GAME THEORY

Bayesian Game

ECE 697AA/597AA

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Overview of Bayesian Game

- Static Bayesian Games
- Bayesian Dynamic Games in Extensive Form
- Detailed Example: Cournot Duopoly Model with Incomplete Information
- Applications in Wireless Networks
  - Packet Forwarding
  - K-Player Bayesian Water-filling
  - Channel Access
  - Bandwidth Auction
- Summary

What is Bayesian Game?

Game in Strategic Form

- ***Complete Information:*** Each player has complete information regarding the elements of the game
- ***Dominant Strategy:*** Iterated deletion of other strategies
- ***Nash Equilibrium:*** Solution of the game in strategic form

Bayesian Game

- A game with incomplete information
- Each player has initial private information, **type**
- ***Bayesian Equilibrium:*** Solution of the Bayesian game

Bayesian Game

Definition

A Bayesian game is a strategic form game with incomplete information. It consists of:

- A set of **players**, N={1, …, n}; for each $i \in$ N,
- An **action** set, $A_i, (A = \times_{i \in N} A_i)$
- A **type** set, $\Theta_i, (\Theta = \times_{i \in N} \Theta_i)$
- A **probability** function, $p_i : \Theta_i \rightarrow \Delta(\Theta_{-i})$
- A **payoff** function, $u_i : A \times \Theta \rightarrow \mathbf{R}$

- The function p_i is player *i*'s **belief** about the types of the other players
- Payoff of player *i*, u_i is defined as a function of action, A and type, Θ
- Belief formulation captures the **uncertainty on the payoffs** of other players

Bayesian Game

Definition

Bayesian game $(N, \{A_i\}_{i \in N}, \{\Theta_i\}_{i \in N}, \{p_i\}_{i \in N}, \{u_i\}_{i \in N})$ is finite if N, A_i, and Θ_i are all finite

Definition (Pure strategy, Mixed strategy)

Given a Bayesian Game $(N, \{A_i\}_{i \in N}, \{\Theta_i\}_{i \in N}, \{p_i\}_{i \in N}, \{u_i\}_{i \in N})$, a pure strategy for player i is a function which maps player i's type into its action set

$$a_i : \Theta_i \to A_i$$

A mixed strategy for player i is

$$\alpha_i : \Theta_i \to \Delta(A_i) : \theta_i \to \alpha_i(. | \theta_i)$$

Bayesian Equilibrium

Definition

A Bayesian equilibrium of a Bayesian game is a mixed strategy profile $\alpha = (\alpha_i)_{i \in N}$, such that for every player $i \in$ N and every type $\theta_i \in \Theta_i$, we have

$$\alpha_i(.|\theta_i) \in \arg\max_{\gamma \in \Delta(A_i)} \sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i} | \theta_i) \sum_{a \in A} \{ \prod_{j \in N \setminus \{i\}} \alpha_j(a_j | \theta_j) \} \gamma(a_i) u_i(a, \theta)$$

- Bayesian equilibrium is one of the mixed strategy profiles which maximizes each players' expected payoff for each type.

- This equilibrium is the solution for the Bayesian game. This equilibrium is the best response to each player's belief about the other player's mixed strategy.

Bayesian Dynamic Game

- Bayesian equilibrium results in complicated equilibria in dynamic game
- Refinement schemes do not always work in incomplete information
- Hence, perfect Bayesian equilibrium (demands optimal subsequent play)

Definition 30 *A perfect Bayesian equilibrium is a strategy profile and a set of beliefs for each player such that:*

1. *at every information set, player i's strategy maximizes its payoff, given strategies of all other players, and player i's beliefs.*
2. *at information sets reached with positive probability when PBE strategy is played, beliefs are formed according to strategy and Bayes' rule when necessary.*
3. *at information sets that are reached with probability zero when PBE strategy is played, beliefs may be arbitrary but must be formed according to Bayes' rule when possible.*

