Homework 2: Astr 5140 (Fall 2021)

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***Problem* 1** ($\mathbf{J} \times \mathbf{B}$ force)**:** Using the two-fluid equations, calculate the Lorentz force per unit volume on a quasi-neutral (MHD) plasma using the definition of $\mathbf{J}$. Separate the electric force ($\mathbf{F}_E$) from the magnetic force ($\mathbf{F}_B$). Show that if the plasma is quasi-neutral, then $\mathbf{F}$ reduces to the standard MHD result.

Solution.

The force equation per unit volume on ions and electrons is

$$n_i m_i \frac{\partial \mathbf{v}_i}{\partial t} = n_i e(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) \tag{1.1a}$$

$$n_e m_e \frac{\partial \mathbf{v}_e}{\partial t} = -n_e e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) \tag{1.1b}$$

Thus, the total force (from the RHS) is $\mathbf{F} = n_i m_i \partial \mathbf{v}_i/\partial t + n_e m_e \partial \mathbf{v}_e/\partial t = \mathbf{F}_E + \mathbf{F}_B$ where

$$\mathbf{F}_E = e(n_i - n_e)\mathbf{E} \qquad \text{and} \qquad \mathbf{F}_B = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e) \times \mathbf{B} = \mathbf{J} \times \mathbf{B} \tag{1.2}$$

where we have used the definition of $\mathbf{J} = e(n_i \mathbf{v}_i - n_e \mathbf{v}_e)$. If the plasma is quasineutral, $n_i = n_e = n$ and the electric force $\mathbf{F}_E = \mathbf{0}$. We can thus write the magnetic force $\mathbf{F}_B$ from (1.1) as

$$\mathbf{F}_B = \mathbf{J} \times \mathbf{B} = \frac{\partial}{\partial t}\big[n(m_i \mathbf{v}_i + m_e \mathbf{v}_e)\big] = \frac{\partial(\rho \mathbf{u})}{\partial t} \tag{1.3}$$

where $\mathbf{u}$ is the one-fluid flow velocity and $\rho = n_i m_i + n_e m_e$ is the average mass in the standard MHD result. $\square$

***Problem* 2** (EM review: Waves in a plasma)**:** Using Maxwell's equations and setting the current to be the electron motion only

$$\frac{\partial \mathbf{J}}{\partial t} = \frac{ne^2}{m_e}\mathbf{E} \tag{2.1}$$

Show that the solution of a transverse light wave becomes:

$$\omega^2 = \omega_{pe}^2 + k^2 c^2 \tag{2.2}$$

where $\omega_{pe}^2 = ne^2/\epsilon_0 m_e$.

Solution.

Differentiating Ampere's Law with time, we get

$$-\nabla \times (\nabla \times \mathbf{E}) = \nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu_0 \frac{\partial J}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{ne^2}{\epsilon_0 m_e}\mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.3}$$

where we have used (2.1). For a Fourier transformed electric field $\mathbf{E}$, $\nabla \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$. If we also assume the field is transverse ($\mathbf{k} \cdot \mathbf{E} = 0$), then the above result becomes

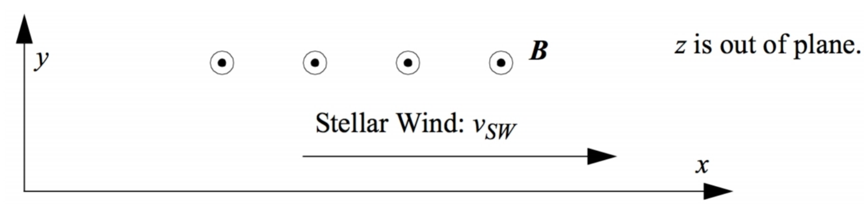
$$-k^2 \mathbf{E} = \left(\frac{\omega_{pe}^2}{c^2} - \frac{\omega^2}{c^2} \right) \mathbf{E} \tag{2.4}$$

where $\omega_{pe}^2 = ne^2/\epsilon_0 m_e$. The dispersion relation thus follows

$$\omega^2 = \omega_{pe}^2 + c^2 k^2 \tag{2.5}$$

□

***Problem* 3** (Pick-up Ion at Mars)**:** Suppose an O atom that escaped from Mars is at rest (in our frame) at $x = 0, y = 0$. It is photo-ionized (charge e) at $t = 0$ in the solar wind ($v_{sw} = 350\,\mathrm{km/s}$ in the x direction) with a magnetic field $\mathbf{B} = 10\,\mathrm{nT}$ in the z direction (see diagram). Assume that the photo-ionization does not move the O^+ ion.



