Homework 2: Phys 7310 (Fall 2021)

Tien Vo

September 8, 2021

***Problem* 2.1** (Image charge for a plane conductor)**:** A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

(c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$, over the whole plane;

(d) the work necessary to remove the charge q from its position to infinity;

(e) the potential energy between the charge q and its image [compare the answer to part (d) and discuss].

(f) Find the answer to part (d) in electron volts for an electron originally one angstrom from the surface.

Solution.

(a) Put an image charge q' at $z = -d$. The total potential is

$$\Phi = k\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{2.1.1}$$

At $z = 0$, $\Phi = 0$ only if $q' = -q$. Thus,

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{2.1.2}$$

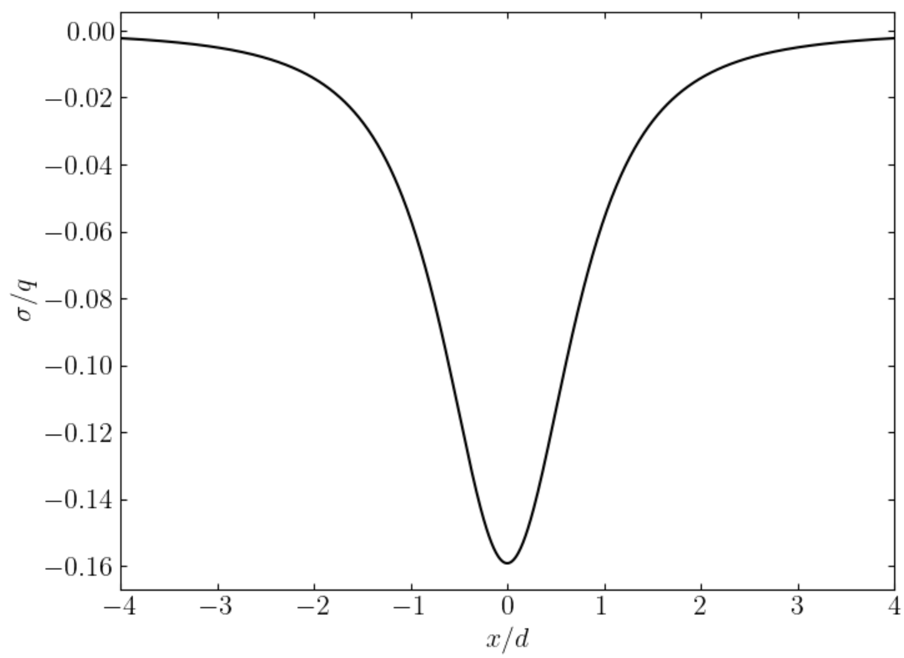
Then we can calculate

$$\begin{aligned}\boldsymbol{\nabla}\Phi = -kq\Bigg\{&(x\hat{\mathbf{x}}+y\hat{\mathbf{y}})\left[\frac{1}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{1}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]\\ &+\left[\frac{z-d}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{z+d}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]\hat{\mathbf{z}}\Bigg\}\end{aligned} \tag{2.1.3}$$

So the surface charge density is

$$\sigma = -\epsilon_0 \, \boldsymbol{\nabla}\Phi\Big|_{z=0} = -\frac{qd}{2\pi}\left(x^2+y^2+d^2\right)^{-3/2} \tag{2.1.4}$$

σ is symmetric over the perpendicular plane. Thus, we need only plot a cross section at $y=0$



(b) Since the distance between the two charges is $2d$, by Coulomb's law, the force acted on the charge q at $z=d$ by the charge $q'=-q$ at $z=-d$ is

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{\mathbf{z}} \tag{2.1.5}$$

(c) From (2.1.4), we can integrate

$$\begin{aligned}
F &= \int_{\mathbb{R}^2} \frac{\sigma^2}{2\epsilon_0} da \\
&= \frac{q^2 d^2}{8\pi^2\epsilon_0}\int_0^\infty \int_0^{2\pi} \frac{r\,dr\,d\theta}{(r^2+d^2)^3} && (r^2 = x^2+y^2) \\
&= \frac{q^2 d^2}{4\pi\epsilon_0}\int_{d^2}^\infty \frac{du}{2u^3} && (u = r^2+d^2) \\
&= \frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2} && (2.1.6)
\end{aligned}$$

This is the same magnitude as that in part (b).

