8. Use functional iteration to find solutions to the following nonlinear systems, accurate to within 10^{-5}, using the l_∞ norm.

a. $$x_2^2 + x_2^2 - x_1 = 0$$
$$x_1^2 - x_2^2 - x_2 = 0.$$

b. $$3x_1^2 - x_2^2 = 0,$$
$$3x_1x_2^2 - x_1^3 - 1 = 0.$$

c. $$x_1^2 + x_2 - 37 = 0,$$
$$x_1 - x_2^2 - 5 = 0,$$
$$x_1 + x_2 + x_3 - 3 = 0.$$

d. $$x_1^2 + 2x_2^2 - x_2 - 2x_3 = 0,$$
$$x_1^2 - 8x_2^2 + 10x_3 = 0,$$
$$\frac{x_1^2}{7x_2x_3} - 1 = 0.$$

9. Use the Gauss-Seidel method to approximate the fixed points in Exercise 7 to within 10^{-5}, using the l_∞ norm.

10. Repeat Exercise 8 using the Gauss-Seidel method.

11. In Exercise 10 of Section 5.9, we considered the problem of predicting the population of two species that compete for the same food supply. In the problem, we made the assumption that the populations could be predicted by solving the system of equations

$$\frac{dx_1(t)}{dt} = x_1(t)(4 - 0.0003x_1(t) - 0.0004x_2(t))$$

and

$$\frac{dx_2(t)}{dt} = x_2(t)(2 - 0.0002x_1(t) - 0.0001x_2(t)).$$

In this exercise, we would like to consider the problem of determining equilibrium populations of the two species. The mathematical criteria that must be satisfied in order for the populations to be at equilibrium is that, simultaneously,

$$\frac{dx_1(t)}{dt} = 0 \quad \text{and} \quad \frac{dx_2(t)}{dt} = 0.$$

This occurs when the first species is extinct and the second species has a population of 20,000 or when the second species is extinct and the first species has a population of 13,333. Can an equilibrium occur in any other situation?

12. Show that a function $\mathbf{F}$ mapping $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is continuous at $\mathbf{x}_0 \in D$ precisely when, given any number $\varepsilon > 0$, a number $\delta > 0$ can be found with property that for any vector norm $\|\cdot\|$,

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)\| < \varepsilon,$$

whenever $\mathbf{x} \in D$ and $\|\mathbf{x} - \mathbf{x}_0\| < \delta$.

13. Let A be an $n \times n$ matrix and $\mathbf{F}$ be the function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$. Use the result in Exercise 12 to show that $\mathbf{F}$ is continuous on $\mathbb{R}^n$.

10.2 Newton's Method

The problem in Example 2 of Section 10.1 is transformed into a convergent fixed-point problem by algebraically solving the three equations for the three variables x_1, x_2, and x_3. It is, however, unusual to be able to find an explicit representation for all the variables. In this section, we consider an algorithmic procedure to perform the transformation in a more general situation.

To construct the algorithm that led to an appropriate fixed-point method in the one-dimensional case, we found a function ϕ with the property that

$$g(x) = x - \phi(x)f(x)$$

gives quadratic convergence to the fixed point p of the function g (see Section 2.4). From this condition Newton's method evolved by choosing $\phi(x) = 1/f'(x)$, assuming that $f'(x) \neq 0$.

A similar approach in the n-dimensional case involves a matrix

$$A(\mathbf{x}) = \begin{bmatrix} a_{11}(\mathbf{x}) & a_{12}(\mathbf{x}) & \cdots & a_{1n}(\mathbf{x}) \\ a_{21}(\mathbf{x}) & a_{22}(\mathbf{x}) & \cdots & a_{2n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ a_{n1}(\mathbf{x}) & a_{n2}(\mathbf{x}) & \cdots & a_{nn}(\mathbf{x}) \end{bmatrix}, \tag{10.5}$$

where each of the entries $a_{ij}(\mathbf{x})$ is a function from $\mathbb{R}^n$ into $\mathbb{R}$. This requires that $A(\mathbf{x})$ be found so that

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - A(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$$

gives quadratic convergence to the solution of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, assuming that $A(\mathbf{x})$ is nonsingular at the fixed point $\mathbf{p}$ of $\mathbf{G}$.

The following theorem parallels Theorem 2.8 on page 80. Its proof requires being able to express $\mathbf{G}$ in terms of its Taylor series in n variables about $\mathbf{p}$.

