16 MAKING SIMPLE DECISIONS

In which we see how an agent should make decisions so that it gets what it wants—on average, at least.

In this chapter, we fill in the details of how utility theory combines with probability theory to yield a decision-theoretic agent—an agent that can make rational decisions based on what it believes and what it wants. Such an agent can make decisions in contexts in which uncertainty and conflicting goals leave a logical agent with no way to decide: a goal-based agent has a binary distinction between good (goal) and bad (non-goal) states, while a decision-theoretic agent has a continuous measure of outcome quality.

Section 16.1 introduces the basic principle of decision theory: the maximization of expected utility. Section 16.2 shows that the behavior of any rational agent can be captured by supposing a utility function that is being maximized. Section 16.3 discusses the nature of utility functions in more detail, and in particular their relation to individual quantities such as money. Section 16.4 shows how to handle utility functions that depend on several quantities. In Section 16.5, we describe the implementation of decision-making systems. In particular, we introduce a formalism called a **decision network** (also known as an **influence diagram**) that extends Bayesian networks by incorporating actions and utilities. The remainder of the chapter discusses issues that arise in applications of decision theory to expert systems.

16.1 COMBINING BELIEFS AND DESIRES UNDER UNCERTAINTY

Decision theory, in its simplest form, deals with choosing among actions based on the desirability of their *immediate* outcomes; that is, the environment is assumed to be episodic in the sense defined on page 43. (This assumption is relaxed in Chapter 17.) In Chapter 3 we used the notation $\text{RESULT}(s_0, a)$ for the state that is the deterministic outcome of taking action a in state s_0. In this chapter we deal with nondeterministic partially observable environments. Since the agent may not know the current state, we omit it and define $\text{RESULT}(a)$ as a *random variable* whose values are the possible outcome states. The probability of outcome s', given evidence observations **e**, is written

$$P(\text{RESULT}(a) = s' \mid a, \mathbf{e}) ,$$

where the a on the right-hand side of the conditioning bar stands for the event that action a is executed.[1]

UTILITY FUNCTION

EXPECTED UTILITY

The agent's preferences are captured by a **utility function**, $U(s)$, which assigns a single number to express the desirability of a state. The **expected utility** of an action given the evidence, $EU(a|\mathbf{e})$, is just the average utility value of the outcomes, weighted by the probability that the outcome occurs:

$$EU(a|\mathbf{e}) = \sum_{s'} P(\textsc{Result}(a) = s' \mid a, \mathbf{e})\, U(s') \,. \tag{16.1}$$

MAXIMUM EXPECTED UTILITY

The principle of **maximum expected utility** (MEU) says that a rational agent should choose the action that maximizes the agent's expected utility:

$$action = \operatorname*{argmax}_a EU(a|\mathbf{e})$$

In a sense, the MEU principle could be seen as defining all of AI. All an intelligent agent has to do is calculate the various quantities, maximize utility over its actions, and away it goes. But this does not mean that the AI problem is *solved* by the definition!

The MEU principle *formalizes* the general notion that the agent should "do the right thing," but goes only a small distance toward a full *operationalization* of that advice. Estimating the state of the world requires perception, learning, knowledge representation, and inference. Computing $P(\textsc{Result}(a) \mid a, \mathbf{e})$ requires a complete causal model of the world and, as we saw in Chapter 14, NP-hard inference in (very large) Bayesian networks. Computing the outcome utilities $U(s')$ often requires searching or planning, because an agent may not know how good a state is until it knows where it can get to from that state. So, decision theory is not a panacea that solves the AI problem—but it does provide a useful framework.

The MEU principle has a clear relation to the idea of performance measures introduced in Chapter 2. The basic idea is simple. Consider the environments that could lead to an agent having a given percept history, and consider the different agents that we could design. *If an agent acts so as to maximize a utility function that correctly reflects the performance measure, then the agent will achieve the highest possible performance score (averaged over all the possible environments).* This is the central justification for the MEU principle itself. While the claim may seem tautological, it does in fact embody a very important transition from a global, external criterion of rationality—the performance measure over environment histories—to a local, internal criterion involving the maximization of a utility function applied to the next state.

16.2 The Basis of Utility Theory

Intuitively, the principle of Maximum Expected Utility (MEU) seems like a reasonable way to make decisions, but it is by no means obvious that it is the *only* rational way. After all, why should maximizing the *average* utility be so special? What's wrong with an agent that

[1] Classical decision theory leaves the current state S_0 implicit, but we could make it explicit by writing $P(\textsc{Result}(a) = s' \mid a, \mathbf{e}) = \sum_s P(\textsc{Result}(s, a) = s' \mid a) P(S_0 = s \mid \mathbf{e})$.

maximizes the weighted sum of the cubes of the possible utilities, or tries to minimize the worst possible loss? Could an agent act rationally just by expressing preferences between states, without giving them numeric values? Finally, why should a utility function with the required properties exist at all? We shall see.

16.2.1 Constraints on rational preferences

These questions can be answered by writing down some constraints on the preferences that a rational agent should have and then showing that the MEU principle can be derived from the constraints. We use the following notation to describe an agent's preferences:

$A \succ B$ the agent prefers A over B.

$A \sim B$ the agent is indifferent between A and B.

$A \succsim B$ the agent prefers A over B or is indifferent between them.

Now the obvious question is, what sorts of things are A and B? They could be states of the world, but more often than not there is uncertainty about what is really being offered. For example, an airline passenger who is offered "the pasta dish or the chicken" does not know what lurks beneath the tinfoil cover.[2] The pasta could be delicious or congealed, the chicken juicy or overcooked beyond recognition. We can think of the set of outcomes for each action as a **lottery**—think of each action as a ticket. A lottery L with possible outcomes $S_1, \ldots, S_n$ that occur with probabilities $p_1, \ldots, p_n$ is written

LOTTERY

$$L = [p_1, S_1;\ p_2, S_2;\ \ldots\ p_n, S_n]\ .$$

In general, each outcome S_i of a lottery can be either an atomic state or another lottery. The primary issue for utility theory is to understand how preferences between complex lotteries are related to preferences between the underlying states in those lotteries. To address this issue we list six constraints that we require any reasonable preference relation to obey:

ORDERABILITY

- **Orderability**: Given any two lotteries, a rational agent must either prefer one to the other or else rate the two as equally preferable. That is, the agent cannot avoid deciding. As we said on page 490, refusing to bet is like refusing to allow time to pass.

  Exactly one of $(A \succ B)$, $(B \succ A)$, or $(A \sim B)$ holds.

TRANSITIVITY

- **Transitivity**: Given any three lotteries, if an agent prefers A to B and prefers B to C, then the agent must prefer A to C.

  $$(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\ .$$

CONTINUITY

- **Continuity**: If some lottery B is between A and C in preference, then there is some probability p for which the rational agent will be indifferent between getting B for sure and the lottery that yields A with probability p and C with probability $1 - p$.

  $$A \succ B \succ C \Rightarrow \exists\, p\ [p, A;\ 1 - p, C] \sim B\ .$$

SUBSTITUTABILITY

- **Substitutability**: If an agent is indifferent between two lotteries A and B, then the agent is indifferent between two more complex lotteries that are the same except that B

[2] We apologize to readers whose local airlines no longer offer food on long flights.

is substituted for A in one of them. This holds regardless of the probabilities and the other outcome(s) in the lotteries.

$$A \sim B \;\Rightarrow\; [p, A;\; 1-p, C] \sim [p, B; 1-p, C]\ .$$

This also holds if we substitute $\succ$ for $\sim$ in this axiom.