(d) Now, from Coulomb's law, the force acted on a charge q at some arbitrary position z by the charge $q' = -q$ at $z = -d$ is

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(z+d)^2}\hat{\mathbf{z}} \tag{2.1.7}$$

Then the work to move q from d to ∞ is, by definition,

$$W = -\int \mathbf{F}\cdot d\mathbf{l} = \frac{q^2}{4\pi\epsilon_0}\int_d^\infty \frac{dz}{(z+d)^2} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} \tag{2.1.8}$$

$\square$

(e) The potential at $z = d$ due to the charge q' placed at $z = -d$ is

$$\Phi(z = d) = -\frac{1}{4\pi\epsilon_0}\frac{q}{2d} \tag{2.1.9}$$

Thus, the potential energy is $U = q\Phi(z = d) = -q^2/8\pi\epsilon_0 d$. This is consistent with part (d), since to remove this charge, we need to provide a work $W = -U$ to remove the charge from the potential well.

(f) With $q = e$ and $d = 1\,\text{Å}$, $U = -7.2\,\text{eV}$.

***Problem* 2.2** (Energy in electric fields)**:** Starting with an expression for potential energy of a configuration of point charges (Jackson (1.50) or (1.51)), we derived an alternate expression for the same thing in terms of the square of the electric field generated by the charges (Jackson (1.54)). However, along the way of the derivation we "forgot" the restriction to remove the interaction of a charge with itself ("self-energy terms"). Here we explore this.

(a) Below (1.51) is the statement to omit $i = j$ terms, so this expression for a *single* point charge is zero. Now set up the integral for formula (1.54) for the electric field of a single point charge q. Show that this self-energy integral is independent of the location of the point charge, but also that it is infinite.

(b) Now consider two point charges q_1 and q_2; Jackson gives their electric fieldin the unnamed equation on the top of page 42. Following Jackson in equations (1.56-1.58), show that the energy (1.54) in this electric field is equal to two (infinite) self-energies plus an interaction term, and show that the interaction term takes the form you'd expect from equation (1.50). To evaluate the integral in (1.58), show the fact Jackson gives under (1.58) is true, and show that this fact plus our familiar expression $\nabla^2\big(1/|\mathbf{x}-\mathbf{y}|\big) = -4\pi\delta^3(\mathbf{x}-\mathbf{y})$ also implies that

$$\boldsymbol{\nabla}\cdot\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} = 4\pi\delta^3(\mathbf{x}-\mathbf{y}) \tag{2.2.1}$$

Thus the electric field expression for the energy of an electrostatic configuration (1.54) is the same as (1.51) up to the self-energy terms of the point charges. Fortunately, these self-energies are independent of position and therefore constant! Unfortunately they are infinite, but you can't have everything.

Solution.

(a) The electric field in a coordinate where the point charge q is at the origin is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{2.2.2}$$

Then the self-energy is

$$W = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} |E|^2 = \frac{q^2}{32\pi^2\epsilon_0}\int_{\mathbb{R}^3}\frac{1}{r^4} \sim \int d\Omega \int_0^\infty \frac{dr}{r^2} \sim -\frac{1}{r}\bigg|_0^\infty = \lim_{r\to 0}\frac{1}{r} \tag{2.2.3}$$

where $\int d\Omega = 4\pi$ is the solid angle. $W \to \infty$ as $r \to 0$. Note that this is irrespective of the location of the charge, since we can always shift into the frame where it is at the origin and evaluate the energy as in (2.2.3).

(b) Given the electric field (1.56, Jackson), we can calculate

$$\begin{aligned}|\mathbf{E}|^2 &= \frac{1}{16\pi^2\epsilon_0^2}\left[\frac{q_1^2|\mathbf{x}-\mathbf{x}_1|^2}{|\mathbf{x}-\mathbf{x}_1|^6} + \frac{q_2^2|\mathbf{x}-\mathbf{x}_2|^2}{|\mathbf{x}-\mathbf{x}_2|^6} + 2\frac{q_1q_2(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}\right] \\ &= \frac{1}{16\pi^2\epsilon_0^2}\left[\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x}_2|^4} + 2\frac{q_1q_2(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}\right]\end{aligned} \tag{2.2.4}$$

It then follows from $w = \epsilon_0|\mathbf{E}|^2/2$ that

$$32\pi^2\epsilon_0 w = \left[\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_2^2}{|\mathbf{x}-\mathbf{x}_2|^4} + 2\frac{q_1q_2(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}\right] \tag{2.2.5}$$

