt{2}}\left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right] \tag{13.2.4}$$

so that

$$\begin{aligned}\left(\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial x}\right)\Bigg|_{y=x} &\sim -\frac{n\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \frac{m\pi}{a}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\\ &\qquad + \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi}{a}\right) + \frac{n\pi}{a}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\\ &= 0\end{aligned} \tag{13.2.5}$$

where the last equality is true only at $y = x$. Thus, the TE solution for the triangular waveguide is (13.2.4). Then from (8.34, Jackson), we can find

$$\nabla_t^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2}\right)\psi = -\gamma_{mn}^2\psi \tag{13.2.6}$$

Thus, $\gamma_{mn} = (\pi/a)\sqrt{m^2+n^2}$. The allowed modes are $m, n \geq 0$, but m and n cannot be both zero because ψ is then not periodic. The cutoff frequency is thus defined by $(m = 0, n = 1)$ or $(m = 1, n = 0)$

$$\omega_{TE} = \frac{\gamma_{10}}{\sqrt{\mu_0\epsilon_0}} = \frac{\gamma_{01}}{\sqrt{\mu_0\epsilon_0}} = \frac{c\pi}{a} \tag{13.2.7}$$

Now, regarding the TM solution for the square waveguide, $\psi = E_z$ has to vanish at $x = 0, a$ and $y = 0, a$. So instead of cosine function, ψ is

$$\psi = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \tag{13.2.8}$$

Now, at the diagonal $y = x$, it must follow also that

$$\psi\Big|_{y=x} = 0 \tag{13.2.9}$$

This condition requires that ψ must be antisymmetric, instead. But ψ as is, is not antisymmetric. So we can rewrite ψ as a linear combination

$$\psi = \frac{E_0}{\sqrt{2}}\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right] \tag{13.2.10}$$

so that at $y = x$,

$$\psi \sim \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = 0 \tag{13.2.11}$$

Thus, (13.2.10) is the TM solution for the triangular waveguide. Similar to the TE case, from (8.34, Jackson),

$$\nabla_t^2\psi = -\frac{\pi^2}{a^2}(m^2 + n^2)\psi = -\gamma_{mn}^2\psi \tag{13.2.12}$$

Thus, γ_{mn} is still the same as the TE case. However, m, n are not allowed to be zero because that leads to a trivial solution. Also, $m \neq n$ because that also leads to a trivial solution due to the antisymmetry of the function if $m = n$. The ground case is then $(m = 1, n = 2)$ or $(m = 2, n = 1)$, in which the cutoff frequency is

$$\omega_{TM} = \frac{\gamma_{12}}{\sqrt{\mu_0\epsilon_0}} = \frac{\gamma_{21}}{\sqrt{\mu_0\epsilon_0}} = \frac{c\pi}{a}\sqrt{5} \tag{13.2.13}$$

$\square$