Homework 7: Phys 5210 (Fall 2021)

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***Problem* 1:** At the bowling alley a bowling ball (a solid uniform sphere of radius R and mass m) was thrown horzontally without spinning with the horizontal velocity v_0. It hit the floor and, thanks to the friction between the ball and the floor, eventually started rolling along without slipping. Find its final velocity v. Neglect air resistance and the rolling friction (so that once it starts rolling without slipping, there's no friction). Which fraction of the initial kinetic energy of the ball was converted into heat during this process?

Solution.

Assume that the frictional force F is constant when the ball starts rolling. The time it takes to accelerate from $\omega_i = 0$ to $\omega_f = \omega$ is

$$t = \frac{\omega}{\alpha} = \frac{\omega I}{FR} \tag{1.1}$$

where $\alpha = FR/I$ is the angular acceleration and $I = (2/5)mR^2$ is the moment of inertia of a uniform sphere of mass m and radius R. In this period, the linear velocity is decelerated to

$$v = v_0 - at = v_0 - \frac{a}{F}\frac{\omega I}{R} = v_0 - \frac{2}{5}\omega R = v_0 - \frac{2}{5}v \Rightarrow v = \frac{5}{7}v_0 \tag{1.2}$$

Then we can calculate the fractional energy loss as

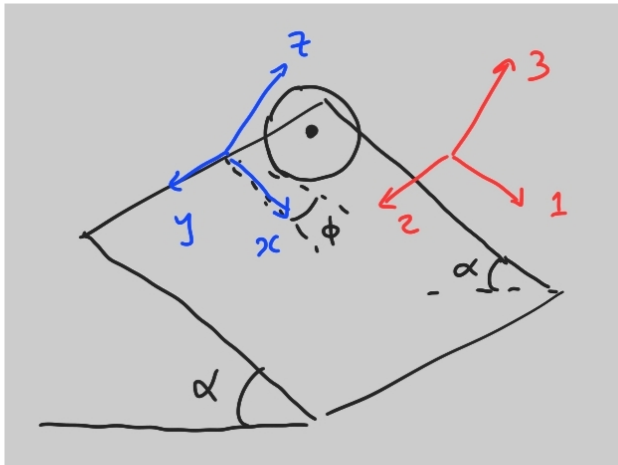
$$\overline{W} = \frac{E_i - E_f}{E_i} = 1 - \frac{1/2mv_0^2}{1/2mv^2 + 1/2I\omega^2} = \frac{49/50mv^2}{7/10mv^2} = \frac{2}{7} \tag{1.3}$$

where we have written $\omega = vR$ since it starts rolling without slipping. □

***Problem* 2** (Goldstein, Problem 5.24)**:** A wheel rolls down a flat inclined surface that makes an angle α with the horizontal (without slipping). The wheel is constrained so that its plane is always perpendicular to the inclined plane, but it may rotate about the axis normal to the surface. Think of a wheel as a symmetric top, with the moment of inertia I_3 defined with respect to the axis perpendicular to the plane of the wheel, and the two other moments of inertia $I_1 = I_2 \neq I_3$. Suppose the wheel starts moving from rest.

Reduce its equations of motion to a single second order differential equation the function $\phi(t)$ satisfies, where ϕ is the azimuthal angle describing the direction where the wheel is rolling. Use the Lagrange function with the appropriate Lagrange multipliers.

Solution.



Define our coordinate system as above. By definition, the angular velocity around the principal axes of the wheel is

$$\Omega_1 = \dot{\phi}\sin\psi, \qquad \Omega_2 = -\dot{\phi}\cos\psi, \qquad \text{and} \qquad \Omega_3 = \dot{\psi} \tag{2.1}$$

if we are to impose the constraint that the wheel is always perpendicular to the plane of the ramp ($\theta = \pi/2$). Because the wheel rolls without slipping, the following constraints must be applied

$$\dot{x} = R\cos\phi\dot{\psi} \qquad \text{and} \qquad \dot{y} = R\sin\phi\dot{\psi} \tag{2.2}$$

Then we can write the Lagrange function (with Lagrange multipliers) as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I_1(\Omega_1^2+\Omega_2^2) + \frac{1}{2}I_3\Omega_3^2 + mgx\sin\alpha - \lambda_1(\dot{x} - R\cos\phi\dot{\psi}) - \lambda_2(\dot{y} - R\sin\phi\dot{\psi}) \\ &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{4}mR^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\psi}^2 + mgx\sin\alpha - \lambda_1(\dot{x} - R\cos\phi\dot{\psi}) - \lambda_2(\dot{y} - R\sin\phi\dot{\psi})\end{aligned} \tag{2.3}$$

where $I_1 = I_2 = 1/2I_3 = 1/2mR^2$ are the moments of inertia of the wheel. Applying Euler-Lagrange equations for x and y, we can write

$$m\ddot{x} = mg\sin\alpha \tag{2.4a}$$

$$m\ddot{y} = 0 \tag{2.4b}$$

These are forces acting on the center of mass of the wheel, as expected from Newtonian mechanics. From (2.2), it follows that

$$(R\cos\phi)\ddot{\psi} - (R\sin\phi)\dot{\phi}\dot{\psi} = g\sin\alpha \tag{2.5a}$$

$$(R\sin\phi)\ddot{\psi} + (R\cos\phi)\dot{\phi}\dot{\psi} = 0 \tag{2.5b}$$

We can then solve this system for

$$\dot{\psi} = -\frac{g\sin\alpha\sin\phi}{R\dot{\phi}} \qquad \text{and} \qquad \ddot{\psi} = \frac{g\sin\alpha\cos\phi}{R} \tag{2.6}$$

Now, applying Euler-Lagrange equations for ϕ and ψ, we get the following system

$$-\sin\phi\dot{\psi}\lambda_1 + \cos\phi\dot{\psi}\lambda_2 = \frac{1}{2}mR\ddot{\phi} \tag{2.7a}$$

$$-\sin\phi\dot{\phi}\lambda_1 + \cos\phi\dot{\phi}\lambda_2 = -mR\ddot{\psi} \tag{2.7b}$$

This implies that $\dot{\phi}\ddot{\phi} = -2\dot{\psi}\ddot{\psi}$. Using the results from (2.6), we can rewrite this into a second order differential equation of the steering angle ϕ

$$\dot{\phi}^2\ddot{\phi} = -\frac{g^2\sin^2\alpha}{R^2}\sin 2\phi \tag{2.8}$$

□