The first two terms are self-energy. Now, we show that the last term is an interaction term, similar to the form of (1.51, Jackson), by integrating

$$W_{\text{int}} = \frac{q_1q_2}{16\pi^2\epsilon_0}\int_{\mathbb{R}^3}\frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}d\mathbf{x} \tag{2.2.6}$$

Letting $\boldsymbol{\rho} = (\mathbf{x}-\mathbf{x}_1)/|\mathbf{x}_1-\mathbf{x}_2|$ and $\mathbf{n} = (\mathbf{x}_1-\mathbf{x}_2)/|\mathbf{x}_1-\mathbf{x}_2|$, it follows that

$$\boldsymbol{\rho}+\mathbf{n} = \frac{\mathbf{x}-\mathbf{x}_1+\mathbf{x}_1-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|} = \frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}_1-\mathbf{x}_2|} \tag{2.2.7}$$

Also, since $\rho_x|\mathbf{x}_1-\mathbf{x}_2| = x - x_{1,x}$ where $x_{1,x}$ is the x component of $\mathbf{x}_1$, we can write $d\rho_x|\mathbf{x}_1-\mathbf{x}_2| = dx$. More generally,

$$d\boldsymbol{\rho}|\mathbf{x}_1-\mathbf{x}_2|^3 = \big(d\rho_x|\mathbf{x}_1-\mathbf{x}_2|\big)\big(d\rho_y|\mathbf{x}_1-\mathbf{x}_2|\big)\big(d\rho_y|\mathbf{x}_1-\mathbf{x}_2|\big) = dxdydz = d\mathbf{x} \tag{2.2.8}$$

Then we can write

$$\begin{aligned}\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|}\int_{\mathbb{R}^3}\frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{|\boldsymbol{\rho}|^3|\boldsymbol{\rho}+\mathbf{n}|^3}d\boldsymbol{\rho} &= \frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^4}\int_{\mathbb{R}^3}\frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)/|\mathbf{x}_1-\mathbf{x}_2|^2}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3/|\mathbf{x}_1-\mathbf{x}_2|^6}d\mathbf{x} \\ &= \int_{\mathbb{R}^3}\frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}d\mathbf{x}\end{aligned} \tag{2.2.9}$$

and finally,

$$W_{\text{int}} = \left[\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|}\right] \times \left[\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{|\boldsymbol{\rho}|^3|\boldsymbol{\rho}+\mathbf{n}|^3} d\boldsymbol{\rho}\right] \tag{2.2.10}$$

Since the term in the first square bracket already has a form resembling (1.51, Jackson), it remains for us to show that

$$\int_{\mathbb{R}^3} \frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{|\boldsymbol{\rho}|^3|\boldsymbol{\rho}+\mathbf{n}|^3} d\boldsymbol{\rho} = 4\pi \tag{2.2.11}$$

However, first we need two lemmas

Lemma A.

$$\boldsymbol{\nabla}_\rho\left(\frac{1}{|\boldsymbol{\rho}+\mathbf{n}|}\right) = -\frac{\boldsymbol{\rho}+\mathbf{n}}{|\boldsymbol{\rho}+\mathbf{n}|^3} \tag{2.2.12}$$

Proof. Let $\mathbf{R} = \boldsymbol{\rho}+\mathbf{n} = R_x\hat{\mathbf{x}} + R_y\hat{\mathbf{y}} + R_z\hat{\mathbf{z}}$, then it follows that $\boldsymbol{\nabla}_\rho = \boldsymbol{\nabla}_R$ and

$$\boldsymbol{\nabla}_R\left(\frac{1}{R}\right) = -\left(\frac{R_x}{R^3}\hat{\mathbf{x}} + \frac{R_y}{R^3}\hat{\mathbf{y}} + \frac{R_z}{R^3}\hat{\mathbf{z}}\right) = -\frac{\mathbf{R}}{R^3} = -\frac{\boldsymbol{\rho}+\mathbf{n}}{|\boldsymbol{\rho}+\mathbf{n}|^3}$$

□

Lemma B.

