Linear Systems of Equations, Matrix Inverses, and Determinants

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Systems of Linear Equations

Many diverse applications are modeled by systems of linear equations, for example: imaging, systems of differential equations, data compression, chemical equations, etc.

Example

The process of photosynthesis: solar energy is converted into forms used by living organisms. More specifically we consider a chemical reaction that occurs in plant leaves: converts CO_2 and water to carbohydrates with a release of oxygen. The equation that describes this:

This gives us a system of linear equations in 4 variables:

Definitions

Definition

A linear equation in the n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

where a_i for $i = 1, \ldots, n$ and b are constant, typically real, numbers.

Definition

A system of m linear equations in n variables or a linear system, is a collection of equations of the form

This is also referred to as an $m \times n$ linear system.

Definitions continued

Definition

A solution to a linear system with n variables is an ordered sequence $(s_1, s_2, ..., s_n)$ such that each equation is satisfied for $x_1 = s_1$, ..., $x_n = s_n$. The general solution or solution set is the set of all possible solutions.

Definition

A system of equations is consistent if there is at least one solution to the system. If there are no solutions the system is inconsistent.

Examples

$$2x - y = 2$$
$$x + 2y = 6$$

We have a system of 2 equations with 2 variables, x and y. A <u>solution</u> consists of values for x and y which simultaneously satisfy each equation. How would you go about solving this system?

Now let's think about this geometrically. What do each of these equations model? What do the plots look like?

Examples Continued

What other combinations of two lines can we have? Draw them.

Come up with examples of linear systems with 2 equations and 2 unknowns that would result in the line combinations you came up with.

Examples Continued

Thinking about these possible line combinations some more what are the solution sets for your examples?

The Elimination Method

The elimination method (aka Gaussian elimination) is an algorithm (or method) for solving linear systems.

Example

How would we solve these?

Elimination Continued

Example

Solve the following system of equations:

$$\begin{aligned} x + y &= 1 \\ -x + y &= 1 \end{aligned}$$

Note. What additional step did you use to solve this system in comparison with our previous examples?

Elimination continued

Definition

An $m \times n$ linear system is in upper triangular form provided that the coefficients $a_{ij} = 0$ whenever $i > j$ and is in lower triangular form provided that the coefficients $a_{ij} = 0$ whenever $j > i$.

In the process of elimination our goal is to take our original system and put it into an equivalent, triangular, form.

Elimination continued

Definition

Two linear systems are <u>equivalent</u> if they have the same solutions.

Example

$$\begin{aligned} x + y &= 1 \\ -x + y &= 1 \end{aligned}$$

Example

Solve the linear system using elimination

$$\begin{aligned} x + y + z &= 4 \\ -x - y + z &= -2 \\ 2x - y + 2z &= 2 \end{aligned}$$

Example continued

Gaussian Elimination

Theorem

Let

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&. \\
&. \\
&. \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m.
\end{aligned}
$$

be an $m \times n$ linear system. Performing any one of the following operations on the linear system produces an equivalent linear system:

1. *Interchanging any two equations.*
2. *Multiplying any equation by a nonzero constant.*
3. *Adding a multiple of one equation to another.*

Practice

Solve the linear system using elimination

$$
\begin{aligned}
x - y + z &= 3 \\
3x + 2y - z &= 0 \\
4x + y &= 3
\end{aligned}
$$

Practice continued

Practice continued

Let's consider systems of three equations with three variables geometrically. Recall that the graph of a plane in three-dimensional space is given by the equation: $ax + by + cz = d$. What are some possible geometric configurations of three planes? What type of solution set do you think these configuration correspond to?

Elimination with Matrices

We saw earlier that converting a linear system to an equivalent triangular system provides an algorithm for solving a linear system. We can streamline this process by utilizing matrices.
Now lets consider a linear system, for example,

- The most important pieces of these equations are the coefficients of the variables and the right-hand side values.
- A matrix allows us to record all these values while using each column to associate with a particular variable and the right-hand side.
- This is called the augmented matrix for the linear system.