Theorem 10.7 Let $\mathbf{p}$ be a solution of $\mathbf{G}(\mathbf{x}) = \mathbf{x}$. Suppose a number $\delta > 0$ exists with

(i) $\partial g_i/\partial x_j$ is continuous on $N_\delta = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{p}\| < \delta\}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$;

(ii) $\partial^2 g_i(\mathbf{x})/(\partial x_j \partial x_k)$ is continuous, and $|\partial^2 g_i(\mathbf{x})/(\partial x_j \partial x_k)| \leq M$ for some constant M, whenever $\mathbf{x} \in N_\delta$, for each $i = 1, 2, ..., n$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, n$;

(iii) $\partial g_i(\mathbf{p})/\partial x_k = 0$, for each $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$.

Then a number $\hat{\delta} \leq \delta$ exists such that the sequence generated by $\mathbf{x}^{(k)} = \mathbf{G}(\mathbf{x}^{(k-1)})$ converges quadratically to $\mathbf{p}$ for any choice of $\mathbf{x}^{(0)}$, provided that $\left\|\mathbf{x}^{(0)} - \mathbf{p}\right\| < \hat{\delta}$. Moreover,

$$\|\mathbf{x}^{(k)} - \mathbf{p}\|_\infty \leq \frac{n^2 M}{2} \|\mathbf{x}^{(k-1)} - \mathbf{p}\|_\infty^2, \quad \text{for each } k \geq 1. \quad \blacksquare$$

To apply Theorem 10.7, suppose that $A(\mathbf{x})$ is an $n \times n$ matrix of functions from $\mathbb{R}^n$ into $\mathbb{R}$ in the form of Eq. (10.5), where the specific entries will be chosen later. Assume, moreover, that $A(\mathbf{x})$ is nonsingular near a solution $\mathbf{p}$ of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, and let $b_{ij}(\mathbf{x})$ denote the entry of $A(\mathbf{x})^{-1}$ in the ith row and jth column.

For $\mathbf{G}(\mathbf{x}) = \mathbf{x} - A(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x})$, we have $g_i(\mathbf{x}) = x_i - \sum_{j=1}^n b_{ij}(\mathbf{x}) f_j(\mathbf{x})$. So

$$\frac{\partial g_i}{\partial x_k}(\mathbf{x}) = \begin{cases} 1 - \displaystyle\sum_{j=1}^n \left(b_{ij}(\mathbf{x}) \frac{\partial f_j}{\partial x_k}(\mathbf{x}) + \frac{\partial b_{ij}}{\partial x_k}(\mathbf{x}) f_j(\mathbf{x}) \right), & \text{if } i = k, \\ -\displaystyle\sum_{j=1}^n \left(b_{ij}(\mathbf{x}) \frac{\partial f_j}{\partial x_k}(\mathbf{x}) + \frac{\partial b_{ij}}{\partial x_k}(\mathbf{x}) f_j(\mathbf{x}) \right), & \text{if } i \neq k. \end{cases}$$

Theorem 10.7 implies that we need $\partial g_i(\mathbf{p})/\partial x_k = 0$, for each $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n$. This means that for $i = k$,

$$0 = 1 - \sum_{j=1}^n b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_i}(\mathbf{p}),$$

that is,

$$\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_i}(\mathbf{p}) = 1. \tag{10.6}$$

When $k \neq i$,

$$0 = -\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_k}(\mathbf{p}),$$

so

$$\sum_{j=1}^{n} b_{ij}(\mathbf{p}) \frac{\partial f_j}{\partial x_k}(\mathbf{p}) = 0. \tag{10.7}$$

The Jacobian Matrix

Define the matrix $J(\mathbf{x})$ by

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}) \end{bmatrix}. \tag{10.8}$$

Then conditions (10.6) and (10.7) require that

$$A(\mathbf{p})^{-1}J(\mathbf{p}) = I, \text{the identity matrix}, \quad \text{so } A(\mathbf{p}) = J(\mathbf{p}).$$

An appropriate choice for $A(\mathbf{x})$ is, consequently, $A(\mathbf{x}) = J(\mathbf{x})$ since this satisfies condition (iii) in Theorem 10.7. The function $\mathbf{G}$ is defined by

$$\mathbf{G}(\mathbf{x}) = \mathbf{x} - J(\mathbf{x})^{-1}\mathbf{F}(\mathbf{x}),$$

and the functional iteration procedure evolves from selecting $\mathbf{x}^{(0)}$ and generating, for $k \geq 1$,

$$\mathbf{x}^{(k)} = \mathbf{G}\big(\mathbf{x}^{(k-1)}\big) = \mathbf{x}^{(k-1)} - J\big(\mathbf{x}^{(k-1)}\big)^{-1}\mathbf{F}\big(\mathbf{x}^{(k-1)}\big). \tag{10.9}$$

This is called **Newton's method for nonlinear systems**, and it is generally expected to give quadratic convergence, provided that a sufficiently accurate starting value is known and that $J(\mathbf{p})^{-1}$ exists. The matrix $J(\mathbf{x})$ is called the **Jacobian** matrix and has a number of applications in analysis. It might, in particular, be familiar to the reader due to its application in the multiple integration of a function of several variables over a region that requires a change of variables to be performed.

