Midterm: Phys 7310 (Fall 2021)

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***Problem* M.1** (Charge density of an electron)**:** An electron in an atom has the charge density

$$\rho(\mathbf{x}) = B\cos^2\theta e^{-r/a} \tag{M.1.1}$$

where a and B are constants.

(a) Calculate all the nonzero multipole moments q_{lm} for this charge distribution.

(b) From the q_{lm} obtain the total charge q, dipole moment $\mathbf{p}$, and quadrupole tensor Q_{ij}. Use these to directly write down the scalar potential $\Phi(\mathbf{x})$ generated by the charge density.

Solution.

(a) From (4.3, Jackson), the multipole moments q_{lm} are

$$\begin{aligned} q_{lm} &= \int Y^*_{lm}(\Omega')r'^l\rho(\mathbf{x}')d^3x' \\ &= B\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_0^\infty dr' r'^{l+2}e^{-r'/a}\int_{-1}^1 d(\cos\theta')P_l^m(\cos\theta')\cos^2\theta'\int_0^{2\pi}e^{-im\phi'} \end{aligned} \tag{M.1.2}$$

Note that for $m \neq 0$, the last integration is proportional to $\exp\{-im\phi'\}\big|_{\phi'=0}^{\phi'=2\pi} = 0$. Thus, q_{lm} are only non-zero for $m = 0$. This is reflected by the azimuthal symmetry in the charge distribution (M.1.1). Then, (M.1.2) becomes

$$\begin{aligned} q_{lm} &= 2\pi B\sqrt{\frac{2l+1}{4\pi}}\int_0^\infty dr' r'^{l+2}e^{-r'/a}\int_{-1}^1 dx x^2 P_l(x) && (x=\cos\theta') \\ &= 2\pi a^{l+3}B\sqrt{\frac{2l+1}{4\pi}}\int_0^\infty du u^{(l+3)-1}e^{-u}\int_{-1}^1 dx x^2 P_l(x) && (u = r'/a) \\ &= 2\pi a^{l+3}B\sqrt{\frac{2l+1}{4\pi}}\Gamma(l+3)\begin{cases} 2/3 & l=0 \\ 4/15 & l=2 \end{cases} && \text{(M.1.3)} \end{aligned}$$

where we have used (3.32, Jackson) for the Legendre polynomial integration. The only two non-zero moments are then

$$q_{00} = \frac{4}{3}\sqrt{\pi}a^3B \qquad \text{and} \qquad q_{20} = 32\sqrt{\frac{\pi}{5}}a^5B \tag{M.1.4}$$

(b) From (4.4, Jackson), we can calculate the charge

$$q = \sqrt{4\pi} q_{00} = \frac{8}{3}\pi a^3 B \tag{M.1.5}$$

From (4.5, Jackson), $\mathbf{p} = \mathbf{0}$ since $q_{lm} = 0$ for $l = 1$. From (4.6, Jackson),

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}} q_{20} = \frac{128\pi}{5} a^5 B \tag{M.1.6}$$

Also, because $q_{21} = -q_{2,-1}^* = 0$, $Q_{13} = Q_{23} = 0$. Similarly, because $q_{22} = q_{2,-2}^* = 0$, $Q_{11} - Q_{22} = Q_{12} = 0$. Finally, requiring that Q_{ij} is traceless, we can calculate

$$Q_{11} = Q_{22} = -\frac{1}{2} Q_{33} = -\frac{64\pi}{5} a^5 B \tag{M.1.7}$$

Then from (4.10), the potential is

$$\begin{aligned}
\Phi &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{1}{2}\left(Q_{11}\frac{x^2}{r^5} + Q_{22}\frac{y^2}{r^5} + Q_{33}\frac{z^2}{r^5}\right)\right] \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{Q_{11}}{2}\frac{x^2 + y^2 - 2z^2}{r^5}\right) \\
&= \frac{a^3 B}{\epsilon_0}\left(\frac{2}{3r} - \frac{8a^2}{5}\frac{x^2 + y^2 - 2z^2}{r^5}\right) \qquad \text{(M.1.8)}
\end{aligned}$$

□

***Problem* M.2** (Charged cylinder)**:** A hollow cylindrical tube of radius R is centered on the negative z-axis, extending from $z = 0$ to $z = -\infty$. It is open on the ends, with no end caps. Constant surface charge densty σ is attached to the surface of the tube.