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{R}}{R^3}\right) = 4\pi\delta^3(\mathbf{R}) \tag{2.2.13}$$

Proof. From (2.2.12), we can take the divergence on both sides

$$\nabla^2\left(\frac{1}{R}\right) = -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{R}}{R^3}\right)$$

However, we know from (1.31, Jackson) that $\nabla^2\big(1/R\big) = -4\pi\delta^3(\mathbf{R})$. Thus, by substitution, (2.2.13) must be true. □

Now, take a spherical volume V with a boundary $S = \partial V$ of radius ρ, we can write the LHS of (2.2.11) as

$$\text{LHS} = -\int_V \frac{\boldsymbol{\rho}}{\rho^3}\cdot\boldsymbol{\nabla}_\rho\left(\frac{1}{|\boldsymbol{\rho}+\mathbf{n}|}\right)d\boldsymbol{\rho} = -\oint_{\partial V}\frac{\boldsymbol{\rho}\cdot d\mathbf{a}}{|\boldsymbol{\rho}|^3|\boldsymbol{\rho}+\mathbf{n}|} + \int_V \frac{1}{|\boldsymbol{\rho}+\mathbf{n}|}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\rho}}{\rho^3}\right)d\boldsymbol{\rho} \tag{2.2.14}$$

via integration by part. Note that the first term has an integrand evaluated at $\rho = \text{const}$ on the surface $S = \partial V$. In the case that $V \to \mathbb{R}^3$, $\rho \to \infty$, and since the integrand scales as $1/\rho^3$, the boundary integral vanishes. Applying (2.2.13) to the second integral, we get

$$\int_{\mathbb{R}^3} \frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\mathbf{n})}{|\boldsymbol{\rho}|^3|\boldsymbol{\rho}+\mathbf{n}|^3} d\boldsymbol{\rho} = 4\pi\int_{\mathbb{R}^3}\frac{\delta^3(\boldsymbol{\rho})}{|\boldsymbol{\rho}+\mathbf{n}|}d\boldsymbol{\rho} = 4\pi \tag{2.2.15}$$

because $|\mathbf{n}| = 1$. Then from (2.2.10), we can write

$$W_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \tag{2.2.16}$$

in the same form as an interaction potential energy.

□

***Problem* 2.3** (Neumann Green's function)**:** For a general Green's function $G(\mathbf{x}, \mathbf{x}')$ for Poisson's equation, the value of the potential $\Phi(\mathbf{x})$ in a volume V with boundary surface S is related to the charge density $\rho(\mathbf{x})$ inside V and the boundary values for Φ and $\partial\Phi/\partial n$ on S, by the Jackson's equation (1.42). When we know the value of Φ on S (Dirichlet boundary conditions), we can choose a Dirichlet Green's function oberying $G_D(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x}'$ on S, and then find a solution for $\Phi(\mathbf{x})$ in V given ρ and our boundary values of Φ.

When we know $\partial\Phi/\partial n$ on S, these are Neumann boundary conditions. Analogously to the Dirichlet case we might think we would like to find a Green's function $G_N(\mathbf{x}, \mathbf{x}')$ obeying $\partial G_N(\mathbf{x}, \mathbf{x}')/\partial n' = 0$ for $\mathbf{x}'$ on S. However, this is not possible!

(a) Any Green's function for Poisson's equation obeys Jackson (1.39) by definition. Show that this implies that for any Poisson Green's function,

$$\oint_S da' \frac{\partial G}{\partial n'} = -4\pi \tag{2.3.1}$$

Observe this makes it impossible for $\partial G_N(\mathbf{x}, \mathbf{x}')/\partial n'$ to be zero for every $\mathbf{x}'$ on S.

(b) Since $\partial G_N/\partial n'$ cannot be zero for every $\mathbf{x}'$ on the boundary, the simplest condition is for $\partial G_N/\partial n'$ to be constant for every $\mathbf{x}'$ on the boundary, as in Jackson (1.45). Show that Jackson (1.45) indeed satisfies the equation from part (a) (this is easy), and show that this leds to the solution for $\Phi(\mathbf{x})$ in Jackson (1.46). Thus knowing the Neumann boundary data $\partial\Phi/\partial n$ doesn't quite get us $\Phi(\mathbf{x})$ everywhere in V, but it gets us the deviation from the average value on the boundary $\Phi(\mathbf{x}) - \langle\Phi\rangle_S$. (Fortunately a constant shift in the potential drops out of physical quantities!)

Solution.