MONOTONICITY

- **Monotonicity**: Suppose two lotteries have the same two possible outcomes, A and B. If an agent prefers A to B, then the agent must prefer the lottery that has a higher probability for A (and vice versa).

$$A \succ B \;\Rightarrow\; (p > q \;\Leftrightarrow\; [p, A;\; 1-p, B] \succ [q, A;\; 1-q, B])\ .$$

DECOMPOSABILITY

- **Decomposability**: Compound lotteries can be reduced to simpler ones using the laws of probability. This has been called the "no fun in gambling" rule because it says that two consecutive lotteries can be compressed into a single equivalent lottery, as shown in Figure 16.1(b).[3]

$$[p, A;\; 1-p, [q, B;\; 1-q, C]] \sim [p, A;\; (1-p)q, B;\; (1-p)(1-q), C]\ .$$

These constraints are known as the axioms of utility theory. Each axiom can be motivated by showing that an agent that violates it will exhibit patently irrational behavior in some situations. For example, we can motivate transitivity by making an agent with nontransitive preferences give us all its money. Suppose that the agent has the nontransitive preferences $A \succ B \succ C \succ A$, where A, B, and C are goods that can be freely exchanged. If the agent currently has A, then we could offer to trade C for A plus one cent. The agent prefers C, and so would be willing to make this trade. We could then offer to trade B for C, extracting another cent, and finally trade A for B. This brings us back where we started from, except that the agent has given us three cents (Figure 16.1(a)). We can keep going around the cycle until the agent has no money at all. Clearly, the agent has acted irrationally in this case.

16.2.2 Preferences lead to utility

Notice that the axioms of utility theory are really axioms about preferences—they say nothing about a utility function. But in fact from the axioms of utility we can derive the following consequences (for the proof, see von Neumann and Morgenstern, 1944):

- **Existence of Utility Function**: If an agent's preferences obey the axioms of utility, then there exists a function U such that $U(A) > U(B)$ if and only if A is preferred to B, and $U(A) = U(B)$ if and only if the agent is indifferent between A and B.

$$U(A) > U(B) \;\Leftrightarrow\; A \succ B$$
$$U(A) = U(B) \;\Leftrightarrow\; A \sim B$$

- **Expected Utility of a Lottery**: The utility of a lottery is the sum of the probability of each outcome times the utility of that outcome.

$$U([p_1, S_1; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)\ .$$

[3] We can account for the enjoyment of gambling by encoding gambling events into the state description; for example, "Have \$10 and gambled" could be preferred to "Have \$10 and didn't gamble."

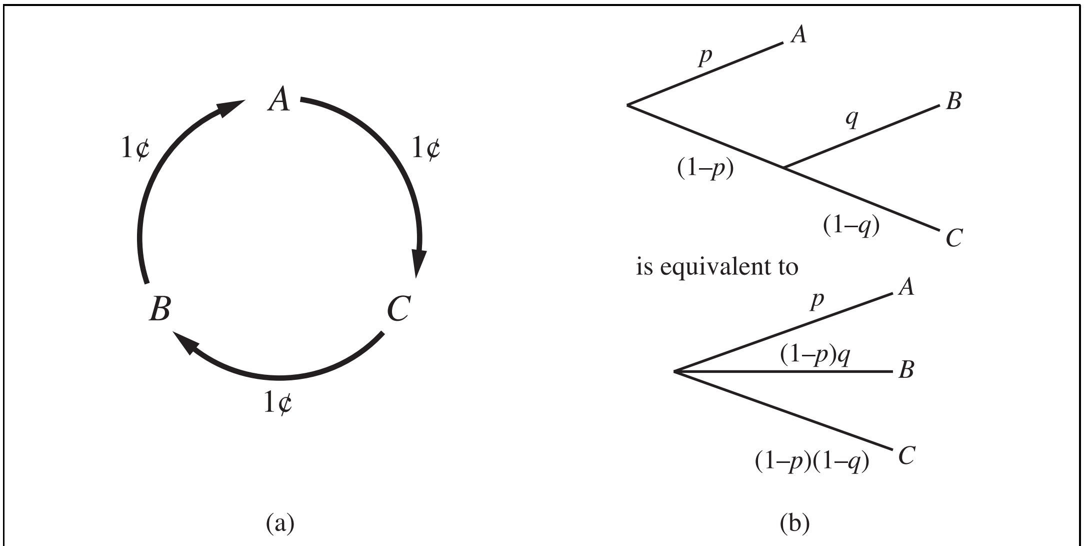


Figure 16.1 (a) A cycle of exchanges showing that the nontransitive preferences $A \succ B \succ C \succ A$ result in irrational behavior. (b) The decomposability axiom.

In other words, once the probabilities and utilities of the possible outcome states are specified, the utility of a compound lottery involving those states is completely determined. Because the outcome of a nondeterministic action is a lottery, it follows that an agent can act rationally—that is, consistently with its preferences—only by choosing an action that maximizes expected utility according to Equation (16.1).

The preceding theorems establish that a utility function *exists* for any rational agent, but they do not establish that it is *unique*. It is easy to see, in fact, that an agent's behavior would not change if its utility function $U(S)$ were transformed according to

$$U'(S) = aU(S) + b \,, \tag{16.2}$$

where a and b are constants and $a > 0$; an affine transformation.[4] This fact was noted in Chapter 5 for two-player games of chance; here, we see that it is completely general.

VALUE FUNCTION

ORDINAL UTILITY FUNCTION

As in game-playing, in a deterministic environment an agent just needs a preference ranking on states—the numbers don't matter. This is called a **value function** or **ordinal utility function**.

It is important to remember that the existence of a utility function that describes an agent's preference behavior does not necessarily mean that the agent is *explicitly* maximizing that utility function in its own deliberations. As we showed in Chapter 2, rational behavior can be generated in any number of ways. By observing a rational agent's preferences, however, an observer can construct the utility function that represents what the agent is actually trying to achieve (even if the agent doesn't know it).

[4] In this sense, utilities resemble temperatures: a temperature in Fahrenheit is 1.8 times the Celsius temperature plus 32. You get the same results in either measurement system.

16.3 UTILITY FUNCTIONS

Utility is a function that maps from lotteries to real numbers. We know there are some axioms on utilities that all rational agents must obey. Is that all we can say about utility functions? Strictly speaking, that is it: an agent can have any preferences it likes. For example, an agent might prefer to have a prime number of dollars in its bank account; in which case, if it had \$16 it would give away \$3. This might be unusual, but we can't call it irrational. An agent might prefer a dented 1973 Ford Pinto to a shiny new Mercedes. Preferences can also interact: for example, the agent might prefer prime numbers of dollars only when it owns the Pinto, but when it owns the Mercedes, it might prefer more dollars to fewer. Fortunately, the preferences of real agents are usually more systematic, and thus easier to deal with.

16.3.1 Utility assessment and utility scales

PREFERENCE ELICITATION

If we want to build a decision-theoretic system that helps the agent make decisions or acts on his or her behalf, we must first work out what the agent's utility function is. This process, often called **preference elicitation**, involves presenting choices to the agent and using the observed preferences to pin down the underlying utility function.

NORMALIZED UTILITIES

Equation (16.2) says that there is no absolute scale for utilities, but it is helpful, nonetheless, to establish *some* scale on which utilities can be recorded and compared for any particular problem. A scale can be established by fixing the utilities of any two particular outcomes, just as we fix a temperature scale by fixing the freezing point and boiling point of water. Typically, we fix the utility of a "best possible prize" at $U(S) = u_\top$ and a "worst possible catastrophe" at $U(S) = u_\bot$. **Normalized utilities** use a scale with $u_\bot = 0$ and $u_\top = 1$.

STANDARD LOTTERY

Given a utility scale between $u_\top$ and $u_\bot$, we can assess the utility of any particular prize S by asking the agent to choose between S and a **standard lottery** $[p, u_\top;\ (1-p), u_\bot]$. The probability p is adjusted until the agent is indifferent between S and the standard lottery. Assuming normalized utilities, the utility of S is given by p. Once this is done for each prize, the utilities for all lotteries involving those prizes are determined.

