Homework 5: Astr 5140 (Fall 2021)

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***Problem* 1** (Poynting's Theorem)**:** Using the energy density as $u = (1/2)(\epsilon_0 E^2 + B^2/\mu_0)$ and Maxwell's equations, derive Poynting's theorem.

Solution.

By definition, the divergence of $\mathbf{S}$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{S} &= \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) \\
&= \frac{1}{\mu_0} \left[(\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{B} - (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{E} \right] \\
&= -\frac{1}{\mu_0} \left[\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \left(\mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathbf{E} \right] \\
&= -\frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{J} \cdot \mathbf{E} \\
&= -\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0} + \epsilon_0 \mathbf{E} \cdot \mathbf{E} \right) - \mathbf{J} \cdot \mathbf{E} \qquad (1.1)
\end{aligned}$$

Thus, we arrive at Poynting's theorem

$$-\frac{\partial u}{\partial t} = \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E} \qquad (1.2)$$

where $u = (1/2)(\epsilon_0 E^2 + B^2/\mu_0)$. □

***Problem* 2** (Energy Conservation at Shocks I)**:** The forth equation of the shock jump conditions involves energy conservation. This problem derives the energy conservation equation considering bulk motion (u_x) and thermal motion ($\mathbf{w}$). Consider a plasma flowing in the x direction with a speed of u_x, which is the bulk or fluid speed. The particles also have thermal energy, with a thermal speed in each dimension (or degree of freedom) defined such that

$$m \left\langle w_{x,y,z}^2 \right\rangle = k_B T \qquad (2.1)$$

The definition of $\mathbf{w}$ is such that $\langle \mathbf{w} \rangle = \mathbf{0}$ and the total thermal energy density in the rest frame of the fluid is

$$\frac{1}{2} nm \left\langle w_x^2 \right\rangle + \frac{1}{2} nm \left\langle w_y^2 \right\rangle + \frac{1}{2} nm \left\langle w_z^2 \right\rangle = \frac{3}{2} k_B nT = \frac{3}{2} P \qquad (2.2)$$

(a) Argue that the flux of thermal energy due to motion in the y direction or z direction is simply

$$u_x\left[\frac{1}{2}nm\left\langle w_y^2\right\rangle+\frac{1}{2}nm\left\langle w_z^2\right\rangle\right]=u_xP \tag{2.3}$$

(b) Now consider the energy flux due to motion of a single particle in the x direction. Explain why it can be described as

$$\frac{1}{2}m(u_x+w_x)^3 \tag{2.4}$$

(c) Show that the energy flux due to the fluid motion in the x direction is $u_x(1/2\rho u_x^2+3/2P)$

(d) Using the defintion that $\gamma=(N+2)/N$ (N is the number of degrees of freedom), show that the particle energy flux is

$$u_x\left[\frac{1}{2}\rho u_x^2+\frac{\gamma}{\gamma-1}P\right] \tag{2.5}$$

Solution.

Recall from HW1 that the 3D velocity distribution function can be written in the form $f(\mathbf{v})=ng_x(v_x)g_y(v_y)g_z(v_z)$ where

$$g_i(v_i)=\sqrt{\frac{m}{2\pi T_i}}\exp\left[-\frac{m(v_i-u_i)^2}{2T_i}\right] \tag{2.6}$$

normalizes to unity with $i\in\{x,y,z\}$. Then the energy flux due to motion in the y or z direction is

$$\begin{aligned}\Phi_{yz}&=\int\frac{1}{2}m(v_y^2+v_z^2)v_xf(\mathbf{v})d^3v\\&=\frac{1}{2}nm\int v_xg_xdv_x\left(\int v_y^2g_ydv_y+\int v_z^2g_zdv_z\right)\\&=u_x\left[\frac{1}{2}nm\left\langle w_y^2\right\rangle+\frac{1}{2}nm\left\langle w_z^2\right\rangle\right]\\&=u_xP\end{aligned} \tag{2.7}$$

