Beam project notes

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Abstract

This contains my derivations of plasma waves generated from electron beams and their corresponding instabilities.

Contents

1 Kinetic plasma dispersion relation

First, we establish the conditions for the formulation of waves in a kinetic plasma. Specifically, the focus is on the instability in beam-plasma processes and beam evolution. In this section, we will heavily rely on Benz textbook for the theoretical discussion in Benz (1993). Under a classical, macroscopic electromagnetic field, the Vlasov equation describes the dynamics of one particle species f_s in configuration space $(\mathbf{r}_s, \mathbf{p}_s = m\mathbf{v}_s)$ is

$$\frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \boldsymbol{\nabla} f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}_s} \tag{1.1}$$

where we have assumed a collisionless distribution f_s of particles. Now, we suppose the system is linearized with a homogenous and stationary unperturbed distribution f_s^0 and write $f_s(\mathbf{r}_s, \mathbf{v}_s, t) = f_s^0(\mathbf{v}_s) + f_s^1(\mathbf{r}_s, \mathbf{v}_s, t)$. Suppose electric oscillations are aligned along the background magnetic field ($\mathbf{k} \parallel \mathbf{B}_0$). Then Eq. (1.1) under a Fourier transform (such that $\boldsymbol{\nabla} \to i\mathbf{k}$ and $\partial_t \to -i\omega$) is

$$-i\omega f_s^1 + i\mathbf{k} \cdot \mathbf{v}_s f_s^1 - i\frac{q_s\phi}{m_s}\mathbf{k} \cdot \frac{\partial f_s^0}{\partial \mathbf{v}_s} = 0 \tag{1.2}$$

where $\phi(\mathbf{r}_s, t)$ is the scalar potential such that $\mathbf{E} = -\boldsymbol{\nabla}\phi$. We can then invert and solve for f_s^1

$$f_s^1 = \frac{q_s\phi}{m_s}\frac{\mathbf{k} \cdot \partial f_s^0/\partial \mathbf{v}_s}{\mathbf{k} \cdot \mathbf{v}_s - \omega} \tag{1.3}$$

which normalizes to the density n_s. From Poisson's equation $k^2 = 4\pi \sum_s q_s n_s$, we can write the general dispersion relation of a kinetic plasma

$$k^2 = \sum_s \frac{4\pi q_s^2}{m_s} \int_{\mathbb{R}^3} d\mathbf{v}_s \frac{\mathbf{k} \cdot \partial f_s^0/\partial \mathbf{v}_s}{\mathbf{k} \cdot \mathbf{v}_s - \omega} \tag{1.4}$$

This corresponds directly with $H\big(k, \omega/k\big)$ from (5.2.16) in Benz (1993). Under an extension to the complex field, suppose $\omega = \Omega_k + i\gamma_k$ and the integral can be evaluated in the curve $C = \mathbb{R}^3 \cap \mathbb{C}^3$. Since it is only of physical interest that $\partial f_s^0/\partial \mathbf{v}_s$ is analytic over $\mathbb{C}^3$, the integrand in Eq. (1.4) is meromorphic with a pole at $\mathbf{k} \cdot \mathbf{v}_s = \omega$. From Sokhotski-Plemeji theorem, the integral evaluates as

$$\int_C d\mathbf{v}_s \frac{\mathbf{k} \cdot \partial f_s^0/\partial \mathbf{v}_s}{\mathbf{k} \cdot \mathbf{v}_s - \omega} = \mathcal{P}\int_C d\mathbf{v}_s \frac{\mathbf{k} \cdot \partial f_s^0/\partial \mathbf{v}_s}{\mathbf{k} \cdot \mathbf{v}_s - \omega} + i\pi \left.\frac{\partial f_s^0}{\partial \mathbf{v}_s}\right|_{\mathbf{k}\cdot\mathbf{v}_s = \omega} \tag{1.5}$$

where $\mathcal{P}$ is the Cauchy principal value and we have assumed $k > 0$.

Now, we consider the formulation of wave modes from the general plasma dispersion relation in Eq. (1.4) applied to Maxwellian particle distributions. For simplicity, let us also assume a one-dimensional system where $C = \mathbb{R} \cap \mathbb{C}$ and

$$f_s^0 = \frac{n_s}{\sqrt{\pi} v_{th,s}} e^{-\eta_s^2} \tag{1.6}$$

where $\eta_s = (v_s - u_s)/v_{th,s}$, u_s is the drift speed, and $v_{th,s} = \sqrt{2T_s/m_s}$ is the thermal speed. Also, let $\lambda_s = (\omega/k - u_s)/v_{th,s}$. Then the dispersion relation now reads $k^2 = \sum_s k_s^2$ where

$$k_s^2 = \frac{4\pi q_s^2}{m_s v_{th,s}} \int_C d\eta_s \frac{\partial f_s^0/\partial \eta_s}{\eta_s - \lambda_s} \tag{1.7}$$

Similar to Eq. (1.5), the integral evaluates as

$$\int_C d\eta_s \frac{\partial f_s^0/\partial \eta_s}{\eta_s - \lambda_s} = \mathcal{P} \int_C d\eta_s \frac{\partial f_s^0/\partial \eta_s}{\eta_s - \lambda_s} + i\pi \left. \frac{\partial f_s^0}{\partial \eta_s} \right|_{\eta_s = \lambda_s} \tag{1.8}$$