In medical, transportation, and environmental decision problems, among others, people's lives are at stake. In such cases, $u_\bot$ is the value assigned to immediate death (or perhaps many deaths). *Although nobody feels comfortable with putting a value on human life, it is a fact that tradeoffs are made all the time.* Aircraft are given a complete overhaul at intervals determined by trips and miles flown, rather than after every trip. Cars are manufactured in a way that trades off costs against accident survival rates. Paradoxically, a refusal to "put a monetary value on life" means that life is often *undervalued*. Ross Shachter relates an experience with a government agency that commissioned a study on removing asbestos from schools. The decision analysts performing the study assumed a particular dollar value for the life of a school-age child, and argued that the rational choice under that assumption was to remove the asbestos. The agency, morally outraged at the idea of setting the value of a life, rejected the report out of hand. It then decided against asbestos removal—implicitly asserting a lower value for the life of a child than that assigned by the analysts.

MICROMORT

Some attempts have been made to find out the value that people place on their own lives. One common "currency" used in medical and safety analysis is the **micromort**, a one in a million chance of death. If you ask people how much they would pay to avoid a risk—for example, to avoid playing Russian roulette with a million-barreled revolver—they will respond with very large numbers, perhaps tens of thousands of dollars, but their actual behavior reflects a much lower monetary value for a micromort. For example, driving in a car for 230 miles incurs a risk of one micromort; over the life of your car—say, 92,000 miles—that's 400 micromorts. People appear to be willing to pay about $10,000 (at 2009 prices) more for a safer car that halves the risk of death, or about $50 per micromort. A number of studies have confirmed a figure in this range across many individuals and risk types. Of course, this argument holds only for small risks. Most people won't agree to kill themselves for $50 million.

QALY

Another measure is the **QALY**, or quality-adjusted life year. Patients with a disability are willing to accept a shorter life expectancy to be restored to full health. For example, kidney patients on average are indifferent between living two years on a dialysis machine and one year at full health.

16.3.2 The utility of money

Utility theory has its roots in economics, and economics provides one obvious candidate for a utility measure: money (or more specifically, an agent's total net assets). The almost universal exchangeability of money for all kinds of goods and services suggests that money plays a significant role in human utility functions.

MONOTONIC PREFERENCE

It will usually be the case that an agent prefers more money to less, all other things being equal. We say that the agent exhibits a **monotonic preference** for more money. This does not mean that money behaves as a utility function, because it says nothing about preferences between *lotteries* involving money.

Suppose you have triumphed over the other competitors in a television game show. The host now offers you a choice: either you can take the $1,000,000 prize or you can gamble it on the flip of a coin. If the coin comes up heads, you end up with nothing, but if it comes up tails, you get $2,500,000. If you're like most people, you would decline the gamble and pocket the million. Are you being irrational?

EXPECTED MONETARY VALUE

Assuming the coin is fair, the **expected monetary value** (EMV) of the gamble is $\frac{1}{2}$($0) + $\frac{1}{2}$($2,500,000) = $1,250,000, which is more than the original $1,000,000. But that does not necessarily mean that accepting the gamble is a better decision. Suppose we use S_n to denote the state of possessing total wealth $\$n$, and that your current wealth is $\$k$. Then the expected utilities of the two actions of accepting and declining the gamble are

$$\begin{aligned} EU(Accept) &= \tfrac{1}{2}U(S_k) + \tfrac{1}{2}U(S_{k+2{,}500{,}000})\ , \\ EU(Decline) &= U(S_{k+1{,}000{,}000})\ . \end{aligned}$$

To determine what to do, we need to assign utilities to the outcome states. Utility is not directly proportional to monetary value, because the utility for your first million is very high (or so they say), whereas the utility for an additional million is smaller. Suppose you assign a utility of 5 to your current financial status (S_k), a 9 to the state $S_{k+2{,}500{,}000}$, and an 8 to the

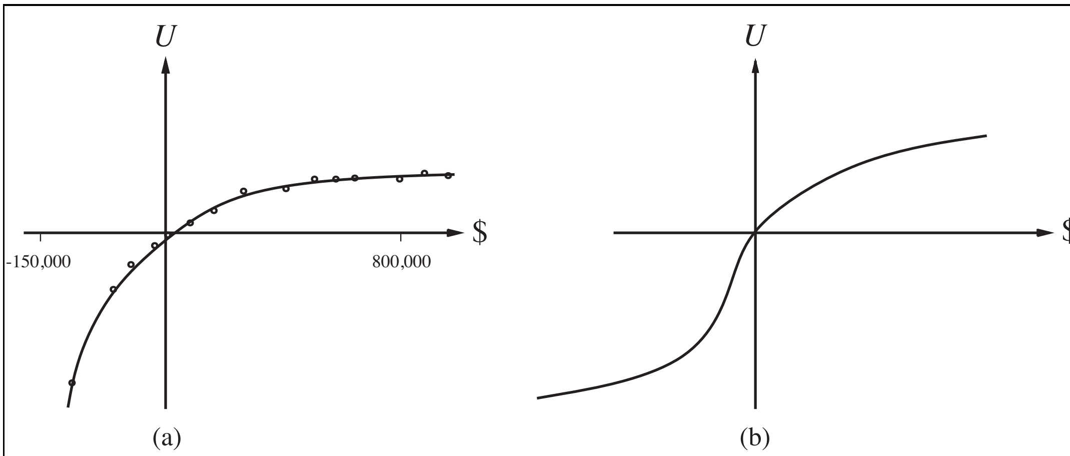


Figure 16.2 The utility of money. (a) Empirical data for Mr. Beard over a limited range. (b) A typical curve for the full range.

state $S_{k+1,000,000}$. Then the rational action would be to decline, because the expected utility of accepting is only 7 (less than the 8 for declining). On the other hand, a billionaire would most likely have a utility function that is locally linear over the range of a few million more, and thus would accept the gamble.

In a pioneering study of actual utility functions, Grayson (1960) found that the utility of money was almost exactly proportional to the *logarithm* of the amount. (This idea was first suggested by Bernoulli (1738); see Exercise 16.3.) One particular utility curve, for a certain Mr. Beard, is shown in Figure 16.2(a). The data obtained for Mr. Beard's preferences are consistent with a utility function

$$U(S_{k+n}) = -263.31 + 22.09 \log(n + 150,000)$$

for the range between $n = -\$150,000$ and $n = \$800,000$.

We should not assume that this is the definitive utility function for monetary value, but it is likely that most people have a utility function that is concave for positive wealth. Going into debt is bad, but preferences between different levels of debt can display a reversal of the concavity associated with positive wealth. For example, someone already $10,000,000 in debt might well accept a gamble on a fair coin with a gain of $10,000,000 for heads and a loss of $20,000,000 for tails.[5] This yields the S-shaped curve shown in Figure 16.2(b).

If we restrict our attention to the positive part of the curves, where the slope is decreasing, then for any lottery L, the utility of being faced with that lottery is less than the utility of being handed the expected monetary value of the lottery as a sure thing:

$$U(L) < U(S_{EMV(L)}) .$$

RISK-AVERSE

That is, agents with curves of this shape are **risk-averse**: they prefer a sure thing with a payoff that is less than the expected monetary value of a gamble. On the other hand, in the "desperate" region at large negative wealth in Figure 16.2(b), the behavior is **risk-seeking**.

RISK-SEEKING

[5] Such behavior might be called desperate, but it is rational if one is already in a desperate situation.

CERTAINTY EQUIVALENT

The value an agent will accept in lieu of a lottery is called the **certainty equivalent** of the lottery. Studies have shown that most people will accept about \$400 in lieu of a gamble that gives \$1000 half the time and \$0 the other half—that is, the certainty equivalent of the lottery is \$400, while the EMV is \$500. The difference between the EMV of a lottery and its certainty equivalent is called the **insurance premium**. Risk aversion is the basis for the insurance industry, because it means that insurance premiums are positive. People would rather pay a small insurance premium than gamble the price of their house against the chance of a fire. From the insurance company's point of view, the price of the house is very small compared with the firm's total reserves. This means that the insurer's utility curve is approximately linear over such a small region, and the gamble costs the company almost nothing.

