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The mechanics of inertial motion on the earth and on a rotating sphere

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Abstract

The Hamiltonian of the horizontal motion on the earth in the absence of friction and pressure gradient force is introduced as a modification of the Hamiltonian of the corresponding motion on a rotating sphere. This approach affords a clear identification of the dynamical effects of the earth's rotation (i.e. Coriolis and centrifugal forces) versus those associated with horizontal component of gravity due to the eccentricity of the earth surface. Using the methods of analytical mechanics on manifolds, we find the exact solution of this 2-degrees-of-freedom integrable system. These considerations permit, in addition, a decomposition of the general trajectory into a motion (oscillation or rotation) along a great circle and the rotation of this great circle relative to the sphere or the earth. In the case of a rotating sphere, the particle rotates along the great circle with constant velocity independent of the great circle's rotation relative to the sphere. In contrast, on the earth the velocity of the particle along the great circle (which can be either rotation or oscillation) varies with time and is coupled with the rotation of the great circle relative to the earth. We also show that on the earth the motion along the great circle can easily be reduced to the classical problem of simple pendulum. The geophysical implication of our results is that the mid-latitude inertial oscillations on the earth — oscillations that remain indefinitely in one hemisphere — originate from the earth's minute eccentricity since they do not exist on a rotating sphere.

Keywords: Inertia; Hamiltonian; Earth; Geophysics

1. Introduction

The dynamics associated with the horizontal inertial motion along the surface of the rotating earth is the primordial problem of geophysics, dynamical meteorology and physical oceanography. (In accordance with the, somewhat inaccurate, geophysical terminology the term 'inertial' is used here to indicate that only the centrifugal, Coriolis and gravitational forces act on the particle when it moves along the geopotential surface.) Despite its fundamental nature in these fields, a complete mechanical description of this problem has not yet been formulated, which left some dynamical issues only very poorly explained. Several attempts were recently made to elucidate some of the subtle aspects of this problem [1,2] but an overall mechanical description is still not available. In the practical application of the same dynamical equations for numerical weather prediction and in ocean current simulations the

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Newtonian form of the equations is employed which is not suitable for mechanical analysis but more convenient for numerical computations.

Traditionally, problems in analytical mechanics were formulated in planar geometry. Recent developments in the field of differential geometry led to the formulation of dynamics on manifolds, extensively elaborated in [3,4]. In recent years new examples of mechanical systems in which the configuration manifolds are curved were added to the classical examples of rigid body and simple and spherical pendulums (see [5,6]). These examples include the motion of vortices on a sphere — another problem that has its origin in geophysical dynamics [7–10].

The horizontal motion on the rotating earth provides natural example of dynamics on curved manifold, which in addition permits exact solution and clear geometric interpretation. The surface of the earth (i.e. the geopotential) is well approximated by an ellipsoid of very small eccentricity. Nevertheless, in the investigation of the horizontal motion along the geopotential surface the spherical approximation is always used and the difference between the ellipsoid and the sphere appears as an additional term in the equations of motion (see e.g. [11]).

In this study, we first investigate the free motion on a rotating sphere, completely ignoring the terms in the equations that represent the earth's slight departure from a perfect sphere. This part provides the formulae for the action–angle variables of this system and the geometric interpretation of the angle variables used subsequently in the investigation of inertial motion on the earth. A comparison between the trajectories on the earth and on a rotating sphere affords a distinction between the dynamic effects of the inertial motion that originate from earth's ellipticity and those that originate from its rotation.

The investigation of the motion under the gravitation force on the surface of the rotating earth has a fairly long history. In his pioneering work from 1917, Whipple [12] solved the equations of motion for small latitudinal departure from the equator in terms of the elliptic functions and plotted trajectories using tabulated values of elliptic functions. In spite of the author's cautionary warning that "it would not be right to use them (i.e. the solutions) in high latitudes" it became clear from works carried out in subsequent decades that the main types of trajectories of the system (except for the polar trajectories) appeared already in Whipple's turn-of-the-past-century work.

The various geophysical trajectories of the inertial system were classified in [13], by integrating numerically the governing momentum equations starting from different locations and velocities. In addition to the classification based on numerical integration, some analytical results were obtained in the vicinity of the equator by employing the equatorial β-plane approximation. Very similar results were also obtained in [14].

The next step in the investigation of the inertial motion was made in [15], where the types of trajectories, in mid-latitudes and near the equator were classified analytically by employing the conserved values of the energy and angular momentum which are set by the initial velocity and latitudes. The solution trajectories were obtained numerically without any approximation whatsoever and matched with the analytical classification. Exact analytical solutions were obtained, for the first time, in [16] and these exact solutions were matched with some of the numerical trajectories calculated in [15].

The inaccuracies associated with the β-plane approximations were the subject of several recent studies [1,2,17,18]. These studies have provided a new, and more accurate, understanding of the origin of some fundamental motions on the rotating earth. One such motion is the inertial circle motion in mid-latitudes where the centrifugal and Coriolis forces balance one another and the trajectory oscillates between two latitudes in the same hemisphere. On the β-plane these inertial circles migrate westward and this migration was attributed to the β effect, i.e. the variation of the Coriolis parameter with latitudes. The emerging scenario from the studies cited above casts a doubt on the suggested linkage between the β effect and the westward migration. An alternate linkage between these migrating circles and the gravitational force was established in [19].

In contrast to all previous works that consider the problem of inertial motion on the earth as a system of differential equation, we approach the same problem from the point of view of Hamiltonian mechanics. It is clear that this approach does not involve any local approximation of the problem such as the β-plane model. On the other hand, in accordance with Liouville's theorem, this approach enables a construction of the global action–angle variables that provide exact solutions of a problem in quadratures. It reduces the complex particle motion on the Earth or on a rotating sphere to a trivial quasi-periodic motion on the surface of a torus. The mechanical consideration of the problem, and particularly the transformation to action–angle variables, yields the following:

1. A clear geometric decomposition of all possible trajectories (i.e. both those that cross the equator and those that remain in one hemisphere) to pendulum-like motion along the great circle and the rotation of this great circle itself. In fact, these two motions are precisely the motions along the two circles that constitute the torus encountered when Liouville's theorem is applied to the rotating sphere.
2. Exact solutions of the system in the form of elliptic integrals. Such solutions were also obtained in [16] but without a clear physically motivated formulation. In addition, our solutions yield the exact formulae for the zonal (westward in mid-latitudes) angular drift, which is derived here for the first time.
3. A distinction between the dynamic effects of the rotation of the earth and those of its ellipticity. It is shown that all cross-equatorial trajectories exist also on a rotating sphere, while the mid-latitude oscillations appear only due to the earth minute ellipticity. This conclusion suggests that the mid-latitudes inertial circles do not exist on a rotating sphere.

2. Dynamics in a rotating frame of reference in polar-coordinates

2.1. The Lagrangian

Let $\mathbf{v}_i$ be the velocity vector of a free particle in 3D space measured from an inertial coordinate system. The Lagrangian of the motion of this particle is

$$L = \tfrac{1}{2}(\mathbf{v}_i, \mathbf{v}_i), \tag{2.1}$$

where (,) is the scalar product of the two vectors.

Denote by $\mathbf{r}$ and $\mathbf{v}$ the position and velocity, respectively, of a particle in a rotating frame and by Ω the vector of angular velocity of the rotating frame ($\omega = |\Omega|$). The velocity of the particle in the inertial coordinate system can be expressed in terms of $\mathbf{r}$, $\mathbf{v}$ and Ω by

$$\mathbf{v}_i = \mathbf{v} + [\Omega, \mathbf{r}], \tag{2.2}$$

where [,] is the cross-product of the two vectors.

Substituting (2.2) into (2.1), one gets the Lagrangian in the rotating frame

$$L = \tfrac{1}{2}|\mathbf{v}|^2 + (\Omega, [\mathbf{r}, \mathbf{v}]) + \tfrac{1}{2}|[\Omega, \mathbf{r}]|^2. \tag{2.3}$$

Let (x, y, z) be the Cartesian coordinates chosen such that the direction of the z-axis coincides with the direction of Ω, and (r, λ, φ) be the polar-coordinates system (here the geographic convention is adopted where φ and λ are the latitude and longitude angles, respectively). The relationship between the two coordinate sets is

$$x = r\cos\varphi\cos\lambda, \qquad y = r\cos\varphi\sin\lambda, \qquad z = r\sin\varphi.$$

Hereafter, we consider the particle motion on the surface of the sphere, so, without loss of generality, $r = 1$ will be assumed hereafter.

In polar-coordinates, $\Omega = (0, 0, \omega)$, $|[\Omega, \mathbf{r}]| = \omega r\cos\varphi$ and $(\Omega, [\mathbf{r}, \mathbf{v}]) = \omega r^2\dot{\lambda}\cos^2\varphi$ so the Lagrangian (2.3) can, for $r = 1$, be expressed as

$$L = \tfrac{1}{2}(\dot{\varphi}^2 + \cos^2\varphi \cdot (\dot{\lambda} + \omega)^2). \tag{2.4}$$

This, polar-coordinate, form of the Lagrangian will be employed in the remainder of this work.

2.2. Hamiltonians and canonical equations

The Lagrangian (2.4) yields the momenta

$$D = \frac{\partial L}{\partial \dot{\lambda}} = \cos^2\varphi(\dot{\lambda} + \omega) \tag{2.5}$$

and

$$P = \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi}. \tag{2.6}$$

Since the derivative in (2.5) is carried out with respect to the angular variable λ it yields the angular momentum relative to the axis of rotation — D.