Since f_s^0 is known, the principal part is

$$\mathcal{P} \int_C d\eta_s \frac{\partial f_s^0/\partial \eta_s}{\eta_s - \lambda_s} = -\frac{2n_s}{\sqrt{\pi} v_{th,s}} \mathcal{P} \int_C d\eta_s \frac{\eta_s}{\eta_s - \lambda_s} e^{-\eta_s^2} \tag{1.9}$$

Note that we have assumed $\omega = \Omega_k + i\gamma_k$ where $\Omega_k, \gamma_k \in \mathbb{R}$. The electric field, for example, can be written as

$$E(z,t) = E_0 e^{i(kz - \omega t)} = E_0 e^{i(kz - \Omega_k t)} e^{-\gamma_k t} \tag{1.10}$$

The real frequency Ω_k contributes to the phase velocity $v_\phi = \Omega_k/k$ of the generated waves. The imaginary term γ_k describes either the damping or growth of the field magnitude depending on its sign. This mechanism is called Landau damping. From the form of Eq. (1.5), it is apparent that this imaginary term comes from the contribution of the pole at the resonant phase velocity v_ϕ. Thus, it is closely connected to particles in resonance with the generated waves. In the wave frame, these particles are able to effectively exchange energy with the wave, either making it grow or damp away exponentially.

2 Plasma waves

In this section, we consider some wave modes permitted by the kinetic plasma dispersion relation and their growth rate in different regimes. The background population usually consists of electrons and ions. Electrons, being the less inertial particles (since $m_e/m_i \sim 1/1836$), are more mobile. They can lose and gain energy to and from the waves much more easily than the ions. Thus, the most dominant plasma wave mode is the electron oscillations and we can consider background ions as stationary.

2.1 Background electrons

Now, background electrons have $u_e = 0$. So $\lambda_e = \omega/kv_{th,e}$. Generally, the electron beam of interest has $u_{be} \gg v_{th,e}$. Thus, we also assume $|\lambda_e| \gg 1$. Under a Taylor expansion, the integral in Eq. (1.9) becomes

$$\mathcal{P} \int_C d\eta_e \frac{\eta_e}{\eta_e - \lambda_e} e^{-\eta_e^2} = -\frac{1}{\lambda_e} \mathcal{P} \int_C d\eta_e e^{-\eta_e^2} \sum_{j=0}^{\infty} \frac{\eta_e^{j+1}}{\lambda_e^j} \approx -\frac{\sqrt{\pi}}{2} \left(\frac{1}{\lambda_e^2} + \frac{3}{2} \frac{1}{\lambda_e^4} \right) \tag{2.1}$$

The contribution from the pole at $\eta_e = \lambda_e$ scales as $e^{-\lambda_e^2}$. Thus, in terms of magnitude, it is negligible and we can write the contribution of the background electrons to the dispersion relation as

$$k_e^2 = \frac{k^2\omega_{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{k^2 v_{th,e}^2}{\omega^2}\right) \tag{2.2}$$

Neglecting the other species' contributions, we can invert the expression and write

$$\omega^2 = \omega_{pe}^2\left(1 + \frac{3}{2}\frac{k^2 v_{th,e}^2}{\omega^2}\right) \tag{2.3}$$

The first term is the solution for a cold plasma, which results in $\omega = \omega_{pe}$. The second term gives the next non-trivial correction for $T_e \neq 0$ which, to the first order, is the Bohm-Gross dispersion relation

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2}k^2 v_{th,e}^2 \tag{2.4}$$

Now, to approximate the imaginary term, we use the cold plasma solution $\omega = \omega_{pe}$ and write the electron contribution to k^2 in full form

$$k_e^2 = \frac{k^2\omega_{pe}^2}{\omega^2}\left(1 + \frac{i\pi}{n_e v_{th,e}}\frac{\omega^2}{k^2}\left.\frac{\partial f_e^0}{\partial \eta_e}\right|_{\eta_e=\lambda_e}\right) \tag{2.5}$$

Again, ignoring the other contributions, we can invert and find

$$\omega = \left(\omega_{pe}^{-2} - \frac{i\pi}{k^2 n_e v_{th,e}}\left.\frac{\partial f_e^0}{\partial \eta_e}\right|_{\eta_e=\lambda_e}\right)^{-1/2} = \left(\omega_{pe}^{-2} - i\epsilon\right)^{-1/2} \tag{2.6}$$

As established above, ϵ is small. So a Taylor expansion brings the solution to the form

$$\omega \sim \omega_{pe}\left(1 + \frac{i\pi}{2}\frac{\omega_{pe}^2}{k^2 n_e v_{th,e}}\left.\frac{\partial f_e^0}{\partial \eta_e}\right|_{\eta_e=\lambda_e}\right) \tag{2.7}$$

Then it follows that

$$\gamma_k = -\sqrt{\pi}\frac{1}{(k\lambda_D)^3}\exp\left[-\frac{1}{(k\lambda_D)^2}\right] < 0 \tag{2.8}$$

where $\lambda_D = v_{th,e}/\omega_{pe}$ is the Debye length. The growth rate is always negative, which means this wave mode is damped everywhere. This is only valid for our assumption regarding λ_e, which is when $k\lambda_D \ll 1$. For $k\lambda_D \geq 0.5$, the rate grows linearly before it decreases exponentially and our assumption of small ϵ is violated.

References

Benz, A. 1993, Plasma Astrophysics: Kinetic Processes in Solar and Stellar Coronae, Vol. 184 (Springer Science and Business Media Dordrecht)