(1) Describe the subsequent motion of the O^+ ion, $x(t), y(t), v_x(t), v_y(t)$, in our rest frame (not the plasma frame). *Hint*: What is the solar wind electric field? Calculate $v_x(t)$ and $v_y(t)$ then integrate. Apply boundary conditions, $x(t = 0) = 0, y(t = 0) = 0, v_x(t = 0) = 0, v_y(t = 0) = 0$ to get an exact solution.

(2) Sketch the O^+ path.

(3) The drift and gyration cause the O^+ ion to slow down and speed up in our rest frame. What is the drift speed? What is the gyration speed? What is the maximum velocity (km/s) and energy (keV) that the O^+ ion reaches?

(4) Using the gyration speed only, what is the perpendicular (to $\mathbf{B}$) temperature of that ion in $^\circ$K? (In 2D, temperature and energy are equal.)

Solution.

(1) Assume an ideal plasma with $\mathbf{B} = B_0\hat{\mathbf{z}}$ and $\mathbf{u} = v_{sw}\hat{\mathbf{x}}$, then the electric field is

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} = v_{sw} B_0 \hat{\mathbf{y}} = E_0 \hat{\mathbf{y}} \tag{3.1}$$

where $E_0 = 3.5\,\mathrm{mV/m}$. Using the Lorentz force equation $\dot{\mathbf{v}} = (e/m)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ where m is the mass of the O^+ ion, we get the following differential equations

$$\dot{v_x} = \frac{eB_0}{m} v_y = \Omega_c v_y \tag{3.2a}$$

$$\dot{v_y} = -\Omega_c \left(v_x - \frac{E_0}{B_0} \right) \tag{3.2b}$$

where $\Omega_c = eB_0/m$ is the unsigned cyclotron frequency. Taking another time derivative of (3.2a), $\ddot{v_x} = \Omega_c \dot{v_y}$ and combining with (3.2b), we can write an ODE in v_x

$$\ddot{v_x} = -\Omega_c^2 \left(v_x - \frac{E_0}{B_0} \right) \tag{3.3}$$

A general solution to this ODE is

$$v_x(t) = -v_\perp \cos\left(\Omega_c t + \delta\right) + \frac{E_0}{B_0} \tag{3.4}$$

It also follows from (3.2a) that

$$v_y(t) = -v_\perp \sin\left(\Omega_c t + \delta\right) \tag{3.5}$$

At $t = 0$, $v_y = 0 \Rightarrow \delta = 0$. Then we can calculate $v_\perp = E_0/B_0 = v_{sw}$ from requiring that $v_x = 0$ at $t = 0$. The solution for the velocity is then

$$v_x(t) = -v_\perp \cos\left(\Omega_c t\right) + \frac{E_0}{B_0} \tag{3.6a}$$

$$v_y(t) = -v_\perp \sin\left(\Omega_c t\right) \tag{3.6b}$$

Integrating with respect to time, we get the position

$$x(t) = x_0 + \frac{E_0}{B_0} t - \frac{v_\perp}{\Omega_c} \sin\left(\Omega_c t\right) \tag{3.7a}$$

$$y(t) = y_0 + \frac{v_\perp}{\Omega_c} \cos\left(\Omega_c t\right) \tag{3.7b}$$

Setting $x(t = 0) = 0$ and $y(t = 0) = 0$, we find $x_0 = 0$ and $y_0 = -v_\perp/\Omega_c$. The trajectory is then

$$x(t) = \frac{E_0}{B_0} t - \frac{v_\perp}{\Omega_c} \sin\left(\Omega_c t\right) \tag{3.8a}$$

$$y(t) = \frac{v_\perp}{\Omega_c} \cos\left(\Omega_c t\right) - \frac{v_\perp}{\Omega_c} \tag{3.8b}$$

(3.8) and (3.6) fully describe the subsequent motion of the ion.

(2) See Figure 1.

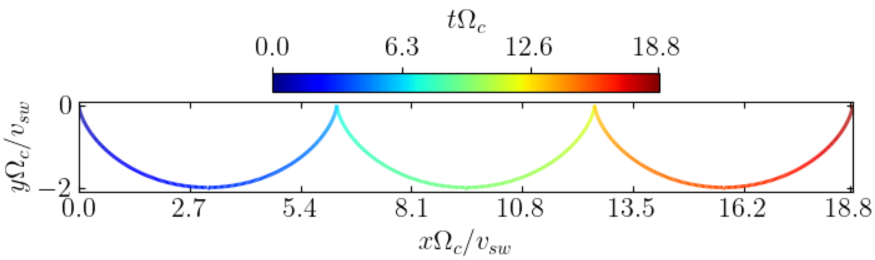


Figure 1: Trajectory of the O^+ ion, colored by time (from blue to red).