The Hamiltonian of free motion on a rotating sphere is obtained by taking the Legendre transformation of the Lagrangian, i.e. $H = \dot{\lambda}D + \dot{\varphi}P - L$, where (φ, P) and (λ, D) are the two pairs of conjugate variables. Thus, the Hamiltonian is given by

$$H = \tfrac{1}{2}(\cos^2\varphi \cdot (\dot{\lambda} + \omega)^2 + \dot{\varphi}^2 - 2\cos^2\varphi\, \omega(\dot{\lambda} + \omega)) \tag{2.7}$$

or

$$H = \frac{1}{2}\left(P^2 + \frac{D^2}{\cos^2\varphi}\right) - \omega D. \tag{2.8}$$

Unlike the angular momentum D, the energy of a moving particle depends on the frame of reference (see e.g. [20]). Denote by E the energy in the inertial frame of reference, then

$$E = H + \omega D = \frac{1}{2}\left(P^2 + \frac{D^2}{\cos^2\varphi}\right). \tag{2.9}$$

The canonical equations corresponding to the Hamiltonian (2.8) are

$$\dot{\varphi} = \frac{\partial H}{\partial P} = P, \quad \dot{P} = -\frac{\partial H}{\partial \varphi} = -\frac{\sin 2\varphi}{2} \cdot \frac{D^2}{\cos^4\varphi}, \quad \dot{\lambda} = \frac{\partial H}{\partial D} = \frac{D}{\cos^2\varphi} - \omega, \qquad \dot{D} = \frac{\partial H}{\partial \lambda} = 0. \tag{2.10}$$

As is commonly accepted in geophysics, we define u and v to be the zonal (i.e. east–west) and meridional (i.e. north–south) velocity components relative to the sphere of radius 1, so these components are given by

$$v = \dot{\varphi}, \qquad u = \cos\varphi \cdot \dot{\lambda}.$$

Using this notation, the Hamiltonian (2.8) can be written as

$$H = \frac{1}{2}\left(P^2 + \frac{D^2}{\cos^2\varphi}\right) - \omega D = \frac{1}{2}(u^2 + v^2) - \frac{1}{2}\omega^2\cos^2\varphi. \tag{2.11}$$

The last term in the Hamiltonian, $U = -\frac{1}{2}\omega^2\cos^2\varphi$, is the potential energy corresponding to the force $F_t = -\mathrm{d}U/\mathrm{d}\varphi = -\omega^2\cos\varphi\sin\varphi$. This force is the tangential component of the centrifugal force $\omega^2\cos\varphi$ on a unit sphere and is directed everywhere towards the equator. In contrast to a rotating sphere, on the earth this tangential force is exactly balanced by the tangential, pole-ward directed, component of gravity, which owes its origin to the ellipsoidal shape of the earth. Fig. 1 provides an illustration to these geometric arguments.

Aside form this centrifugal force, all other forces acting on particles on the earth are identical with those acting on particles on a sphere. In geophysics, this slight modification of gravity by the centrifugal force is the sole contribution of the earth's ellipsoidal shape and aside from it the earth is considered to be the sphere. This is possible since

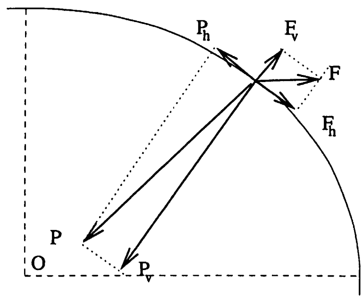

Fig. 1. The forces on the geopotential. In this figure, F is the centrifugal force, P the gravitational force, F_h, P_h, F_v, P_v are their tangential and vertical components, respectively.

the earth's eccentricity (i.e. the deviation of the ratio between its long and short radii from 1.0) is only 0.0033. Accordingly, on the earth the free particle motion along the ellipsoidal surface is replaced by the forced motion on the sphere with the Hamiltonian

$$H = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}\left(P^2 + \frac{D^2}{\cos^2\varphi} + \omega^2\cos^2\varphi\right) - \omega D. \tag{2.12}$$

This Hamiltonian can also be derived directly from the momentum equations of geophysical dynamics [17].

The energy in the inertial frame of reference is related to the Hamiltonian by $E = H + \omega D$. Therefore, in the case of inertial motion on the earth

$$E = \frac{1}{2}\left(P^2 + \frac{D^2}{\cos^2\varphi} + \omega^2\cos^2\varphi\right). \tag{2.13}$$

Both Hamiltonian functions on a rotating sphere (2.11) and on the earth (2.12) are independent of the λ coordinate and, therefore, this coordinate is cyclic. Hence, in both cases there exist two integrals of motion: H and D, which classifies these cases as integrable Hamiltonian systems.

The canonical equations on the earth corresponding to the Hamiltonian (2.12) are

$$\dot\varphi = \frac{\partial H}{\partial P} = P, \qquad \dot P = -\frac{\partial H}{\partial \varphi} = \frac{\sin 2\varphi}{2}\cdot\left(\omega^2 - \frac{D^2}{\cos^4\varphi}\right), \qquad \dot\lambda = \frac{\partial H}{\partial D} = \frac{D}{\cos^2\varphi} - \omega,$$
$$\dot D = -\frac{\partial H}{\partial \lambda} = 0. \tag{2.14}$$

Only the equation for P differs from its counterpart on the sphere in (2.10).

3. Polar trajectories on a rotating sphere and on the earth

Since in the general case the calculations and subsequent analyses are quite complex, it is constructive to start out with a simple case where the basic features of the dynamics on the earth and on a rotating sphere (and the essential differences between them) are demonstrated straightforwardly. This simple case is obtained when the trajectories pass through the poles, namely, for $D = 0$.

Setting $D = 0$ in systems (2.10) and (2.14) one gets that the equations for λ and D de-couple from the equations for φ and P and that $\dot\lambda = -\omega$. Since the angular zonal velocity, $\dot\lambda$, does not vary with φ, a particle with $D = 0$ in both the earth and a rotating sphere moves along a great circle that passes through the poles and rotates about the rotation axis with angular velocity $-\omega$.

On a rotating sphere the equation for P becomes: $\dot{P} = 0$, so $\dot{\varphi} = P =$ constant. The φ-coordinate simply increases linearly with time.

These conclusions are obvious if we consider the motion of a particle with $D = 0$ from an inertial frame of reference. In the inertial frame, the zonal velocity, u, vanishes on the equator, so that the particle simply moves in this frame along the great circle with constant speed.

In contrast to this obvious scenario on the sphere, setting $D = 0$ in (2.14), one gets on the earth that φ changes with time according to equation: $\ddot{\varphi} = \frac{1}{2}\omega^2 \sin 2\varphi$. To gain a better appreciation of the dynamics of this case, it is useful to transform the $(-\pi/2, \pi/2)$ interval over which φ varies to the standard $(0, 2\pi)$ interval. Thus, defining $\xi = 2\varphi + \pi$, one gets that ξ satisfies the equation of a simple (mathematical) pendulum (i.e. a particle that moves without friction on a circle in the vertical plane subject to the downward acting gravity):

$$\ddot{\xi} = -\omega^2 \sin \xi.$$

The pendulum has two types of motion: oscillation about the lowest point $\xi = 0$ and rotation around the center of the circle. In addition, it has a stable fixed point at $\xi = 0$ and an unstable fixed point at $\xi = \pi$ (see [22]).

As we have mentioned above if $D = 0$ a particle on the earth, as well as on the sphere, resides all the time on the great circle that rotates around the earth's axis and passes through the poles. Hence, there exist the following types of motion of the particle on the earth with zero angular momentum D: oscillation around the poles $\varphi = \pm\pi/2(\xi = 0, \xi = 2\pi)$, rotation around the center of the great circle: a stable fixed point at each of the poles and an unstable fixed point at the equator $\varphi = 0(\xi = \pi)$. This is in sharp contrast to the motion on the sphere where the particle moves along the great circle with constant speed.

Thus, on the earth, the particle motion is a combination of a pendulum-like motion along the great circle and a rotation of the great circle itself around the earth's axis.

Fig. 3e illustrates the particle trajectories when $D = 0$ in (φ, λ) coordinates on a sphere (which are valid on the earth as well) and Fig. 4c illustrates the same on the earth. It should be noted that the λ coordinate has a jump of π when the trajectory passes through the poles.

As will be shown subsequently this very simple geometric scenario re-appears in both the sphere and the earth in the general case (i.e. when $D \neq 0$).

These simple considerations bring about another issue, which will also be encountered in the general case. Both on the earth and a rotating sphere the coordinate φ does not provide a convenient dynamical description. As has been shown here, on the earth, φ has to be transformed to ξ in order to obtain the equation of a simple pendulum. In addition, a solution of the differential equations on both a rotating sphere and the earth that passes through a pole has an angle with respect to the great circle increasing (or decreasing) monotonically as the particle rotates around the great circle there, but an accurate dynamical description requires a reflection of φ at the poles (accompanied by a jump of π in λ). The underlying reason for this is of course, the singularity of the spherical coordinates at the poles so we can expect that a transformation of φ will be required in the general case too. The transformation to the action–angle variables resolves both problems.

4. Free motion on a rotating sphere

4.1. Action–angle variables

According to Liouville's theorem each integral surface $H =$ constant and $D =$ constant of a two-dimensional integrable system is topologically equivalent to a two-dimensional torus. This equivalence implies that a canonical transformation of variables (P, D, φ, λ) to action–angle variables (I, J, ψ, μ) exists, where angles ψ, μ are the angle variables of a torus and actions I, J are the corresponding conjugate variables. The Hamiltonian of the system in these variables depends on the action variables only and, hence,

$$\psi(t) = \omega_I \cdot t, \qquad \mu(t) = \omega_J \cdot t,$$

where $\omega_I = \partial H/\partial I$ and $\omega_J = \partial H/\partial J$.

In the inertial frame, a free particle on a sphere moves along a great circle Q with velocity $\pm\sqrt{2E}$ (E is defined by (2.9)), irrespective of whether or not the sphere rotates. This follows from the fact that the only force acting on the particle is the normal force and, hence, the acceleration is directed towards the center of a sphere. Since the radius of the sphere is 1.0, the velocity $\pm\sqrt{2E}$ may be considered as angular velocity on the great circle. If, in addition, the sphere rotates with an angular velocity ω then in the inertial frame the circle Q itself rotates relative to a sphere with the opposite angular velocity $-\omega$. Therefore, the motion in the rotating frame is composed of two motions: rotation along the circle Q with angular velocity $\pm\sqrt{2E}$ and rotation of the circle Q relative to a sphere with angular velocity $-\omega$. Both velocities remain constant throughout the entire trajectory.

Let μ be a λ-value of the highest point P (not to be confused with the momentum P) of the great circle Q and ψ be an angle between this point and a particle position $M(\varphi, \lambda)$ measured along the great circle (see Fig. 2). The two angles ψ and μ define the position of a particle on a sphere given an angle φ_0 that the great circle Q forms with the equatorial plane. Angular coordinates ψ and μ of a free particle are changed linearly with velocities $\omega_I = \sqrt{2E}$ and $\omega_J = -\omega$. Therefore, ψ and μ are the angle variables of the system described by the Hamiltonian (2.8).