(b) For a single particle, we replace the distribution function f with the Dirac delta function $\delta^3(\mathbf{v}'-\mathbf{v})$ where $\mathbf{v}=\mathbf{u}+\mathbf{w}=u_x\hat{\mathbf{x}}+\mathbf{w}$. Then the energy flux per unit volume V due to motion in the x direction is

$$\begin{aligned}\frac{\Phi_x}{V}&=\int\frac{1}{2}mv_x'^3\delta^3(\mathbf{v}'-\mathbf{v})d^3v'\\&=\frac{1}{2}m\int v_x'^3\delta(v_x'-(u_x+w_x))dv_x\int\delta(v_y'-w_y)dv_y\int\delta(v_z'-w_z)dv_z\\&=\frac{1}{2}m(u_x+w_x)^3\end{aligned} \tag{2.8}$$

(c) For a distribution of particles, the energy flux in the x direction is then

$$
\begin{aligned}
\Phi_x &= \int \frac{1}{2} m v_x^3 f(\mathbf{v}) d^3 v \\
&= \frac{1}{2} n m \int v_x^3 g_x dv_x \int g_y dv_y \int g_z dv_z \\
&= \frac{1}{2} n m \sqrt{\frac{m}{2\pi T_x}} \int v_x^3 \exp\left[-\frac{m(v_x - u_x)^2}{2T_x}\right] dv_x \\
&= \frac{1}{2} n m \sqrt{\frac{m}{2\pi T_x}} \int (u_x + w_x)^3 e^{-m w_x^2/2T_x} dw_x && (w_x = v_x - u_x) \\
&= \frac{1}{2} n m \left[u_x^3 + 3u_x^2 \langle w_x \rangle + 3u_x \langle w_x^2 \rangle + \langle w_x^3 \rangle\right] \\
&= u_x \left(\frac{1}{2}\rho u_x^2 + \frac{3}{2} n k_B T\right) \\
&= u_x \left(\frac{1}{2}\rho u_x^2 + \frac{3}{2} P\right) && (2.9)
\end{aligned}
$$

where $\langle w_x \rangle = \langle w_x^3 \rangle = 0$ because the integrand is odd over an even domain ($\mathbb{R}$) and we have written $\rho = nm$.

(d) From (2.2), we can generally write the thermal energy as $E_{\text{therm}} = (N/2)P$ where N is the number of degrees of freedom. Then the energy flux due to motion other than x can be written as the difference between the total energy flux and the x contribution

$$
\Phi_{\text{not } x} = u_x \left[\frac{N}{2} P - \frac{1}{2} n m \langle w_x^2 \rangle\right] = \frac{N-1}{2} u_x P \tag{2.10}
$$

Combining this with part (c), the total energy flux is

$$
\Phi_{\text{total}} = \Phi_x + \Phi_{\text{not } x} = u_x \left(\frac{1}{2}\rho u_x^2 + \frac{N+2}{2} P\right) = u_x \left(\frac{1}{2}\rho u_x^2 + \frac{\gamma}{\gamma - 1} P\right) \tag{2.11}
$$

where γ is as defined in the problem. □

***Problem* 3** (Energy Conservation at Shocks II)**:** Use Poynting's theorem and the ideal generalized Ohm's law to calculate the electromagnetic energy flux for a quasi-perpendicular shock. Combine your result with the previous problem for the energy equation of shocks.

Solution.

Let $\mathbf{u}_1 = u_1 \hat{\mathbf{x}}$ and $\mathbf{B}_1 = B_1 \hat{\mathbf{z}}$ be the upstream fluid flow and magnetic field. Then by ideal Ohm's law, the electric field is $\mathbf{E}_1 = -\mathbf{u}_1 \times \mathbf{B}_1 = u_1 B_1 \hat{\mathbf{y}}$. The Poynting vector, or energy flux density, is

$$
\mathbf{S}_1 = \frac{\mathbf{E}_1 \times \mathbf{B}_1}{\mu_0} = \frac{u_1 B_1^2}{\mu_0} \hat{\mathbf{x}} \tag{3.1}
$$