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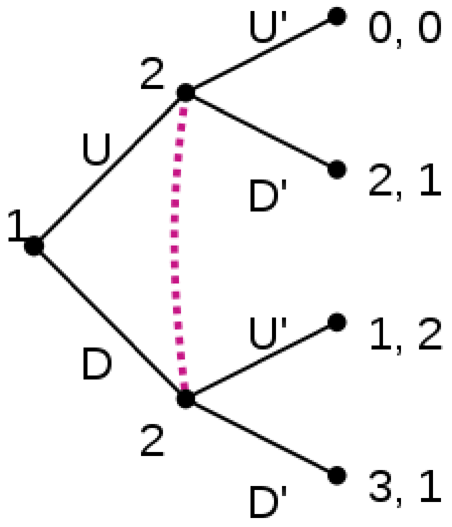
Simple Example

- **Information is imperfect** since *player 2* does not know what *player 1* does when he comes to play.
- If both players are rational and both know that too, play in the game will be as follows according to perfect Bayesian equilibrium:

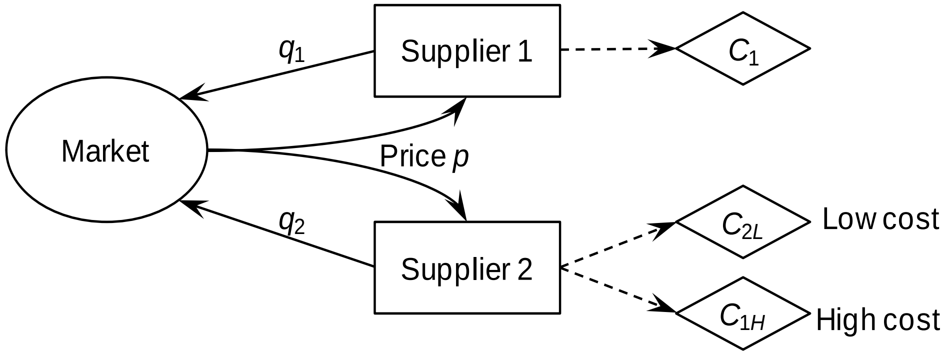
- *Player 1* likes to fool *player 2* into thinking he has played *U*, when he has actually played *D*, so that *player 2* will play *D'* and *player 1* will receive 3.
- In fact, there is a perfect Bayesian equilibrium where *player 1* plays *D* and *player 2* plays *U'* and *player 2* holds the belief that *player 1* will definitely play *D* (i.e *player 2* places a probability of 1 on the node reached if *player 1* plays *D*).
- In this equilibrium, every strategy is rational given the beliefs held and every belief is consistent with the strategies played. In this case, the perfect Bayesian equilibrium is the **only Nash equilibrium**.



Player 2 cannot observe *player 1*'s move

Example: Cournot Duopoly

Cournot duopoly model



(1) Players (2 firms): $N = \{1, 2\}$

(2) Action set (outcome of firms): $q_i \in \mathbf{R}_+, (i = 1, 2)$

(3) Type set: $\theta_1 = \{1\},\ \theta_2 = \{3/4,\ 5/4\}$

(4) Probability function:

$$p(\theta_2 = 3/4 \mid \theta_1) = 1/2,\ p(\theta_2 = 5/4 \mid \theta_1) = 1/2$$

(5) Profit function:

$$u_1(q_1, q_2, \theta_1, \theta_2) = q_1(\theta_1 - q_1 - q_2)$$

$$u_2(q_1, q_2, \theta_1, \theta_2) = q_2(\theta_2 - q_1 - q_2)$$

Example: Cournot Duopoly

Bayesian equilibrium for pure strategy:

- The Bayesian equilibrium is a maximal point of expected payoff of firm 2, EP_2:

$$EP_2 = u_2 \qquad \frac{\partial EP_2}{\partial q_2}(q_1^*, q_2^*) = \theta_2 - q_1^* - 2q_2^* = 0$$

$$q_2^*(\theta_2) = (\theta_2 - q_1^*)/2, \ (\theta_2 = 3/4, 5/4)$$

- The expected payoff of firm 1, EP_1, is given as follows:

$$EP_1 = \frac{1}{2} q_1(\theta_1 - q_1 - q_2(3/4)) + \frac{1}{2} q_1(\theta_1 - q_1 - q_2(5/4))$$