Elimination with Matrices continued

- We may combine all the coefficients into one matrix, called the coefficient matrix A, write all unknowns as a vector $\vec{x}$, and write the right hand side of our system as a vector $\vec{b}$
- We therefore see we may represent the linear system of equations as a single matrix equation
- **Note.** For an $m \times n$ linear system the augmented matrix is of size $m \times (n+1)$.
- **Note.** We use a zero to record any missing terms in the matrix representation of a linear system.
- Now we use the augmented matrix to solve our linear system. We can perform equation operations as corresponding operations to the rows of the augmented matrix representing the linear system.

Example

$$
\begin{aligned}
x - y + z &= 3 \\
3x + 2y - z &= 0 \\
4x + y &= 3
\end{aligned}
$$

Example continued

Gaussian Elimination with matrices

Theorem

Any one of the following operations performed on the augmented matrix, corresponding to a linear system, produces an augmented matrix corresponding to an equivalent linear system.

1. *Interchanging two rows.*
2. *Multiplying any row by a nonzero constant.*
3. *Adding a multiple of one row to another.*

The operations listed are called row operations.

Definition

An $m \times n$ matrix A is called row equivalent to an $m \times n$ matrix B if B can be obtained from A by a sequence of row operations.

Example

Consider the following linear system:

$$
\begin{aligned}
4x_1 - 8x_2 - 3x_3 + 2x_4 &= 13\\
3x_1 - 4x_2 - x_3 - 3x_4 &= 5\\
2x_1 - 4x_2 - 2x_3 + 2x_4 &= 6
\end{aligned}
$$

Example continued

Practice

Consider the following linear system of equations

$$\begin{aligned} x_1 - x_2 - 2x_3 - 2x_4 - 2x_5 &= 3 \\ 3x_1 - 2x_2 - 2x_3 - 2x_4 - 2x_5 &= -1 \\ -3x_1 + 2x_2 + x_3 + x_4 - x_5 &= -1 \end{aligned}$$

- Rewrite the following system of equations as a matrix vector equation in the form $A\vec{x} = \vec{b}$.
- Rewrite the following system as an augmented matrix and use elimination to solve.

Practice continued

Gauss-Jordan Elimination

We can actually do further elimination than what we have been doing. In this way we can incorporate the work of back substitution into our matrix row operations.

Definition

An $m \times n$ matrix is in <u>row echelon form</u> if

1. Every row with all zero entries is below every row with nonzero entries.
2. If rows $1, 2, ..., k$ are rows with nonzero entries and if the leading nonzero entry (<u>pivot</u>) in row i occurs in column c_i, for $i = 1, 2, ..., k$ then $c_1 < c_2 < ... < c_k$.

It looks like this:

Gauss-Jordan Elimination continued

Definition

An $m \times n$ matrix is in reduced row echelon form if, in addition to the conditions above, we have

1. The first nonzero entry (pivot) of each row is one.
2. Each column that contains a pivot has all other entries zero.

It looks like this:

Example

Consider the linear system:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 1 \\ 2x_1 + 5x_2 - 8x_3 &= 4 \\ -2x_1 - 4x_2 + 6x_3 &= -2 \end{aligned}$$

Example continued

Practice

State whether the following augmented matrices are in REF or RREF. If it is not in RREF explain why not.

(a) $$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

(b) $$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) $$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Practice continued

Put the following matrix in RREF. When you are finished think about what pseudo code would look like for implementing this elimination, pay attention to where you start, is there a way to create an algorithm that works for any matrix?

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ -1 & -2 & -1 & -1 \\ -1 & -5 & 8 & -3 \end{bmatrix}$$

Gauss-Jordan Elimination Algorithm

1. Set $j = 0$ and $r = 1$
2. Increase j by 1. If j now equals $n + 1$ then stop.
3. Examine the entries of A in column j located in rows r through m. If all entries are zero go to step 2.
4. Choose a row from rows r through m with a nonzero entry in column j. Let i denote the index for this row.
5. Use the first row operation to swap rows i and r.
6. Use the second row operation to convert the entry in row r an column j to a 1.
7. Use the third row operation with row r to convert every other entry of column j to zero.
8. Increase r by 1.
9. Go to step 2.

