Homework 12: Phys 7320 (Spring 2022)

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***Problem* 12.1** (The Lagrangian for a charged particle)**:** The Lagrangian for a charged particle of mass m and charge e with position $\mathbf{r}$ and velocity $\mathbf{u} \equiv d\mathbf{r}/dt$ moving in scalar and vector potentials Φ and $\mathbf{A}$ is

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{u^2}{c^2}} - e\Phi + \frac{e}{c}\mathbf{u} \cdot \mathbf{A}. \tag{12.1.1}$$

(a) Show that the Euler-Lagrange equations for this Lagrangian indeed give rise to the Lorentz force law. *Hint*: I suggest using index notation, and remember the potentials $\Phi, \mathbf{A}$ can depend on both $\mathbf{r}$ and t, which as far as the particle is concerned means they depend on time in two ways: $\Phi(\mathbf{r}(t), t)$ and $\mathbf{A}(\mathbf{r}(t), t)$.

(b) Go though the steps (12.13)-(12.17) in Jackson to derive the Hamiltonian

$$\mathcal{H} = \sqrt{(c\mathbf{P} - e\mathbf{A})^2 + m^2c^4} + e\Phi. \tag{12.1.2}$$

Along the way derive the canonical/conjugate momentum $\mathbf{P}$ (which is different from the familiar "kinematic momentum" $\mathbf{p} = \gamma m\mathbf{u}$), and invert it to find an expression for $\mathbf{u}$ in terms of $\mathbf{P}$ and $\mathbf{A}$.

If you want extra practice, you can show that Hamilton's equations for this Hamiltonian also give rise to the Lorentz force law, with some steps similar to part (a).

Solution.

(a) First, we write $\mathcal{L}$ in Einstein notation with $j \in \{x, y, z\}$

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{u_j^2}{c^2}} - e\Phi + \frac{e}{c}u_jA_j. \tag{12.1.3}$$

Then it is clear that

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = \frac{\partial \mathcal{L}}{\partial u_i} = \frac{mu_i}{\sqrt{1 - u_j^2/c^2}} + \frac{e}{c}A_i \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \gamma m\mathbf{u} + \frac{e}{c}\mathbf{A} = \mathbf{p} + \frac{e}{c}\mathbf{A}. \tag{12.1.4}$$

Thus, the LHS of Euler-Lagrange equation is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = \frac{\partial \mathbf{p}}{\partial t} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c}(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{A}. \tag{12.1.5}$$

Now, for the RHS of Euler-Lagrange equation, we can write

$$\boldsymbol{\nabla}\mathcal{L} = -e\boldsymbol{\nabla}\Phi + \frac{e}{c}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{A}) = -e\boldsymbol{\nabla}\Phi + \frac{e}{c}\big[(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{A})\big], \tag{12.1.6}$$

where we have used the vector calculus identity $\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$. Now, combining (12.1.5) and (12.1.6), we can write

$$\frac{\partial\mathbf{p}}{\partial t} = e\left(-\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) + e\frac{\mathbf{u}}{c}\times(\boldsymbol{\nabla}\times\mathbf{A}) = e\left(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}\right), \tag{12.1.7}$$

which is Lorentz force law.

(b) From (12.1.4), we have already found the canonical momentum

$$\mathbf{P} = \gamma m\mathbf{u} + \frac{e}{c}\mathbf{A}. \tag{12.1.8}$$

Thus,

$$\mathbf{u} = \frac{c\mathbf{P} - e\mathbf{A}}{\gamma mc} \Rightarrow u^2 = \frac{(c\mathbf{P} - e\mathbf{A})^2}{\gamma^2 m^2 c^2}. \tag{12.1.9}$$

But also, the Lorentz factor by definition is

$$\gamma^2 = \frac{1}{1 - u^2/c^2} = \frac{\gamma^2 m^2 c^4}{\gamma^2 m^2 c^4 - (c\mathbf{P} - e\mathbf{A})^2}. \tag{12.1.10}$$

Inverting, we get

$$\gamma = \sqrt{1 + \left(\frac{\mathbf{P}}{mc} - \frac{e\mathbf{A}}{mc^2}\right)^2}. \tag{12.1.11}$$

Thus, $\mathbf{u}$ in terms of $\mathbf{P}$ and $\mathbf{A}$ is

$$\mathbf{u} = \frac{c\mathbf{P} - e\mathbf{A}}{\sqrt{m^2c^2 + \big(\mathbf{P} - e\mathbf{A}/c\big)^2}}. \tag{12.1.12}$$