Example: Cournot Duopoly

- Bayesian equilibrium is also the maximal point of expected payoff, EP_1:

$$\frac{\partial EP_1}{\partial q_1}(q_1^*, q_2^*) = 1 - 2q_1^* - \frac{1}{2}\{q_2^*(3/4) + q_2^*(5/4)\} = 0$$

$$q_1^* = \frac{2 - q_2^*(3/4) - q_2^*(5/4)}{4}$$

- Solving the above equations, we can get Bayesian equilibrium as follows:

$$q_1^* = \frac{1}{3},\ q_2^*(3/4) = \frac{11}{24},\ q_2^*(5/4) = \frac{5}{24}.$$

Example: Sheriff's Dilemma

A sheriff faces an armed suspect. Both must simultaneously decide whether to shoot the other or not.

Suspect type: "criminal" or "civilian".
Sheriff has only one type

The suspect knows its type and the Sheriff's type.
The Sheriff does not know the suspect's type (**incomplete info**.)

Probability *p* that the suspect is a criminal

Probability *1-p* that the suspect is a civilian

Question: what is the probability that the Sheriff will shoot?

Normal form game

Type = "Civilian"

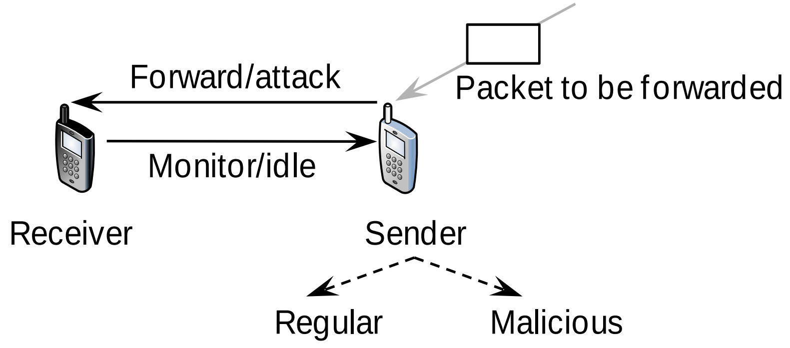
		Sheriff's action	
		Shoot	Not
Suspect's action	Shoot	-3, -1	-1, -2
	Not	-2, -1	0, 0

Type = "Criminal"

		Sheriff's action	
		Shoot	Not
Suspect's action	Shoot	0, 0	2, -2
	Not	-2, -1	-1,1

Example: Packet Forwarding

- System model; Pure strategy
  - ***Sender type:*** Malicious or regular
  - ***Payoff:*** Malicious vs. regular



	Monitor	Stay idle
Attack	$(-G_A - C_A, G_A - C_M)$	$(G_A - C_A, -G_A)$
Forward	$(-C_F, -C_M)$	$(-C_F, 0)$

	Monitor	Stay idle
Forward	$(-C_F, -C_M)$	$(-C_F, 0)$

  - *Malicious sender:* $\mathbf{C_A}$- Cost of attack, $\mathbf{G_A}$- Attack success
  - *Regular sender:* $\mathbf{C_F}$- Cost of forwarding
  - *Receiver:* $\mathbf{C_M}$- Cost of monitoring, $\mathbf{-G_A}$- Cost of being attacked
- Mixed strategy
  - *Malicious sender:* Attack with a certain **probability**
  - *Regular sender:* Forward packet
  - *Receiver:* Monitor with a certain probability

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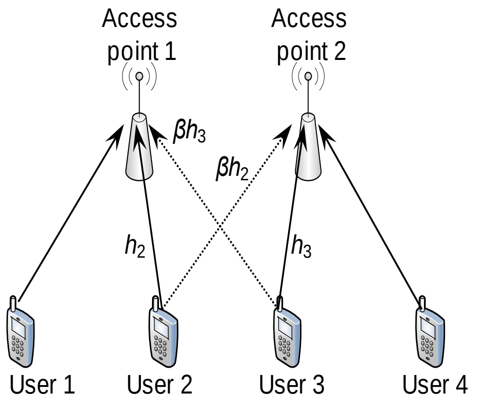
Example: Channel Access