INSURANCE PREMIUM

Notice that for *small* changes in wealth relative to the current wealth, almost any curve will be approximately linear. An agent that has a linear curve is said to be **risk-neutral**. For gambles with small sums, therefore, we expect risk neutrality. In a sense, this justifies the simplified procedure that proposed small gambles to assess probabilities and to justify the axioms of probability in Section 13.2.3.

RISK-NEUTRAL

16.3.3 Expected utility and post-decision disappointment

The rational way to choose the best action, a^*, is to maximize expected utility:

$$a^* = \underset{a}{\operatorname{argmax}}\, EU(a|\mathbf{e}) \,.$$

If we have calculated the expected utility correctly according to our probability model, and if the probability model correctly reflects the underlying stochastic processes that generate the outcomes, then, on average, we will get the utility we expect if the whole process is repeated many times.

In reality, however, our model usually oversimplifies the real situation, either because we don't know enough (e.g., when making a complex investment decision) or because the computation of the true expected utility is too difficult (e.g., when estimating the utility of successor states of the root node in backgammon). In that case, we are really working with *estimates* $\widehat{EU}(a|\mathbf{e})$ of the true expected utility. We will assume, kindly perhaps, that the estimates are **unbiased**, that is, the expected value of the error, $E(\widehat{EU}(a|\mathbf{e}) - EU(a|\mathbf{e})))$, is zero. In that case, it still seems reasonable to choose the action with the highest estimated utility and to expect to receive that utility, on average, when the action is executed.

UNBIASED

Unfortunately, the real outcome will usually be significantly *worse* than we estimated, even though the estimate was unbiased! To see why, consider a decision problem in which there are k choices, each of which has true estimated utility of 0. Suppose that the error in each utility estimate has zero mean and standard deviation of 1, shown as the bold curve in Figure 16.3. Now, as we actually start to generate the estimates, some of the errors will be negative (pessimistic) and some will be positive (optimistic). Because we select the action with the *highest* utility estimate, we are obviously favoring the overly optimistic estimates, and that is the source of the bias. It is a straightforward matter to calculate the distribution of the maximum of the k estimates (see Exercise 16.11) and hence quantify the extent of our disappointment. The curve in Figure 16.3 for $k = 3$ has a mean around 0.85, so the average disappointment will be about 85% of the standard deviation in the utility estimates.

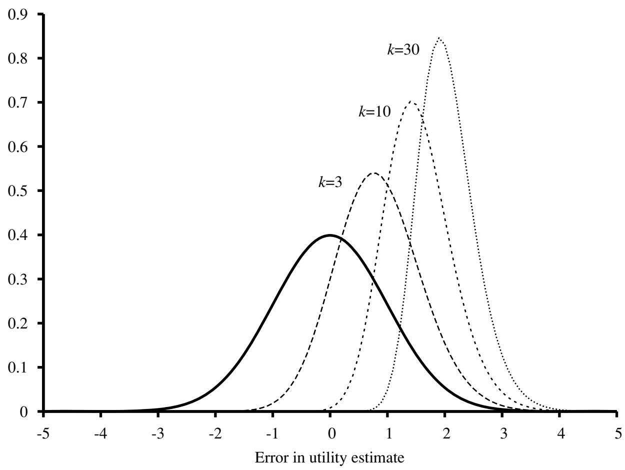


Figure 16.3 Plot of the error in each of k utility estimates and of the distribution of the maximum of k estimates for $k=3$, 10, and 30.

With more choices, extremely optimistic estimates are more likely to arise: for $k=30$, the disappointment will be around twice the standard deviation in the estimates.

OPTIMIZER'S CURSE

This tendency for the estimated expected utility of the best choice to be too high is called the **optimizer's curse** (Smith and Winkler, 2006). It afflicts even the most seasoned decision analysts and statisticians. Serious manifestations include believing that an exciting new drug that has cured 80% patients in a trial will cure 80% of patients (it's been chosen from $k=$ thousands of candidate drugs) or that a mutual fund advertised as having above-average returns will continue to have them (it's been chosen to appear in the advertisement out of $k=$ dozens of funds in the company's overall portfolio). It can even be the case that what appears to be the best choice may not be, if the variance in the utility estimate is high: a drug, selected from thousands tried, that has cured 9 of 10 patients is probably *worse* than one that has cured 800 of 1000.

The optimizer's curse crops up everywhere because of the ubiquity of utility-maximizing selection processes, so taking the utility estimates at face value is a bad idea. We can avoid the curse by using an explicit probability model $\mathbf{P}(\widehat{EU} \mid EU)$ of the error in the utility estimates. Given this model and a prior $\mathbf{P}(EU)$ on what we might reasonably expect the utilities to be, we treat the utility estimate, once obtained, as evidence and compute the posterior distribution for the true utility using Bayes' rule.

16.3.4 Human judgment and irrationality

NORMATIVE THEORY

DESCRIPTIVE THEORY

Decision theory is a **normative theory**: it describes how a rational agent *should* act. A **descriptive theory**, on the other hand, describes how actual agents—for example, humans—really do act. The application of economic theory would be greatly enhanced if the two coincided, but there appears to be some experimental evidence to the contrary. The evidence suggests that humans are "predictably irrational" (Ariely, 2009).

The best-known problem is the Allais paradox (Allais, 1953). People are given a choice between lotteries A and B and then between C and D, which have the following prizes:

$$\begin{array}{llll} A: & 80\% \text{ chance of } \$4000 & C: & 20\% \text{ chance of } \$4000 \\ B: & 100\% \text{ chance of } \$3000 & D: & 25\% \text{ chance of } \$3000 \end{array}$$

Most people consistently prefer B over A (taking the sure thing), and C over D (taking the higher EMV). The normative analysis disagrees! We can see this most easily if we use the freedom implied by Equation (16.2) to set $U(\$0) = 0$. In that case, then $B \succ A$ implies that $U(\$3000) > 0.8\,U(\$4000)$, whereas $C \succ D$ implies exactly the reverse. In other words, there is no utility function that is consistent with these choices. One explanation for the apparently irrational preferences is the **certainty effect** (Kahneman and Tversky, 1979): people are strongly attracted to gains that are certain. There are several reasons why this may be so. First, people may prefer to reduce their computational burden; by choosing certain outcomes, they don't have to compute with probabilities. But the effect persists even when the computations involved are very easy ones. Second, people may distrust the legitimacy of the stated probabilities. I trust that a coin flip is roughly 50/50 if I have control over the coin and the flip, but I may distrust the result if the flip is done by someone with a vested interest in the outcome.[6] In the presence of distrust, it might be better to go for the sure thing.[7] Third, people may be accounting for their emotional state as well as their financial state. People know they would experience **regret** if they gave up a certain reward (B) for an 80% chance at a higher reward and then lost. In other words, if A is chosen, there is a 20% chance of getting no money *and feeling like a complete idiot*, which is worse than just getting no money. So perhaps people who choose B over A and C over D are not being irrational; they are just saying that they are willing to give up \$200 of EMV to avoid a 20% chance of feeling like an idiot.

CERTAINTY EFFECT

REGRET

A related problem is the Ellsberg paradox. Here the prizes are fixed, but the probabilities are underconstrained. Your payoff will depend on the color of a ball chosen from an urn. You are told that the urn contains 1/3 red balls, and 2/3 either black or yellow balls, but you don't know how many black and how many yellow. Again, you are asked whether you prefer lottery A or B; and then C or D:

$$\begin{array}{llll} A: & \$100 \text{ for a red ball} & C: & \$100 \text{ for a red or yellow ball} \\ B: & \$100 \text{ for a black ball} & D: & \$100 \text{ for a black or yellow ball}\ . \end{array}$$

It should be clear that if you think there are more red than black balls then you should prefer A over B and C over D; if you think there are fewer red than black you should prefer the opposite. But it turns out that most people prefer A over B and also prefer D over C, even though there is no state of the world for which this is rational. It seems that people have **ambiguity aversion**: A gives you a 1/3 chance of winning, while B could be anywhere between 0 and 2/3. Similarly, D gives you a 2/3 chance, while C could be anywhere between 1/3 and 3/3. Most people elect the known probability rather than the unknown unknowns.

