Homework 1: Phys 5210 (Fall 2021)

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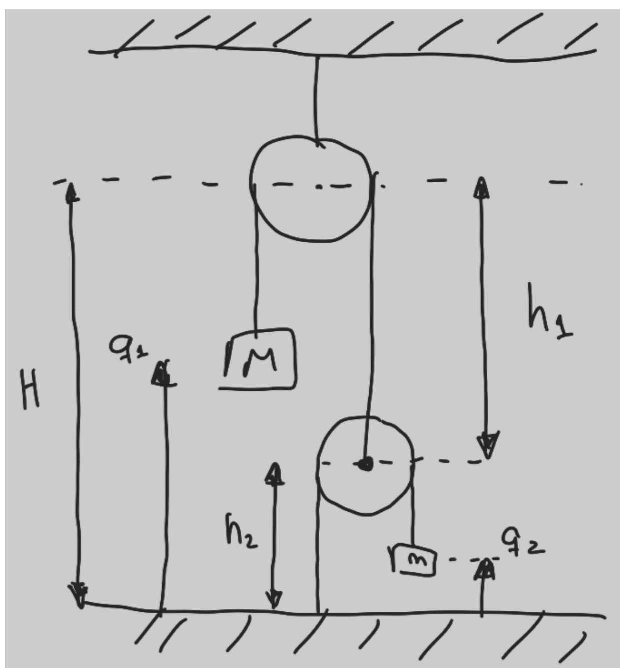
***Problem* 1:** Masses M and m are connected to a system of strings and pulleys as shown. The strings are massless and inextensible, and the pulleys are massless and frictionless. The "floor" and the "ceiling" shown in the figure are fixed and cannot move. Find the accelerations of m and M. The bottom and top panels are fixed in place and cannot move.



Solution.

Let q_1, q_2, h_1, h_2 be as described in the figure. $h_1 + h_2 = H$ is a constant. Also, let L_1, L_2 be the length of the strings hung on the first and second pulley. The following contraints follow

$$L_1 - h_1 = H - q_1 \qquad \text{and} \qquad 2h_2 - q_2 = L_2 \tag{1.1}$$

We can combine these two constraints into

$$q_2 = 4H - L_2 - 2L_1 - 2q_1 = C - 2q_1 \tag{1.2}$$

where C is a constant. Thus, the system only has one degree of freedom, where $\dot{q}_2 = -2\dot{q}_1$ and $\ddot{q}_2 = -2\ddot{q}_1$. The Lagrange function is thus

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}M\dot{q}_1{}^2 + \frac{1}{2}m\dot{q}_2{}^2 - Mgq_1 - mgq_2 \\ &= \frac{1}{2}(M+4m)\dot{q}_1{}^2 - Mgq_1 - mgC + 2mgq_1 \end{aligned} \tag{1.3}$$

From Euler-Lagrange equation, we can write

$$(M+4m)\ddot{q}_1 = (2m-M)g \tag{1.4}$$

The acceleration of the masses are then

$$\ddot{q}_1 = \frac{2m-M}{4m+M}g \tag{1.5a}$$

$$\ddot{q}_2 = -\frac{4m-2M}{4m+M}g \tag{1.5b}$$

□

***Problem* 2:** A tunnel was dug between New York and Los Angeles such that a train which starts at rest in New York and moves along the tunnel by the gravitational pull only arrives in Los Angeles in the shortest possible time, neglecting air resistance and any other kind of friction. Following the discussion of the brachistochrone problem in Section 2.2 of Goldstein, calculate how far below the surface would the deepest point of the tunnel be if the distance between New York and Los Angeles is 4800 km apart, while the radius of the Earth is approximately 6400 km. Calculate also the time it would take for this train to travel between New York and Los Angeles. Mass of the Earth is $M \approx 6 \times 10^{24}$ kg.

Solution.

Assume the mass density, ρ, is uniform. Then at a radius $r \leq R$ where R is the Earth's radius, the enclosed mass within a volume $4\pi r^3/3$ is

$$M = M_E \frac{r^3}{R^3} \tag{2.1}$$

where M_E is the total mass of the Earth. Now, the gravitational force on a train at r with mass m is

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\boldsymbol{\nabla}V \tag{2.2}$$

So the gravitational potential is

$$V = -\int_0^r \mathbf{F}\cdot\hat{\mathbf{r}}dr' = \frac{GM_E}{R^3}\int_0^r r'dr' = \frac{GM_E m}{2R^3}r^2 = Cr^2 \tag{2.3}$$

