Final: Astr 5140 (Fall 2021)

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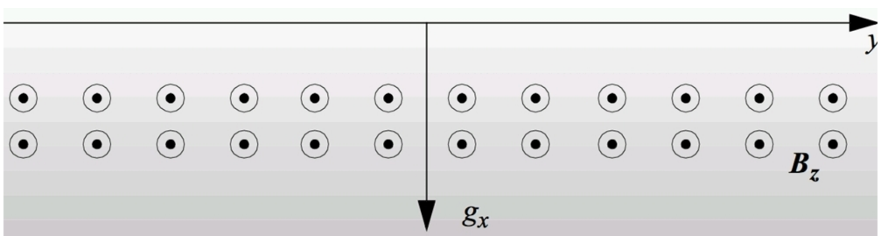
***Problem* 1** (Particle Motion in a Gravitational Field)**:** Examine the diagram below. A plasma in a gravitational field (x direction) is supported by a magnetic field (B_z) in the z direction. Set $\mathbf{E} = \mathbf{0}$.



(a) Start with the basic equations of motion of a charged particle in a magnetic field $\mathbf{v} = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}}$ and gravitational force. Break the Lorentz equation into its components,

$$\frac{\partial v_x}{\partial t} = \frac{qB_z}{m} v_y + g \qquad \text{and} \qquad \frac{\partial v_y}{\partial t} = -\frac{qB_z}{m} v_x \tag{1.1}$$

where $\mathbf{B} = B\hat{\mathbf{z}}$. Average over time (alternatively, ignore gyration) to arrive at the gravitational drift.

(b) Calculate the current $\mathbf{J}$ for a constant plasma density ρ_0.

(c) Show that the current from gravitational drift is equivalent to that in the MHD force equation.

(d) Setting the single fluid velocity to $\mathbf{u} \approx \mathbf{v}_{Di}$, is $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$? ($\mathbf{v}_{Di}$ is the ion gravitational drift).

(e) Show generalized Ohm's law must include the Hall term to be consistent with the gravitational drift in the limit of $m_i \gg m_e$. Should Hall MHD be used in the presence of strong gravitational field?

Solution.

(a) Given $\mathbf{g} = g\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, the force equation is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + q\mathbf{v} \times \mathbf{B} \Rightarrow \dot{\mathbf{v}} = g\hat{\mathbf{x}} + \Omega_c(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}}) \tag{1.2}$$

where we have written $\Omega_c = qB/m$. So the velocity components satisfy the following differential equations

$$\dot{v}_x = g + \Omega_c v_y \qquad \text{and} \qquad \dot{v}_y = -\Omega_c v_x \tag{1.3}$$

Taking another time derivative and combining the two equations, we get the following differential equation for v_x

$$\ddot{v}_x = \Omega_c \dot{v}_y = -\Omega_c^2 v_x \tag{1.4}$$

The general solution for v_x is

$$v_x = v_\perp \cos\left(\Omega_c t + \delta\right) \tag{1.5}$$

where $v_\perp, \delta$ are initial conditions dependent constants. We can then find v_y

$$v_y = \frac{\dot{v}_x}{\Omega_c} - \frac{g}{\Omega_c} = -v_\perp \sin\left(\Omega_c t + \delta\right) - \frac{g}{\Omega_c} \tag{1.6}$$

Averaging over one period, $\left\langle \sin(\Omega t + \delta) \right\rangle = 0$, so there remains a drift in the y direction

$$v_D = \left\langle v_y \right\rangle = -\frac{g}{\Omega_c} \tag{1.7}$$

So the gravitational drift is

$$\mathbf{v}_D = -\frac{g}{\Omega_c}\hat{\mathbf{y}} = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B}}{B^2} = -\frac{mg}{qB}\hat{\mathbf{y}} \tag{1.8}$$

(b) The current due to the electron drift is

$$\mathbf{J}_e = -ne\mathbf{v}_{De} = nm_e\frac{\mathbf{g}\times\mathbf{B}}{B^2} \tag{1.9}$$

and the current due to the ion drift is

$$\mathbf{J}_i = ne\mathbf{v}_{Di} = nm_i\frac{\mathbf{g}\times\mathbf{B}}{B^2} \tag{1.10}$$