AMBIGUITY AVERSION

[6] For example, the mathematician/magician Persi Diaconis can make a coin flip come out the way he wants every time (Landhuis, 2004).

[7] Even the sure thing may not be certain. Despite cast-iron promises, we have not yet received that \$27,000,000 from the Nigerian bank account of a previously unknown deceased relative.

FRAMING EFFECT

Yet another problem is that the exact wording of a decision problem can have a big impact on the agent's choices; this is called the **framing effect**. Experiments show that people like a medical procedure that it is described as having a "90% survival rate" about twice as much as one described as having a "10% death rate," even though these two statements mean exactly the same thing. This discrepancy in judgment has been found in multiple experiments and is about the same whether the subjects were patients in a clinic, statistically sophisticated business school students, or experienced doctors.

ANCHORING EFFECT

People feel more comfortable making *relative* utility judgments rather than absolute ones. I may have little idea how much I might enjoy the various wines offered by a restaurant. The restaurant takes advantage of this by offering a $200 bottle that it knows nobody will buy, but which serves to skew upward the customer's estimate of the value of all wines and make the $55 bottle seem like a bargain. This is called the **anchoring effect**.

If human informants insist on contradictory preference judgments, there is nothing that automated agents can do to be consistent with them. Fortunately, preference judgments made by humans are often open to revision in the light of further consideration. Paradoxes like the Allais paradox are greatly reduced (but not eliminated) if the choices are explained better. In work at the Harvard Business School on assessing the utility of money, Keeney and Raiffa (1976, p. 210) found the following:

> Subjects tend to be too risk-averse in the small and therefore . . . the fitted utility functions exhibit unacceptably large risk premiums for lotteries with a large spread. . . . Most of the subjects, however, can reconcile their inconsistencies and feel that they have learned an important lesson about how they want to behave. As a consequence, some subjects cancel their automobile collision insurance and take out more term insurance on their lives.

EVOLUTIONARY PSYCHOLOGY

The evidence for human irrationality is also questioned by researchers in the field of **evolutionary psychology**, who point to the fact that our brain's decision-making mechanisms did not evolve to solve word problems with probabilities and prizes stated as decimal numbers. Let us grant, for the sake of argument, that the brain has built-in neural mechanism for computing with probabilities and utilities, or something functionally equivalent; if so, the required inputs would be obtained through accumulated experience of outcomes and rewards rather than through linguistic presentations of numerical values. It is far from obvious that we can directly access the brain's built-in neural mechanisms by presenting decision problems in linguistic/numerical form. The very fact that different wordings of the *same decision problem* elicit different choices suggests that the decision problem itself is not getting through. Spurred by this observation, psychologists have tried presenting problems in uncertain reasoning and decision making in "evolutionarily appropriate" forms; for example, instead of saying "90% survival rate," the experimenter might show 100 stick-figure animations of the operation, where the patient dies in 10 of them and survives in 90. (Boredom is a complicating factor in these experiments!) With decision problems posed in this way, people seem to be much closer to rational behavior than previously suspected.

16.4 MULTIATTRIBUTE UTILITY FUNCTIONS

Decision making in the field of public policy involves high stakes, in both money and lives. For example, in deciding what levels of harmful emissions to allow from a power plant, policy makers must weigh the prevention of death and disability against the benefit of the power and the economic burden of mitigating the emissions. Siting a new airport requires consideration of the disruption caused by construction; the cost of land; the distance from centers of population; the noise of flight operations; safety issues arising from local topography and weather conditions; and so on. Problems like these, in which outcomes are characterized by two or more attributes, are handled by **multiattribute utility theory**.

MULTIATTRIBUTE UTILITY THEORY

We will call the attributes $\mathbf{X} = X_1, \ldots, X_n$; a complete vector of assignments will be $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$, where each x_i is either a numeric value or a discrete value with an assumed ordering on values. We will assume that higher values of an attribute correspond to higher utilities, all other things being equal. For example, if we choose $AbsenceOfNoise$ as an attribute in the airport problem, then the greater its value, the better the solution.[8] We begin by examining cases in which decisions can be made *without* combining the attribute values into a single utility value. Then we look at cases in which the utilities of attribute combinations can be specified very concisely.

16.4.1 Dominance

Suppose that airport site S_1 costs less, generates less noise pollution, and is safer than site S_2. One would not hesitate to reject S_2. We then say that there is **strict dominance** of S_1 over S_2. In general, if an option is of lower value on all attributes than some other option, it need not be considered further. Strict dominance is often very useful in narrowing down the field of choices to the real contenders, although it seldom yields a unique choice. Figure 16.4(a) shows a schematic diagram for the two-attribute case.

STRICT DOMINANCE

That is fine for the deterministic case, in which the attribute values are known for sure. What about the general case, where the outcomes are uncertain? A direct analog of strict dominance can be constructed, where, despite the uncertainty, all possible concrete outcomes for S_1 strictly dominate all possible outcomes for S_2. (See Figure 16.4(b).) Of course, this will probably occur even less often than in the deterministic case.

STOCHASTIC DOMINANCE

Fortunately, there is a more useful generalization called **stochastic dominance**, which occurs very frequently in real problems. Stochastic dominance is easiest to understand in the context of a single attribute. Suppose we believe that the cost of siting the airport at S_1 is uniformly distributed between \$2.8 billion and \$4.8 billion and that the cost at S_2 is uniformly distributed between \$3 billion and \$5.2 billion. Figure 16.5(a) shows these distributions, with cost plotted as a negative value. Then, given only the information that utility decreases with

[8] In some cases, it may be necessary to subdivide the range of values so that utility varies monotonically within each range. For example, if the $RoomTemperature$ attribute has a utility peak at 70°F, we would split it into two attributes measuring the difference from the ideal, one colder and one hotter. Utility would then be monotonically increasing in each attribute.

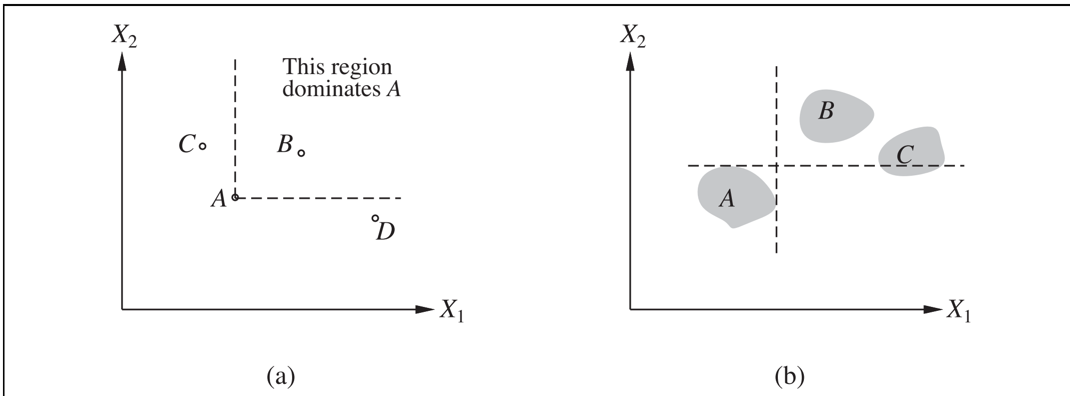


Figure 16.4 Strict dominance. (a) Deterministic: Option A is strictly dominated by B but not by C or D. (b) Uncertain: A is strictly dominated by B but not by C.