Employing simple geometric considerations, it is straightforward to show that

$$\psi = \mp \arccos \frac{\sin\varphi}{\sin\varphi_0}, \tag{4.1}$$

$$\mu = \lambda \pm \arccos \frac{\tan\varphi}{\tan\varphi_0}, \tag{4.2}$$

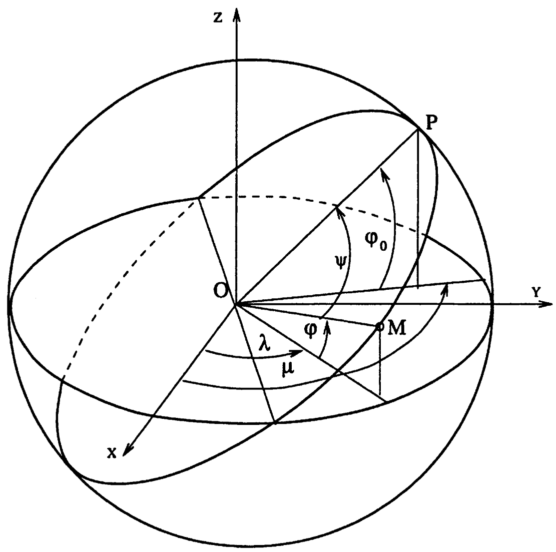

Fig. 2. The various geometric terms involved in the transformation from the latitude φ and longitude λ, to the μ, ψ angles. The great circle passing through particle position M rotates relative the sphere with frequency $-\omega$ while the point M itself moves along its perimeter with velocity $\mathbf{v}$.

where the upper sign in both formulae corresponds to positive P and the lower sign to negative P. Expression (4.1) implies that when the latitude φ varies over the interval $(\varphi_0, 0)$, the angle variable ψ varies over the intervals $(0, \pi/2)$ (and similarly for μ in (4.2)).

The angles φ_0 and μ define the orientation of the circle Q relative to the unit sphere while the angle ψ determines the position of point M on this circle relative to the maximal point P. These three angles are related to the, well-known, Euler angles of analytical mechanics (θ, ψ, ϕ) (see [3]) by $\theta = \pi/2 - \varphi_0$, ψ, and $\phi = \pi/2 - \mu$.

The angle φ_0 is defined from the condition $P = \dot{\varphi} = 0$ satisfied at the highest point of the circle Q which is also an extreme φ-coordinate along the trajectory. Therefore, from (2.9), we get

$$P^2(\varphi) = 2E - \frac{D^2}{\cos^2\varphi_0} = 0,$$

and, hence,

$$\cos^2\varphi_0 = \frac{D^2}{2E}. \tag{4.3}$$

Thus, when mapping coordinates (P, D, φ, λ) of a given point M into action–angle variables one must construct the circle Q passing through M and forming an angle φ_0 with the equatorial plane of the sphere. The λ-coordinate of the highest point P of this circle is the angle variable μ while angle POM is the angle variable ψ.

In fact, there are two circles Q satisfying the above conditions. The second circle being a mirror image of the first one through the plane that contains z-axis and the radius OM. These two circles provide two different values of ψ and μ, which correspond to upper and lower signs in their expressions (4.1) and (4.2). The proper sign is determined by a combination of the signs of P and D.

The action variables I and J are defined as functions of H and D by the condition that the transformation from the variables (P, D, φ, λ) to the variables (I, J, ψ, μ) is a canonical one. This is done by constructing a generating function that provides (the details are given in Appendix A) the following expressions:

$$I = \pm\sqrt{2E}, \qquad J = D. \tag{4.4}$$

Therefore taking (4.3) into account, one gets

$$\cos\varphi_0 = \left|\frac{J}{I}\right|. \tag{4.5}$$

Upon expressing $\sin\varphi_0$ and $\tan\varphi_0$ in terms of $\cos\varphi_0$ and substituting (4.5), we finally arrive at the following transformation from the (P, D, φ, λ) variables to the action–angle variables:

$$I = \pm\sqrt{P^2 + \frac{D^2}{\cos^2\varphi}}, \qquad J = D, \qquad \psi = \mp\arccos\frac{\sin\varphi}{\sqrt{1 - J^2/I^2}}, \qquad \mu = \lambda \pm \arccos\frac{\tan\varphi}{\sqrt{I^2/J^2 - 1}}. \tag{4.6}$$

The transformation of the Hamiltonian function (2.8) to the action–angle variables takes the simple expression

$$H = \tfrac{1}{2}I^2 - \omega J \tag{4.7}$$

and the evolution of the system in the transformed variables is given by

$$\dot{I} = -\frac{\partial H}{\partial \psi} = 0, \qquad \dot{J} = -\frac{\partial H}{\partial \mu} = 0, \qquad \dot{\psi} = \frac{\partial H}{\partial I} = I, \qquad \dot{\mu} = \frac{\partial H}{\partial J} = -\omega. \tag{4.8}$$

4.2. Action–angle variables and Liouville's theorem

The geometric interpretation of the action–angle variables given above is fully consistent with Liouville's theorem for integrable systems. According to this theorem, the integral surface $T_{IJ} : I = \text{constant}, J = \text{constant}$ in phase space (more accurately — the co-tangent bundle of configuration manifold, see [3]) is equivalent topologically to a torus and the variables ψ and μ are the angles of this torus. Phase space, as a whole, is naturally transformed into the configuration manifold (which in our case is the original sphere) by mapping a point in phase space to the point in configuration manifold with the same coordinates (i.e. disregarding the momenta). The results of the previous section imply that all phase space points with given values of μ, I and J are mapped onto the great circle Q defined by the angles $\varphi_0 = \arccos(I/J)$ and μ. Therefore, the torus T_{IJ} is mapped by that natural transformation onto the union of these circles, i.e. the latitude band on the sphere: $\hat{T}_{IJ} = \{-\varphi_0 \leq \varphi \leq \varphi_0\}$. As we have mentioned above, two different points on the torus with opposing signs of momentum P are mapped into a single point on the sphere. If we consider the band $\hat{T}_{IJ}$ to be a two-sided surface (i.e. a torus with zero thickness) with an outer side corresponding to positive P and an inner side corresponding to negative P, then it can be identified with torus T_{IJ} itself.

It should also be noted that although I and $-I$ produce two different tori, they are mapped into the same subset $\hat{T}_{IJ}$ of the sphere. The motion in these two tori is in the opposite directions but with the same speed.

4.3. Trajectories and drift

The decomposition of the motion described above helps explain very naturally the topological types of trajectories of a free particle on a sphere and as we will show in the next section these trajectories occur in the inertial motion on the earth as well.

An important characteristic of a particle motion is its so-called drift Dr_λ defined as the average angular zonal velocity: $Dr_\lambda = \Delta\lambda/T$, where $\Delta\lambda$ is the zonal angle between two points on the trajectory with identical φ and v values and T is the time elapsed between these points. The drift can simply be represented by the sum of the average angular particle velocity along the circle Q and the angular velocity of the circle Q relative to the sphere. The first term equals $2\pi/T_I = \omega_I$ (T_I is the period of rotation along the circle Q) and the second term equals $-\omega$, so that

$$Dr_\lambda = \omega_I - \omega = \pm\sqrt{2E} - \omega. \tag{4.9}$$

Fig. 3 illustrates the possible types of trajectories starting from $\lambda(0) = 0$, $\varphi(0) = 0$ and $v(0) = 0.3$ on a rotating sphere. These numerically computed trajectories are interpreted in terms of the decomposition outlined above and the zonal drift as follows.

By considering the motion with constant speed along an inclined (relative to equator) great circle Q, one immediately realizes that the zonal velocity u obtains its minimal value on the equator. Following the definition of the zonal velocity $u = \dot{\lambda}/\cos\varphi$ and (2.5), one gets that on the equator $u = D - \omega$. Therefore, if $D \geq \omega$ then $u \geq 0$ on the equator and, hence, $u \geq 0$ everywhere as in Fig. 3a.

If D is slightly smaller than ω then the particle moves in the negative λ-direction on the equator but away from the equator it moves in the positive direction so that the drift $|\mathbf{v}| - \omega$, ($|\mathbf{v}| = \sqrt{u(0)^2 + v(0)^2}$) remains positive as in Fig. 3b.

When $Dr_\lambda = 0$($D = \sqrt{\omega^2 - v(0)^2}$) the trajectory has a "figure-eight" shape (Fig. 3c). Upon decreasing D further, one gets the trajectory of the same type as in Fig. 3b but with negative drift (Fig. 3d).

If $D = 0$, then according to (4.3) $\varphi_0 = \pi/2$, hence, the great circle Q passes over the poles where the λ-coordinate undergoes a jump of π. According to (2.10), the zonal angular velocity $\dot{\lambda}$ is equal to ω everywhere along the trajectory

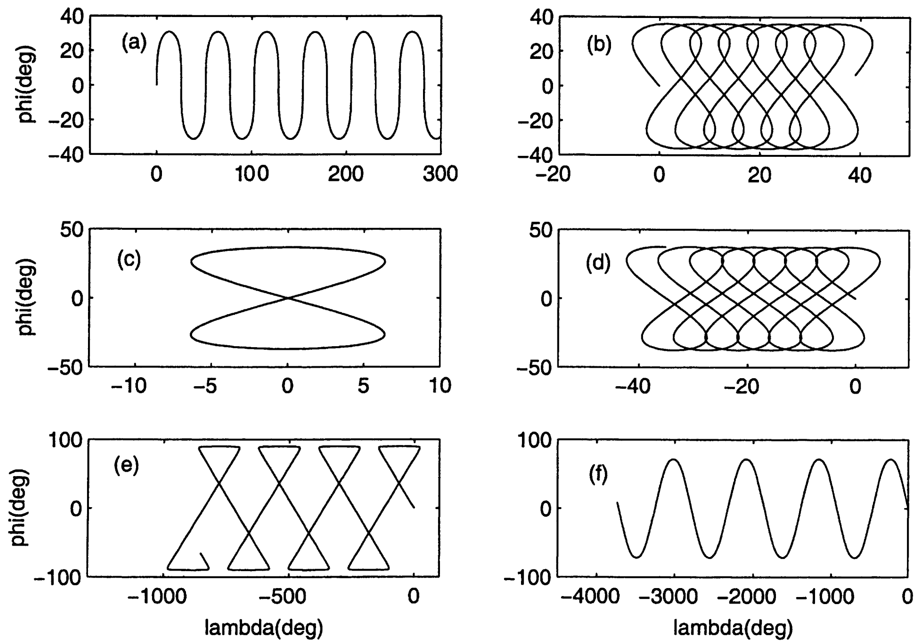

Fig. 3. The types of trajectories on a rotating sphere. The rotation velocity is $\omega = 0.5$ and the initial conditions of all trajectories are $\lambda(0) = 0, \varphi(0) = 0$ and $P(0) = 0.3$. The angular momentum D in the different trajectories/panels is: (a) $D = 0.5$ (i.e. $D = \omega$), (b) $D = 0.41$, (c) $D = 0.4$ (i.e. $\sqrt{2E} = \omega$), (d) $D = 0.39$, (e) $D = 0.0$, (f) $D = -0.1$

and the meridional angular velocity $\dot{\phi}$ is equal to $\pm|\mathbf{v}|$, where the sign changes at the poles (Fig. 3e and the discussion in Section 3).

