

# Homework 12: Phys 7320 (Spring 2022)

Tien Vo

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**Problem 12.1** (The Lagrangian for a charged particle): The Lagrangian for a charged particle of mass  $m$  and charge  $e$  with position  $\mathbf{r}$  and velocity  $\mathbf{u} \equiv d\mathbf{r}/dt$  moving in scalar and vector potentials  $\Phi$  and  $\mathbf{A}$  is

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} - e\Phi + \frac{e}{c} \mathbf{u} \cdot \mathbf{A}. \quad (12.1.1)$$

(a) Show that the Euler-Lagrange equations for this Lagrangian indeed give rise to the Lorentz force law. *Hint:* I suggest using index notation, and remember the potentials  $\Phi, \mathbf{A}$  can depend on both  $\mathbf{r}$  and  $t$ , which as far as the particle is concerned means they depend on time in two ways:  $\Phi(\mathbf{r}(t), t)$  and  $\mathbf{A}(\mathbf{r}(t), t)$ .

(b) Go through the steps (12.13)-(12.17) in Jackson to derive the Hamiltonian

$$\mathcal{H} = \sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2c^4} + e\Phi. \quad (12.1.2)$$

Along the way derive the canonical/conjugate momentum  $\mathbf{P}$  (which is different from the familiar “kinematic momentum”  $\mathbf{p} = \gamma m\mathbf{u}$ ), and invert it to find an expression for  $\mathbf{u}$  in terms of  $\mathbf{P}$  and  $\mathbf{A}$ .

If you want extra practice, you can show that Hamilton’s equations for this Hamiltonian also give rise to the Lorentz force law, with some steps similar to part (a).

*Solution.*

(a) First, we write  $\mathcal{L}$  in Einstein notation with  $j \in \{x, y, z\}$

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{u_j^2}{c^2}} - e\Phi + \frac{e}{c} u_j A_j. \quad (12.1.3)$$

Then it is clear that

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = \frac{\partial \mathcal{L}}{\partial u_i} = \frac{mu_i}{\sqrt{1 - u_j^2/c^2}} + \frac{e}{c} A_i \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \gamma m\mathbf{u} + \frac{e}{c} \mathbf{A} = \mathbf{p} + \frac{e}{c} \mathbf{A}. \quad (12.1.4)$$

Thus, the LHS of Euler-Lagrange equation is

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{\partial \mathbf{p}}{\partial t} + \frac{e}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c} (\mathbf{u} \cdot \nabla) \mathbf{A}. \quad (12.1.5)$$

Now, for the RHS of Euler-Lagrange equation, we can write

$$\nabla \mathcal{L} = -e \nabla \Phi + \frac{e}{c} \nabla (\mathbf{u} \cdot \mathbf{A}) = -e \nabla \Phi + \frac{e}{c} [(\mathbf{u} \cdot \nabla) \mathbf{A} + \mathbf{u} \times (\nabla \times \mathbf{A})], \quad (12.1.6)$$

where we have used the vector calculus identity  $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$ . Now, combining (12.1.5) and (12.1.6), we can write

$$\frac{\partial \mathbf{p}}{\partial t} = e \left( -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right) + e \frac{\mathbf{u}}{c} \times (\nabla \times \mathbf{A}) = e \left( \mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right), \quad (12.1.7)$$

which is Lorentz force law.

(b) From (12.1.4), we have already found the canonical momentum

$$\mathbf{P} = \gamma m \mathbf{u} + \frac{e}{c} \mathbf{A}. \quad (12.1.8)$$

Thus,

$$\mathbf{u} = \frac{c\mathbf{P} - e\mathbf{A}}{\gamma mc} \Rightarrow u^2 = \frac{(c\mathbf{P} - e\mathbf{A})^2}{\gamma^2 m^2 c^2}. \quad (12.1.9)$$

But also, the Lorentz factor by definition is

$$\gamma^2 = \frac{1}{1 - u^2/c^2} = \frac{\gamma^2 m^2 c^4}{\gamma^2 m^2 c^4 - (c\mathbf{P} - e\mathbf{A})^2}. \quad (12.1.10)$$

Inverting, we get

$$\gamma = \sqrt{1 + \left( \frac{\mathbf{P}}{mc} - \frac{e\mathbf{A}}{mc^2} \right)^2}. \quad (12.1.11)$$

Thus,  $\mathbf{u}$  in terms of  $\mathbf{P}$  and  $\mathbf{A}$  is

$$\mathbf{u} = \frac{c\mathbf{P} - e\mathbf{A}}{\sqrt{m^2 c^2 + (\mathbf{P} - e\mathbf{A}/c)^2}}. \quad (12.1.12)$$

Then, starting from (12.15, Jackson),

$$\begin{aligned} \mathcal{H} &= \mathbf{P} \cdot \mathbf{u} - \mathcal{L} \\ &= \frac{\mathbf{u}}{c} \cdot (c\mathbf{P} - e\mathbf{A}) + \frac{mc^2}{\gamma} + e\Phi \\ &= \frac{(c\mathbf{P} - e\mathbf{A})^2}{\sqrt{m^2 c^4 + (c\mathbf{P} - e\mathbf{A})^2}} + \frac{m^2 c^4}{\sqrt{m^2 c^4 + (c\mathbf{P} - e\mathbf{A})^2}} + e\Phi \\ &= \sqrt{m^2 c^4 + (c\mathbf{P} - e\mathbf{A})^2} + e\Phi. \end{aligned} \quad (12.1.13)$$

□

**Problem 12.2** (Equivalent Lagrangians): (a) Use the Principle of Least Action (really the principle of extremal action) to show that if the Lagrangian  $\mathcal{L}$  is changed by adding the time derivative of some function of the coordinates and time, then the Euler-Lagrange equations are unchanged. The new and old Lagrangians are said to be *equivalent*. Generalize this to a statement about what change to a Lagrangian *density*  $\mathcal{L}$  leaves the EL equations unchanged.

(b) Show that under a gauge transformation, the Lagrangian for a charged particle given in the previous problem becomes an equivalent Lagrangian, thus showing the equations of motion do not change.

*Solution.*

□

**Problem 12.3** ( $SO(2)$  symmetry of two real scalar fields.): Consider the dynamics of two (real) scalar fields  $\phi_1(\mathbf{x}, t)$  and  $\phi_2(\mathbf{x}, t)$  specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V(\phi_1, \phi_2), \quad (12.3.1)$$

where the potential  $V$  depends only on the combination  $\phi_1^2 + \phi_2^2$ . In class, we will study this case with the real scalars combined into a single complex scalar; here we will leave them as two real scalars. Let's make a definite choice for the potential:

$$V(\phi_1, \phi_2) = \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}m^2\phi_2^2 + \frac{\lambda}{2}(\phi_1^2 + \phi_2^2)^2. \quad (12.3.2)$$

- (a) Calculate the equations of motion (Euler-Lagrange equations) for both  $\phi_1$  and  $\phi_2$ .
- (b) Show that the  $SO(2)$  transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (12.3.3)$$

where  $\alpha$  is a constant, is a symmetry of the Lagrangian. Is this a rotation in physical space? What space does this “rotation” act on?

- (c) According to Noether's theorem, the existence of this symmetry means there is a corresponding conserved current  $J^\mu$ . Find  $J^\mu$  in terms of  $\phi_1$  and  $\phi_2$  (you may drop an overall constant  $\alpha$ ) and show that it is conserved,  $\partial_\mu J^\mu = 0$ , when you use the equations of motion.

*Solution.*

□