- System model
  - Bayesian since do not know the other's channel (or type)
  - Transmission success: $\gamma_i = \dfrac{Ph_i}{\beta \sum_{j \neq i} Ph_j + \sigma^2} \geq \gamma_{\text{thr}}$
  - Throughput: $R_i(\mathbf{s}_{-i}) = \begin{cases} \log(1+\gamma_i), & \text{if } \gamma_i \geq \gamma_{\text{thr}}, \\ 0, & \text{otherwise}, \end{cases}$
  - Utility: $U_i(s_i, \mathbf{s}_{-i}) = \begin{cases} 0, & \text{if } s_i = \text{backoff}, \\ R_i(\mathbf{s}_{-i}) - C, & \text{if } s_i = \text{transmit}, \end{cases}$
  - Bayesian NE: $s_i^*(h_i) = \arg\max_{s_i \in \mathcal{S}_i} E\left(U(s_i, \mathbf{s}_{-i}(\mathbf{h}_{-i}), h_i)\right)$



Interference among transmissions

- Threshold strategy
  - Interpretation of opportunistic spectrum access from the game theory point of view

$$s_i(h_i) = \begin{cases} \text{transmit}, & \text{if } h_i > h_{\text{thr},i}, \\ \text{backoff}, & \text{otherwise}. \end{cases}$$

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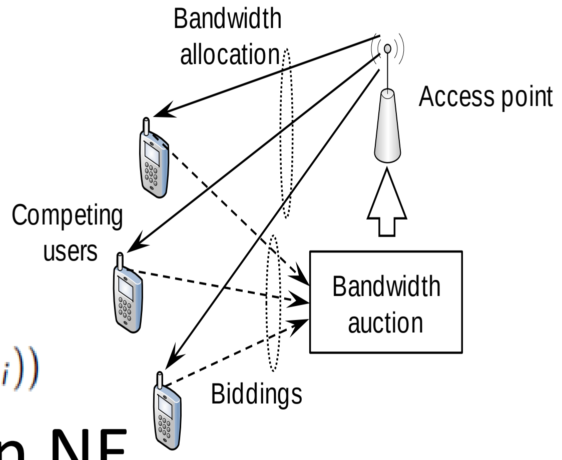
Example: Bandwidth Auction

- System model
  - Allocated bandwidth $$g_i = \frac{s_i t_i}{\sum_{j=1}^{N} s_j t_j} B,$$
  - $\mathbf{s_i}$ - Bid, **B** - Total bandwidth, $\mathbf{t_i}$ - Time duration of connection, **U** - utility
  - Pdf of $\mathbf{t_i}$ of other nodes $$\alpha_T(t) = \frac{1}{\beta} \exp(-t/\beta) > 0,$$
  - Bayesian NE $$s_i^*(t_i) \quad = \quad \arg\max_{s_i} E\left(\mathscr{U}_i(g_i, s_i, \mathbf{s}_{-i}, t_i, \mathbf{t}_{-i})\right)$$



- Iterative algorithm to obtain Bayesian NE

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Initialize iteration counter k = 1.
Access point receives s_i[k], t_i[k], and g_thr,i from all vehicular nodes.
Access point computes g_i[k − 1] from (4.56) and allocates bandwidth to the
vehicular nodes.
repeat
    k ← k + 1 .
    s_i^*[k] ← arg max_{s_i} E (𝒰_i(g_i[k − 1], s_i[k − 1], s_−i[k − 1], t_i[k − 1], t_−i)).
    Vehicular node i sends s_i^*[k] to access point.
    Access point computes g_i[k] from (4.56) and allocates bandwidth to the
    vehicular nodes.
until max_i |s_i^*[k] − s_i^*[k − 1]| ≤ ϵ .
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Summary

- Games with incomplete information (i.e., Bayesian game) can be used to analyze situations where a player does not know the preference (i.e., payoff) of his opponents.
- This is a common situation in wireless communications and networking where there is **no centralized controller** to maintain the information of all users. Also, the users may not reveal the private information to others.
- The details of the Bayesian game framework were studied.
- Some applications of the Bayesian game framework in wireless communications and networking were discussed.