Homework 3: Phys 7310 (Fall 2021)

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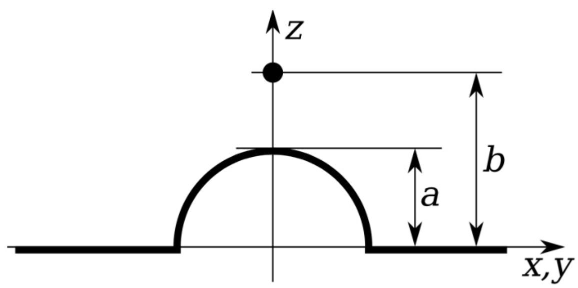
***Problem* 3.1** (A plate with a half-sphere)**:**
An infinite grounded conducting plate has a bulge in form of a haf-sphere with radius a. A



point charge q is put on the symmetry axis of the system at a distance $b > a$ from the center of the half-sphere (see figure). Using the method of image, calculate the potential Φ above the plate, the force on the point charge q, and the total charge induced on the half-sphere.

***Problem* 3.2** (A line charge)**:** In this problem, we consider an infinite line with a constant linear charge density λ (charge per unit length). Let the line charge sit at the origin $(x, y) = (0, 0)$ in the xy-plane, and stretch infinitely in both directions along the z-axis.

Draw an appropriate Gaussian surface and find the electric field a distance r away from the line charge. Show that a potential for this electric field takes the form

$$\Phi(\mathbf{x}) = \alpha \ln \frac{r}{R} \tag{3.2.1}$$

where α is a constant you should determine in terms of λ and other things, and R is a *totally arbitrary* constant with units of length that we include to make the units in the log dimensionless; explain why R drops out of physically measurable quantities. Finally, if there is another line charge with linear charge density λ' a distance r away from the first charge, find the force per unit length experienced by this second line charge.

***Problem* 3.3** (Line charges and images)**:** A straight line charge with constant linear charge density λ is located perpendicular to the xy-plane in the first quadrant at (x_0, y_0). The intersecting planes $x = 0$, $y \geq 0$, and $y = 0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x,y) = (\lambda/4\pi\epsilon_0)\ln\bigl(R^2/r^2\bigr)$ where $r^2 = (x-x_0)^2 + (y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential eectric field vanish on the boundary surfaces.

(b) Determine the surface charge density σ on the plane $y = 0, x \geq 0$. Plot σ/λ versus x for $(x_0 = 2, y_0 = 1), (x_0 = 1, y_0 = 1), (x_0 = 1, y_0 = 2)$.

(c) Show that the total charge (per unit length in z) on the plane $y = 0, x \geq 0$ is

$$Q_x = -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right) \tag{3.3.1}$$

What is the total charge on the plane $x = 0$?

(d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{x^2+y^2}$ and $\rho_0 = \sqrt{x_0^2+y_0^2}$] the leading term in the potential is

$$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4} \tag{3.3.2}$$

Interpret.

***Problem* 3.4** (Green's function in Cartesian coordinates)**:** (a) Show that the Green function $G(x,y;x',y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 \leq x \leq 1, 0 \leq y \leq 1$, has an expansion

$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin\bigl(n\pi x'\bigr) \tag{3.4.1}$$

where $g_n(y,y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right)g_n(y,y') = -4\pi\delta(y'-y) \qquad \text{and} \qquad g_n(y,0) = g_n(y,1) = 0 \tag{3.4.2}$$

(b) Taking for $g_n(y,y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions, $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin\bigl(n\pi x'\bigr)\sinh(n\pi y_<)\sinh\left[n\pi(1-y_>)\right] \tag{3.4.3}$$

where $y_<(y_>)$ is the smaller (larger) of y and y'.