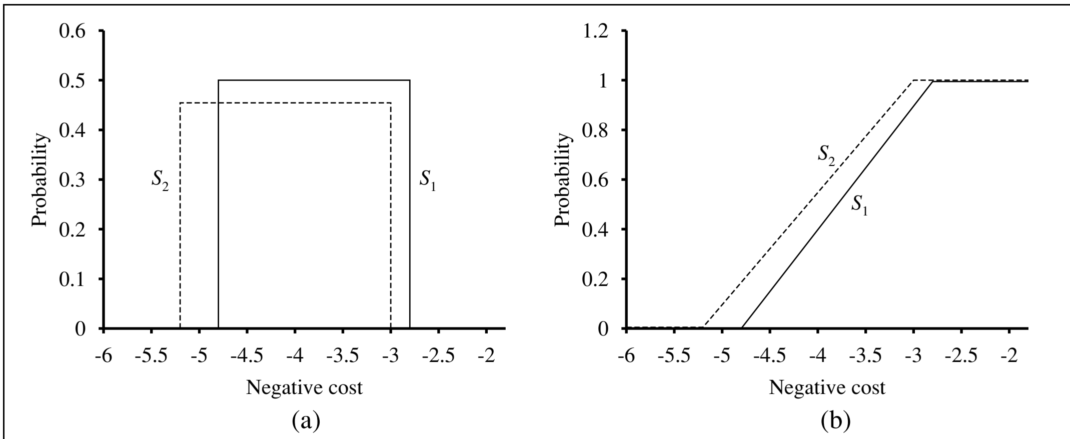


Figure 16.5 Stochastic dominance. (a) S_1 stochastically dominates S_2 on cost. (b) Cumulative distributions for the negative cost of S_1 and S_2.

cost, we can say that S_1 stochastically dominates S_2 (i.e., S_2 can be discarded). It is important to note that this does *not* follow from comparing the expected costs. For example, if we knew the cost of S_1 to be *exactly* \$3.8 billion, then we would be *unable* to make a decision without additional information on the utility of money. (It might seem odd that *more* information on the cost of S_1 could make the agent *less* able to decide. The paradox is resolved by noting that in the absence of exact cost information, the decision is easier to make but is more likely to be wrong.)

The exact relationship between the attribute distributions needed to establish stochastic dominance is best seen by examining the **cumulative distributions**, shown in Figure 16.5(b). (See also Appendix A.) The cumulative distribution measures the probability that the cost is less than or equal to any given amount—that is, it integrates the original distribution. If the cumulative distribution for S_1 is always to the right of the cumulative distribution for S_2, then, stochastically speaking, S_1 is cheaper than S_2. Formally, if two actions A_1 and A_2 lead to probability distributions $p_1(x)$ and $p_2(x)$ on attribute X, then A_1 stochastically dominates A_2 on X if

$$\forall\, x \quad \int_{-\infty}^{x} p_1(x')\, dx' \leq \int_{-\infty}^{x} p_2(x')\, dx' \ .$$

The relevance of this definition to the selection of optimal decisions comes from the following property: *if A_1 stochastically dominates A_2, then for any monotonically nondecreasing utility function $U(x)$, the expected utility of A_1 is at least as high as the expected utility of A_2.* Hence, if an action is stochastically dominated by another action on all attributes, then it can be discarded.

The stochastic dominance condition might seem rather technical and perhaps not so easy to evaluate without extensive probability calculations. In fact, it can be decided very easily in many cases. Suppose, for example, that the construction transportation cost depends on the distance to the supplier. The cost itself is uncertain, but the greater the distance, the greater the cost. If S_1 is closer than S_2, then S_1 will dominate S_2 on cost. Although we will not present them here, there exist algorithms for propagating this kind of qualitative information among uncertain variables in **qualitative probabilistic networks**, enabling a system to make rational decisions based on stochastic dominance, without using any numeric values.

QUALITATIVE PROBABILISTIC NETWORKS

16.4.2 Preference structure and multiattribute utility

Suppose we have n attributes, each of which has d distinct possible values. To specify the complete utility function $U(x_1, \ldots, x_n)$, we need d^n values in the worst case. Now, the worst case corresponds to a situation in which the agent's preferences have no regularity at all. Multiattribute utility theory is based on the supposition that the preferences of typical agents have much more structure than that. The basic approach is to identify regularities in the preference behavior we would expect to see and to use what are called **representation theorems** to show that an agent with a certain kind of preference structure has a utility function

REPRESENTATION THEOREM

$$U(x_1, \ldots, x_n) = F[f_1(x_1), \ldots, f_n(x_n)] \ ,$$

where F is, we hope, a simple function such as addition. Notice the similarity to the use of Bayesian networks to decompose the joint probability of several random variables.

Preferences without uncertainty

Let us begin with the deterministic case. Remember that for deterministic environments the agent has a value function $V(x_1, \ldots, x_n)$; the aim is to represent this function concisely. The basic regularity that arises in deterministic preference structures is called **preference independence**. Two attributes X_1 and X_2 are preferentially independent of a third attribute X_3 if the preference between outcomes $\langle x_1, x_2, x_3 \rangle$ and $\langle x_1', x_2', x_3 \rangle$ does not depend on the particular value x_3 for attribute X_3.

PREFERENCE INDEPENDENCE

Going back to the airport example, where we have (among other attributes) $Noise$, $Cost$, and $Deaths$ to consider, one may propose that $Noise$ and $Cost$ are preferentially independent of $Deaths$. For example, if we prefer a state with 20,000 people residing in the flight path and a construction cost of \$4 billion over a state with 70,000 people residing in the flight path and a cost of \$3.7 billion when the safety level is 0.06 deaths per million passenger miles in both cases, then we would have the same preference when the safety level is 0.12 or 0.03; and the same independence would hold for preferences between any other pair of values for $Noise$ and $Cost$. It is also apparent that $Cost$ and $Deaths$ are preferentially independent of $Noise$ and that $Noise$ and $Deaths$ are preferentially independent of $Cost$. We say that the set of attributes $\{Noise, Cost, Deaths\}$ exhibits **mutual preferential independence** (MPI). MPI says that, whereas each attribute may be important, it does not affect the way in which one trades off the other attributes against each other.

MUTUAL PREFERENTIAL INDEPENDENCE

Mutual preferential independence is something of a mouthful, but thanks to a remarkable theorem due to the economist Gérard Debreu (1960), we can derive from it a very simple form for the agent's value function: *If attributes $X_1, \ldots, X_n$ are mutually preferentially independent, then the agent's preference behavior can be described as maximizing the function*

$$V(x_1, \ldots, x_n) = \sum_i V_i(x_i) ,$$

where each V_i is a value function referring only to the attribute X_i. For example, it might well be the case that the airport decision can be made using a value function

$$V(noise, cost, deaths) = -noise \times 10^4 - cost - deaths \times 10^{12} .$$

ADDITIVE VALUE FUNCTION

A value function of this type is called an **additive value function**. Additive functions are an extremely natural way to describe an agent's preferences and are valid in many real-world situations. For n attributes, assessing an additive value function requires assessing n separate one-dimensional value functions rather than one n-dimensional function; typically, this represents an exponential reduction in the number of preference experiments that are needed. Even when MPI does not strictly hold, as might be the case at extreme values of the attributes, an additive value function might still provide a good approximation to the agent's preferences. This is especially true when the violations of MPI occur in portions of the attribute ranges that are unlikely to occur in practice.

To understand MPI better, it helps to look at cases where it *doesn't* hold. Suppose you are at a medieval market, considering the purchase of some hunting dogs, some chickens, and some wicker cages for the chickens. The hunting dogs are very valuable, but if you don't have enough cages for the chickens, the dogs will eat the chickens; hence, the tradeoff between dogs and chickens depends strongly on the number of cages, and MPI is violated. The existence of these kinds of interactions among various attributes makes it much harder to assess the overall value function.