Thus, the total current is

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e = n(m_e + m_i)\frac{\mathbf{g}\times\mathbf{B}}{B^2} = \rho_0\frac{\mathbf{g}\times\mathbf{B}}{B^2} = -\frac{\rho_0 g}{B}\hat{\mathbf{y}} \tag{1.11}$$

(c) Recall that the current in the force equation $\mathbf{J}\times\mathbf{B}$ comes from the term

$$n_s q_s \mathbf{u}_s \times \mathbf{B} \tag{1.12}$$

in the two-fluid equations where $\mathbf{u}_s$ is the drift of the particle species. Here, the only drift is due to gravitation. Thus, by definition, $\mathbf{J} = \sum_s n_s q_s \mathbf{u}_s$ in the force equation is the same as that in part (b).

(d) Since $\mathbf{u} = \mathbf{v}_{Di}$, from (1.8),

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = -\frac{m_i g}{e}\hat{\mathbf{y}}\times\hat{\mathbf{z}} = -\frac{m_i g}{e}\hat{\mathbf{x}} \neq 0 \tag{1.13}$$

where M is the ion mass.

(d) By definition, the Hall term is

$$\frac{\mathbf{J} \times \mathbf{B}}{ne} = -\frac{\rho_0 g}{ne}\hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\frac{\rho_0 g}{ne}\hat{\mathbf{x}} \tag{1.14}$$

But $\rho_0 \approx nm_i$ when $m_i \gg m_e$, so

$$\frac{\mathbf{J} \times \mathbf{B}}{ne} = -\frac{m_i g}{e}\hat{\mathbf{x}} \tag{1.15}$$

This is the same as the RHS in (1.13). So we must require that the generalized Ohm's law have the Hall term

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{J} \times \mathbf{B}}{ne} \tag{1.16}$$

In the presence of strong gravitational field, Hall MHD should be used because the plasma is certainly not ideal, as indicated by part (d). □

***Problem* 2** (Petersen Conductance)**:** (a) Derive the steady-state Petersen conductance assuming that only ions contribute. Simplify the problem by breaking the fluid equation

$$\frac{\partial \mathbf{u}_i}{\partial t} = \frac{e}{m_i}(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nu_{\text{in}}\mathbf{u}_i \tag{2.1}$$

into its x and y components. Assign $\mathbf{B}$ to be in the z direction and $\mathbf{E}$ to be in the x direction.

(b) Plot the Petersen conductance as a function of ν_{in}. Show that the Petersen conductance peaks when $\nu_{\text{in}} = \omega_{ci}$. Hint, a peak or valley occurs when $\partial\sigma_i/\partial\nu_{\text{in}} = 0$.

Solution.

(a) For a steady state, $\partial\mathbf{u}_i/\partial t = 0$. Now, breaking the fluid equation into x and y, we get

$$\frac{e}{m_i}\big(E_x + u_y B_z\big) - \nu_{\text{in}}u_x = 0 \qquad \text{and} \qquad -\frac{e}{m_i}u_x B_z - \nu_{\text{in}}u_y = 0 \tag{2.2}$$

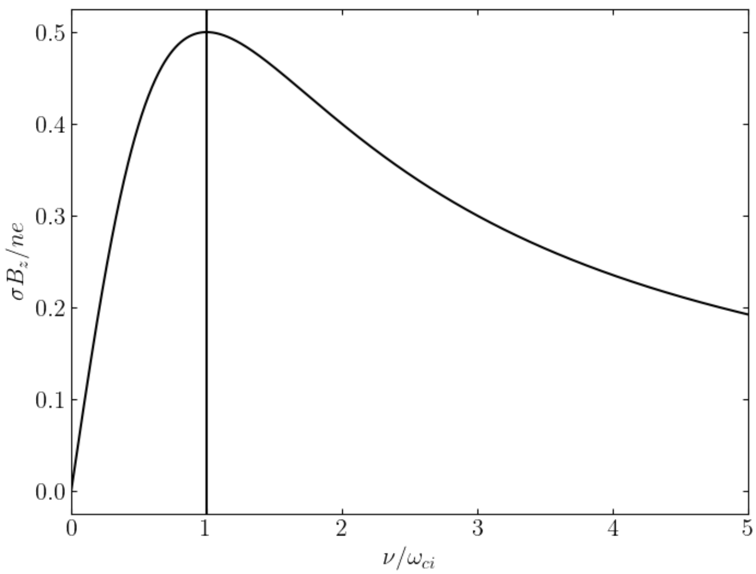
Combining these two equations, we can eliminate u_y

