Homework 1: Phys 5040 (Spring 2022)

Tien Vo

Due: January 27, 2022

***Problem* 1:** Prove that a subgroup $H \subset G$ is normal iff $gH = Hg$ for all $g \in G$. (That is, you are showing here that normal subgroups are those for which left and right cosets agree.)

Solution.

$\square$

***Problem* 2:** Explain why the following is true: for a collection of sets denoted A_i, there is no natural function from one of the A_i's to the product set, but for a collection of groups G_i, there is a natural injective homomorphism from each G_i to the product group. (The key here is the rather loose meaning of "natural".) In the latter case you should write down what this injective homomorphism is.

Solution.

$\square$

***Problem* 3:** Let G be a finite group with $|G| = p$, for p a prime number. Prove that G is isomorphic to $\mathbb{Z}_p$.

Solution.

$\square$

***Problem* 4:** Let A be a set and $\mathrm{Perm}(A)$ be the group of invertible functions from $A \to A$. That is $\mathrm{Perm}(A)$ is the subset of $\mathrm{Fun}(A, A)$ consisting of invertible functions, and is a group as discussed in the notes.

Now let G be a group. Prove that a group action of G on A determines a natural homomorphism $G \to \mathrm{Perm}(A)$, and vice versa.

Yes, the word natural is not precise so you aren't aasked to prove naturalness – the point is that part of this problem is to figure out what the homomorphism is, and it should be natural in the sense that it should "do the same thing as the group action". You have to prove that what you find is in fact a homomorphism. Note you have to go both ways here: (1) Given a group action, find a natural homomorphism and prove it's a homomorphism. (2) Given a homomorphism from $G \to \mathrm{Perm}(A)$, obtain a natural group action (proving it's in fact a group action).

Remarks (you don't have to do anything with this except read it): This result is helpful if you want to generaize the action of a group on a set to a group acting on a set with some additional structure. For instance, an action of a group G on another group H can be defined simply as a homomorphism $G \to \mathrm{Aut}(H)$. The idea is that we replaced $\mathrm{Perm}(H)$ with $\mathrm{Aut}(H)$, because these are invertible functions from H to itself that know about and respect the group structure on H.

Similarly, the action of a group G on a topological space X is a homomorphism $G \to \mathrm{Homeo}(X)$, the group of homeomorphisms from X to itself. (If the group G is a topological group, then we additionally require that the map $G \times X \to X$ is continuous in the product topology on $G \times X$).

Solution.

□

***Problem* 5:** Let G be a group and $H \subset G$ a subgroup (not necessarily normal). Let G/H be the set of left cosets. Show that G/H is naturally a left G-set (i.e. it has a natural group action of G, acting on the left). Does G/H have a natural G-right action? Explain why or why not.

Solution.

□

***Problem* 6:** Take $m, n \in \mathbb{Z}$ with m and n relatively prime (this means m and n do not share any prime factors). Prove that $Z_{m,n} \cong Z_m \times \mathbb{Z}_n$.

Solution.

□

***Problem* 7:** Prove Proposition 2.1.7 in the lecture notes (i.e. prove the two definitions of continuity given in Section 2.1 are equivalent). Here you should only use results from the notes in 2.1 or earlier.

Solution.

□