Homework 4: Astr 5140 (Fall 2021)

Tien Vo

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***Problem* 1** (Linearization)**:** The idea behind "linearization" is that each of the MHD quantities, $\rho, \mathbf{u}, P$, and $\mathbf{B}$ can be broken into two parts

$$\rho = \rho_0 + \rho_1, \qquad P = P_0 + P_1, \qquad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \qquad \mathbf{u} = \mathbf{u}_1 \tag{1.1}$$

where the subscript '0' indicates a constant background value and the subscript '1' indicates a small perturbation, e.g., $|\rho_1| \ll \rho_0$. We work in a uniform plasma in a frame with $\mathbf{u}_0 = \mathbf{0}$. Furthermore, it is assumed that the perturbations are oscillatory:

$$\rho_1, P_1, \mathbf{B}_1, \mathbf{u}_1 \sim e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{1.2}$$

Linearization implies that any 2nd order term can be neglected, for example, $\rho_1 \mathbf{u}_1 = \mathbf{0}$, and $\mathbf{u}_1 \cdot \mathbf{B}_1 = 0$.

(a) Show that the continuity equation can be written as

$$\omega \rho_1 = \rho_0 \mathbf{k} \cdot \mathbf{u}_1 \tag{1.3}$$

(b) Show that the equations of state, $P \sim \rho^\gamma$ and $P = nT$ (T is considered a constant and is in units of energy), imply that

$$\boldsymbol{\nabla} P = \frac{i\mathbf{k}\gamma T \rho_1}{m} \tag{1.4}$$

(c) Show that the force equation becomes

$$\omega \rho_0 \mathbf{u}_1 = \mathbf{k}\frac{\gamma T}{m}\rho_1 + \mathbf{k}\frac{\mathbf{B}_0 \cdot \mathbf{B}_1}{\mu_0} - \frac{\mathbf{k} \cdot \mathbf{B}_0}{\mu_0}\mathbf{B}_1 \tag{1.5}$$

(d) Show that generalized Ohm's law, $\partial \mathbf{B}/\partial t = \boldsymbol{\nabla} \times \mathbf{u} \times \mathbf{B}$ becomes

$$\omega \mathbf{B}_1 = \mathbf{B}_0(\mathbf{k} \cdot \mathbf{u}_1) - (\mathbf{k} \cdot \mathbf{B}_0)\mathbf{u}_1 \tag{1.6}$$

Solution.

(a) The continuity equation is

$$\frac{\partial(\rho_0+\rho_1)}{\partial t}+\boldsymbol{\nabla}\cdot(\rho_0\mathbf{u}_1+\rho_1\mathbf{u}_1)=0\Rightarrow\frac{\partial\rho_1}{\partial t}+\rho_0\boldsymbol{\nabla}\cdot\mathbf{u}_1=0 \tag{1.7}$$

where we have omitted second order terms. Now, letting $\partial_t\to -i\omega$ and $\boldsymbol{\nabla}\to i\mathbf{k}$ yields

$$\omega\rho_1=\rho_0\mathbf{k}\cdot\mathbf{u}_1 \tag{1.8}$$

as desired.

(b) Given that $P\sim\rho^\gamma$, we can write

$$P=C\rho^\gamma=nT \tag{1.9}$$

where C is a constant. Taking the gradient yields

$$\boldsymbol{\nabla}P=\gamma C\rho^{\gamma-1}\boldsymbol{\nabla}\rho=\frac{\gamma nT}{\rho}\boldsymbol{\nabla}\rho=\frac{\gamma T}{m}\boldsymbol{\nabla}\rho \tag{1.10}$$

where we have written $\rho=nm$. Since $\boldsymbol{\nabla}P_0=\boldsymbol{\nabla}\rho_0=0$, this becomes

$$\boldsymbol{\nabla}P_1=\frac{i\mathbf{k}\gamma T\rho_1}{m} \tag{1.11}$$

where we have also substituted $\boldsymbol{\nabla}\to i\mathbf{k}$.