(3) From (3.6), the drift speed (E_0/B_0) and the gyration speed ($v_\perp$) are the same and they are both equal to $v_{sw} = 350\,\mathrm{km/s}$. From (3.6), the speed of the ion is

$$v = \sqrt{2(1-\cos(\Omega_c t))}v_\perp \tag{3.9}$$

This is maximum when $\Omega_c t = \pi$ ($v_y = 0$; so the ion is at the turning points in y), and $v_{\max} = 2v_\perp = 700\,\mathrm{km/s}$. The rest mass of an oxygen ion is $m \approx 14.92\,\mathrm{GeV}$. Thus, the kinetic energy of this ion travelling at $700\,\mathrm{km/s}$ is $E = (\gamma - 1)m \approx 40.7\,\mathrm{keV}$ where γ is the Lorentz factor.

(4) The perpendicular temperature is $T_\perp = (1/2)mv_\perp^2/k_B \approx 1.2 \times 10^8\,\mathrm{K}$. □

***Problem* 4** (Current sheet)**:** A current sheet is such that $\mathbf{J}$ is in the y direction and $\mathbf{B}$ is in the z direction. $\mathbf{B}, \mathbf{J}$, and P vary only with x (see diagram). Derive a solution for $\mathbf{B}$, and $\mathbf{J}$ under the condition $\mathbf{J} \sim P$ that is valid for $-L < x < L$. Make sure that your solution satisfies the boundary conditions of $\mathbf{B}(x = L) = -B_0\hat{\mathbf{z}}$, and $\mathbf{B}(x = 0) = \mathbf{0}$. L is a characteristic length. Sketch your results.

Hint: There is more than one possible solution – just give any valid solution. Be careful, the condition $\mathbf{J}^2 \sim P$ is NOT the same as in the Harris solution.

Solution.

From the force equation, the total pressure is constant

$$\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) = 0 \tag{4.1}$$

Thus, the kinetic pressure is

$$P(x) = \frac{B_\infty^2 - B^2(x)}{2\mu_0} \tag{4.2}$$

where B_∞ is the value of the magnetic field at the boundary and we have also set $P_\infty = 0$. From Ampere's Law, $\partial B/\partial x = -\mu_0 J_y = \mu_0 A\sqrt{P}$ where A is some constant such that $J_y^2 = A^2 P$. Combining this with (4.2), we can write

$$\frac{\partial B}{\partial x} = -A\sqrt{\frac{\mu_0}{2}}\sqrt{B_\infty^2 - B^2(x)} \tag{4.3}$$

Since this is a separable differential equation, we can integrate for B

$$-A\sqrt{\frac{\mu_0}{2}}\int dx = \int \frac{dB}{\sqrt{B_\infty^2 - B^2}} \tag{4.4}$$

By a transformation $B = B_\infty \sin\theta$, (4.4) becomes

$$-A\sqrt{\frac{\mu_0}{2}}x = \int d\theta = \sin^{-1}\left(\frac{B}{B_\infty}\right) \tag{4.5}$$

Inverting, we can write the magnetic field as

$$B(x) = -B_\infty \sin\left(A\sqrt{\frac{\mu_0}{2}}x\right) \tag{4.6}$$

This immediately satisfies $B(0) = 0$. However, we have to set

$$B_\infty = \frac{B_0}{\sin\left(A\sqrt{\mu_0/2}L\right)} \tag{4.7}$$

So that $B(L) = -B_0$. The full solution for $\mathbf{B}$ is then

$$\mathbf{B} = -B_0\frac{\sin\left(A\sqrt{\mu_0/2}x\right)}{\sin\left(A\sqrt{\mu_0/2}L\right)}\hat{\mathbf{z}} \tag{4.8}$$

The current is thus

$$\mathbf{J} = J_y\hat{\mathbf{y}} = -\frac{1}{\mu_0}\frac{\partial B}{\partial x}\hat{\mathbf{y}} = -\frac{AB_0}{\sqrt{2\mu_0}}\frac{\cos\left(A\sqrt{\mu_0/2}x\right)}{\sin\left(A\sqrt{\mu_0/2}L\right)}\hat{\mathbf{y}} \tag{4.9}$$

□

***Problem* 5** (Magnetic diffusion)**:** Consider a magnetic field $\mathbf{B} = B_z(x,t)\hat{\mathbf{z}}$ where $B_z(x,t = 0) = B_0\cos(k_1x) + B_0\cos(k_2x)$ in a resistive plasma with $k_2 \gg k_1$.

(a) Find the solution for $B_z(x,t)$.

(b) Sketch (accurate plot not needed) the solution for $t = 0$ and $t > 0$. What happens to the high-k wave?

Solution.