The Jacobian matrix first appeared in a 1815 paper by Cauchy, but Jacobi wrote *De determinantibus functionalibus* in 1841 and proved numerous results about this matrix.

A weakness in Newton's method arises from the need to compute and invert the matrix $J(\mathbf{x})$ at each step. In practice, explicit computation of $J(\mathbf{x})^{-1}$ is avoided by performing the operation in a two-step manner. First, a vector $\mathbf{y}$ is found that satisfies $J(\mathbf{x}^{(k-1)})\mathbf{y} = -\mathbf{F}(\mathbf{x}^{(k-1)})$. Then the new approximation, $\mathbf{x}^{(k)}$, is obtained by adding $\mathbf{y}$ to $\mathbf{x}^{(k-1)}$. Algorithm 10.1 uses this two-step procedure.

ALGORITHM 10.1

Newton's Method for Systems

To approximate the solution of the nonlinear system $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ given an initial approximation $\mathbf{x}$:

INPUT number n of equations and unknowns; initial approximation $\mathbf{x} = (x_1, \ldots, x_n)^t$, tolerance TOL; maximum number of iterations N.

OUTPUT approximate solution $\mathbf{x} = (x_1, \ldots, x_n)^t$ or a message that the number of iterations was exceeded.

Step 1 Set $k = 1$.

Step 2 While ($k \le N$) do Steps 3–7.

- *Step 3* Calculate $\mathbf{F}(\mathbf{x})$ and $J(\mathbf{x})$, where $J(\mathbf{x})_{i,j} = (\partial f_i(\mathbf{x})/\partial x_j)$ for $1 \le i, j \le n$.
- *Step 4* Solve the $n \times n$ linear system $J(\mathbf{x})\mathbf{y} = -\mathbf{F}(\mathbf{x})$.
- *Step 5* Set $\mathbf{x} = \mathbf{x} + \mathbf{y}$.
- *Step 6* If $||\mathbf{y}|| < TOL$ then OUTPUT ($\mathbf{x}$);
  (*The procedure was successful.*)
  STOP.
- *Step 7* Set $k = k + 1$.

Step 8 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was unsuccessful.*)
STOP.

Example 1 The nonlinear system

$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,$$
$$x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0,$$
$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

was shown in Example 2 of Section 10.1 to have the approximate solution $(0.5, 0, -0.52359877)^t$. Apply Newton's method to this problem with $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$.

Solution Define

$$\mathbf{F}(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))^t,$$

where

$$f_1(x_1, x_2, x_3) = 3x_1 - \cos(x_2x_3) - \frac{1}{2},$$
$$f_2(x_1, x_2, x_3) = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06,$$

and

$$f_3(x_1, x_2, x_3) = e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3}.$$

The Jacobian matrix $J(\mathbf{x})$ for this system is

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_3 \sin x_2 x_3 & x_2 \sin x_2 x_3 \\ 2x_1 & -162(x_2 + 0.1) & \cos x_3 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}.$$

Let $\mathbf{x}^{(0)} = (0.1, 0.1, -0.1)^t$. Then $\mathbf{F}(\mathbf{x}^{(0)}) = (-0.199995, -2.269833417, 8.462025346)^t$ and

$$J(\mathbf{x}^{(0)}) = \begin{bmatrix} 3 & 9.999833334 \times 10^{-4} & 9.999833334 \times 10^{-4} \\ 0.2 & -32.4 & 0.9950041653 \\ -0.09900498337 & -0.09900498337 & 20 \end{bmatrix}.$$

Solving the linear system, $J(\mathbf{x}^{(0)})\mathbf{y}^{(0)} = -\mathbf{F}(\mathbf{x}^{(0)})$ gives

$$\mathbf{y}^{(0)} = \begin{bmatrix} 0.3998696728 \\ -0.08053315147 \\ -0.4215204718 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \mathbf{y}^{(0)} = \begin{bmatrix} 0.4998696782 \\ 0.01946684853 \\ -0.5215204718 \end{bmatrix}.$$

Continuing for $k = 2, 3, \ldots,$ we have

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

where

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -\left(J\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right)\right)^{-1} \mathbf{F}\left(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}\right).$$