(a) Using Gauss theorem, we can integrate

$$\int_V \nabla^{2\prime} G d^3x' = \int_V \boldsymbol{\nabla}' \cdot (\boldsymbol{\nabla}' G) d^3x' = \oint_S \boldsymbol{\nabla}' G \cdot \hat{\mathbf{n}} da' = \oint_S \frac{\partial G}{\partial n'} da' \tag{2.3.2}$$

where $\partial G/\partial n' = (\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}')G$ is the directional derivative in the normal direction $\hat{\mathbf{n}}$. From (1.39, Jackson), $\int_V \nabla^{2\prime} G d^3x' = -4\pi \int_V \delta^3(\mathbf{x} - \mathbf{x}') d^3x' = -4\pi$. Thus, it follows that

$$\oint_S da' \frac{\partial G}{\partial n'} = -4\pi \tag{2.3.3}$$

(b) If $\partial G/\partial n'$ is a constant, then we can pull it out of the integral in (2.3.3)

$$\frac{\partial G}{\partial n'} \oint_S da' = \frac{\partial G}{\partial n'} S = -4\pi \tag{2.3.4}$$

where $S = \oint_S da'$ is the surface area of the bounding surface. Thus,

$$\frac{\partial G}{\partial n'} = -\frac{4\pi}{S} \tag{2.3.5}$$

Now, we can write from (1.42, Jackson)

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'}da' - \frac{1}{4\pi}\frac{\partial G}{\partial n'}\oint_S \Phi(\mathbf{x}')da' \\ &= \frac{1}{S}\oint_S \Phi(\mathbf{x}')da' + \frac{1}{4\pi\epsilon_0}\int_V \rho(x')G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'}da' \\ &= \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(x')G(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi(\mathbf{x}')}{\partial n'}da' \qquad (2.3.6)\end{aligned}$$

where $\langle\Phi\rangle_S = (1/S)\oint_S \Phi(\mathbf{x}')da'$ is the average value of the potential on the boundary surface. □

***Problem* 2.4** (Green's function for a plane conductor)**:** Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

(a) Write down the appropriate Green function $G(\mathbf{x},\mathbf{x}')$.

(b) If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z).

(c) Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \qquad (2.4.1)$$

(d) Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2+z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2} + \ldots\right] \qquad (2.4.2)$$

Verify that the results of part (c) and (d) are consistent with each other in their common range of validity.

Solution.

(a) Generalizing from Problem 2.1(a), a potential for a point chage placed at $\mathbf{x} = (x, y, z)$ is

$$\Phi(\mathbf{x}') = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} - \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'+z)^2}}\right] \qquad (2.4.3)$$

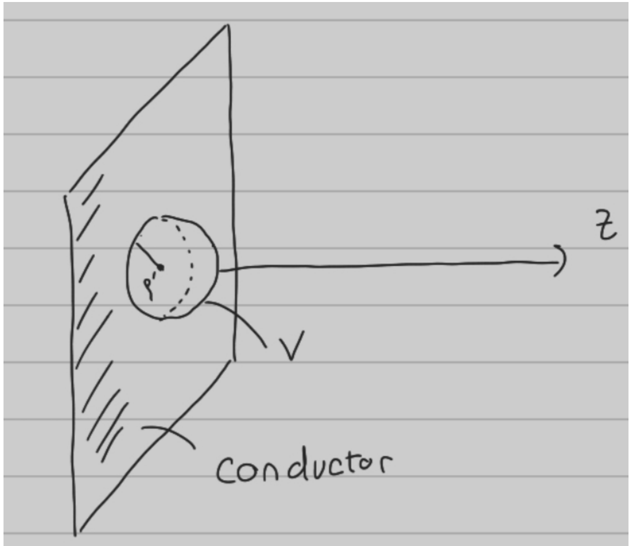
Since $\Phi = (q/4\pi\epsilon_0)G$ for a point charge, we can write the Green function as

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} - \frac{1}{\sqrt{(x'-x)^2+(y'-y)^2+(z'+z)^2}} \qquad (2.4.4)$$

(b) Then we can differentiate

$$\frac{\partial G}{\partial z'} = \frac{z - z'}{\left[(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right]^{3/2}} + \frac{z + z'}{\left[(x'-x)^2 + (y'-y)^2 + (z'+z)^2\right]^{3/2}} \tag{2.4.5}$$

Now, we choose a volume V lying on a conductor at $z = 0$ and is a hemisphere with a radius $\rho' > a$ for $z > 0$ (see figure)



Then as $\rho' \to \infty$, this volume covers the half-space $z \geq 0$ where all the charge distribution is. From (1.42, Jackson),