If $D \leq 0$ then both the great circle and the particle on it move in the negative λ-direction and thus the whole motion is also executed in the negative λ-direction (Fig. 3f).

5. Inertial motion on the rotating earth

As was pointed out in Section 2.2, the horizontal motion of particles on the earth under the gravitation force differs from the free motion on a sphere due to the omission of the centrifugal force, which is balanced by the tangential component of the gravitation. This has several important consequences, which are the focus of this section. The calculation of the action–angle transformation is much more complicated than in case of a rotating sphere (so it makes sense to tackle it at the end of the paper) and relies on the analysis of the roots of the momentum equations and not on the reconstruction of the generating function.

We, therefore, address this aspect first.

5.1. The roots of the momentum equations

From (2.12), we get

$$P^2(\varphi) = 2E - \frac{D^2}{\cos^2\varphi} - \omega^2 \cos^2\varphi, \tag{5.1}$$

where E is the energy in the inertial frame of reference — (2.13). Since $\dot{\varphi} = P$, the roots of momentum equation $P(\varphi) = 0$ are the extremum points of $\varphi(t)$, so the particle changes its direction of meridional motion at these points. Thus, the roots of $P(\varphi)$ are important for determining the geometry of the trajectories. For free motion on a rotating sphere, the last term in (5.1) is absent and, therefore, the appropriate momentum equation has a single root (4.3), while for the rotating earth, Eq. (5.1) implies that there are several roots.

If $D = 0$, the momentum equation is linear with respect to $\cos^2\varphi$, so the only root is

$$\cos\varphi_0 = \frac{\sqrt{2E}}{\omega}.$$

If $D \neq 0$, then $\cos^2\varphi > 0$ (otherwise the denominator in (5.1) is equal to zero) and the roots of $P(\varphi)$ are the roots of the quadratic equation:

$$W(\cos^2\varphi) = -\cos^2\varphi \cdot P^2(\varphi) = -2E\cos^2\varphi + D^2 + \omega^2\cos^4\varphi = 0. \tag{5.2}$$

Define $x = \cos^2\varphi$, then

$$W(x) = \omega^2 x^2 - 2Ex + D^2 = 0. \tag{5.3}$$

The two roots of this equation, x_0 and x_1 $(x_0 < x_1)$ are

$$x_0 = \frac{E - \sqrt{E^2 - \omega^2 D^2}}{\omega^2}, \qquad x_1 = \frac{E + \sqrt{E^2 - \omega^2 D^2}}{\omega^2}. \tag{5.4}$$

Since $H = \frac{1}{2}(u^2 + v^2) \geq 0$, it follows that $E = H + \omega D \geq \omega D$, so that the discriminant in (5.4) is non-negative. Hence Eq. (5.4) has two real roots and since $x = \cos^2\varphi > 0$ for $D \neq 0$, these roots must satisfy the inequalities

$$0 < x_0, \quad x_1 \leq 1. \tag{5.5}$$

It follows from (2.13) that $E \geq 0$, therefore, $E > \sqrt{E^2 - \omega^2 D^2}$ for $D \neq 0$, hence positiviteness of both roots x_0, x_1 is guaranteed. The condition $x_0, x_1 \leq 1$ may hold for both roots, one root, or none of them.

In order to determine the conditions for this inequality to be satisfied, we note that the function $W(x)$ in (5.3) is a parabola with a minimum, hence, it is negative only in the interval between the roots.

Assume that $W(1) = \omega^2 - 2E + D^2 < 0$. Then $x_0 < 1$ and $x_1 > 1$ and there exists only one root of momentum equation $W(x) = 0$ that satisfies (5.5).

According to the Viet theorem on the roots of quadratic equations $x_0 x_1 = D^2/\omega^2$ and if $|D/\omega| \geq 1$ then no more than one root exists that satisfies (5.5).

Assume that $W(1) > 0$. Then either $x_0, x_1 > 1$ or $x_0, x_1 < 1$. Since $x_0 x_1 = D^2/\omega^2$, it follows that if $|D/\omega| > 1$, then both x_0 and x_1 are larger than 1 (i.e. these roots do not represent roots of $P(\varphi)$) while if $|D/\omega| < 1$, then both roots of $W(x)$ satisfy (5.5). If $E = \omega D$, then $H = 0$ and these two roots coincide.

Assume now that $W(1) = 0$, i.e. 1.0 is a root of $W(x)$. In this case, if $|D/\omega| \geq 1$, then $x_0 = 1$, $x_1 \geq 1$ (one root). If $|D/\omega| < 1$, then $x_0 < 1$, $x_1 = 1$ (two roots) and $D = \omega x_0$.

Taking into account the equalities

$$W(1) = \omega^2 - 2E + D^2 = \omega^2 - 2(H + \omega D) + D^2 = (D - \omega)^2 - 2H,$$

we can sum up these findings regarding the number of roots of (5.3) satisfying (5.5) as follows.

1. If $|D| < \omega$ and $H \leq \frac{1}{2}(\omega - D)^2$ there exist two roots. Three different sub-cases are possible:
   1.1. if either $H = 0$, $D > 0$ or $H = -2\omega D$, $D < 0$, then the two roots coalesce to a single root;
   1.2. if $0 < H < \frac{1}{2}(\omega - D)^2$, then both roots are different and smaller than 1;

1.3. if $H > \frac{1}{2}(\omega - D)^2$, then $x_1 = 1$ (i.e. this root resides on the equator) and $D = \omega x_0 = \omega \cos\varphi_0$.
2. If $H > \frac{1}{2}(\omega - D)^2$ or $D \geq \omega$, then only one root exists.
3. If $|D| \geq \omega$ and $H < \frac{1}{2}(\omega - D)^2$, there are no roots.

Since $x = \cos^2\varphi$, every root of $W(x) = 0$ that satisfies (5.5) corresponds to two roots ($-\varphi$ and φ) of the momentum equation (5.2).

5.2. Trajectories

Having determined the roots of the momentum equation, we can proceed to classifying the types of trajectories with different values of integrals H and D, and angular velocity ω.

Since $P^2(\varphi) = -\cos^2\varphi\, W(\cos^2\varphi) > 0$, where W is given in (5.3), the trajectory resides entirely in the φ-interval: $W(\cos^2\varphi) < 0$, i.e. between values of φ corresponding to the roots of $W(x)$ that satisfy (5.5). As was mentioned above the roots of $P(\varphi)$ are the extrema of $\varphi(t)$. Therefore, the corresponding trajectory changes its direction of meridional motion at latitudes $\pm\varphi_0$ and $\pm\varphi_1$, where $\cos\varphi_0 = x_0$ and $\cos\varphi_1 = x_1$.

Note also that the first two equations in system (2.14), the $(\varphi, \dot{\varphi})$ system, do not depend of λ and thus may be solved separately. The analysis in the previous section relates is, therefore, relevant to this subsystem. Below, we refer to the $(\varphi, \dot{\varphi})$ system as the reduced system while the "whole" system consists of the four equations in system (2.14).

The various possible types of trajectory correspond to the values of the roots of momentum equation described in the previous section as follows:

Case 2. In the case of a single root $H > \frac{1}{2}(\omega - D)^2$, the change of direction of the meridional motion occurs only at the points $-\varphi_0$ and φ_0. On the equator $P^2(0) = 2(H - \frac{1}{2}(\omega - D)^2) > 0$, hence, a trajectory has to cross the equator in this case. These high-energy trajectories are topologically identical to the trajectories of the free motion on a rotating sphere (shown in Fig. 3). Naturally they are called cross-equatorial trajectories.

The three scenarios of case 1 of the preceding section do not exist on a sphere and the trajectories corresponding to them are shown in Fig. 4, which is interpreted as follows.

Case 1a. When the two roots coincide the solution of the reduced system degenerates into a stable fixed φ-point and the trajectory of the whole system depends on a value of H. If $H = 0$, then this fixed φ-point is also a fixed point of the whole (λ, φ) system, since $H = (u^2 + v^2) = 0$ so that $u = v = 0$ and the trajectory becomes a point. If $H = -2\omega D$ then this fixed φ-point corresponds to westward motion along a latitude circle with constant velocity $u = -\sqrt{2\omega D}$ as in Fig. 4e.

Case 1b. The trajectories with $0 < H < \frac{1}{2}(\omega - D)^2$ when $|D| < \omega$ reside between two different extrema of φ in one hemisphere (i.e. they have no analogy in the motion on a rotating sphere) and they are called single hemisphere trajectories. The four trajectories shown in panels b, c, d and f of Fig. 4 represent four values (in decreasing order) of D.

Case 1c. If the larger root is 1.0 then $D = \omega\cos\varphi_0$ and $x_1 = \cos^2\varphi_1 = 1$, so that $\varphi_1 = 0$. This implies that $P(0) = \dot{\varphi}(0) = 0$ and the corresponding trajectory approaches the equator asymptotically as time tends to ∞ (Fig. 4a). This trajectory corresponds to the separatrix curve in phase space of the reduced system that separates the cross-equatorial trajectories from the single hemisphere ones. An additional trajectory, corresponding to an unstable fixed φ-point exists in this case at $\varphi = 0$. It describes the motion along the equator with the velocity $u = D - \omega < 0$.

If $D = 0$ then $E = H$, $P^2(\varphi) = 2H - \omega^2\cos^2\varphi \geq 0$, hence, $\cos^2\varphi < \cos^2\varphi_0 = 2H/\omega^2$. The trajectory, in this case, resides in the interval $0 \leq \varphi \leq \pi/2$ and passes through the point $\varphi = \pi/2$. If $2H/\omega^2 < 1$ then we have the single hemisphere trajectory that oscillates between φ_0 and the pole (Fig. 4c). If $2H/\omega^2 > 1$, then the trajectory crosses the equator passing through both poles (as in Fig. 3e). If $2H/\omega^2 > 1$ then the trajectory approaches the equator asymptotically or the particle moves along the equator with velocity $u = -\omega$ as in Fig. 4a but with the upper branch originating at the pole.