Thus, at the kth step, the linear system $J\left(\mathbf{x}^{(k-1)}\right)\mathbf{y}^{(k-1)} = -\mathbf{F}\left(\mathbf{x}^{(k-1)}\right)$ must be solved, where

$$J\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3 & x_3^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} & x_2^{(k-1)} \sin x_2^{(k-1)} x_3^{(k-1)} \\ 2x_1^{(k-1)} & -162\left(x_2^{(k-1)} + 0.1\right) & \cos x_3^{(k-1)} \\ -x_2^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & -x_1^{(k-1)} e^{-x_1^{(k-1)} x_2^{(k-1)}} & 20 \end{bmatrix},$$

$$\mathbf{y}^{(k-1)} = \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix},$$

and

$$\mathbf{F}\left(\mathbf{x}^{(k-1)}\right) = \begin{bmatrix} 3x_1^{(k-1)} - \cos x_2^{(k-1)} x_3^{(k-1)} - \frac{1}{2} \\ \left(x_1^{(k-1)}\right)^2 - 81\left(x_2^{(k-1)} + 0.1\right)^2 + \sin x_3^{(k-1)} + 1.06 \\ e^{-x_1^{(k-1)} x_2^{(k-1)}} + 20x_3^{(k-1)} + \frac{10\pi - 3}{3} \end{bmatrix}.$$

The results using this iterative procedure are shown in Table 10.3. ■

Table 10.3

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\lVert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \rVert_\infty$
0	0.1000000000	0.1000000000	−0.1000000000	
1	0.4998696728	0.0194668485	−0.5215204718	0.4215204718
2	0.5000142403	0.0015885914	−0.5235569638	1.788×10^{-2}
3	0.5000000113	0.0000124448	−0.5235984500	1.576×10^{-3}
4	0.5000000000	8.516×10^{-10}	−0.5235987755	1.244×10^{-5}
5	0.5000000000	-1.375×10^{-11}	−0.5235987756	8.654×10^{-10}

The previous example illustrates that Newton's method can converge very rapidly once a good approximation is obtained that is near the true solution. However, it is not always easy to determine good starting values, and the method is comparatively expensive to employ. In the next section, we consider a method for overcoming the latter weakness. Good starting values can usually be found using the Steepest Descent method, which will be discussed in Section 10.4.

Using Maple for Initial Approximations

The graphing facilities of Maple can assist in finding initial approximations to the solutions of 2×2 and often 3×3 nonlinear systems. For example, the nonlinear system

$$x_1^2 - x_2^2 + 2x_2 = 0, \quad 2x_1 + x_2^2 - 6 = 0$$

has two solutions, $(0.625204094, 2.179355825)^t$ and $(2.109511920, -1.334532188)^t$. To use Maple we first define the two equations

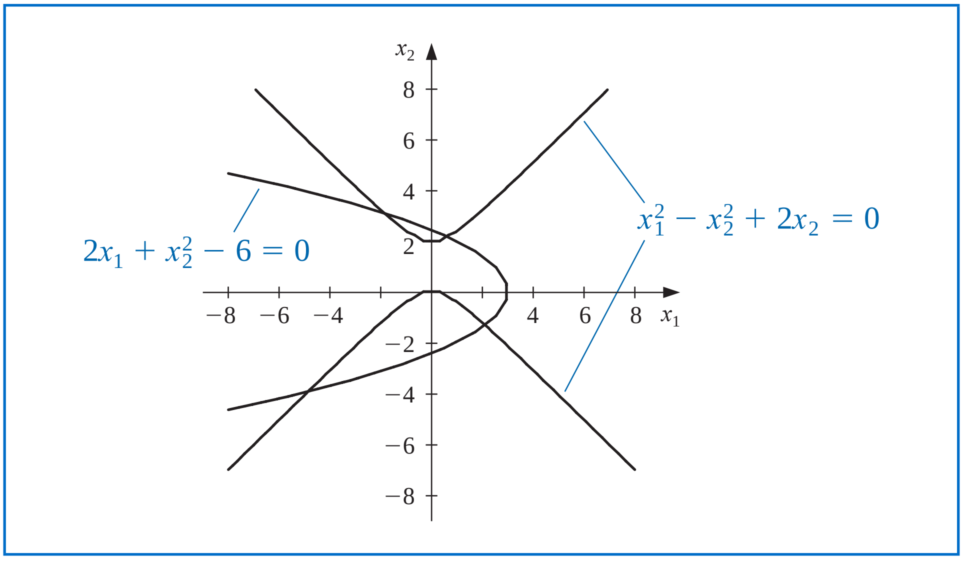
$eq1 := x1^2 - x2^2 + 2x2 = 0;\ eq2 := 2x1 + x2^2 - 6 = 0;$

To obtain a graph of the two equations for $-3 \le x_1, x_2 \le 3$, enter the commands

$with(plots): implicitplot(\{eq1, eq2\}, x1 = -6..6,\ x2 = -6..6);$

From the graph shown in Figure 10.2, we are able to estimate that there are solutions near $(2.1, -1.3)^t$, $(0.64, 2.2)^t$, $(-1.9, 3.0)^t$, and $(-5.0, -4.0)^t$. This gives us good starting values for Newton's method.

