Homework 10: Phys 5210 (Fall 2021)

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***Problem* 1** (Goldstein, 9.30)**:** (a) Prove that the Poisson bracket of two constants of the motion is itself a constant of motion even if the constants are explicitly dependent on time. Constants of motions are two functions of position, momenta and time, $u(q,p,t)$ and $v(q,p,t)$ such that

$$\frac{du}{dt} = \{\mathcal{H}, u\} + \frac{\partial u}{\partial t} = 0, \qquad \frac{dv}{dt} = \{\mathcal{H}, v\} + \frac{\partial v}{\partial t} = 0 \tag{1.1}$$

(b) Show that if both the Hamiltonian $\mathcal{H}$ and a quantity $u(q,p,t)$ are constants of motion, the nth partial derivative of u with respect to time must also be a constant of motion.

(An illustration of this result is the motion of a free particle with $\mathcal{H} = p^2/2m$) where there exists a constant of motion $x - pt/m$, whose derivative with respect to time t is also a constant of motion.

Solution.

(a) First, since $\{u, v\}$ is also a function of q, p, and t, we can write

$$\frac{d}{dt}\{u, v\} = \big\{\mathcal{H}, \{u, v\}\big\} + \frac{\partial}{\partial t}\{u, v\} \tag{1.2}$$

Now, the Poisson bracket has a cyclic property

$$\big\{\mathcal{H}, \{u, v\}\big\} + \big\{v, \{\mathcal{H}, u\}\big\} - \big\{u, \{\mathcal{H}, v\}\big\} = 0 \tag{1.3}$$

So we can write

$$\big\{\mathcal{H}, \{u, v\}\big\} = \big\{u, \{\mathcal{H}, v\}\big\} - \big\{v, \{\mathcal{H}, u\}\big\} = \left\{v, \frac{\partial u}{\partial t}\right\} - \left\{u, \frac{\partial v}{\partial t}\right\} \tag{1.4}$$

from (1.1). Now, by definition of the Poisson bracket,

$$\begin{aligned}\frac{\partial}{\partial t}\{u, v\} &= \frac{\partial}{\partial q}\left(\frac{\partial u}{\partial t}\right)\frac{\partial v}{\partial p} + \frac{\partial u}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial t}\right) - \frac{\partial}{\partial p}\left(\frac{\partial u}{\partial t}\right)\frac{\partial v}{\partial q} - \frac{\partial u}{\partial p}\frac{\partial}{\partial q}\left(\frac{\partial v}{\partial t}\right) \\ &= \left\{\frac{\partial u}{\partial t}, v\right\} + \left\{u, \frac{\partial v}{\partial t}\right\}\end{aligned} \tag{1.5}$$

Thus,

$$\frac{d}{dt}\{u, v\} = \left\{v, \frac{\partial u}{\partial t}\right\} - \left\{u, \frac{\partial v}{\partial t}\right\} - \left\{v, \frac{\partial u}{\partial t}\right\} + \left\{u, \frac{\partial v}{\partial t}\right\} = 0 \tag{1.6}$$

(b) Given that $d\mathcal{H}/dt = \partial\mathcal{H}/\partial t = 0$ and $du/dt = 0$, we can write

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial^n u}{\partial t^n}\right) &= \left\{\mathcal{H}, \frac{\partial^n u}{\partial t^n}\right\} + \frac{\partial^{n+1}}{\partial t^{n+1}}u \\
&= \frac{\partial^n}{\partial t^n}\{\mathcal{H}, u\} + \frac{\partial^{n+1}}{\partial t^{n+1}}u \\
&= -\frac{\partial^{n+1}}{\partial t^{n+1}}u + \frac{\partial^{n+1}}{\partial t^{n+1}}u \\
&= 0
\end{aligned} \tag{1.7}$$

□

Problem 2 (Goldstein, 10.16)**:** A particle of mass m is constrained to move on a curve in the vertical plane defined by the parametric equations

$$x = l(2\phi + \sin(2\phi)), \qquad y = l(1 - \cos(2\phi)) \tag{2.1}$$

There's the usual constant gravitational force acting in the vertical y direction. By the method of action-angle variable, find the frequency of oscillations for all initial conditions such that the maximum of ϕ is less than or equal to $\pi/2$.

Solution.

Given (2.1), the velocity is

$$\dot{x} = 2l(1 + \cos 2\phi)\dot{\phi} \qquad \text{and} \qquad \dot{y} = 2l \sin 2\phi\dot{\phi} \tag{2.2}$$

Then the Lagrange function is

$$\begin{aligned}
\mathcal{L} &= 2ml^2\dot{\phi}^2\left[(1 + \cos 2\phi)^2 + \sin^2\phi\right] - mgl(1 - \cos 2\phi) \\
&= 4ml^2\dot{\phi}^2(1 + \cos 2\phi) - mgl(1 - \cos 2\phi) \\
&= 8ml^2\dot{\phi}^2\cos^2\phi - 2mgl\sin^2\phi
\end{aligned} \tag{2.3}$$

The canonical momentum is then

$$p = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 12ml^2\dot{\phi}\cos^2\phi \tag{2.4}$$

Then the Hamiltonian is

$$\begin{aligned}
\mathcal{H} &= p\dot{\phi} - 8ml^2\dot{\phi}^2\cos^2\phi + 2mgl\sin^2\phi \\
&= \frac{p^2}{16ml^2\cos^2\phi} - 8ml^2\cos^2\phi\frac{p^2}{256m^2l^4\cos^4\phi} + 2mgl\sin^2\phi \\
&= \frac{p^2}{32ml^2\cos^2\phi} + 2mgl\sin^2\phi
\end{aligned} \tag{2.5}$$

Since $\mathcal{H}$ is time-independent, we can also write $\mathcal{H} = E = 2mgl$ where E is the energy at the point where $\phi = \phi_{\text{max}} = \pi/2$ and $\dot{\phi} = 0$. Then, define the new adiabatic invariant

$$
\begin{aligned}
P &= 4\int_0^{\pi/2} p d\phi \\
&= 4\int_0^{\pi/2} d\phi \sqrt{32ml^2\cos^2\phi(E - 2mgl\sin^2\phi)} \\
&= 4\sqrt{32ml^2E}\int_0^{\pi/2} d\phi\cos\phi\sqrt{1-\sin^2\phi} \\
&= 4\sqrt{2}\pi\sqrt{ml^2E}
\end{aligned}
\tag{2.6}
$$

Then it follows that

$$
\mathcal{H} = \frac{1}{32\pi^2}\frac{P^2}{ml^2} \qquad \text{and} \qquad P = 8\pi ml\sqrt{gl} \tag{2.7}
$$

Then the new coordinate Φ conjugate to P is cyclic and the frequency is

$$
\dot{\Phi} = \frac{\partial\mathcal{H}}{\partial P} = \frac{1}{16\pi^2}\frac{P}{ml^2} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \tag{2.8}
$$

□