So the electromagnetic energy flux through an area $d\mathbf{A}$ normal to the fluid flow is just $\mathbf{S}_1 \cdot d\mathbf{A}$. From Poynting's theorem,

$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t} \tag{3.2}$$

since $\mathbf{J} \sim \nabla \times \mathbf{B} = \mathbf{0}$. Since $\int u d^3x$ is the electromagnetic energy stored in the field, this says that the rate of depletion of the energy density is dependent on the Poynting vector. However, since energy is conserved, this depleted energy has to be transferred to the particles. Combining with the result from Problem 2, we can write

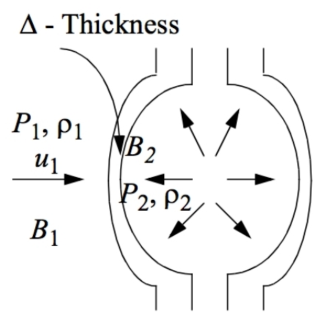
$$\frac{\partial}{\partial x}\left(S_x + \Phi_{\text{particle}}\right) = \frac{\partial}{\partial x}\left[u_x\left(\frac{1}{2}\rho u_x^2 + \frac{\gamma}{\gamma-1}P + \frac{B^2}{\mu_0}\right)\right] = 0 \tag{3.3}$$

or more generally,

$$\nabla \cdot \left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma}{\gamma-1}P\right)\mathbf{u} + \mathbf{S}\right] = 0 \tag{3.4}$$

□

***Problem* 4** (Super Nova Shells)**:** A super nova shell has two shocks, a forward shock which plows into a weakly-magnetized interstellar medium and a reverse shock in which the accelerated remnants of the star plow into the shock.



Conservation of momentum (or sometimes called pressure balance) sets one of the jump conditions of the forward shock

$$P_1 + \rho_1 u_1^2 + \frac{B_1^2}{2\mu_0} = P_2 + \rho_2 u_2^2 + \frac{B_2^2}{2\mu_0} \tag{4.1}$$

(a) The late-stage forward shock of a strongly-magnetized supernova shell (ρ_2, P_2, u_2, and B_2) moves at high M_s into a weaky-magnetized interstellar medium (ρ_1, P_1, u_1, and B_1), having plowed the magnetic field into a shell. Simplify the momentum equation by keeping only one dominant term on each side of the equation. Explain why.

(b) Use magnetic flux conservation to derive a simple, geometric-based relationship between the shell thickness (Δ), the radius of the shell (R), B_1, and B_2.

(c) Combine the results above to derive a relation between the relative shell thickness (Δ/R) and the Alfvén Mach speed. If the ISM is weakly magnetized, do we expect thin ($\Delta/R \ll 1$) or thick ($\Delta/R \sim 1$) shells? Explain.

Solution.

(a) Since $M_s \gg 1$, $\rho u^2 \gg P$. The flow dominates the pressure term. So both P_1 and P_2 can be neglected. Since the shell is highly magnetized, the magnetic field term dominates. The opposite occurs in the ISM. So then we can write

$$\rho_1 u_1^2 = \frac{B_2^2}{2\mu_0} \tag{4.2}$$

(b) Assuming the shell is spherical, the magnetic flux of the ISM through it is just $\Phi_{\text{ISM}} = \pi R^2 B_1$, since a sphere has a cross-sectional area of πR^2. However, inside the thickness Δ of the shell, the cross section is $2\pi R\Delta$. So the flux of the magnetic field inside the shell is $2\pi R\Delta B_2$. Then by conservation of the magnetic flux,

$$\pi R^2 B_1 = 2\pi R\Delta B_2 \Rightarrow \frac{B_2}{B_1} = \frac{R}{2\Delta} \tag{4.3}$$