Figure 10.2



The problem is more difficult in three dimensions. Consider the nonlinear system

$$2x_1 - 3x_2 + x_3 - 4 = 0, \quad 2x_1 + x_2 - x_3 + 4 = 0, \quad x_1^2 + x_2^2 + x_3^2 - 4 = 0.$$

Define three equations using the Maple commands

$eq1 := 2x1{-}3x2{+}x3{-}4 = 0; eq2 := 2x1{+}x2{-}x3{+}4 = 0; eq3 := x1^2{+}x2^2{+}x3^2{-}4 = 0;$

The third equation describes a sphere of radius 2 and center $(0, 0, 0)$, so $x1$, $x2$, and $x3$ are in $[-2, 2]$. The Maple commands to obtain the graph in this case are

with(*plots*): *implicitplot3d*($\{eq1, eq2, eq3\}$, $x1 = -2..2$, $x2 = -2..2$, $x3 = -2..2$);

Various three-dimensional plotting options are available in Maple for isolating a solution to the nonlinear system. For example, we can rotate the graph to better view the sections of the surfaces. Then we can zoom into regions where the intersections lie and alter the display form of the axes for a more accurate view of the intersection's coordinates. For this problem, a reasonable initial approximation is $(x_1, x_2, x_3)^t = (-0.5, -1.5, 1.5)^t$.

EXERCISE SET 10.2

1. Use Newton's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to compute $\mathbf{x}^{(2)}$ for each of the following nonlinear systems.

a. $$4x_1^2 - 20x_1 + \frac{1}{4}x_2^2 + 8 = 0, \qquad \frac{1}{2}x_1x_2^2 + 2x_1 - 5x_2 + 8 = 0.$$

b. $$\sin(4\pi x_1x_2) - 2x_2 - x_1 = 0, \qquad \left(\frac{4\pi - 1}{4\pi}\right)(e^{2x_1} - e) + 4ex_2^2 - 2ex_1 = 0.$$

c. $$x_1(1 - x_1) + 4x_2 = 12, \qquad (x_1 - 2)^2 + (2x_2 - 3)^2 = 25.$$

d. $$5x_1^2 - x_2^2 = 0, \qquad x_2 - 0.25(\sin x_1 + \cos x_2) = 0.$$

2. Use Newton's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to compute $\mathbf{x}^{(2)}$ for each of the following nonlinear systems.

a. $$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0, \qquad 4x_1^2 - 625x_2^2 + 2x_2 - 1 = 0, \qquad e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

b. $$x_1^2 + x_2 - 37 = 0, \qquad x_1 - x_2^2 - 5 = 0, \qquad x_1 + x_2 + x_3 - 3 = 0.$$

c. $$15x_1 + x_2^2 - 4x_3 = 13, \qquad x_1^2 + 10x_2 - x_3 = 11, \qquad x_2^3 - 25x_3 = -22.$$

d. $$10x_1 - 2x_2^2 + x_2 - 2x_3 - 5 = 0, \qquad 8x_2^2 + 4x_3^2 - 9 = 0, \qquad 8x_2x_3 + 4 = 0.$$

3. Use the graphing facilities of Maple to approximate solutions to the following nonlinear systems.

a. $$4x_1^2 - 20x_1 + \frac{1}{4}x_2^2 + 8 = 0, \qquad \frac{1}{2}x_1x_2^2 + 2x_1 - 5x_2 + 8 = 0.$$

b. $$\sin(4\pi x_1x_2) - 2x_2 - x_1 = 0, \qquad \left(\frac{4\pi - 1}{4\pi}\right)(e^{2x_1} - e) + 4ex_2^2 - 2ex_1 = 0.$$

c. $$x_1(1 - x_1) + 4x_2 = 12, \qquad (x_1 - 2)^2 + (2x_2 - 3)^2 = 25.$$

d. $$5x_1^2 - x_2^2 = 0, \qquad x_2 - 0.25(\sin x_1 + \cos x_2) = 0.$$

4. Use the graphing facilities of Maple to approximate solutions to the following nonlinear systems within the given limits.