Algorithm continued

Now let's redo this using the algorithm given on the previous slide:

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ -1 & -2 & -1 & -1 \\ -1 & -5 & 8 & -3 \end{bmatrix}$$

Algorithm continued

Types of Solution Sets

Recall that we said earlier that when solving a linear system of equations the only three solution scenarios are: unique solution, no solution, or infinitely many solutions.

Theorem

Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is row equivalent to A and in RREF with r non-zero rows. Then the system is inconsistent if and only if (iff) column $(n+1)$ of B is a pivot column.

Example

Write down an example augmented matrix (which is 3 by 4 in size) in RREF with a pivot in the 4th column.

Types of Solution Sets

Theorem

Suppose A is the augmented matrix of a consistent system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Then $r \leq n$. If $r = n$ then the system has a unique solution, and if $r < n$ then the system has infinitely many solutions.

Let's come up with some examples to demonstrate the ideas of this theorem:

Types of Solution Sets

Theorem

Suppose we have a consistent linear system with m equations and n variables. If $n > m$ then the system has infinitely many solutions.

Why is this true?

The Inverse of a Square Matrix and Linear Systems

Consider solving the following equation

$$ax = b$$

where a, x, b are all scalars. How do we do this?

Similarly we would like to solve systems of linear equations:

Matrix Inverses continued

Definition

A number x that satisfies $xa = 1$ is called the inverse of a. An inverse $(1/a)$ exists if and only if $a \neq 0$, and is unique.

Definition

A matrix X that satisfies $XA = AX = I$ is called a **inverse** of A. If an inverse exists we say A is **invertible**.

Question. Why do you think we are only considering square matrices?

Example

Find an inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Matrix Inverse Facts

Theorem

The inverse of a matrix, if it exists, is unique.

Note. The unique inverse of A is denoted A^{-1}. When the inverse of A exists we call A invertible or nonsingular. Otherwise we say A is noninvertible or singular.

Theorem

The inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *exists if and only if (iff)* $ad - bc \neq 0$. *In this case the inverse of the matrix is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example

Find the inverse of the following matrices using the last theorem.

- $$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- $$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Solving systems with matrix inverses

Recall we may represent a linear system of equations as a single matrix equation

Note if A is invertible then we may solve this equation using A^{-1}, i.e.

Example

Solve the following system of equations using the inverse of the coefficient matrix:

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -x_1 + 6x_2 &= 0 \end{aligned}$$

More inverse facts

Theorem

If the $n \times n$ matrix A is invertible then for every vector $\vec{b}$ with n components the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

Question:. Why is this true? What can we say about the solutions to a system when A is not invertible?

Homogeneous Systems

Definition

A <u>homogeneous linear system</u> is a system of the form $A\vec{x} = \vec{0}$.

Question: Will a homogeneous linear system ever be inconsistent?

Note. If a homogenous linear system $A\vec{x} = \vec{0}$ is such that A is invertible then by our previous theorem and note what can we conclude?

Inverses for Matrices Larger than 2×2

Let A be an $n \times n$ matrix and let B be another $n \times n$ matrix. Now lets denote the column vectors of B as $\vec{B}_1, ..., \vec{B}_n$ and the row vectors of A as $\vec{A}_1, ..., \vec{A}_n$. Now looking at AB we have

Now if B is to be A's inverse we must have that

Each of these is a matrix equation

which must have a unique solution if A is to be invertible.

Inverses for Matrices Larger than 2×2 continued

So we have n linear systems

We can actually solve all of these linear systems simultaneously by row reducing the following $n \times 2n$ augmented matrix

Note. A has an inverse iff it is row equivalent to I.

