Homework 12: Phys 7320 (Spring 2022)

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Due: April 20, 2022

***Problem* 12.1** (The Lagrangian for a charged particle)**:** The Lagrangian for a charged particle of mass m and charge e with position $\mathbf{r}$ and velocity $\mathbf{u} \equiv d\mathbf{r}/dt$ moving in scalar and vector potentials Φ and $\mathbf{A}$ is

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{u^2}{c^2}} - e\Phi + \frac{e}{c}\mathbf{u} \cdot \mathbf{A}. \tag{12.1.1}$$

(a) Show that the Euler-Lagrange equations for this Lagrangian indeed give rise to the Lorentz force law. *Hint*: I suggest using index notation, and remember the potentials $\Phi, \mathbf{A}$ can depend on both $\mathbf{r}$ and t, which as far as the particle is concerned means they depend on time in two ways: $\Phi(\mathbf{r}(t), t)$ and $\mathbf{A}(\mathbf{r}(t), t)$.

(b) Go though the steps (12.13)-(12.17) in Jackson to derive the Hamiltonian

$$\mathcal{H} = \sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2c^4} + e\Phi. \tag{12.1.2}$$

Along the way derive the canonical/conjugate momentum $\mathbf{P}$ (which is different from the familiar "kinematic momentum" $\mathbf{p} = \gamma m\mathbf{u}$), and invert it to find an expression for $\mathbf{u}$ in terms of $\mathbf{P}$ and $\mathbf{A}$.

If you want extra practice, you can show that Hamilton's equations for this Hamiltonian also give rise to the Lorentz force law, with some steps similar to part (a).

Solution.

□

***Problem* 12.2** (Equivalent Lagrangians)**:** (a) Use the Principle of Least Action (really the principle of extremal action) to show that if the Lagrangian $\mathcal{L}$ is changed by adding the time derivative of some function of the coordinates and time, then the Euler-Lagrange equations are unchanged. The new and old Lagrangians are said to be *equivalent*. Generalize this to a statement about what change to a Lagrangian *density* $\mathcal{L}$ leaves the EL equations unchanged.

(b) Show that under a gauge transformation, the Lagrangian for a charged particle given in the previous problem becomes an equivalent Lagrangian, thus showing the equations of motion do not change.

Solution.

$\square$

***Problem* 12.3** ($SO(2)$ symmetry of two real scalar fields.)**:** Consider the dynamics of two (real) scalar fields $\phi_1(\mathbf{x}, t)$ and $\phi_2(\mathbf{x}, t)$ specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2), \tag{12.3.1}$$

where the potential V depends only on the combination $\phi_1^2 + \phi_2^2$. In class, we will study this case with the real scalars combined into a single compex scalar; here we will leave them as two real scalars. Let's make a definite choice for the potential:

$$V(\phi_1, \phi_2) = \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}m^2\phi_2^2 + \frac{\lambda}{2}\left(\phi_1^2 + \phi_2^2\right)^2. \tag{12.3.2}$$

(a) Calculate the equations of motion (Euler-Lagrange equations) for both ϕ_1 and ϕ_2.
(b) Show that the $SO(2)$ transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{12.3.3}$$

where α is a constant, is a symmetry of the Lagrangian. Is this a rotation in physical space? What space does this "rotation" act on?

(c) According to Noether's theorem, the existence of this symmetry means there is a corresponding conserved current J^μ. Find J^μ in terms of ϕ_1 and ϕ_2 (you may drop an overall constant α) and show that it is conserved, $\partial_\mu J^\mu = 0$, when you use the equations of motion.

Solution.

□