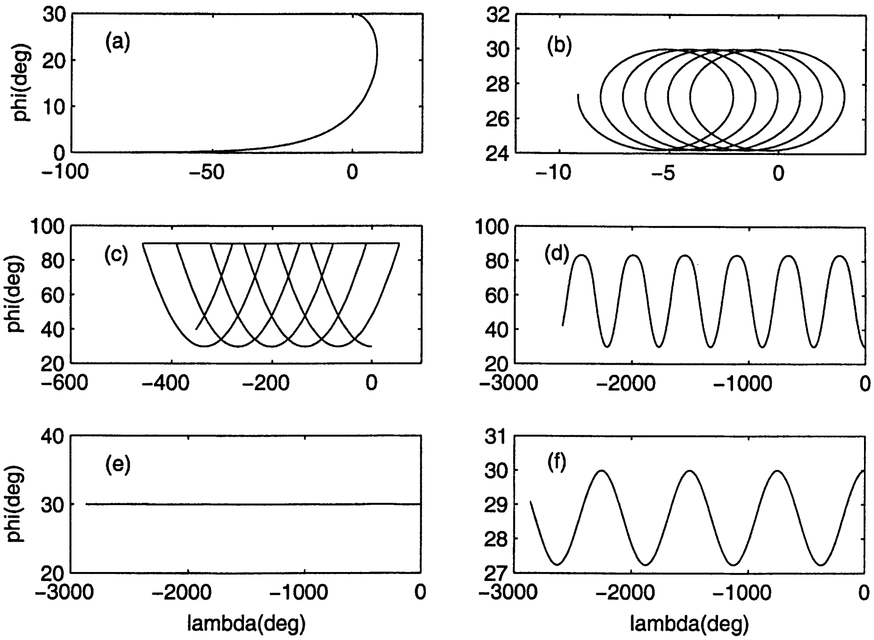

Fig. 4. The single hemisphere trajectories on the earth. The rotation velocity $\omega = 0.5$ and the initial conditions $\lambda(0) = 0$, $\varphi(0) = 30^\circ$, $P(0) = 0.0$ in all panels. The angular momentum D in different panels is: (a) $0.5\cos 30^\circ$ (i.e. $\omega\cos\varphi_0$); (b) $0.5\cos^2 30^\circ + 0.02$ (i.e. slightly larger than the value that makes 30° the stable φ-point). This increase in the value D corresponds to a stable fixed point of the whole system located at $\varphi = 27.27^\circ$ as is evident from the central latitude of the oscillations; (c) 0.0. The trajectory passes at the pole while drifting eastward; (d) -0.05. Since $D \neq 0$ the trajectory never reaches the pole; (e) $-0.5\cos^2 30^\circ$ (i.e $-\omega\cos\varphi_0$ — the stable fixed point of the reduced (φ, v)-system); (f) $-0.5\cos^2 30^\circ - 0.01$, a value slightly smaller than its value in panel (e) so that -30° is not a fixed point of the reduced, φ, system. The (eastward drifting) oscillation about the fixed point are clearly evident.

5.3. Action–angle variables

As in the case of the motion on a rotating sphere the action–angle variables on the rotating earth play a crucial role in the geometrical interpretation of inertial motion. However, in contrast to a rotating sphere the calculations of this variables for the earth cannot be done simply by constructing the generation function and its derivatives. Our alternate method is based on the roots of momentum equation, and the details of the calculations can be found in Appendix B to which we refer the interested reader. Here, we briefly summarize the results of calculations, leaving their interpretation to the next section. To derive the formulae in the following sections from their counterparts in Appendix B, one must substitute the variables a and b of the formulae in Appendix B by their expressions in terms of the roots x_0 and x_1 of the momentum equation (5.3). The rates of change of the two angle variables (which on the earth have no geometric interpretation but on the sphere these rates are the angular velocities) are called the frequencies, designated by ω_D and ω_I.

5.3.1. Cross-equatorial trajectories

For the cross-equatorial motion (which is topologically equivalent to the motion on a rotating sphere) the angle variable θ describing the meridional oscillations is given by

$$\theta = \frac{\pi}{2}\frac{F(\psi, k)}{K(k)},$$

where $F(\psi, k)$ and $K(k)$ are the elliptic integral and complete elliptic integral of the first kind, respectively, ψ is the angle variable of free motion on the sphere given by (4.1) and $k^2 = (1 - x_0)/(x_1 - x_0)$. The velocity corresponding to the θ angle is

$$\omega_I = \pm \frac{\pi\omega\sqrt{x_1 - x_0}}{2K(k)}, \tag{5.6}$$

which should be compared with the frequency (angular velocity) on the sphere $\pm\sqrt{2E}$.

The angle variable describing the longitude motion is given by

$$\nu(\psi) = \lambda - \eta(\psi) + (\omega + \omega_D)t(\psi), \tag{5.7}$$

where

$$\eta(\psi) = \frac{D}{\omega x_0\sqrt{x_1 - x_0}} \Pi(\psi, n, k) \tag{5.8}$$

($\Pi(\psi, n, k)$ is the elliptic integral of the third kind), $t(\psi) = (1/\omega\sqrt{x_1 - x_0})F(\psi, k)$, $n = (1 - x_0)/x_0$, ω_I is given by (5.6) and

$$\omega_D = \frac{\eta(2\pi)}{T_1} - \omega_I - \omega \quad \left(T_I = \frac{2\pi}{\omega_I}\right). \tag{5.9}$$

5.3.2. Single-hemisphere trajectories

The angle variables corresponding to the single-hemisphere trajectories have no counterpart on a rotating sphere and (obviously) they also differ from those of the cross-equatorial trajectories. The angle variable θ describing the meridional oscillations is given by

$$\theta = \pi \frac{F(\tau, k)}{K(k)},$$

where $\tau = \arcsin(\sin\psi/k)$, $k^2 = (x_1 - x_0)/(1 - x_0)$. The corresponding frequency is

$$\omega_I = \frac{\pi\omega\sqrt{1 - x_0}}{K(k)}. \tag{5.10}$$

The second angle variable describing the longitude motion is given by

$$\nu(\tau) = \lambda - \eta(\tau) + (\omega + \omega_D)t(\tau), \tag{5.11}$$

where $\eta(\tau) = D/\omega(x_0\sqrt{1 - x_0})\Pi(\psi, n, k)$, $t(\tau) = (1/\omega\sqrt{1 - x_0})F(\tau, k)$, $n = (x_1 - x_0)/x_0$, and

$$\omega_D = \frac{\eta(2\pi)}{T_I} - \omega. \tag{5.12}$$

In both the single hemisphere, and the cross-equatorial, trajectories the action variable J coincides with D as in the case of the free motion on a sphere. As for the action variable I it is of little use because if its complexity: it includes elliptic integrals of all three kinds.

5.4. Decomposition of the motion

As in the motion on a rotating sphere, the motion on the rotating earth can be decomposed into two components: the motion along the great circle Q with frequency ω_I and the rotation of the circle Q itself relative to the earth with frequency ω_D. Although, these two frequencies are constants, the rates of change of the two geographic angles

φ (latitude) and λ (longitude) are time-dependent. The angles that change linearly with time on the trajectory are θ(i.e. $\dot{\theta} = \omega_I$) and ν (i.e. $\dot{\nu} = \omega_D$).

According to (5.9), the rotation frequency of the circle Q relative to the rotating earth, ω_D, consists of three terms. To understand the meaning of the first term, $\eta(2\pi)/T_I$, suppose that the point with coordinates (D, I, ν, θ) belongs to the trajectory that begins at the point $(\lambda = 0, \psi = 0)$. Then $\nu = \omega_D t$ and substituting this value into (5.7), we obtain: $0 = \lambda - \eta(\psi) + \omega t$ so that in the inertial coordinate system

$$\lambda = \eta(\psi). \tag{5.13}$$

Thus, $\eta(\psi)$ is the λ-coordinate of a particle as a function of variable ψ in the inertial frame. Hence, $\eta(2\pi)$ is the net gain in λ-coordinate over an entire period of meridional (i.e. φ) oscillation of a particle in the inertial coordinate system. Thus, according to the definition of drift in Section 4.3, the first term in (5.9) defined as Dr_i

$$Dr_i = \frac{\eta(2\pi)}{t(2\pi)} = \frac{\eta(2\pi)}{T_I} \tag{5.14}$$

is the particle zonal drift in the inertial frame. The drift Dr_i consists of both the motion of the particle relative to the great circle and the motion of the great circle itself relative to the inertial frame. To obtain the angular velocity of the great circle one has to subtract from Dr_i the motion of the particle relative to the great circle: ω_I. This explains the origin of the first and second terms in (5.9): $Dr_i - \omega_I$: it is simply the average angular velocity of the great circle in the inertial frame. The third term, $-\omega$, accounts for the transformation from the inertial frame to the earth (which rotates with frequency ω in the inertial frame).

These considerations hold for the single-hemisphere trajectories, too. In this case, however, the average angular velocity along the great circle vanishes because the particle oscillates about the stable point. Therefore, the expression (5.12) for the frequency ω_D does not contain the ω_I term associated with the average particle's velocity along the great circle. Thus, ω_D is the sum of the particle drift in the inertial frame Dr_i and the speed $(-\omega)$ that originates from the rotation of the earth.

5.4.1. Trajectories and drift

In the preceding section, it was shown that $Dr_i = \eta(2\pi)/T_I$ is the average λ-velocity (drift) in the inertial coordinate system. Hence, taking into account (5.14), we get that in the rotating coordinate system, the average λ-velocity of the particle is

$$Dr_\lambda = Dr_i - \omega = \frac{\eta(2\pi)}{T_I} - \omega. \tag{5.15}$$

The cross-equatorial trajectories have the same topology as the trajectories on a rotating sphere (Fig. 3). Since $\dot{\lambda} = \partial H/\partial D = D/\cos^2\varphi - \omega$ (see (2.14)) if $D > \omega$ then the particle moves with positive velocity in the λ-direction (Fig. 3a) at all times, while when $D < 0$ it moves with negative velocity in the λ-direction (Fig. 3f). As in the case of a rotating sphere, $Dr_\lambda > 0$ guarantees that the drift (i.e. average motion over long enough times) is directed eastward (Fig. 3b) (and when $Dr_\lambda < 0$ it is directed westward, Fig. 3d). If $Dr_\lambda = 0$, we get the figure-eight trajectory. (Fig. 3c).