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Practice

Solve the linear system by finding the inverse of the coefficient matrix.

$$\begin{aligned} -2x - 2y - z &= 0 \\ -x - y &= -1 \\ -y + 2z &= 2 \end{aligned}$$

Left inverses

You may be wondering if we can do anything similar when our system's coefficient matrix is not square. The answer is sometimes.

Definition

A matrix X that satisfies $XA = I$ is called a **left inverse** of A. If a left inverse exists we say A is **left-invertible**.

Example

The matrix

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

has two different left inverses:

$$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix} \quad C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

Left inverse and column independence

Proposition

A matrix is left-invertible if and only if its columns are linearly independent.

Note. What if A is wide, i.e. $m < n$?

If A has a left inverse C and $Ax = b$ (system of linear equations), then

$$Ax = b$$

Example: system solved with a left inverse

Example

Find solution(s) to the system using info about the left inverses of A:

$$\begin{aligned} -3x_1 - 4x_2 &= 1 \\ 4x_1 + 6x_2 &= -2 \\ 1x_1 + 1x_2 &= 0 \end{aligned}$$

Overdetermined system with no solution

It is possible for the matrix A to have a left inverse but the system $Ax = b$ has no solution. In this case we want to verify that

$$A(Cb) = b$$

Example

$$\begin{bmatrix} 5 & 2 \\ 10 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$$

Right inverses

Definition

A matrix X that satisfies $AX = I$ is a **right inverse** of A. If a right inverse exists we say that A is **right-invertible**.

This is equivalent to saying A is right-invertible if and only if A^T is left-invertible. Therefore we have the following proposition.

Proposition

A is right invertible if and only if its rows are linearly independent.

What shape can a right-invertible matrix be?

Solving linear equations with a right inverse

Suppose A has a right inverse B. Consider a square or underdetermined system of equations $Ax = b$.

Determinants

- Earlier we saw how important the number $ad - bc$ was when finding the inverse of a 2×2 matrix.
- The number has a name, the determinant of A, and provides special information.
- A is invertible iff the determinant does not equal 0.

Definition

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denoted by $|A|$ or $\det(A)$, is given by

$$|A| = \det(A) = ad - bc$$

Example

Find the determinant of $A = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$.

3×3 Determinants

Definition

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$|A| = \det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example

Find the determinant of $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$.

Determinants continued

- Note for 3×3 matrices (and larger) the determinant still tells us if a matrix is invertible/nonsingular or not
- You can prove that fact simply by doing row reduction on a general matrix A, dividing by $\det(A)$ will come up in a necessary row operation
- Often the calculation of the determinant of a matrix larger than 2×2 is called a <u>cofactor expansion</u>
- In our previous slide we are performing an expansion along row 1, what does that mean?

Cofactor Expansions

It turns out you may use any row or column to expand along using the following sign conventions:

Example

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}.$$

Cofactor Expansions

- When would changing from the first row make sense?

Example

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 4 \\ 1 & 3 & -1 \end{bmatrix}$$

Cofactor Expansions

Definition

If A is a square matrix, then the <u>minor</u> M_{ij}, associated with the entry a_{ij}, is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from the matrix A. The <u>cofactor</u> of a_{ij} is $C_{ij} = (-1)^{i+j} \det(M_{ij})$.

Theorem

Let A be an $n \times n$ matrix. Then the determinant of A equals the cofactor expansion along any row or column of the matrix. That is, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$

$$\det(A) = a_{i1}C_{i1} + \ldots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}$$

and

$$\det(A) = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.$$

Cofactor Expansions

Cofactor expansions work for any size matrix, the calculation just becomes more tedious:

Example

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 0 & 5 & 6 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

Practice

Determine the determinant of the following matrices and state whether or not the matrix is invertible

1. $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Invertible Matrix Theorem

Theorem

Let A be a square matrix. The following statements are equivalent.

1. *The matrix A is invertible.*
2. *The linear system $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$.*
3. *The homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.*
4. *The column vectors of A are linearly independent.*
5. *The matrix A is row equivalent to the identity matrix.*
6. *The determinant of the matrix A is nonzero.*