(a) Find and evaluate an integral for the electrostatic potential $\Phi(z)$ everywhere along the z-axis. In evaluating the integral, you will find a term that is infinite, but independent of z; drop this term, and explain briefly why it is physically reasonable to do so.

(b) Use the result of part (a) to find an expression for $\Phi(r, \theta)$ for any θ with $r \ll R$ in terms of Legendre polynomials, keeping the first two terms in the expansion in r/R.

Solution.

(a) In the following, we shall use the solution for the potential of a uniformly charged ring in Section 3.3 of Jackson for this problem. Let us also redefine $c \to r'$, $a \to R$, and $r \to z'$. Then $r'^2 = R^2 + z'^2$ where z' is the position of the ring and r' is the distance from points on the ring to the z axis. The charge is $dq' = \sigma 2\pi R dz'$ so that the cylinder is uniformly charged on the surface for $z' = 0$ to $z' \to -\infty$.

Now, if $z < R$, then $r' > z$ by definition. From the results of Jackson 3.3, the contribution from a thin ring located at z' is

$$d\Phi_< = \frac{dq'}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{z^l}{r'^{l+1}} P_l(\cos\alpha') \tag{M.2.1}$$

where $\tan\alpha' = R/z'$. Let $x = \cos\alpha'$. Then the total potential at z is

$$\begin{aligned} \Phi_{z<R} &= \frac{\sigma R}{2\epsilon_0} \sum_{l=0}^{\infty} z^l \int_{-\infty}^{0} dz' (R^2 + z'^2)^{-\frac{l+1}{2}} P_l(\cos\alpha') \\ &= \frac{\sigma R}{2\epsilon_0} \left[\int_{-\infty}^{0} \frac{dz'}{\sqrt{R^2 + z'^2}} + \sum_{l=1}^{\infty} \left(\frac{z}{R}\right)^l \int_0^1 dx (1 - x^2)^{\frac{l-2}{2}} P_l(x) \right] \end{aligned} \tag{M.2.2}$$

The first term is infinite. However, it is independent of z. So we can set it to zero since the potential has to be finite at $z = R$ (because it is continuous at every finite distance). Then we can use Mathematica to evaluate the integral in the infinite series

$$\Phi_{z<R}(z) = \frac{\sigma R}{2\epsilon_0} \sum_{l=1}^{\infty} \left(\frac{z}{R}\right)^l \frac{P_{l-1}(0)}{l} \tag{M.2.3}$$

Note that all even terms vanish here.

Similarly, for $z \geq R$, we need to split the integral into two because there is a domain where $R \leq r' \leq z$, or $-\sqrt{z^2 - R^2}/z < x < 0$. The potential contribution in this domain is

$$d\Phi_> = \frac{dq'}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r'^l}{z^{l+1}} P_l(x) \tag{M.2.4}$$