(c) Now, the Alfvén Mach number is defined as

$$M_A^2 = \frac{1}{2}\frac{\rho_1 u_1^2}{B_1^2/2\mu_0} = \frac{1}{2}\frac{B_2^2}{B_1^2} = \frac{1}{8}\frac{R^2}{\Delta^2} \Rightarrow \frac{\Delta}{R} = \frac{1}{2\sqrt{2}M_A} \tag{4.4}$$

If the ISM is weakly magnetized, $M_A \gg 1$, so $\Delta/R \ll 1$. This is because the electric field inside the shell accelerates ions to high enough energy and the magnetic field ejects them out into the ISM. So the shell can't be very thick. □

***Problem* 5** (Energy in Magnetic Reconnection)**:** Derive a simple expression for energy conservation in magnetic reconnection. Justify the approximations that are made.

(a) The incoming energy flux can be written as

$$u_{\text{in}}\left(\frac{\rho_{\text{in}}u_{\text{in}}^2}{2} + \frac{\gamma}{\gamma-1}P_{\text{in}} + \frac{B_{\text{in}}^2}{\mu_0}\right) \tag{5.1}$$

where $\mathbf{B}_{\text{in}}$ is the component parallel to the incoming surface. Write a similar expression for the out-going energy flux. Derive an approximate expression balancing the incoming and out-going energy flux given a diffusion region size as indicated above.

(b) Use the continuity equation to show that $\Delta\rho_{\text{in}}u_{\text{in}} = \delta\rho_{\text{out}}u_{\text{out}}$.

(c) Show that if $u_{\text{in}} \ll V_A$ then the kinetic energy of the inflow, $u_{\text{in}}(1/2\rho_{\text{in}}u_{\text{in}}^2)$, can be neglected.

(d) Argue from a geometric standpoint or by using $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ that

$$\left|\frac{B_{\text{out}}}{B_{\text{in}}}\right| \approx \frac{\delta}{\Delta} \tag{5.2}$$

so that the out-going magnetic energy is negligible (no need to be overly rigorous).

(e) Show that the above equation reduces to

$$\frac{B_{\text{in}}^2}{\mu_0\rho_{\text{in}}} + \frac{\gamma}{\gamma-1}\frac{P_{\text{in}}}{\rho_{\text{in}}} \approx \frac{u_{\text{out}}^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_{\text{out}}}{\rho_{\text{out}}} \tag{5.3}$$

(f) Through observation and through kinetic analysis, one can demonstrate that the outflow velocity is often at the Alfvén speed. If one ignores the contribution of the pressure terms, is the energy equation balanced? If not, what can you conclude about the temperatures or pressures of the inflow versus the outflow.

Solution.

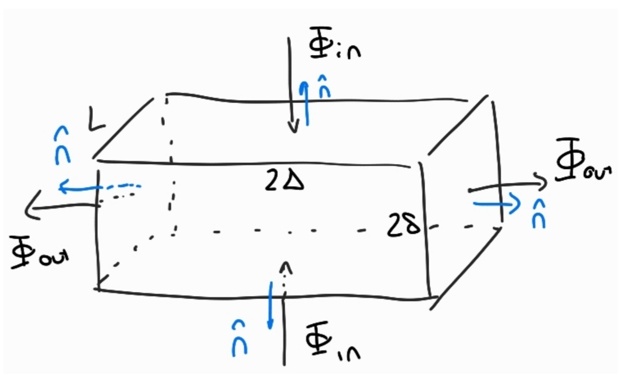
(a) From (5.1), the out-going energy flux is

$$u_{\text{out}}\left(\frac{\rho_{\text{out}}u_{\text{out}}^2}{2}+\frac{\gamma}{\gamma-1}P_{\text{out}}+\frac{B_{\text{out}}^2}{\mu_0}\right) \tag{5.4}$$

Let us use the general expression in (3.4), $\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_{\text{total}}=0$ where $\boldsymbol{\Phi}_{\text{total}}$ is the energy density flux of both the particles and the electromagnetic field. By Gauss theorem,

$$\int \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}_{\text{total}}d^3x=\oint \boldsymbol{\Phi}_{\text{total}}\cdot d\mathbf{A}=0 \tag{5.5}$$

where the final equality assumes that the energy transport only happens between the particles and the field.