(c) The force equation is

$$\begin{aligned}
&\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=-\boldsymbol{\nabla}P-\boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right)+\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B}}{\mu_0}\\
\Rightarrow\quad&(\rho_0+\rho_1)\frac{\partial\mathbf{u}_1}{\partial t}=-\frac{i\mathbf{k}\gamma T}{m}\rho_1-\boldsymbol{\nabla}\left[\frac{(\mathbf{B}_0+\mathbf{B}_1)\cdot(\mathbf{B}_0+\mathbf{B}_1)}{2\mu_0}\right]+\frac{(\mathbf{B}_0+\mathbf{B}_1)\cdot\boldsymbol{\nabla}\mathbf{B}_1}{\mu_0}\\
\Rightarrow\quad&\rho_0\frac{\partial\mathbf{u}_1}{\partial t}=-\frac{i\mathbf{k}\gamma T}{m}\rho_1-\boldsymbol{\nabla}\left(\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right)+\frac{\mathbf{B}_0\cdot\boldsymbol{\nabla}\mathbf{B}_1}{\mu_0}
\end{aligned} \tag{1.12}$$

where we have used the result from part (c) and eliminated high order terms. Now, letting $\partial_t\to-i\omega$ and $\boldsymbol{\nabla}\to i\mathbf{k}$ yields

$$\omega\rho_0\mathbf{u}_1=\mathbf{k}\frac{\gamma T}{m}\rho_1+\mathbf{k}\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}-\frac{\mathbf{k}\cdot\mathbf{B}_0}{\mu_0}\mathbf{B}_1 \tag{1.13}$$

(d) Similarly, from Ohm's law, we can write

$$-i\omega\mathbf{B}_1=i\mathbf{k}\times(\mathbf{u}_1\times\mathbf{B}_0)\Rightarrow\omega\mathbf{B}_1=-(\mathbf{k}\cdot\mathbf{B}_0)\mathbf{u}_1+(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{B}_0 \tag{1.14}$$

where we have used the vector calculus identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$. $\square$

***Problem* 2** (Oblique Alfvén wave)**:** Derive the dispersion relation of an oblique Alfvén wave in a homogeneous plasma. Let $\mathbf{B}_0$ lie in the z direction, $\mathbf{B}_1$ and $\mathbf{u}_1$ in the x direction, and have $\mathbf{k}$ in the zy plane. What direction is the group velocity ($d\omega/d\mathbf{k}$)? You may start with linearized equations that have been derived in class.

Solution.

Let $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, $\mathbf{B}_1 = B_1\hat{\mathbf{x}}$, $\mathbf{u}_1 = u_1\hat{\mathbf{x}}$, and $\mathbf{k} = k(\sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$. Then from (1.14), in the x direction,

$$\omega B_1 = -kB_0\cos\theta u_1 \tag{2.1}$$

Also, from (1.13) in the x direction,

$$\omega\rho_0 u_1 = -\frac{kB_0\cos\theta}{\mu_0}B_1 = \frac{k^2B_0^2\cos^2\theta}{\mu_0\omega}u_1 \Rightarrow \frac{\omega^2}{k^2} = \frac{B_0^2}{\mu_0\rho_0}\cos^2\theta = V_A^2\cos^2\theta \tag{2.2}$$

Or we can also write $\omega = k_zV_A$. Then it is apparent that the group velocity is

$$\mathbf{v}_g = \frac{d\omega}{d\mathbf{k}} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial k_x} + \hat{\mathbf{y}}\frac{\partial}{\partial k_y} + \hat{\mathbf{z}}\frac{\partial}{\partial k_z}\right)\omega = V_A\hat{\mathbf{z}} \tag{2.3}$$

$\square$

***Problem* 3** (Rayleigh-Taylor instability)**:** Derive the growth rate of the Rayleigh-Taylor instability for k_y perpendicular to B_z and constant gravity g_x in the $-x$ direction. Assume the plasma is incompressible. Use the linearized, Fourier-transformed, ideal MHD equations

$$i\omega\rho_1 = \mathbf{u}\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\mathbf{u}_1 \tag{3.1a}$$
$$i\omega P_1 = \mathbf{u}\cdot\boldsymbol{\nabla}P_0 + \gamma P_0\boldsymbol{\nabla}\cdot\mathbf{u}_1 \tag{3.1b}$$
$$i\omega\mathbf{B}_1 = \mathbf{u}_1\cdot\boldsymbol{\nabla}\mathbf{B}_0 + \mathbf{B}_0\boldsymbol{\nabla}\cdot\mathbf{u}_1 \tag{3.1c}$$

For the momentum equation, remove the tension term and expand

$$-i\omega\rho_0\mathbf{u}_1 = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \rho\mathbf{g} = -\boldsymbol{\nabla}P_1 - \boldsymbol{\nabla}\left(\frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) + \rho_1\mathbf{g} \tag{3.2}$$

Solution.