Then the total potential is

$$
\begin{aligned}
\Phi_{z\geq R} &= \int d\Phi_< + \int d\Phi_> \\
&= \frac{\sigma R}{2\epsilon_0}\sum_{l=0}^{\infty}\left[\int_{-\sqrt{z^2-R^2}} dz' \frac{r'^l}{z^{l+1}}P_l(x) + \int_{-\infty}^{-\sqrt{z^2-R^2}} dz' \frac{z^l}{r'^{l+1}}P_l(x)\right] \\
&= \frac{\sigma R}{2\epsilon_0}\sum_{l=0}^{\infty}\left[\left(\frac{R}{z}\right)^{l+1}\int_{-\sqrt{z^2-R^2}/z}^{0} dx(1-x^2)^{-\frac{l+3}{2}}P_l(x)\right. \\
&\qquad\left. + \left(\frac{z}{R}\right)^{l}\int_{-1}^{-\sqrt{z^2-R^2}/z} dx(1-x^2)^{\frac{l-2}{2}}P_l(x)\right] \\
&= \frac{\sigma R}{2\epsilon_0}\left\{\sum_{l=0}^{\infty}\left[\left(\frac{R}{z}\right)^{l+1}\frac{P_{l+1}(0)}{l+1} - \frac{P_{l+1}(-u)}{l+1}\right] + \sum_{l=1}^{\infty}\frac{P_{l-1}(-u)}{l}\right\} \qquad (u\equiv\sqrt{z^2-R^2}/z) \\
&= \frac{\sigma R}{2\epsilon_0}\left\{\frac{\sqrt{z^2-R^2}}{z} + \sum_{l=1}^{\infty}\left[\left(\frac{R}{z}\right)^{l+1}\frac{P_{l+1}(0)}{l+1} - \frac{P_{l+1}(u)}{l+1} + \frac{P_{l-1}(u)}{l}\right]\right\} \qquad \text{(M.2.5)}
\end{aligned}
$$

where we have again used Mathematica to solve the two integrals in the third equality. The $l = 0$ term in the series with $(z/R)^l$ is also omitted for the same reason as before. In summary,

$$
\Phi_{z<R} = \frac{\sigma R}{2\epsilon_0}\sum_{l=1}^{\infty}\left(\frac{z}{R}\right)^{l}\frac{P_{l-1}(0)}{l} \qquad \text{(M.2.6a)}
$$

$$
\Phi_{z\geq R} = \frac{\sigma R}{2\epsilon_0}\left\{\frac{\sqrt{z^2-R^2}}{z} + \sum_{l=1}^{\infty}\left[\left(\frac{R}{z}\right)^{l+1}\frac{P_{l+1}(0)}{l+1} - \frac{P_{l+1}(u)}{l+1} + \frac{P_{l-1}(u)}{l}\right]\right\} \qquad \text{(M.2.6b)}
$$

Note that at $z = R$, $u = 0$. So (M.2.6a) and (M.2.6b) are equal and the potential is continuous. Also, at $z \to \infty$, $\Phi_{z\geq R} \to \sigma R/2\epsilon_0$, as expected of the potential inside an infinite cylinder.

(b) From (M.2.6a), we let $z \to r$ and add the polar basis function (Legendre polynomials) to write the potential at (r, θ)

$$
\begin{aligned}
\Phi_{r\ll R}(r,\theta) &= \frac{\sigma R}{2\epsilon_0}\sum_{l=1}^{\infty}\left(\frac{r}{R}\right)^{l}\frac{P_{l-1}(0)}{l}P_l(\cos\theta) \\
&\approx \frac{\sigma R}{2\epsilon_0}\left[\frac{r}{R}\cos\theta - \frac{1}{6}\left(\frac{r}{R}\right)^{3}P_3(\cos\theta)\right] \qquad \text{(M.2.7)}
\end{aligned}
$$

□

***Problem* M.3** (Dielectric sphere)**:** A dielectric sphere with radius a and permittivity ϵ has a fixed density of free charge attached to its surface:

$$\sigma_{\text{free}} = \sigma_0 \cos\theta, \tag{M.3.1}$$

with σ_0 a constant. The sphere sits centered at the origin in otherwise empty space.

(a) Find the potential $\Phi(\mathbf{x})$ everywhere in space.

(b) For the region inside the sphere, find the electric field $\mathbf{E}$, electric displacement $\mathbf{D}$ and polarization $\mathbf{P}$. (Hint: it is useful to start by passing to Cartesian coordinates.)

(c) Find the polarization surface charge density σ_{pol} on the surface of the sphere. Is there bulk polarization charge density ρ_{pol} within the sphere? Explain.

Solution.