(a) The 1D diffusion equation is

$$\frac{\partial B_z}{\partial t} = \frac{1}{\sigma\mu_0}\frac{\partial^2 B_z}{\partial x^2} \tag{5.1}$$

where σ is the electric conductivity of the plasma such that

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \frac{\mathbf{J}}{\sigma} \tag{5.2}$$

The differential equation (5.1) can be solved by separation of variables. Let us write $B_z(x,t) = X(x)T(t)$ and plug it back into (5.1)

$$\frac{T'(t)}{T(t)} = \frac{1}{\sigma\mu_0}\frac{X''(x)}{X(x)} \tag{5.3}$$

The LHS is completely dependent on time, while the RHS is dependent on the position x. Thus, they must equate to a constant $-\gamma$. We then have the following two ordinary differential equations

$$\frac{dT}{dt} = -\gamma T(t) \tag{5.4a}$$

$$\frac{d^2X(x)}{dx^2} = -\gamma\sigma\mu_0 X(x) \tag{5.4b}$$

The general solution to (5.4a) is $T(t) = e^{-\gamma t}$, while the general solution to (5.4b) is

$$X(x) = a\sin(kx) + b\cos(kx) \tag{5.5}$$

such that

$$k^2 = \gamma\sigma\mu_0 \tag{5.6}$$

However, note that at $t = 0$, $B(x,0)$ is an even function in x. Thus, there cannot be any sine dependence and a must be zero. Then the magnetic field can be written as a linear combination of these orthogonal functions

$$B_z(x,t) = \sum_{n\in\mathbb{N}} b_n e^{-\gamma_n t}\cos(k_n x) \tag{5.7}$$

where γ_n, k_n follow the relation in (5.6). At $t = 0$, we have

$$B_z(x,0) = B_0\cos(k_1 x) + B_0\cos(k_2 x) = \sum_{n\in\mathbb{N}} b_n\cos(k_n x) \tag{5.8}$$

By symmetry, we must require that $b_n = 0$ where $n \neq 1$ or 2 and that $b_{1,2} = B_0$. The solution for the magnetic field is then

$$B_z(x,t) = B_0 e^{-k_1^2 t/\sigma\mu_0}\cos(k_1 x) + B_0 e^{-k_2^2 t/\sigma\mu_0}\cos(k_2 x) \tag{5.9}$$

(b) Let the normalized position be $\overline{x} = k_2 x$ and normalized time be $\overline{t} = k_2^2 t/\sigma\mu_0$. Also, let $\epsilon = k_1/k_2 \ll 1$. Then from (5.9), the normalized magnetic field $\overline{B} = B_z/B_0$ is

$$\overline{B} = e^{-\epsilon^2\overline{t}}\cos(\epsilon\overline{x}) + e^{-\overline{t}}\cos(\overline{x}) \tag{5.10}$$

In this form, it is clear that the decay rate in the low k wave (first term) is much smaller than that in the high k wave (second term). Thus, we expect the high k wave to damp away much sooner than the low k wave. This is demonstrated in Figure 2 where we have plotted the magnetic field $\overline{B}$ with $\epsilon = 0.1$. $\square$

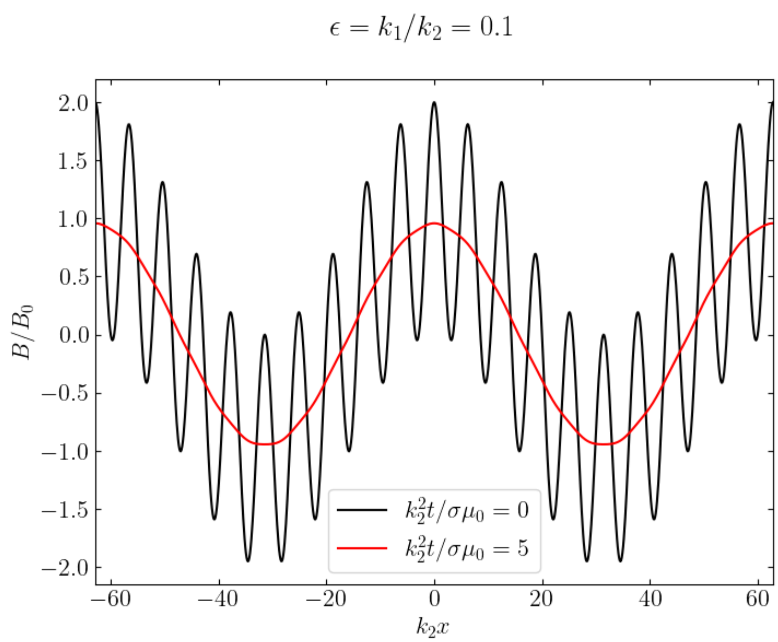


Figure 2: Magnetic field solution in spatial domain at $\bar{t} = 0$ (black) and $\bar{t} = 5$ (red).