Given the momentum equation (3.2), we can write in the x and y directions

$$\rho_0\frac{\partial u_x}{\partial t} = -\frac{\partial}{\partial x}\left(P_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) + \rho_1 g \tag{3.3a}$$
$$\rho_0\frac{\partial u_y}{\partial t} = -\frac{\partial}{\partial y}\left(P_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) = -ik_y\left(P_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}\right) \tag{3.3b}$$

Substituting (3.3b) into (3.3a), we get

$$\rho_0 \frac{\partial u_x}{\partial t} = -\frac{\partial}{\partial x}\left(\rho_0 \frac{\partial u_y}{\partial t}\frac{1}{-ik_y}\right) + \rho_1 g \tag{3.4}$$

Since the plasma is incompressible, $\partial u_x/\partial x + ik_y u_y = 0$. Plugging this into (3.4) and performing a temporal Fourier transform, we can write

$$\omega^2 \rho_0 u_x = \rho_0 \frac{\omega^2}{k_y^2}\frac{\partial^2 u_x}{\partial x^2} + g u_x \frac{\partial \rho_0}{\partial x} \tag{3.5}$$

At large k_y, the first term vanishes, then the dispersion relation becomes

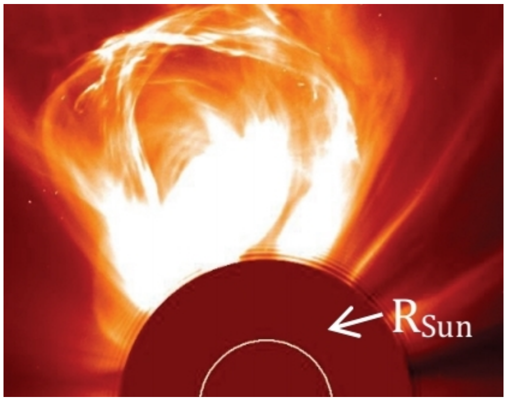
$$\omega^2 = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial x} \tag{3.6}$$

Note that the term $\partial\rho_0/\partial x$ is negative, since the density decreases with the height. So ω is purely imaginary. The growth rate is thus

$$\gamma = \sqrt{\left|\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial x}\right|} \tag{3.7}$$

□

Problem 4 (Flux ropes and CMEs)**:** A very large magnetic flux rope, 4000 km in radius, is formed just below the Sun's surface so that there is virtually no material in it. The particle pressure surrounding the flux rope is 10^5 Pa (1 Pa=1 N/m^2).



(a) What is the magnetic field strength of the flux rope required for local pressure balance?
(b) The buoyancy of the flux rope causes it to rise. The flux rope breaks through the Sun's surface and rises high into the corona. The surrounding pressure drops to 10^{-5} Pa. What is the magnetic field strength at this point?
(c) What is the radius of the flux rope when it is high in the corona? Hint: Magnetic flux, BA where A is area, is conserved. How does its size compare to the radius of the Sun. $R_{\text{sun}} = 7 \times 10^5$ km.

(d) The flux rope breaks out into the solar wind (see image). The surrounding pressure is 10^{-7} Pa. What is the magnetic field of the flux rope?

(e) What is the radius of the flux rope as it breaks out into the solar wind?

(f) Examine the CME image. Estimate the size of the CME based on the solar radius. Does your calculation of the size of this flux rope match what is seen in the CME image?

Solution.

(a) The pressure internal and external to the flux rope is balanced. So we have

$$P_{\text{ext}} = \frac{B^2}{2\mu_0} \Rightarrow B = \sqrt{2\mu_0 P_{\text{ext}}} \approx 0.5\,\text{T} \tag{4.1}$$

(b) Using $P_{\text{ext}} = 10^{-5}$ Pa, the magnetic field at this point is $B \approx 5\,\mu\text{T}$.

(c) The magnetic flux is conserved, so we can calculate the radius high in the corona

$$B_{\text{low}}\pi R_{\text{low}}^2 = B_{\text{high}}\pi R_{\text{high}}^2 \Rightarrow R_{\text{high}} = R_{\text{low}}\sqrt{\frac{B_{\text{low}}}{B_{\text{high}}}} \approx 1.27\times10^6\,\text{km} \approx 1.8R_{\text{sun}} \tag{4.2}$$

(d) Similar to part (b), the magnetic field is 501 nT.

(e) From part (c), the radius as it breaks out into the solar wind is

$$R_{sw} = R_{\text{low}}\sqrt{\frac{B_{\text{low}}}{B_{\text{sw}}}} \approx 5.7R_{\text{sun}} \tag{4.3}$$

(f) This is on the same order of magnitude as the radius of the sun. So it is comparable to the size seen in the image ($\approx 3-4R_{\text{sun}}$). □