(a) This charge density has azimuthal symmetry. So we can use the general solution (3.33, Jackson) to the Laplace equation

$$\Phi = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta) \tag{M.3.2}$$

for the potential. Inside the sphere ($r < a$), the radial term with negative power has to vanish because it contains the origin. Outside the sphere ($r \geq a$), the radial term with positive power has to vanish because the potential tends to zero at infinity.

$$\Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{M.3.3a}$$

$$\Phi_{\text{out}} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \tag{M.3.3b}$$

At $r = a$, the potential is continuous

$$\Phi_{\text{in}}\Big|_{r=a} = \Phi_{\text{out}}\Big|_{r=a} \Rightarrow \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta) \tag{M.3.4}$$

Because the Legendre polynomials P_l are orthogonal, the coefficients must be equal and we can write

$$B_l = A_l a^{2l+1} \tag{M.3.5}$$

Also, at $r = a$, the electric displacement follows the boundary condition

$$D_{\text{out}} - D_{\text{in}} = \left[-\epsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial r} + \epsilon \frac{\partial \Phi_{\text{in}}}{\partial r} \right]\Bigg|_{r=a} = \sigma_{\text{free}} \tag{M.3.6}$$

From (M.3.3) and (M.3.5), we then have

$$\sum_{l=0}^{\infty} \left[\epsilon l + \epsilon_0 (l+1) \right] a^{l-1} A_l P_l(\cos\theta) = \sigma_0 \cos\theta = \sigma_0 P_1(\cos\theta) \tag{M.3.7}$$

The only non-trivial term must then be $l = 1$ where

$$A_l = \frac{\sigma_0}{\epsilon + 2\epsilon_0} \Rightarrow B_l = \frac{\sigma_0}{\epsilon + 2\epsilon_0} a^{2l+1} \tag{M.3.8}$$

Then the potential inside and outside the sphere is

$$\Phi_{\text{in}} = A_1 r \cos\theta = \frac{\sigma_0}{\epsilon + 2\epsilon_0} r \cos\theta \tag{M.3.9a}$$

$$\Phi_{\text{out}} = \frac{B_1}{r^2} \cos\theta = \frac{\sigma_0}{\epsilon + 2\epsilon_0} \frac{a^3}{r^2} \cos\theta \tag{M.3.9b}$$

(b) In Cartesian coordinates, the potential (M.3.9a) inside the sphere is

$$\Phi_{\text{in}}(x, y, z) = \frac{\sigma_0}{\epsilon + 2\epsilon_0} z \tag{M.3.10}$$

By definition, the electric field is thus

$$\mathbf{E}_{\text{in}} = -\boldsymbol{\nabla}\Phi_{\text{in}} = -\frac{\sigma_0}{\epsilon + 2\epsilon_0} \hat{\mathbf{z}} \tag{M.3.11}$$

Then from (4.37, Jackson), the electric displacement is

$$\mathbf{D} = \epsilon \mathbf{E} = -\frac{\epsilon}{\epsilon + 2\epsilon_0} \sigma_0 \hat{\mathbf{z}} \tag{M.3.12}$$

and from (4.36, Jackson), the polarization is

$$\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E} = -\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sigma_0 \hat{\mathbf{z}} = -\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sigma_0 \Big(\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}} \Big) \tag{M.3.13}$$

(c) From (4.46, Jackson), the polarization surface charge density at $r = a$ is

$$\sigma_{\text{pol}} = \mathbf{P} \cdot \hat{\mathbf{r}} \Big|_{r=a} = -\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sigma_0 \cos\theta \tag{M.3.14}$$

There is no polarization volume charge density inside the sphere because the polarization $\mathbf{P}$ is uniform. We can also show this explicitly by calculating

$$\boldsymbol{\nabla} \cdot \mathbf{P} = -\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \sigma_0 \left[\frac{\cos\theta}{r^2} \frac{\partial(r^2)}{\partial r} - \frac{1}{r \sin\theta} \frac{\partial(\sin^2\theta)}{\partial\theta} \right] \sim \frac{2\cos\theta}{r} - \frac{2\sin\theta\cos\theta}{r\sin\theta} = 0 \tag{M.3.15}$$