A concrete numerical verification of the analytical expression for the drift (5.15) obtains in the case of polar trajectories — $D = 0$. In this case, (5.8) implies that $\eta(\psi) = 0$ so according to (5.15) $DR_\lambda = -\omega$ which is precisely the mean rate of change in λ shown in Fig. 3e.

A harder test for the applicability of the analytical estimates of the zonal drift to our numerical computation is encountered in the single hemisphere trajectories that exist only on the Earth. These trajectories were analyzed in

Section 5.2 and are shown in Fig. 4. A tedious manipulation of the elliptic integral in (5.15) shows that for these single hemisphere trajectories

$$\frac{\eta(2\pi)}{T_I} < \omega \tag{5.16}$$

and that when the two roots of the momentum equations, x_0, x_1 are close to each other: $x_0 - x_1 \ll 1$, i.e. (φ_0 and φ_1 are near the center point of the trajectory shown in Fig. 3b)

$$Dr_\lambda = -\frac{E - \omega D}{2(\omega - D)}. \tag{5.17}$$

The inequality (5.16) together with (5.15) imply that the drift of the single hemisphere trajectories is always negative, i.e. always directed westward. This conclusion is confirmed by the numerical calculations shown in Fig. 4, where all the trajectories drift westward as expected. The simple quantitative estimate (5.17) was first derived in [21] based on expansion of the Hamiltonian near its elliptic fixed point and confirmed numerically even when the departure from the fixed point (i.e. the radius of the inertial oscillation) is 0.1.

6. Inertial motion on the earth as a simple pendulum

One of the classical examples of analytical mechanics is the simple pendulum (see Section 3). The energy equation of the pendulum of unit mass on the circle of radius 1 is

$$\tfrac{1}{2}\dot{\theta}^2 - n^2 \cos\theta = n^2 p, \tag{6.1}$$

where $n^2 p$ is the energy and θ the angle relative to the vertical downward pointing direction (see [22]). The pendulum has four type of motions:

1. stable equilibrium at the lowest point $\theta = 0$ of the circle $(p = -1)$;
2. oscillations (also called librations) around this point $(-1 < p < 1)$;
3. unstable equilibrium at the upper point of the circle $\theta = \pi (p = 1)$;
4. rotation around the center of the circle $(p > 1)$.

These are exactly the four types of motion described above that a free particle on the earth displays in its motion along the circle Q (see Sections 5.1 and 5.2). In fact, the meridional movement of the particle can be reduced to the pendulum as follows.

Setting

$$s_0 = 1 - x_0 \sin^2\varphi_0, \qquad s_1 = 1 - x_1, \qquad s = 1 - x = \sin^2\varphi, \tag{6.2}$$

we can rewrite the function $W(x)$ in (5.2) as

$$W(x) = W(1 - s) = -\omega^2 (s_0 - s)(s - s_1). \tag{6.3}$$

Substituting $P = \dot{\varphi}$ into (5.2) and (6.2) into (6.3), we have

$$\cos^2\varphi \cdot \dot{\varphi}^2 = \omega^2 (\sin\varphi_0^2 - \sin^2\varphi)(\sin^2\varphi - s_1), \tag{6.4}$$

setting

$$\cos\psi = \frac{\sin\varphi}{\sin\varphi_0}, \tag{6.5}$$

and comparing it with (4.1), one gets that ψ is the angle of the circle Q, counted from the upper point of this circle. Differentiating (6.5) with respect to time, we get $\sin\varphi_0 \sin\psi \cdot \dot{\psi} = \cos\varphi \cdot \dot{\psi}$, so (6.4) can be written as

$$\cos^2\psi \cdot \dot{\psi}^2 = \omega^2 \sin^2\varphi_0 \left(1 - \left(\frac{\sin\varphi}{\sin\varphi_0}\right)^2\right)\left(\left(\frac{\sin\varphi}{\sin\varphi_0}\right)^2 - \frac{s_1}{\sin\varphi_0^2}\right).$$

Substituting (6.5) into the last equality, we get

$$\dot{\psi}^2 = n^2(\cos^2\psi - q), \tag{6.6}$$

where

$$q = \frac{s_1}{s_0} = \frac{s_1}{\sin^2\varphi_0}, \qquad n = \omega \sin\varphi_0.$$

Using the formula $2\cos^2\psi = 1 + \cos(2\psi)$, we can rewrite (6.6) as

$$\tfrac{1}{2}\dot{\theta}^2 - n^2\cos\theta = n^2(1 - 2q), \tag{6.7}$$

where $\theta = 2\psi$. The last equation is just the energy equation for the pendulum when we identify $1 - 2q$ with p. The transformation $\theta(\psi) = 2\psi$ results in mapping two symmetric points (with respect to the center of sphere) into a single point on the circle $Q : \theta(\psi + \pi) = 2\pi + \theta(\psi)$. Since, the equations of motion (2.14) are invariant with respect to this transformation it does not alter the number of admissible solutions.

There exists a direct correspondence between the types of pendulum trajectories and the types of trajectories on the rotating earth.

If $p = -1$ then $q = 1$, thus $s_1 = s_2$. This is case 1.1 of Section 5.1 when the two roots coincide. The stable fixed point of the pendulum corresponds to the stable fixed φ-point on the rotating earth.

If $-1 < p < 1$ then $0 < q < 1$. This is case 1.2 of Section 5.1 when the two points are different. The oscillations in the case of a pendulum correspond to the single hemisphere trajectory of the particle on earth that oscillates between φ_1 and φ_0.

If $p = 1$ then $q = 0$, and $s_1 = 0$. Hence, $x_1 = 1 - s_1 = 1$, which occurs in case 1.3 of Section 5.1. The unstable fixed point of the pendulum corresponds to the unstable rotation along the equator.

If $p > 1$ then $q < 0$ and $s_1 < 0$. The rotation of the pendulum around the center corresponds to the cross-equatorial trajectory.

Thus, the value of q provides us with the most general classification of the trajectories on the rotating earth.

7. Geophysical implications

The decomposition of the motion, in both the earth and a rotating sphere, into two angular components that can be interpreted as a motion along a great circle and the rotation of the great circle relative to the sphere/earth suggests the following view of the various factors affecting the inertial motion on the earth.

1. Along the surface of a non-rotating sphere (the sole contribution of the central gravitational force is to keep the particle on the surface of the sphere) the only possible trajectory of a freely moving particle is a great circle that passes through the particle's initial coordinates and is tangent to the particle's initial velocity vector. Obviously, this trajectory crosses the equator (unless it coincides with it) and the tangential velocity along the great circle is the initial speed (i.e. absolute value of the initial velocity vector). In the absence of potential energy for this horizontal motion, conservation of (kinetic) energy mandates that the pole-ward velocity component attains its

maximal absolute value on the equator (where the zonal velocity component is minimal) and it vanishes at the point on the great circle located the farthest from the equator (i.e. point P in Fig. 2). The motion along this trajectory has the same topology as the trajectories shown in Fig. 3a,f.

2. When the sphere is rotating with frequency ω relative to an inertial frame of reference then, in the inertial frame, this rotation is simply added to the simple motion described in 1. The vectorial sum of velocities associated with these two motions can result in all the trajectories shown in Fig. 3 depending on the particle's initial conditions (i.e. coordinates and velocity). It is clear from the decomposition into these two motions that any inertial trajectory on a rotating sphere has to cross the equator despite the presence of Coriolis force that varies with latitude (i.e. the β-effect).
3. On the non-rotating earth, with its 0.003 eccentricity, the simple scenario outlined in 1 has to be modified due to the presence of a tangential (pole-ward directed) component of gravity — Fig. 1. This tangential force vanishes both on the equator and at the poles and it varies with latitude as $\sin(2\varphi)$ so the potential associated with the tangential component of gravity decreases monotonically with the latitude as $\cos(2\varphi)$. Thus, conservation of total (kinetic plus potential) energy on the non-rotating earth mandates that the kinetic energy attains its minimal value on the equator (where the gravitational potential is maximal) and its maximal value at the pole-ward-most point of the trajectory (e.g. point P in Fig. 2), where the gravitational potential is minimal. This causes a change in the magnitude of the velocity vector in addition to the change in its orientation encountered in the non-rotating sphere. Therefore, on the non-rotating earth there are two contributions to the change in northward velocity component along the trajectory and one of them is associated with the change in kinetic energy at the expense of the potential energy. As a result of the pole-ward decrease in potential energy on the earth, it is possible to construct a trajectory there that begins at some latitude φ_0 with small enough velocity directed towards the equator that will never reach the equator! The increase in the particle's potential (gravitational) energy along its equator-ward motion can result in a vanishing of its initial kinetic energy at some latitude φ_f located between φ_0 and the equator. This is the non-rotating counterpart of the mid-latitudes inertial circle on the rotating earth that does not crosses the equator. However, when the initial velocity is large enough (high energy), the trajectory crosses the equator and the eccentricity of earth will only constitute a correction to the great circle on a sphere (i.e. 1).
4. When the ellipsoidal earth rotates with frequency ω the single hemisphere trajectories outlined in 3 are compounded by the rotation to yield the single hemisphere trajectories shown in Fig. 4. The equator-crossing trajectories on earth are simply the high energy counterparts of the trajectories on a rotating sphere, shown in Fig. 3.

The above analysis suggests a four-step scenario describing the inertial oscillations on the earth that starts from the non-rotating sphere to which rotation is trivially added, followed by addition of the effect of earth's eccentricity. The implication of this scenario on the inertial motion on the earth is that inertial circles in mid-latitudes originate from the eccentricity of earth and not only from its rotation. By the same token, on the β-plane these circles do not migrate westward unless the gravitational potential changes along the surface of the earth — i.e. unless the geopotential is elliptical.