Preferences with uncertainty

When uncertainty is present in the domain, we also need to consider the structure of preferences between lotteries and to understand the resulting properties of utility functions, rather than just value functions. The mathematics of this problem can become quite complicated, so we present just one of the main results to give a flavor of what can be done. The reader is referred to Keeney and Raiffa (1976) for a thorough survey of the field.

UTILITY INDEPENDENCE

MUTUALLY UTILITY INDEPENDENT

The basic notion of **utility independence** extends preference independence to cover lotteries: a set of attributes **X** is utility independent of a set of attributes **Y** if preferences between lotteries on the attributes in **X** are independent of the particular values of the attributes in **Y**. A set of attributes is **mutually utility independent** (MUI) if each of its subsets is utility-independent of the remaining attributes. Again, it seems reasonable to propose that the airport attributes are MUI.

MULTIPLICATIVE UTILITY FUNCTION

MUI implies that the agent's behavior can be described using a **multiplicative utility function** (Keeney, 1974). The general form of a multiplicative utility function is best seen by looking at the case for three attributes. For conciseness, we use U_i to mean $U_i(x_i)$:

$$U = k_1U_1 + k_2U_2 + k_3U_3 + k_1k_2U_1U_2 + k_2k_3U_2U_3 + k_3k_1U_3U_1 \\ + k_1k_2k_3U_1U_2U_3 \ .$$

Although this does not look very simple, it contains just three single-attribute utility functions and three constants. In general, an n-attribute problem exhibiting MUI can be modeled using n single-attribute utilities and n constants. Each of the single-attribute utility functions can be developed independently of the other attributes, and this combination will be guaranteed to generate the correct overall preferences. Additional assumptions are required to obtain a purely additive utility function.

16.5 DECISION NETWORKS

INFLUENCE DIAGRAM

DECISION NETWORK

In this section, we look at a general mechanism for making rational decisions. The notation is often called an **influence diagram** (Howard and Matheson, 1984), but we will use the more descriptive term **decision network**. Decision networks combine Bayesian networks with additional node types for actions and utilities. We use airport siting as an example.

16.5.1 Representing a decision problem with a decision network

In its most general form, a decision network represents information about the agent's current state, its possible actions, the state that will result from the agent's action, and the utility of that state. It therefore provides a substrate for implementing utility-based agents of the type first introduced in Section 2.4.5. Figure 16.6 shows a decision network for the airport siting problem. It illustrates the three types of nodes used:

CHANCE NODES

- **Chance nodes** (ovals) represent random variables, just as they do in Bayesian networks. The agent could be uncertain about the construction cost, the level of air traffic and the potential for litigation, and the $Deaths$, $Noise$, and total $Cost$ variables, each of which also depends on the site chosen. Each chance node has associated with it a conditional distribution that is indexed by the state of the parent nodes. In decision networks, the parent nodes can include decision nodes as well as chance nodes. Note that each of the current-state chance nodes could be part of a large Bayesian network for assessing construction costs, air traffic levels, or litigation potentials.

DECISION NODES

- **Decision nodes** (rectangles) represent points where the decision maker has a choice of

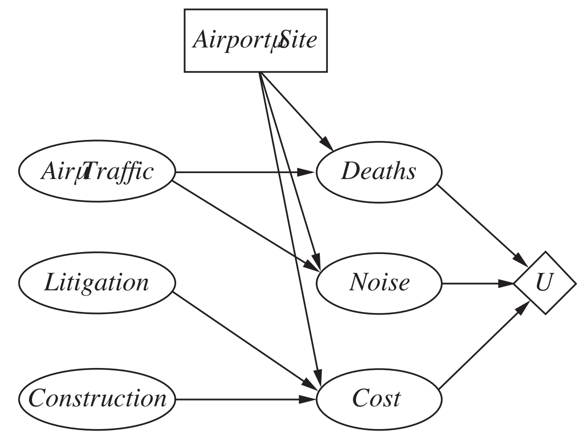


Figure 16.6 A simple decision network for the airport-siting problem.

actions. In this case, the $AirportSite$ action can take on a different value for each site under consideration. The choice influences the cost, safety, and noise that will result. In this chapter, we assume that we are dealing with a single decision node. Chapter 17 deals with cases in which more than one decision must be made.

UTILITY NODES

- **Utility nodes** (diamonds) represent the agent's utility function.[9] The utility node has as parents all variables describing the outcome that directly affect utility. Associated with the utility node is a description of the agent's utility as a function of the parent attributes. The description could be just a tabulation of the function, or it might be a parameterized additive or linear function of the attribute values.

A simplified form is also used in many cases. The notation remains identical, but the chance nodes describing the outcome state are omitted. Instead, the utility node is connected directly to the current-state nodes and the decision node. In this case, rather than representing a utility function on outcome states, the utility node represents the *expected* utility associated with each action, as defined in Equation (16.1) on page 611; that is, the node is associated with an **action-utility function** (also known as a **Q-function** in reinforcement learning, as described in Chapter 21). Figure 16.7 shows the action-utility representation of the airport siting problem.

ACTION-UTILITY FUNCTION

Notice that, because the $Noise$, $Deaths$, and $Cost$ chance nodes in Figure 16.6 refer to future states, they can never have their values set as evidence variables. Thus, the simplified version that omits these nodes can be used whenever the more general form can be used. Although the simplified form contains fewer nodes, the omission of an explicit description of the outcome of the siting decision means that it is less flexible with respect to changes in circumstances. For example, in Figure 16.6, a change in aircraft noise levels can be reflected by a change in the conditional probability table associated with the $Noise$ node, whereas a change in the weight accorded to noise pollution in the utility function can be reflected by

[9] These nodes are also called **value nodes** in the literature.

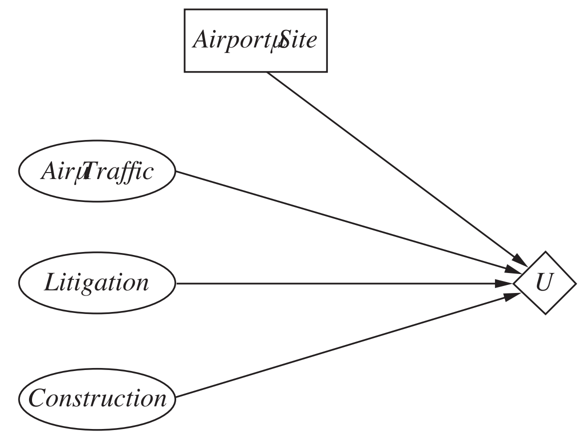


Figure 16.7 A simplified representation of the airport-siting problem. Chance nodes corresponding to outcome states have been factored out.

a change in the utility table. In the action-utility diagram, Figure 16.7, on the other hand, all such changes have to be reflected by changes to the action-utility table. Essentially, the action-utility formulation is a *compiled* version of the original formulation.

16.5.2 Evaluating decision networks

Actions are selected by evaluating the decision network for each possible setting of the decision node. Once the decision node is set, it behaves exactly like a chance node that has been set as an evidence variable. The algorithm for evaluating decision networks is the following:

1. Set the evidence variables for the current state.
2. For each possible value of the decision node:
   (a) Set the decision node to that value.
   (b) Calculate the posterior probabilities for the parent nodes of the utility node, using a standard probabilistic inference algorithm.
   (c) Calculate the resulting utility for the action.
3. Return the action with the highest utility.

This is a straightforward extension of the Bayesian network algorithm and can be incorporated directly into the agent design given in Figure 13.1 on page 484. We will see in Chapter 17 that the possibility of executing several actions in sequence makes the problem much more interesting.

16.6 THE VALUE OF INFORMATION

In the preceding analysis, we have assumed that all relevant information, or at least all available information, is provided to the agent before it makes its decision. In practice, this is hardly ever the case. *One of the most important parts of decision making is knowing what questions to ask.* For example, a doctor cannot expect to be provided with the results of *all possible* diagnostic tests and questions at the time a patient first enters the consulting room.[10] Tests are often expensive and sometimes hazardous (both directly and because of associated delays). Their importance depends on two factors: whether the test results would lead to a significantly better treatment plan, and how likely the various test results are.