$$\frac{e}{m_i}\left(E_x - \frac{e u_x B_z^2}{\nu_{\text{in}} m_i}\right) - \nu_{\text{in}}u_x = 0 \Rightarrow u_x = \frac{\nu_{\text{in}}}{\nu_{\text{in}}^2 + \omega_{ci}^2}\frac{eE_x}{m_i} \tag{2.3}$$

where $\omega_{ci} = eB_z/m_i$. Now, by Ohm's law, $J_x = \sigma E_x = neu_x$. Thus, we can eliminate u_x/E_x from (2.3) and write

$$\sigma = \frac{ne^2}{m_i}\frac{\nu_{\text{in}}}{\nu_{\text{in}}^2 + \omega_{ci}^2} \tag{2.4}$$

(b) We plot the normalized conductance below.



The maximum satifies

$$\frac{\partial \sigma}{\partial \nu_{\text{in}}} \sim \frac{1}{\nu_{\text{in}}^2 + \omega_{ci}^2} - \frac{2\nu_{\text{in}}^2}{(\nu_{\text{in}}^2 + \omega_{ci}^2)^2} = 0 \Rightarrow \nu_{\text{in}}^2 + \omega_{ci}^2 - 2\nu_{\text{in}}^2 = 0 \Leftrightarrow \nu_{\text{in}} = \omega_{ci} \tag{2.5}$$

This is consistent with the peak in the plot (vertical line). □

***Problem* 3** (Oblique slow mode)**:** At oblique angles, plasmas have three characteristic modes that are distinguished by the role of the magnetic field. We want to derive the slow mode with $\mathbf{k}$ at an arbitrary angle to $\mathbf{B}_0$ in a strongly magnetized plasma. The slow mode is dominated by particle pressure and, since the plasma is strongly magnetized, there should be only a small perturbation in $\mathbf{B}$. Start with our linearized wave equations

$$\omega\rho_1 = \rho_0 \mathbf{k} \cdot \mathbf{u}_1 \tag{3.1a}$$

$$\omega\rho_0\mathbf{u}_1 = \mathbf{k}\frac{\gamma T}{m}\rho_1 + \mathbf{k}\frac{(\mathbf{B}_0 \cdot \mathbf{B}_1)}{\mu_0} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\mu_0}\mathbf{B}_1 \tag{3.1b}$$

$$\omega\mathbf{B}_1 = \mathbf{B}_0(\mathbf{k} \cdot \mathbf{u}_1) - (\mathbf{k} \cdot \mathbf{B}_0)\mathbf{u}_1 \tag{3.1c}$$

Let $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, $\mathbf{k} = k(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}})$, and $\mathbf{u}_1 = u_1(\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{z}})$. Careful! An oblique wave is neither longitudinal nor transverse.

(a) Examine (3.1c). If $\mathbf{B}_1$ is small, what is the angle ϕ from $\mathbf{B}_0$ is $\mathbf{u}_1$? Remember, θ is arbitrary. Show that $\mathbf{u}_1$ must be nearly parallel to $\mathbf{B}_0$.

(b) In (3.1b), one can immediately see that there is no solution for large θ if $\mathbf{B}_1$ is exactly 0. As a rough approximation, take the dot product of (3.1b) with $\mathbf{B}_0$ to derive a dispersion relation for the oblique sound wave. Assume $\phi = 0$ but θ is arbitrary after the dot product.

(c) Now let's do the problem more accurately. Start with (3.1c) and separate it into the x and z components.

(d) Next, evaluate (3.1b) in the x direction. Show that u_{1x} depends on two terms; one is proportional to c_s^2 (sound speed) and the other is proportional to V_A^2 (Alfvén speed). Show that if $V_A^2 \gg c_s^2$, then $|\phi| \ll |\theta|$, so our simple derivation is correct. Hint: Do not try to solve this problem exactly!

Solution.