Appendix A. Action–angle variables of the free particle on a sphere

The two pairs of action–angle variables (I, ψ) and (J, μ) can be explicitly derived from the generating function

$$S(H, D, \varphi, \lambda) = \int_{(\varphi_0,\lambda_0)}^{(\varphi,\lambda)} (P\,\mathrm{d}\varphi + D\,\mathrm{d}\lambda)\,\mathrm{d}s, \tag{A.1}$$

where the integration is carried out along the integral surface: $H = \text{constant}$, $D = \text{constant}$ in phase space along a curve that begins at φ_0, λ_0 and ends at φ, λ. The action variables I and J are defined by

$$I = \frac{1}{2\pi}\oint_{C_I}(P\,\mathrm{d}\varphi + D\,\mathrm{d}\lambda), \qquad J = \frac{1}{2\pi}\oint_{C_2}(P\,\mathrm{d}\varphi + D\,\mathrm{d}\lambda), \tag{A.2}$$

where C_1 and C_2 are the generating cycles of the torus $H = \text{constant}$, $D = \text{constant}$ and the direction of integration along them determines the sign of I and J. For a complete discussion of the generating function and its relationship to the generating cycle see [3].

In order to obtain the angle variables (ψ, μ) from the corresponding action variables (I, J) one has first to express H and D as functions of I and J using (A.2). Once this is accomplished the angle variables are given by

$$\psi = \frac{\partial S}{\partial I}, \qquad \mu = \frac{\partial S}{\partial J}. \tag{A.3}$$

The function $P(H, D, \varphi, \lambda)$ is obtained by inverting the Hamiltonian $H(P, D, \lambda, \varphi)$ in (2.8)

$$P = \pm\sqrt{2E - \frac{D^2}{\cos^2\varphi}}, \tag{A.4}$$

where $E = H + \omega D$ is the particle energy in the inertial frame of reference — (2.9). The choice of the sign in front of the square root in (A.4) depends on the sign of the momentum P at the given point φ. Since, according to (2.6), $P = \dot{\varphi}$, it is positive when φ increases and negative when φ decreases.

Substituting (A.4) into (A.1), we get on the interval $\varphi_0 \geq \varphi \geq -\varphi_0$

$$S = \pm\int_{\varphi_0}^{\varphi}\left(\sqrt{2E - \frac{D^2}{\cos^2\varphi}}\right)\mathrm{d}\varphi + \int_{\lambda_0}^{\lambda} D\,\mathrm{d}\lambda. \tag{A.5}$$

Let λ_0 be 0. The generating function (A.5) can, therefore, be written as

$$S = \pm\int_{\varphi_0}^{\varphi}\frac{2E - D^2/\cos^2\varphi}{\sqrt{2E - D^2/\cos^2\varphi}}\,\mathrm{d}\varphi + D\lambda = \mp(S_1 + S_2) + D\lambda,$$

where

$$S_1 = 2E\int_{\varphi}^{\varphi_0}\frac{\cos\varphi}{\sqrt{2E\cos^2\varphi - D^2}}\,\mathrm{d}\varphi, \qquad S_2 = -D^2\int_{\varphi}^{\varphi_0}\frac{1}{\cos^2\varphi\sqrt{2E - D^2(\tan^2\varphi + 1)}}\,\mathrm{d}\varphi. \tag{A.6}$$

Substituting $t = \cos\varphi$ in S_1, one gets

$$S_1 = 2E\int_{\varphi}^{\varphi_0}\frac{\mathrm{d}t}{\sqrt{2E(1-t^2) - D^2}} = \sqrt{2E}\int_{\varphi}^{\varphi_0}\frac{\mathrm{d}t}{\sqrt{a^2 - t^2}}\,\mathrm{d}\varphi = \sqrt{2E}\arcsin\frac{\sin\varphi}{a}\bigg|_{\varphi}^{\varphi_0},$$

where $a^2 = (2E - D^2)/2E = \sin^2\varphi_0$. Substituting $t = \tan\varphi$ in S_2, one gets

$$S_2 = -D^2\int_{\varphi}^{\varphi_0}\frac{\mathrm{d}t}{\sqrt{(2E - D^2) - D^2t^2}} = -D\int_{\varphi}^{\varphi_0}\frac{\mathrm{d}t}{\sqrt{b^2 - t^2}} = -D\arcsin\frac{\tan\varphi}{b}\bigg|_{\varphi}^{\varphi_0},$$

where $b^2 = (2E - D^2)/D^2 = \tan^2\varphi_0$. Since, $\arcsin 1 = \pi/2$, we finally obtain

$$S = \mp\left(\sqrt{2E}\arccos\frac{\sin\varphi}{\sin\varphi} - D\arccos\frac{\tan\varphi}{\tan\varphi_0}\right) + \lambda D. \tag{A.7}$$

The action variables I and J depend on the choice of the generating cycles C_1 and C_2 of the torus $H = \text{constant}$, $D = \text{constant}$ in (A.2). Let the cycle C_2 be the closed curve in phase space (P, D, φ, λ): $\varphi = 0$, $0 \leq \lambda < 2\pi$, and P is defined by (A.4). Then according to (A.2)

$$J = \frac{1}{2\pi}\Delta_{C_2} S = D, \tag{A.8}$$

where $\Delta_{C_2} S$ is the increment of the generating function S along the cycle C_2.

Let C_1 be a circle in phase space made up of the two half-circles A and B

$$A: \quad \lambda = \arccos\frac{\tan\varphi}{\tan\varphi_0}, \qquad B: \quad \lambda = -\arccos\frac{\tan\varphi}{\tan\varphi_0}, \tag{A.9}$$

where $-\varphi_0 \leq \varphi \leq \varphi_0$ for both A and B.

In contrast to cycle C_2, P is not a constant on C_1 and varies, instead, according to (A.4). This expression does not determine the sign of P, which is chosen to be positive on the half-circle A and negative on the half-circle B. This choice ensures that the function S is both continuous and monotonic throughout the entire cycle C_1. Substitution of the relationship between λ and φ given in (A.9) into (A.7) eliminates the second and third terms (i.e. those proportional to D) there, which leaves the first term as the sole contributor to the increment of the function S on cycle C_1. The direction of integration on cycle C_1 is chosen to be from the point $-\varphi_0$ to $+\varphi_0$ along the half-circle A and backward along half-circle B. Since, $\arccos(\sin\varphi/\sin\varphi_0)$ increases by π when φ increases from $-\varphi_0$ to $+\varphi_0$ and by $-\pi$ when φ decreases from $+\varphi_0$ to $-\varphi_0$, the overall increment along the entire cycle C_1 is equal 2π so (A.7) implies

$$\Delta_{C_1} S = 2\pi\sqrt{2E}. \tag{A.10}$$

Substituting (A.10) into (A.2), we get the final result

$$I = \frac{1}{2\pi}\Delta_{C_1} S = \pm\sqrt{2E}, \tag{A.11}$$

where the negative sign in this expression corresponds to integration in (A.2) along cycle C_1 in the opposite direction.

Having calculated the expressions for the two action variables I and J, we can rewrite the generating function (A.7) as

$$S = \mp I \arccos\frac{\sin\varphi}{\sin\varphi_0} + J\left(\lambda \pm \arccos\frac{\tan\varphi}{\tan\varphi_0}\right), \tag{A.12}$$

where according to (4.3), (A.8) and (A.10), φ_0 is given by

$$\cos\varphi_0 = \left|\frac{J}{I}\right|. \tag{A.13}$$

By substituting the explicit expression for the generating function (A.12) into (A.3), we get once more the explicit expressions (4.1) and (4.2) for the two angle variables.

Appendix B. Action–angle variables of the inertial motion on the earth

B.1. The general case

The action–angle variables (4.6) of the motion on a rotating sphere do not coincide with those on the rotating earth. Nevertheless, these variable are important for deciphering the dynamics on the rotating earth, too. The generating function (A.1), which was derived for a rotating sphere, applies to the rotating earth too but according to (2.12)

$$P = \pm\sqrt{2E - \frac{D^2}{\cos^2\varphi} + \omega^2\cos^2\varphi}, \quad E = H + \omega D \tag{B.1}$$

instead of (A.4).

The expression (2.5) for the angular momentum D on the rotating sphere remains unchanged for the earth (i.e. system (2.12)). Indeed upon expressing D from the equations for $\dot{\lambda}$ in (2.14), we get (2.5).

The standard method used in calculating the action–angle variables (4.6) on the sphere, does not work as-is for the system (2.12). While it is possible to get an expression for I with formula (A.2), it is of little use due to its complexity since it includes elliptic integrals of all three kinds. By comparison, the variable J in system (2.12) coincides with D just as it does in system (2.8). Indeed, formulae (A.1) and (A.2) hold for system (2.12) but the expression for P is different. Choosing the cycle C_2 as previously, we get the same expression (A.8) for J since the integral of $P(\varphi)$ vanishes identically.

Let (θ, I), (ν, D) be the conjugate pairs of action–angle variables for system (2.12). By definition the angular velocity of the first pair, $\omega_I = \dot{\theta}$, is given by

$$\omega_I = \frac{\partial H}{\partial I} = \frac{\partial E}{\partial I} = \left(\frac{\partial I}{\partial E}\right)^{-1}. \tag{B.2}$$

Therefore, according to (A.1) and (A.3)

$$\theta = \frac{\partial S}{\partial I} = \frac{\partial S}{\partial I}\cdot\frac{\partial E}{\partial I} = \frac{\partial S}{\partial E}\left(\frac{\partial I}{\partial E}\right)^{-1} = \omega_I\frac{\partial S}{\partial E}, \tag{B.3}$$

where

$$\frac{\partial S}{\partial E} = \pm\int_{\varphi}^{\varphi_0}\frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}}. \tag{B.4}$$

Let C be the generating cycle of the angle variable ν. Then according to (A.2) and (B.1)

$$\frac{\partial I}{\partial E} = \pm\frac{1}{2\pi}\oint_C\frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}}. \tag{B.5}$$

Using (A.1) and (A.3) with $\lambda_0 = 0$, we get the second angle variable:

$$\begin{aligned}\nu &= \frac{\partial S}{\partial D} = \lambda + \int_{\varphi_0}^{\varphi}\frac{(\partial/\partial D)(2(H+\omega D) - D^2/\cos^2\varphi)}{2\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}}\,\mathrm{d}\varphi \\ &= \lambda + \left(\frac{\partial H}{\partial D} + \omega\right)\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}} - D\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\cos^2\varphi\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}}.\end{aligned}$$

The different terms in this expression of ν originate from three different sources written simply as

$$\nu(\varphi) = \lambda - \eta(\varphi) + (\omega + \omega_D)t(\varphi), \tag{B.6}$$

where

$$\omega_D = \frac{\partial H}{\partial D}, \tag{B.7}$$

$$\eta = D\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\cos^2\varphi\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}} \tag{B.8}$$

and

$$t = \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}}. \tag{B.9}$$

Let the variables φ and λ in Eqs. (B.6) and (B.9) change along the cycle C and denote by $\Delta_C t$, $\Delta_C\lambda$ and $\Delta_C\eta$ the increments of t, λ and η, respectively, along the cycle C. Then, according to (B.2), (B.5) and (B.9)

$$\Delta_C t = 2\pi\frac{\partial I}{\partial E} = 2\pi\left(\frac{\partial E}{\partial I}\right)^{-1} = \frac{2\pi}{\omega_I} = T_I,$$

where ω_I is defined by (B.16) and T_I is the period corresponding to the angle variable θ.