$\square$

***Problem* M.4** (Potential in a rectangular tube)**:** (a) Consider a semi-infinite rectangular tube region defined by $0 < x < a, 0 < y < a$ and $0 < z < \infty$. Construct a Dirichlet Green's function for this region (with $z \to \infty$ treated as part of the boundary) of the form

$$G(\mathbf{x}, \mathbf{x}') = \left(\frac{2}{a}\right)^2 \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi x'}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi y'}{a}\right) g_{m,n}(z, z') \quad \text{(M.4.1)}$$

by deriving an equation for $g_{m,n}(z, z')$ and solving it with suitable boundary conditions (first at $z = 0$ and $z \to \infty$, then at $z = z'$) as well as requiring symmetry between z and z'. For simplicity, let $k^2 \equiv \pi^2(m^2 + n^2)/a^2$.

(b) The potential on the $z = 0$ square is held at a constant value V, while the rest of the walls of the tube are grounded, and the potential vanishes at infinity. Find an expression for $\Phi(\mathbf{x})$ everywhere in the region, and find the limiting behavior at large z. To do this you may either use the Green's function you constructed in part (a), or use separation of variables and impose the boundary conditions.

Solution.

(a) From (1.39, Jackson), the Green function (M.4.1) follows $\boldsymbol{\nabla}'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$. So we can write

$$\begin{aligned}\left(\frac{2}{a}\right)^2 \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi x'}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi y'}{a}\right) \left[\frac{\partial^2}{\partial z'^2} - \frac{\pi^2(m^2+n^2)}{a^2}\right] g_{m,n}(z, z') \\ = -4\pi\delta(x - x')\delta(y - y')\delta(z - z') \qquad \text{(M.4.2)}\end{aligned}$$

By completeness, the delta functions can be expanded into

$$\text{RHS} = -4\pi \left(\frac{2}{a}\right)^2 \sum_{m,n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi x'}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi y'}{a}\right) \delta(z - z') \quad \text{(M.4.3)}$$

Recall from Homework 3, the orthogonality condition is

$$\delta_{m,n} = \left\langle \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi x}{a}\right) \middle| \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi x}{a}\right) \right\rangle = \frac{2}{a} \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx \quad \text{(M.4.4)}$$

Then, we can take the inner product of both the LHS of (M.4.2) and the RHS of (M.4.3) with $\sqrt{2/a}\sin(m'\pi x/a)$, $\sqrt{2/a}\sin(m'\pi x'/a)$, $\sqrt{2/a}\sin(n'\pi y/a)$, and $\sqrt{2/a}\sin(n'\pi y'/a)$ to yield

$$\sum_{m,n=1}^{\infty} \delta_{m,m'}\delta_{m,m'}\delta_{n,n'}\delta_{n,n'} \left(\frac{\partial^2}{\partial z'^2} - k^2\right) g_{m,n}(z, z') = -4\pi \sum_{m,n=1}^{\infty} \delta_{m,m'}\delta_{m,m'}\delta_{n,n'}\delta_{n,n'}\delta(z - z') \quad \text{(M.4.5)}$$

where we have defined $k^2 = \pi^2(m^2 + n^2)/a^2$. It follows that

$$\left(\frac{\partial^2}{\partial z'^2} - k^2\right) g_{m,n}(z, z') = -4\pi\delta(z - z') \quad \text{(M.4.6)}$$

where we have dropped the primes on m and n. Recall from Problem 3.4 in Homework 3, the general solution to this differential equation is

$$g_{m,n}(z,z') = \begin{cases} A_{m,n}e^{kz'} + B_{m,n}e^{-kz'} & z' < z \\ C_{m,n}e^{kz'} + D_{m,n}e^{-kz'} & z' \geq z \end{cases} \tag{M.4.7}$$