Then, starting from (12.15, Jackson),

$$\begin{aligned}
\mathcal{H} &= \mathbf{P}\cdot\mathbf{u} - \mathcal{L} \\
&= \frac{\mathbf{u}}{c}\cdot(c\mathbf{P} - e\mathbf{A}) + \frac{mc^2}{\gamma} + e\Phi \\
&= \frac{(c\mathbf{P} - e\mathbf{A})^2}{\sqrt{m^2c^4 + (c\mathbf{P} - e\mathbf{A})^2}} + \frac{m^2c^4}{\sqrt{m^2c^4 + (c\mathbf{P} - e\mathbf{A})^2}} + e\Phi \\
&= \sqrt{m^2c^4 + (c\mathbf{P} - e\mathbf{A})^2} + e\Phi.
\end{aligned} \tag{12.1.13}$$

□

***Problem* 12.2** (Equivalent Lagrangians)**:** (a) Use the Principle of Least Action (really the principle of extremal action) to show that if the Lagrangian $\mathcal{L}$ is changed by adding the time derivative of some function of the coordinates and time, then the Euler-Lagrange equations are unchanged. The new and old Lagrangians are said to be *equivalent*. Generalize this to a statement about what change to a Lagrangian *density* $\mathcal{L}$ leaves the EL equations unchanged.

(b) Show that under a gauge transformation, the Lagrangian for a charged particle given in the previous problem becomes an equivalent Lagrangian, thus showing the equations of motion do not change.

Solution.

(a) Writing the new Lagrangian as $\overline{\mathcal{L}} = \mathcal{L} + \partial\Gamma(\mathbf{r}, t)/\partial t$, the action is

$$\overline{S} = \int_{t_i}^{t_f} dt\overline{\mathcal{L}} = \int_{t_i}^{t_f} dt\mathcal{L} + \Gamma\Big|_{t_i}^{t_f} = S + \Gamma(\mathbf{r}_f, t_f) - \Gamma(\mathbf{r}_i, t_i). \tag{12.2.1}$$

Thus,

$$\frac{\delta\overline{S}}{\delta\mathbf{r}(t)} = \frac{\delta S}{\delta\mathbf{r}(t)} = 0, \tag{12.2.2}$$

if the original action S is extremized, since $\Gamma\big|_{t_i}^{t_f}$ is constant under variations of the path $\mathbf{r}(t)$ such that $\delta\mathbf{r}(t_i) = \delta\mathbf{r}(t_f) = \mathbf{0}$.

(b) Under the gauge transformation

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\Lambda \qquad \text{and} \qquad \Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \tag{12.2.3}$$

such that Λ follows (6.18, Jackson), the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= -mc^2\sqrt{1-\beta^2} - e\Phi + \frac{e}{c}\mathbf{u}\cdot\mathbf{A} \\
&= -mc^2\sqrt{1-\beta^2} - e\Phi' + \frac{e}{c}\mathbf{u}\cdot\mathbf{A}' - \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right)\left(\frac{e}{c}\Lambda\right) \\
&= \mathcal{L}' - \frac{d\Gamma}{dt},
\end{aligned} \tag{12.2.4}$$

where $\Gamma = (e/c)\Lambda$, and the new Lagrangian is

$$\mathcal{L}' = -mc^2\sqrt{1-\beta^2} - e\Phi' + \frac{e}{c}\mathbf{u}\cdot\mathbf{A}'. \tag{12.2.5}$$

Thus, $\mathcal{L}'$ only differ from $\mathcal{L}$ by $d\Gamma/dt$, and is equivalent to it, leaving the equation of motion unchanged. $\square$

***Problem* 12.3** ($SO(2)$ symmetry of two real scalar fields.)**:** Consider the dynamics of two (real) scalar fields $\phi_1(\mathbf{x}, t)$ and $\phi_2(\mathbf{x}, t)$ specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2), \tag{12.3.1}$$

where the potential V depends only on the combination $\phi_1^2 + \phi_2^2$. In class, we will study this case with the real scalars combined into a single compex scalar; here we will leave them as two real scalars. Let's make a definite choice for the potential:

$$V(\phi_1, \phi_2) = \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}m^2\phi_2^2 + \frac{\lambda}{2}\left(\phi_1^2 + \phi_2^2\right)^2. \tag{12.3.2}$$

(a) Calculate the equations of motion (Euler-Lagrange equations) for both ϕ_1 and ϕ_2.
(b) Show that the $SO(2)$ transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{12.3.3}$$

where α is a constant, is a symmetry of the Lagrangian. Is this a rotation in physical space? What space does this "rotation" act on?

(c) According to Noether's theorem, the existence of this symmetry means there is a corresponding conserved current J^μ. Find J^μ in terms of ϕ_1 and ϕ_2 (you may drop an overall constant α) and show that it is conserved, $\partial_\mu J^\mu = 0$, when you use the equations of motion.

Solution.

□