The circle C is the generating cycle for the angle θ, therefore, the increment of ν vanishes at its final point

$$0 = \Delta_C\lambda - \Delta_C\eta + (\omega + \omega_D)T_I,$$

so that

$$\omega_D = \frac{\Delta_C\eta}{T_I} - \frac{\Delta_C\lambda}{T_I} - \omega. \tag{B.10}$$

The cycle C depends on the type of trajectory that depends, in turn, on the number of roots of $P(\varphi)$. We consider separately the cases of the cross-equatorial, and single hemisphere, trajectories.

B.2. Cross-equatorial trajectories

For the cross-equatorial trajectory $x_1 > 1$, hence, $s_1 = 1 - x_1 < 0$ and $s_0 = 1 - x_0 > 0$. Setting

$$a^2 = s_0 = \sin^2\varphi_0, \qquad b^2 = -s_1, \qquad x = \sin\varphi, \tag{B.11}$$

one gets from (B.4) and (6.3)

$$\frac{\partial S}{\partial E} = \pm\frac{1}{\omega}\int_x^a \frac{\mathrm{d}s}{\sqrt{(a^2 - x^2)(x^2 + b^2)}} = \pm\frac{1}{\omega\sqrt{a^2 + b^2}}F(\psi, k), \tag{B.12}$$

where $F(\psi, k) = \int_0^{\psi}\mathrm{d}\varphi/\sqrt{1 - k^2\sin^2\varphi}$ is the elliptic integral of the first kind in the Legendre form, $\psi = \arccos(x/a)$ is the angle variable for system (2.8) and

$$k^2 = \frac{a^2}{a^2 + b^2} \tag{B.13}$$

(see e.g. [23]). Let the cycle C in (B.5) be equal to C_1 defined in (A.9). Then according to (B.5) and (B.12)

$$\frac{\partial I}{\partial E} = \pm\frac{1}{\pi}\int_{-\sin\varphi_0}^{\sin\varphi_0} \frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}} = \pm\frac{2}{\pi\omega\sqrt{a^2 + b^2}}F\left(\frac{\pi}{2}, k\right) = \pm\frac{2}{\pi\omega\sqrt{a^2 + b^2}}K(k), \tag{B.14}$$

where $K(k) = F(\pi/2, k)$ is the complete elliptic integral of the first kind.

Finally using (B.3), we get

$$\theta = \frac{\pi}{2}\frac{F(\psi, k)}{K(k)}. \tag{B.15}$$

Using (B.2) and (B.14), the angular velocity, ω_I, corresponding to the action–angle pair (θ, I) can be written as

$$\omega_I = \frac{\partial E}{\partial I} = \left(\frac{\partial I}{\partial E}\right)^{-1} = \pm\frac{\pi\omega\sqrt{a^2+b^2}}{2K(k)}. \tag{B.16}$$

Expression (B.15) shows that the angle variable θ depends only on the angle variable ψ that originates in the free motion on a rotating sphere. In particular, if $\omega = 0$ then $k = 0$, hence, $F(\psi, k) = \psi$, $K(k) = \pi/2$ and $\theta = \psi$. This follows from the fact that the Hamiltonians (2.12) and (2.8) are identical when $\omega = 0$ since there is no centrifugal acceleration to be neglected.

In order to get the second angle variable ν, we use the expression (B.6). The function t on the left-hand side of (B.9) (considered as function of ψ) in this expression coincides with (B.4) and is, therefore, given by (B.12)

$$t(\psi) = \frac{1}{\omega\sqrt{a^2+b^2}}F(\psi, k). \tag{B.17}$$

The function η in (B.8) may also be represented as an elliptic integral. Using the notation of (B.11), we have

$$\eta = \frac{D}{\omega}\int_x^a \frac{\mathrm{d}x}{(1-x^2)\sqrt{(a^2-x^2)(x^2+b^2)}}.$$

Substituting $x = a\cos\psi$ then yields

$$\eta(\psi) = \frac{D}{\omega(1-a^2)\sqrt{(a^2+b^2)}}\Pi(\psi, n, k), \tag{B.18}$$

where

$$\Pi(\psi, n, k) = \int_0^\psi \frac{\mathrm{d}\psi}{(1+n\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}$$

is the elliptic integral of the third kind, $n = a^2/(1-a^2) = \tan^2\varphi_0$, k is defined by (B.13) and $\psi = \arccos(x/a)$ is the angle variable of the free motion on a rotating sphere.

The cycle C in (B.10) coincides with generating cycle C_1 in (A.9) for the angle θ, therefore, $\Delta_C\lambda = 2\pi$ and $\Delta_C\eta = \eta(2\pi)$. Hence

$$\omega_D = \frac{\eta(2\pi)}{T_I} - \omega_I - \omega. \tag{B.19}$$

Taking into account (5.4), (6.2) and (B.11), we get

$$\omega_D = \left(\frac{E}{D} + \sqrt{\left(\frac{E}{D}\right)^2 - \omega^2}\right)\frac{\Pi(n, k)}{K(k)} - \omega_I - \omega. \tag{B.20}$$

It follows from (B.6) that the angle variable ν is given by

$$\nu(\psi) = \lambda - \eta(\psi) + (\omega + \omega_D)t(\psi), \tag{B.21}$$

where η, t and ω_D are defined by formulae (B.17), (B.18) and (B.20), respectively.

B.3. Single-hemisphere trajectories

In the case of single hemisphere trajectories $x_1 \leq 1$, hence, $s_1 = 1 - x_1 \geq 0$. Setting

$$a^2 = s_0 = \sin^2\varphi_0, \qquad b^2 = s_1 = \sin^2\varphi_1, \qquad x = \sin\varphi \tag{B.22}$$

and using formula (6.3), one gets

$$\frac{\partial S}{\partial E} = \frac{1}{\omega}\int_x^a \frac{\mathrm{d}s}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{\omega a}F(\tau,k), \tag{B.23}$$

where

$$\tau = \arcsin\left(\frac{\sin\psi}{k}\right), \qquad \psi = \arccos\frac{x}{a}, \qquad k^2 = \frac{a^2-b^2}{a^2} \tag{B.24}$$

(see [23]). Substituting a and b from (B.22) yields: $k = \sin\psi_1$, where $\psi_1 = \arccos(\sin\varphi_1/a)$, so that

$$\tau = \arcsin\left(\frac{\sin\psi}{\sin\psi_1}\right). \tag{B.25}$$

The expression (B.25) scales the angle ψ so that τ varies over the interval $(0, \pi/2)$ when ψ varies over the interval $(0, \psi_1)$.

Let the cycle C in (B.5) be $\lambda = 0$, $\varphi_1 \le \varphi \le \varphi_0$, where increasing φ from φ_1 to φ_0 implies that P is positive and decreasing φ from φ_0 to φ_1 implies that P is negative. Then, using (B.23) and (B.24), we have

$$\frac{\mathrm{d}I}{\mathrm{d}E} = \frac{1}{\pi}\int_{\sin\varphi_1}^{\sin\varphi_0} \frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}} = \pm\frac{1}{\pi\omega a}F\left(\frac{\pi}{2},k\right) = \frac{1}{\pi\omega a}K(k). \tag{B.26}$$

Finally, taking (B.3) into account, we find that

$$\theta = \pi\frac{F(\tau,k)}{K(k)}, \tag{B.27}$$

where τ is defined by (B.25) and

$$k^2 = \frac{a^2-b^2}{a^2} = \frac{2\omega^2 E}{\omega^2 - E + \sqrt{E^2-\omega^2 D^2}}. \tag{B.28}$$

The expression (B.26) also yields the angular velocity corresponding to the action I

$$\omega_I = \frac{\mathrm{d}E}{\mathrm{d}I} = \left(\frac{\mathrm{d}I}{\mathrm{d}E}\right)^{-1} = \frac{\pi\omega a}{K(k)}. \tag{B.29}$$

In order to calculate the angle variable ν in (B.6), we note that the functions $\eta(\varphi)$ and $t(\varphi)$ in (B.8) and (B.9), respectively, can now be evaluated as follows:

$$t(\tau) = \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{2E - D^2/\cos^2\varphi - \omega^2\cos^2\varphi}} = \frac{1}{\omega a}F(\tau,k), \tag{B.30}$$

where τ and k are defined in (B.24). Also, substituting (B.25) and (B.24) into (B.8) yields for $\eta(\tau)$

$$\begin{aligned}\eta(\tau) &= \frac{D}{\omega}\int_x^a \frac{\mathrm{d}x}{(1-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{D}{\omega a(1-a^2)}\int_0^\tau \frac{\mathrm{d}\tau}{(1+n\sin^2\tau)\sqrt{1-k^2\sin^2\tau}} \\ &= \frac{D}{\omega a(1-a^2)}\Pi(\tau,n,k),\end{aligned} \tag{B.31}$$

where $\Pi(\tau,n,k)$ is the elliptic integral of the third kind, $n = (a^2-b^2)/(1-a^2)$, k is defined in (B.28), $\tau = \arcsin\sin\psi/k$ and $\psi = \arccos(x/a)$ is the angle variable for the motion on a rotating sphere.

The increment $\Delta_C\lambda$ vanishes for the cycle C. Therefore, according to (B.10)

$$\omega_D = \frac{\eta(2\pi)}{T_I} - \omega, \tag{B.32}$$

so that taking (5.4), (6.2) and (B.11) into account, we finally obtain

$$\omega_D = \frac{D\Pi(n,k)}{2\cos^2\varphi_0 K(k)_I} - \omega, \tag{B.33}$$

where

$$\cos^2\varphi_0 = \frac{E - \sqrt{E^2 - \omega^2 D^2}}{\omega^2}. \tag{B.34}$$

According to (B.6), the resulting expression of ν is, therefore,

$$\nu(\tau) = \lambda - \eta(\tau) + (\omega + \omega_D)t(\tau), \tag{B.35}$$

where η, t and ω_D are defined by formulae (B.30)–(B.32).

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