Homework 4: Phys 5210 (Fall 2021)

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***Problem* 1:** Two particles move about each other in circular orbits under the influence of gravitational forces (so that the potential energy is $U = -\alpha/r$), with a period τ.

Their motion is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other.

Prove that they collide after a time $\tau/(4\sqrt{2})$.

Solution.

From the general solution derived in class, a circular orbit has a radius

$$R = \frac{1}{x_+} = \frac{1}{x_-} = \frac{l^2}{\mu\alpha} \tag{1.1}$$

where $l = \mu R^2 \omega$ is the angular momentum. Then the period is

$$\tau = \frac{2\pi}{\omega} = 2\pi \frac{l^3}{\mu\alpha^2} \tag{1.2}$$

Now, the particle is set to free fall with an intial energy $E_0 = -\alpha/R$. By energy conservation,

$$\frac{1}{2}\mu\dot{r}^2 - \frac{\alpha}{r} = E_0 = -\frac{\alpha}{R} \tag{1.3}$$

Thus, we can write the differential equation

$$dt = \sqrt{\frac{\mu}{2\alpha}} \left(\frac{1}{r} - \frac{1}{R}\right)^{-1/2} dr \tag{1.4}$$

Integrating both sides, we can find the time for the particles collide as

$$\tau' = \sqrt{\frac{\mu}{2\alpha}} \int_0^R \left(\frac{1}{r} - \frac{1}{R}\right)^{-1/2} dr = \frac{\pi}{2}\sqrt{\frac{\mu}{2\alpha}} R^{3/2} = \frac{\pi}{2\sqrt{2}} \frac{l^3}{\mu\alpha^2} = \frac{\tau}{4\sqrt{2}} \tag{1.5}$$

□

***Problem* 2:** A central force potential is given by

$$U(r) = \begin{cases} 0, & r > a \\ -U_0, & r < a \end{cases} \tag{2.1}$$

Here $U_0 > 0$. We would like to study the scattering in this potential.

(a) If a particle approaches the potential with the kinetic energy E and the impact parameter s, find the angle φ and $\tilde{\varphi}$ as shown in the Figure above. The figure represents the potential as a circle of radius a, with the trajectory of a particle shown in blue. Use conservation of energy and conservation of angular momentum to find $\tilde{\varphi}$.

(b) Determine the scattering angle θ in terms of φ and $\tilde{\varphi}$.

(c) Express s as a function of θ. This step is tricky as it is easy to find θ in terms of s, harder the other way around. To do that, take the ratio $\sin \frac{\tilde{\varphi}}{\sin\varphi}$. On the one hand, use the results of part (a) to figure out what it is equal to. On the other hand, express $\tilde{\varphi}$ in terms of φ and θ, and work to solve the resulting equation for φ. Finally, use the previous result relating φ to s.

(d) Finally, find the differential cross section $d\sigma/d\Omega$ where $d\Omega = 2\pi\sin(\theta)d\theta$ for a particle scattering in this potential.

Solution.

(a) The angular momentum is defined as $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. At the point of impact (when the particle enters the potential field U), the radius is $r = a$. Also, let v_0 be the initial velocity. Then the initial angular momentum is

$$L = mav_0 \sin\varphi \tag{2.2}$$

Let v be the velocity briefly after it has entered U. The radius is still a, but the angle between $\mathbf{r}$ and $\mathbf{p}$ is $\tilde{\varphi}$ and the angular momentum is

$$L = mav \sin\varphi \tag{2.3}$$

From conservation of the angular momentum, we can write

$$v = v_0 \frac{\sin\varphi}{\sin\tilde{\varphi}} \tag{2.4}$$

From energy conservation and the above result, we can write

$$E_0 = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - U_0 = \frac{1}{2}mv_0^2\frac{\sin^2\varphi}{\sin^2\tilde{\varphi}} - U_0 \tag{2.5}$$

Inverting, we can write

$$\sin\tilde{\varphi} = \left(1 + \frac{U_0}{E_0}\right)^{-1/2} \sin\varphi \tag{2.6}$$

From geometry, $\sin\varphi = s/a$, so we can write $\tilde{\varphi}$ as

$$\tilde{\varphi} = \sin^{-1}\left[\frac{s}{a}\left(1 + \frac{U_0}{E_0}\right)^{-1/2}\right] \tag{2.7}$$

(b) From geometry, we can also write $\theta = 2(\varphi - \tilde{\varphi}) \Rightarrow \tilde{\varphi} = \varphi - \theta/2$.

(c) From part (b), we can rewrite (2.6) as

$$\sin\left(\varphi - \frac{\theta}{2}\right) = \sin\varphi\cos\left(\frac{\theta}{2}\right) - \cos\varphi\sin\left(\frac{\theta}{2}\right) = \left(1 + \frac{U_0}{E_0}\right)^{-1/2}\sin\varphi \tag{2.8}$$

Solving for φ, we have

$$\tan\varphi = f(\theta) = \frac{\sin\big(\theta/2\big)}{\cos\big(\theta/2\big) - (1 + U_0/E_0)^{-1/2}} = \frac{s}{\sqrt{a^2 - s^2}} \tag{2.9}$$

Inverting, we can write s as a function of θ

$$s^2 = a^2\frac{f^2(\theta)}{1 + f^2(\theta)} = a^2\frac{\sin^2(\theta/2)}{1 + C^2 - 2C\cos\big(\theta/2\big)} \tag{2.10}$$

where $C = \sqrt{E_0/(E_0 + U)}$.

(d) Taking the derivative of (2.10) with Mathematica, we can write the differential cross section as

$$\sigma_{\text{diff}} = \frac{1}{2\sin\theta}\frac{ds^2}{d\theta} = \frac{a^2}{4}\frac{1 + C^2 - C[\cos\big(\theta/2\big) + \sec\big(\theta/2\big)]}{[1 + C^2 - 2C\cos\big(\theta/2\big)]^2} \tag{2.11}$$

Let $n = 1/C$, this can be simplified into

$$\begin{aligned}
\sigma_{\text{diff}} &= \frac{a^2}{4}\frac{1 + \frac{1}{n^2} - \frac{1}{n}\Big[\cos\big(\theta/2\big) + \sec\big(\theta/2\big)\Big]}{[1 + \frac{1}{n^2} - \frac{2}{n}\cos\big(\theta/2\big)]^2} \\
&= \frac{n^2a^2}{4}\frac{n^2 + 1 - n\cos\big(\theta/2\big) - \frac{n}{\cos(\theta/2)}}{[n^2 + 1 - 2n\cos\big(\theta/2\big)]^2} \\
&= \frac{n^2a^2}{4}\frac{(n\cos\big(\theta/2\big) - 1)(n - \cos\big(\theta/2\big))}{(n^2 + 1 - 2n\cos\big(\theta/2\big))^2}
\end{aligned} \tag{2.12}$$

$\square$