Note that we have rewritten the sinh and cosh basis into exponentials for their apparent behavior at ∞. Given a finite $z \in \mathbb{R}$, $g_{m,n}$ has to vanish at $z' \to \infty$. So $C_{m,n}$ must be zero. Also, imposing the Dirichlet boundary condition at $z' = 0$, $g_{m,n}(z, 0) = A_{m,n} + B_{m,n} = 0$. So we now have

$$g_{m,n}(z,z') = \begin{cases} 2A_{m,n}\sinh(kz') & z' < z \\ D_{m,n}e^{-kz'} & z' \geq z \end{cases} \tag{M.4.8}$$

At $z' = z$, $g_{m,n}$ has to be continuous. So we can write

$$D_{m,n} = 2A_{m,n}\sinh(kz)e^{kz} \tag{M.4.9}$$

Also, we proved in Homework 3.4 from (M.4.6) that $g_{m,n}$ has to satisfy the jump condition

$$\lim_{\epsilon\to 0}\left(\left.\frac{\partial g_{m,n}}{\partial z'}\right|_{z'=z+\epsilon} - \left.\frac{\partial g_{m,n}}{\partial z'}\right|_{z'=z-\epsilon} \right) = -4\pi \Rightarrow D_{m,n}e^{-kz} + 2A_{m,n}\cosh(kz) = \frac{4\pi}{k} \tag{M.4.10}$$

From (M.4.9) and (M.4.10), we can solve for

$$A_{m,n} = \frac{2\pi}{k}e^{-kz} \qquad \text{and} \qquad D_{m,n} = \frac{4\pi}{k}\sinh(kz) \tag{M.4.11}$$

Then the final solution for $g_{m,n}$ is

$$g_{m,n}(z,z') = \begin{cases} \frac{4\pi}{k}\sinh(kz')e^{-kz} & z' < z \\ \frac{4\pi}{k}\sinh(kz)e^{-kz'} & z' \geq z \end{cases} \tag{M.4.12}$$

or we can write more succinctly as

$$g_{m,n}(z,z') = \frac{4\pi}{k}\sinh(kz_<)e^{-kz_>} \tag{M.4.13}$$

where $z_< = \min(z, z')$ and $z_> = \max(z, z')$. From (M.4.12), it can be verified that $g_{m,n}$ is symmetric between z and z'

$$g_{m,n}(z',z) = \begin{cases} \frac{4\pi}{k}\sinh(kz)e^{-kz'} & z < z' \\ \frac{4\pi}{k}\sinh(kz')e^{-kz} & z \geq z' \end{cases} \Rightarrow g_{m,n}(z,z') = g_{m,n}(z',z) \tag{M.4.14}$$

The full Green function can then be written as

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= 4\pi\left(\frac{2}{a}\right)^2 \sum_{m,n=1}^{\infty} \frac{\sinh(kz_<)e^{-kz_>}}{k} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi y'}{a}\right) \\ &= \frac{16}{a}\sum_{m,n=1}^{\infty} \frac{\sinh(kz_<)e^{-kz_>}}{\sqrt{m^2+n^2}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi y'}{a}\right) \end{aligned} \tag{M.4.15}$$

(b) From (1.44, Jackson), with $\Phi(\mathbf{x}') = V$ on the surface S at $z' = 0$ with the normal vector $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, $z_< = z'$, $z_> = z$ and the potential is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \left.\frac{\partial G}{\partial z'}\right|_{z'=0} da' \\
&= V \left(\frac{2}{a}\right)^2 \sum_{m,n=1}^{\infty} e^{-kz} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \left[\int_0^a \sin\left(\frac{m\pi x'}{a}\right) dx'\right] \left[\int_0^a \sin\left(\frac{n\pi y'}{a}\right) dy'\right] \\
&= \frac{4V}{\pi^2} \sum_{m,n=1}^{\infty} \frac{\left[1-(-1)^m\right]\left[1-(-1)^n\right]}{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \exp\left(-\frac{\pi\sqrt{m^2+n^2}z}{a}\right)
\end{aligned} \tag{M.4.16}$$

where we have used Mathematica to evaluate the integrals. At large z, the potential decays exponentially and tends to zero, as expected from the boundary condition. □