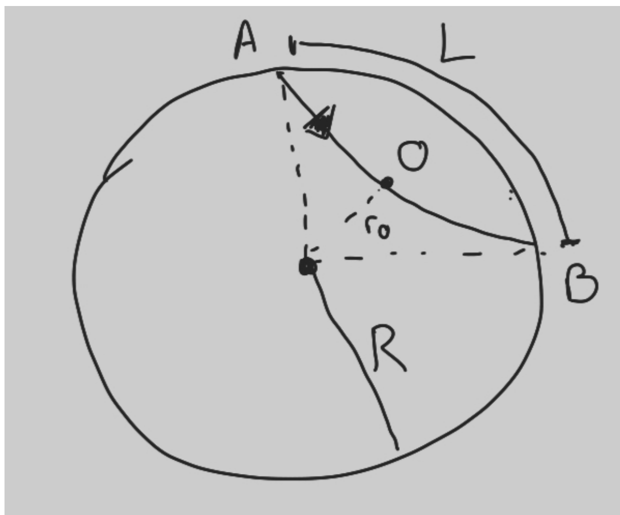
where we have defined the constant $C \equiv GM_e m/2R^3$. The total energy is thus

$$E = \frac{1}{2}mv^2 + Cr^2 \tag{2.4}$$

At departure on the surface ($r = R$), the train has zero speed, so $E_0 = CR^2$. Then we can solve for the velocity

$$v = \sqrt{\frac{2C}{m}}\sqrt{R^2 - r^2} \tag{2.5}$$

The train descends into the Earth to get from A (New York) to B (Los Angeles). At some point O ($r = r_0$), it reaches the deepest point ($r = r_0$) into the Earth (see figure). Now,



since the force is conservative, the time it takes from A to O must be the same as that from O to B. This time is

$$t_{O\to B} = \int_O^B \frac{ds}{v} \tag{2.6}$$

The length element in polar coordinates is $ds^2 = dr^2 + r^2 d\theta^2$. Setting $dr/d\theta = r'$, we can write

$$t_{O\to B} = \sqrt{\frac{m}{2C}} \int_{\theta_O}^{\theta_B} d\theta \frac{\sqrt{r^2 + r'^2}}{v} = \sqrt{\frac{m}{2C}} \int_{\theta_O}^{\theta_B} d\theta \sqrt{\frac{r^2 + r'^2}{R^2 - r^2}} \tag{2.7}$$

Let the integrand be

$$f(r, r') = \sqrt{\frac{r^2 + r'^2}{R^2 - r^2}} \Rightarrow \frac{\partial f}{\partial r'} = \frac{1}{f}\frac{r'}{R^2 - r^2} \tag{2.8}$$

The time $t_{O\to B}$ is minimized when

$$\tilde{E} = f - r'\frac{\partial f}{\partial r'} = f\left[1 - \frac{r'^2}{f^2(R^2 - r^2)}\right] = \frac{r^2}{\sqrt{(R^2 - r^2)(r^2 + r'^2)}} \tag{2.9}$$

is constant. We can thus evaluate $\tilde{E}$ at $r = r_0$, where $r' = 0$ (because it is the lowest point)

$$\tilde{E}\bigg|_{r=r_0} = \frac{r_0}{\sqrt{R^2 - r_0^2}} \tag{2.10}$$

From (2.9) and (2.10), we can invert to find the differential equation

$$r' = \frac{dr}{d\theta} = \frac{Rr}{r_0}\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}} \tag{2.11}$$

Integrating, we get

$$\begin{aligned}
\int_{\theta_O}^{\theta_B} d\theta &= \frac{r_0}{R}\int_{r_0}^{R} \frac{dr}{r}\sqrt{\frac{R^2 - r^2}{r^2 - r_0^2}} \\
&= -\frac{r_0}{R}\frac{r_0 - R}{r_0}\sin^{-1}\left[\frac{\sqrt{(r_0^2 - R^2)^2}}{r_0^2 - R^2}\right] && \text{(from Mathematica)} \\
&= -\frac{r_0 - R}{R}\sin^{-1}(-1) \\
&= \frac{\pi}{2}\left(\frac{r_0}{R} - 1\right) && (2.12)
\end{aligned}$$

Note that $\theta_B < \theta_O$, so the RHS is also negative because $r_0 < R$. By geometry, $\theta_O - \theta_B = \Delta\theta = L/2R$ where L is the arc length from A to B, so we can invert to find

$$r_0 = R - \frac{L}{\pi} = 6,400\,\text{km} - \frac{4,800\,\text{km}}{\pi} = 4,872\,\text{km} \tag{2.13}$$

Now that r_0 is known, we can plug (2.11) into (2.7) to get

$$\begin{aligned}
t_{O\to B} &= \sqrt{\frac{m}{2C}}\int_{r_0}^{R} dr\frac{d\theta}{dr}\sqrt{\frac{r^2 + r'^2}{R^2 - r^2}} \\
&= \frac{\sqrt{R^2 - r_0^2}}{R}\sqrt{\frac{m}{2C}}\int_{r_0}^{R}\frac{rdr}{\sqrt{(r^2 - r_0^2)(R^2 - r^2)}} \quad = \frac{\pi}{2}\sqrt{1 - \left(\frac{r_0}{R}\right)^2}\sqrt{\frac{R^3}{GM_E}} \approx 824\,\text{s}
\end{aligned} \tag{2.14}$$

Then the total time it takes to get from A to B is $t_{A\to B} = 2t_{O\to B} \approx 27.5\,\text{min}$. □