Since the dimensions of the reconnection site are small enough that $\boldsymbol{\Phi}_{\text{in}}$ and $\boldsymbol{\Phi}_{\text{out}}$ are constant on the surface (see the figure above), the surface integral is approximately

$$\oint \boldsymbol{\Phi}_{\text{total}}\cdot d\mathbf{A}\approx -2\Phi_{\text{in}}2\Delta L+2\Phi_{\text{out}}2\delta L=0 \tag{5.6}$$

So we can relate the incoming and out-going energy flux as

$$\Delta u_{\text{in}}\left(\frac{\rho_{\text{in}}u_{\text{in}}^2}{2}+\frac{\gamma}{\gamma-1}P_{\text{in}}+\frac{B_{\text{in}}^2}{\mu_0}\right)=\delta u_{\text{out}}\left(\frac{\rho_{\text{out}}u_{\text{out}}^2}{2}+\frac{\gamma}{\gamma-1}P_{\text{out}}+\frac{B_{\text{out}}^2}{\mu_0}\right) \tag{5.7}$$

(b) Similar to the arguments in part (a), the continuity equation states that $\boldsymbol{\nabla}\cdot(\rho\mathbf{u})=0$. So we can integrate on the surface of the reconnection site

$$\oint(\rho\mathbf{u})\cdot d\mathbf{A}\approx -4\rho_{\text{in}}u_{\text{in}}+4\delta\rho_{\text{out}}u_{\text{out}}=0\Rightarrow \Delta\rho_{\text{in}}u_{\text{in}}=\delta\rho_{\text{out}}u_{\text{out}} \tag{5.8}$$

(c) Factoring out ρ_{in} in the LHS of (5.7), we get

$$\Delta\rho_{\text{in}}u_{\text{in}}\left(\frac{u_{\text{in}}^2}{2}+\frac{\gamma}{\gamma-1}\frac{P_{\text{in}}}{\rho_{\text{in}}}+V_A^2\right)\approx\Delta\rho_{\text{in}}u_{\text{in}}\left(\frac{\gamma}{\gamma-1}\frac{P_{\text{in}}}{\rho_{\text{in}}}+V_A^2\right) \tag{5.9}$$

if $u_{\text{in}} \ll V_A$.

(d) Since $\mathbf{B}_{\text{in}}$ is perpendicular to the normal vector of the upper and lower surface in the Figure in part (a) and similarly for $\mathbf{B}_{\text{out}}$, when evaluating the surface integral, we get

$$\oint \boldsymbol{\nabla}\cdot\mathbf{B}\cdot d\mathbf{A}\approx -4\delta B_{\text{in}}+4\Delta B_{\text{out}}=0\Rightarrow\left|\frac{B_{\text{out}}}{B_{\text{in}}}\right|=\frac{\delta}{\Delta} \tag{5.10}$$

(e) Part (b) makes the pre-factors in (5.7) cancel. Part (c) and (d) make the terms u_{in} and B_{out}^2 vanish. So combining these results, (5.7) reduces to

$$\frac{\gamma}{\gamma-1}\frac{P_{\text{in}}}{\rho_{\text{in}}}+\frac{B_{\text{in}}^2}{\mu_0\rho_textin}\approx\frac{u_{\text{out}}^2}{2}+\frac{\gamma}{\gamma-1}\frac{P_{\text{out}}}{\rho_{\text{out}}} \tag{5.11}$$

(f) Ignoring the pressure terms, we get a non-sensical result that $u_{\text{out}}^2 = 2V_A^2 \neq V_A^2$. Thus, using ideal gas law $P = nT$, we can write

$$\frac{\gamma}{\gamma-1}(T_{\text{out}}-T_{\text{in}})=\frac{mV_A^2}{2}\Rightarrow T_{\text{out}}-T_{\text{in}}=\frac{\gamma-1}{\gamma}\frac{mV_A^2}{2} \tag{5.12}$$

□