(a) If $\mathbf{B}_1$ is small, then the LHS is $\sim \mathbf{0}$ and we can write $\mathbf{u}_1 \propto \mathbf{B}_0$. So ϕ must be close to zero. In other words, $\mathbf{u}_1$ is nearly parallel to $\mathbf{B}_0$.

(b) Taking the dot product with $\mathbf{B}_0$, we get

$$\begin{aligned}
& \omega\rho_0(\mathbf{u}_1 \cdot \mathbf{B}_0) = c_s^2\rho_1(\mathbf{k}\cdot\mathbf{B}_0) + \frac{(\mathbf{B}_0\cdot\mathbf{B}_1)(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0} - \frac{(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{B}_0\cdot\mathbf{B}_1)}{\mu_0} \\
\Rightarrow \quad & \omega\rho_0 B_0 u_1 \cos\phi = c_s^2 \rho_1 k B_0 \cos\theta \\
\Leftrightarrow \quad & \frac{\omega}{k} = c_s^2 \frac{\rho_1}{\rho_0 u_1}\cos\theta
\end{aligned} \tag{3.2}$$

where $\phi = 0$. But from (3.1a), we can also write $(\rho_1/\rho_0 u_1) = (k/\omega)\cos\theta$. Plugging this in, we get the dispersion relation

$$\omega = k c_s \cos\theta = k_{\|} c_s \tag{3.3}$$

(c) Separating (3.1c) into x and z, we get

$$\omega B_{1x} = -B_0 k_z u_x \qquad \text{and} \qquad \omega B_{1z} = B_0(k_x u_x + k_z u_z) - B_0 k_z u_z = B_0 k_x u_x \tag{3.4}$$

(d) Then the x component of (3.1b) is

$$\begin{aligned}
& \omega\rho_0 u_x = \rho_1 k_x c_s^2 + k_x \frac{B_0 B_{1z}}{\mu_0} - \frac{k_z B_0}{\mu_0} B_{1x} \\
\Rightarrow \quad & u_x = \frac{\rho_1}{\rho_0}\frac{k_x}{\omega} c_s^2 + \frac{k_x}{\omega}\frac{B_0}{\mu_0\rho_0} B_{1z} - \frac{k_z}{\omega}\frac{B_0}{\mu_0\rho_0} B_{1x} \\
& \quad = \frac{k_x}{\omega^2}(k_x u_x + k_z u_z) c_s^2 + \frac{k_x^2}{\omega^2} V_A^2 u_x + \frac{k_z^2}{\omega^2} V_A^2 u_x
\end{aligned} \tag{3.5}$$

Then we can solve for

$$\frac{u_x}{u_z} = \tan\phi = \frac{k_x k_z c_s^2}{\omega^2 - k_x^2 c_s^2 - k^2 V_A^2} \tag{3.6}$$

Thus, when $c_s^2 \ll V_A^2$, $\tan\phi \to 0$, which means $\phi \to 0$. □

***Problem* 4** (Force free current sheet)**:** Assume a current $\mathbf{J}$ in the (yz) plane that flows along $\mathbf{B}$ (also in the (yz) plane). The only gradients are in the x direction. There is no external magnetic field. For boundary conditions

$$\mathbf{B}(x \to \infty) = -B_0\hat{\mathbf{z}}, \qquad \mathbf{B}(x \to -\infty) = B_0\hat{\mathbf{z}}, \qquad \mathbf{J}(x \to \pm\infty) = 0 \tag{4.1}$$

(a) Find a solution for $\mathbf{B}$ with $\alpha = \alpha_0 \operatorname{sech}\big(x/x_0\big)$, where x_0 is a "characteristic" of the thickness of the current sheet. Sketch B_y, B_z. Sketch or describe the magnetic pressure. There are several possible solutions. Choose the simplest one. Hint: Let $\alpha_0 = 1/x_0$.

(b) Compare this solution to that of a Harris current sheet (do not derive again – use the results from homework). Discuss the similarities and differences.

Solution.