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d\mathbf{x}' + \frac{1}{4\pi} \oint_{\partial V} \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} \right] da' \\
&= -\frac{1}{4\pi} \oint_{\partial V} \Phi(\mathbf{x}') \frac{\partial G}{\partial n'} da' \\
&= \frac{V}{4\pi} \oint_{\partial V} \left. \frac{\partial G}{\partial z'} \right|_{z'=0} da'
\end{aligned} \tag{2.4.6}$$

where we have written $\partial G / \partial n' = -\partial G / \partial z'$ by the choice of surface ∂V (recall that $\rho' > a$). Now, as $\rho' \to \infty$, we can write from (2.4.5)

$$\Phi(\mathbf{x}) = \frac{V}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{\left[\rho^2 + z^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')\right]^{3/2}} \tag{2.4.7}$$

through a coordinate transformation $(x, y) = \rho(\cos\phi, \sin\phi)$ and $(x', y') = \rho'(\cos\phi', \sin\phi')$.

(c) Setting $\rho = 0$, the integral (2.4.7) can be easily evaluated

$$\Phi = \frac{V}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho'}{(\rho'^2 + z^2)^{3/2}} = V\left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right] \tag{2.4.8}$$

where we have used Mathematica in the last step.

(d) Factoring out $\epsilon = (\rho^2 + z^2)^{-1}$, we can rewrite the integrand I in (2.4.7) as

$$\begin{aligned}
I &= \epsilon^{3/2}\left[1 + \epsilon\Big(\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big)\right]^{-3/2} \\
&\approx \epsilon^{3/2}\left[1 - \frac{3}{2}\epsilon\Big(\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big) + \frac{15}{8}\epsilon^2\Big(\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big)^2 + \dots\right] \qquad (2.4.9)
\end{aligned}$$

where we have expanded in order of the small parameter ϵ. Integrating each term in I up to ϵ^3, we get

$$\begin{aligned}
\Phi &= \frac{V}{2\pi} z\epsilon^{3/2} \int_0^a \int_0^{2\pi} \rho' d\rho' d\phi' \times \\
&\quad \left[1 - \frac{3}{2}\epsilon\Big(\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big) + \frac{15}{8}\epsilon^2\Big(\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big)^2\right] \\
&= \frac{V}{2\pi}\frac{z}{(\rho^2 + z^2)^{3/2}}\Bigg[\pi a^2 - \frac{3}{2}\epsilon \int_0^a \int_0^{2\pi} \rho' d\rho' \Big[\rho'^2 - 2\rho\rho'\cos\big(\phi - \phi'\big)\Big] \\
&\quad + \frac{15}{8}\epsilon^2 \int_0^a \int_0^{2\pi} \rho' d\rho' \Big[\rho'^4 + 4\rho^2\rho'^2\cos^2(\phi - \phi') - 4\rho\rho'\cos\big(\phi - \phi'\big)\Big]\Bigg] \qquad (2.4.10)
\end{aligned}$$

Using the fact that the ϕ-averaged $\big\langle\cos(\phi - \phi')\big\rangle = 0$ and $\big\langle\cos^2(\phi - \phi')\big\rangle = \pi$ over a domain $[0, 2\pi]$, we can simplify the integral

$$\begin{aligned}
\Phi &= \frac{V}{2\pi}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[\pi a^2 - \frac{3\pi a^4}{4}\epsilon + \frac{5\pi a^2}{8}(3\rho^2 a^2 + a^4)\epsilon^2\right] \\
&= \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4}\epsilon + \frac{5}{8}(3\rho^2 a^2 + a^4)\epsilon^2\right] \qquad (2.4.11)
\end{aligned}$$

This is what we wish to shown in (2.4.2). For $\rho = 0$, this reduces to

$$\Phi = V\left[\frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{3}{8}\left(\frac{a}{z}\right)^4 + \frac{5}{16}\left(\frac{a}{z}\right)^6\right] \qquad (2.4.12)$$

From part (c), we can write

$$\begin{aligned}
\Phi &= V\left[1 - \frac{1}{\sqrt{1 + (a/z)^2}}\right] \\
&\approx V\left\{1 - \left[1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{3}{8}\left(\frac{a}{z}\right)^4 - \frac{5}{16}\left(\frac{a}{z}\right)^6 + \dots\right]\right\} \\
&= V\left[\frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{3}{8}\left(\frac{a}{z}\right)^4 + \frac{5}{16}\left(\frac{a}{z}\right)^6\right] \qquad (2.4.13)
\end{aligned}$$

From (2.4.11) and (2.4.13), we conclude the answers for part (c) and (d) agree in their common range of validity. □