(a) As per the hint, let $\alpha = (1/x_0)\,\mathrm{sech}\big(x/x_0\big)$. Force free current sheet satisfies

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} = \alpha \mathbf{B} \tag{4.2}$$

Thus, the magnetic field components satisfy the following system of differential equations

$$\frac{\partial B_z}{\partial x} = -\alpha B_y \qquad \text{and} \qquad \frac{\partial B_y}{\partial x} = \alpha B_z \tag{4.3}$$

Taking another x derivative and combining the two equations, we get the following differential equation in B_z

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{1}{x_0}\tanh\left(\frac{x}{x_0}\right)\frac{\partial B_z}{\partial x} + \frac{1}{x_0^2}\,\mathrm{sech}^2\left(\frac{x}{x_0}\right) B_z = 0 \tag{4.4}$$

The general solution is obtained from Mathematica

$$B_z = \frac{c_1}{\cosh\big(x/x_0\big)} + c_2 \tanh\left(\frac{x}{x_0}\right) \tag{4.5}$$

with c_1, c_2 constants. Now, we can let $c_1 = 0$ and $c_2 = -B_0$ so that B_z matches the boundary conditions. It follows that

$$B_y = -\frac{1}{\alpha}\frac{\partial B_z}{\partial x} = B_0\,\mathrm{sech}\left(\frac{x}{x_0}\right) \tag{4.6}$$

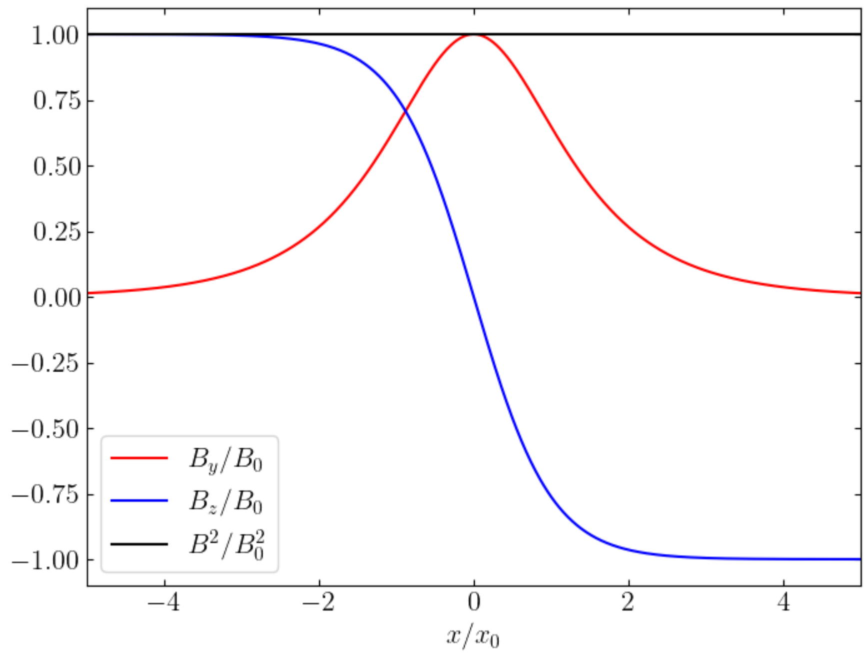
In full, the magnetic field is

$$\mathbf{B} = B_0\left[\mathrm{sech}\left(\frac{x}{x_0}\right)\hat{\mathbf{y}} - \tanh\left(\frac{x}{x_0}\right)\hat{\mathbf{z}}\right] \tag{4.7}$$

and the current is

$$\mathbf{J} = \frac{\alpha}{\mu_0}\mathbf{B} = \frac{B_0}{\mu_0 x_0}\,\mathrm{sech}\left(\frac{x}{x_0}\right)\left[\mathrm{sech}\left(\frac{x}{x_0}\right)\hat{\mathbf{y}} - \tanh\left(\frac{x}{x_0}\right)\hat{\mathbf{z}}\right] \tag{4.8}$$

So $\mathbf{J}(x \to \pm\infty) = 0$. This also matches the boundary conditions. In the following, we plot the components B_y (red), B_z (blue), and the normalized magnetic pressure $(B^2/2\mu_0)/(B_0^2/2\mu_0)$ (black).



(b) The magnetic pressure is constant here, but not in the Harris current sheet. B_z has the same shape as that in the Harris current sheet. However, B_y is non-zero here. Similarly, J_y has the same shape as that in the Harris current sheet, but J_z is non-zero here. □

***Problem* 5** (Isothermal shock)**:** In some astrophysical situations, radiative cooling is so efficient that the post-shock temperature relaxes to the pre-shock temperature within a thin layer. Under this rare condition, the shock can be treated isothermal. In this problem, we derive the compression ratio for a perpendicular isothermal shock.

(a) Show the energy equation for a perpendicular isothermal shock in the limit $\gamma \to 1$ is: $u_1 P_1 = u_2 P_2$.

(b) Write down the remaining jump conditions for a perpendicular isothermal shock.

(c) Eliminate ρ_2, u_2, P_2, and B_2 from the equation for the pressure balance. The result should only have ρ_1, u_1, P_1, B_1, and the compression ratio $r = \rho_2/\rho_1$.

(d) Remove the trivial solution ($r = 1$) by dividing by $1 - r$. You should arrive at a quadratic equation in r (you may have to multiply by r).

(e) Solve for the compression ratio r as a function of u_1 in the limit of high Alfvén Mach isothermal shock. Hint: you do not need to solve the quadratic equation. Reform the equation in terms of M_A and M_S and assume $M_A \gg 1$.

(f) Now solve for the compression ratio as a function of u_1 for $B = 0$. How does this solution differ from the magnetized case?

(g) Briefly discuss the effect of a finite magnetic field in high-Mach isothermal shock.

Solution.

(a) The energy equation for a perpendicular shock is, ignoring the ram energy and magnetic field energy

$$\frac{\gamma}{\gamma-1}u_1 P_1 = \frac{\gamma}{\gamma-1}u_2 P_2 \tag{5.1}$$

Using l'Hôpital's rule, both $\gamma/(\gamma-1) \to 1$ when $\gamma \to 1$. So we get $u_1 P_1 = u_2 P_2$.

(b) The remaining jump conditions are

$$\rho_1 u_1 = \rho_2 u_2 \tag{5.2a}$$

$$u_1 B_1 = u_2 B_2 \tag{5.2b}$$

$$P_1 + \rho_1 u_1^2 + \frac{B_1^2}{2\mu_0} = P_2 + \rho_2 u_2^2 + \frac{B_2^2}{2\mu_0} \tag{5.2c}$$

(c,d) From (5.2c), we can write

$$\begin{aligned}
& P_1\left(1 - \frac{P_2}{P_1}\right) + \rho_1 u_1^2\left(1 - \frac{\rho_2}{\rho_1}\frac{u_2^2}{u_1^2}\right) + \frac{B_1^2}{2\mu_0}\left(1 - \frac{B_2^2}{B_1^2}\right) = 0 \\
\Rightarrow \quad & P_1(1-r) + \rho_1 u_1^2\left(1 - \frac{1}{r}\right) + \frac{B_1^2}{2\mu_0}\left(1 - r^2\right) = 0 \\
\Leftrightarrow \quad & rP_1 - \rho_1 u_1^2 + \frac{B_1^2}{2\mu_0}\left(r + r^2\right) = 0 \qquad (5.3)
\end{aligned}$$

(e) Dividing (5.3) by $\rho_1 u_1^2$, we get

$$\frac{r}{M_S^2} - 1 + \frac{r + r^2}{2M_A^2} = 0 \tag{5.4}$$

For $M_A \gg 1$, the quadratic term vanishes and we can write $r = M_S^2 = (\rho_1/P_1)u_1^2$.

(f) In the unmagnetized case, (5.2c) becomes

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \tag{5.5}$$

Similarly, we eliminate P_2, ρ_2, u_2

$$P_1\left(1 - \frac{P_2}{P_1}\right) + \rho_1 u_1^2\left(1 - \frac{\rho_2}{\rho_1}\frac{u_2^2}{u_1^2}\right) = P_1(1-r) + \rho_1 u_1^2\left(1 - \frac{1}{r}\right) = 0 \tag{5.6}$$

Dividing by $\rho_1 u_1^2$ yields

$$\frac{1}{M_S^2} - \frac{1}{r} = 0 \Rightarrow r = M_S^2 = \frac{\rho_1}{P_1}u_1^2 \tag{5.7}$$

(g) So a finite magnetic field has no effect on an isothermal shock. □