INFORMATION VALUE THEORY

This section describes **information value theory**, which enables an agent to choose what information to acquire. We assume that, prior to selecting a "real" action represented by the decision node, the agent can acquire the value of any of the potentially observable chance variables in the model. Thus, information value theory involves a simplified form of sequential decision making—simplified because the observation actions affect only the agent's **belief state**, not the external physical state. The value of any particular observation must derive from the potential to affect the agent's eventual physical action; and this potential can be estimated directly from the decision model itself.

16.6.1 A simple example

Suppose an oil company is hoping to buy one of n indistinguishable blocks of ocean-drilling rights. Let us assume further that exactly one of the blocks contains oil worth C dollars, while the others are worthless. The asking price of each block is C/n dollars. If the company is risk-neutral, then it will be indifferent between buying a block and not buying one.

Now suppose that a seismologist offers the company the results of a survey of block number 3, which indicates definitively whether the block contains oil. How much should the company be willing to pay for the information? The way to answer this question is to examine what the company would do if it had the information:

- With probability $1/n$, the survey will indicate oil in block 3. In this case, the company will buy block 3 for C/n dollars and make a profit of $C - C/n = (n-1)C/n$ dollars.
- With probability $(n-1)/n$, the survey will show that the block contains no oil, in which case the company will buy a different block. Now the probability of finding oil in one of the other blocks changes from $1/n$ to $1/(n-1)$, so the company makes an expected profit of $C/(n-1) - C/n = C/n(n-1)$ dollars.

Now we can calculate the expected profit, given the survey information:

$$\frac{1}{n} \times \frac{(n-1)C}{n} + \frac{n-1}{n} \times \frac{C}{n(n-1)} = C/n\ .$$

Therefore, the company should be willing to pay the seismologist up to C/n dollars for the information: the information is worth as much as the block itself.

The value of information derives from the fact that *with* the information, one's course of action can be changed to suit the *actual* situation. One can discriminate according to the situation, whereas without the information, one has to do what's best on average over the possible situations. In general, the value of a given piece of information is defined to be the difference in expected value between best actions before and after information is obtained.

[10] In the United States, the only question that is always asked beforehand is whether the patient has insurance.

16.6.2 A general formula for perfect information

VALUE OF PERFECT INFORMATION

It is simple to derive a general mathematical formula for the value of information. We assume that exact evidence can be obtained about the value of some random variable E_j (that is, we learn $E_j = e_j$), so the phrase **value of perfect information** (VPI) is used.[11]

Let the agent's initial evidence be **e**. Then the value of the current best action α is defined by

$$EU(\alpha|\mathbf{e}) = \max_a \sum_{s'} P(\textsc{Result}(a) = s' \mid a, \mathbf{e})\, U(s') ,$$

and the value of the new best action (after the new evidence $E_j = e_j$ is obtained) will be

$$EU(\alpha_{e_j}|\mathbf{e}, e_j) = \max_a \sum_{s'} P(\textsc{Result}(a) = s' \mid a, \mathbf{e}, e_j)\, U(s') .$$

But E_j is a random variable whose value is *currently* unknown, so to determine the value of discovering E_j, given current information **e** we must average over all possible values e_{jk} that we might discover for E_j, using our *current* beliefs about its value:

$$VPI_{\mathbf{e}}(E_j) = \left(\sum_k P(E_j = e_{jk}|\mathbf{e})\ EU(\alpha_{e_{jk}}|\mathbf{e}, E_j = e_{jk})\right) - EU(\alpha|\mathbf{e}) .$$

To get some intuition for this formula, consider the simple case where there are only two actions, a_1 and a_2, from which to choose. Their current expected utilities are U_1 and U_2. The information $E_j = e_{jk}$ will yield some new expected utilities U_1' and U_2' for the actions, but before we obtain E_j, we will have some probability distributions over the possible values of U_1' and U_2' (which we assume are independent).

Suppose that a_1 and a_2 represent two different routes through a mountain range in winter. a_1 is a nice, straight highway through a low pass, and a_2 is a winding dirt road over the top. Just given this information, a_1 is clearly preferable, because it is quite possible that a_2 is blocked by avalanches, whereas it is unlikely that anything blocks a_1. U_1 is therefore clearly higher than U_2. It is possible to obtain satellite reports E_j on the actual state of each road that would give new expectations, U_1' and U_2', for the two crossings. The distributions for these expectations are shown in Figure 16.8(a). Obviously, in this case, it is not worth the expense of obtaining satellite reports, because it is unlikely that the information derived from them will change the plan. With no change, information has no value.

Now suppose that we are choosing between two different winding dirt roads of slightly different lengths and we are carrying a seriously injured passenger. Then, even when U_1 and U_2 are quite close, the distributions of U_1' and U_2' are very broad. There is a significant possibility that the second route will turn out to be clear while the first is blocked, and in this

[11] There is no loss of expressiveness in requiring perfect information. Suppose we wanted to model the case in which we become somewhat more certain about a variable. We can do that by introducing *another* variable about which we learn perfect information. For example, suppose we initially have broad uncertainty about the variable $Temperature$. Then we gain the perfect knowledge $Thermometer = 37$; this gives us imperfect information about the true $Temperature$, and the uncertainty due to measurement error is encoded in the sensor model $\mathbf{P}(Thermometer \mid Temperature)$. See Exercise 16.17 for another example.

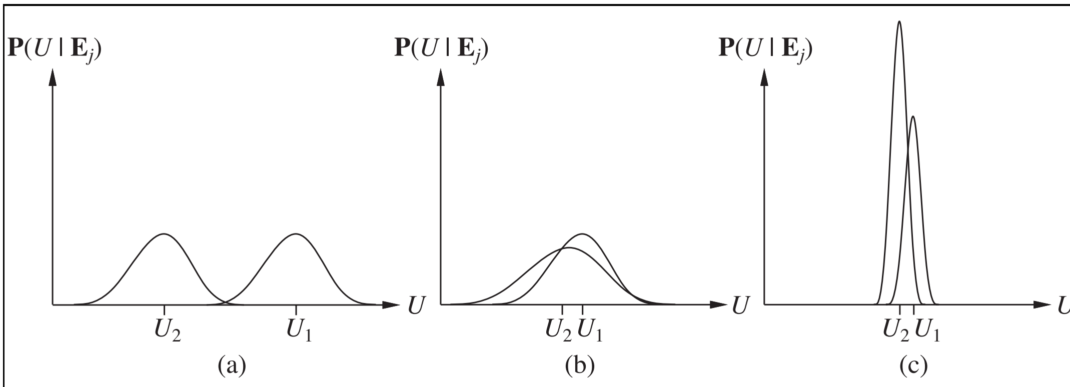


Figure 16.8 Three generic cases for the value of information. In (a), a_1 will almost certainly remain superior to a_2, so the information is not needed. In (b), the choice is unclear and the information is crucial. In (c), the choice is unclear, but because it makes little difference, the information is less valuable. (Note: The fact that U_2 has a high peak in (c) means that its expected value is known with higher certainty than U_1.)

case the difference in utilities will be very high. The VPI formula indicates that it might be worthwhile getting the satellite reports. Such a situation is shown in Figure 16.8(b).

Finally, suppose that we are choosing between the two dirt roads in summertime, when blockage by avalanches is unlikely. In this case, satellite reports might show one route to be more scenic than the other because of flowering alpine meadows, or perhaps wetter because of errant streams. It is therefore quite likely that we would change our plan if we had the information. In this case, however, the difference in value between the two routes is still likely to be very small, so we will not bother to obtain the reports. This situation is shown in Figure 16.8(c).

In sum, *information has value to the extent that it is likely to cause a change of plan and to the extent that the new plan will be significantly better than the old plan.*

16.6.3 Properties of the value of information

One might ask whether it is possible for information to be deleterious: can it actually have negative expected value? Intuitively, one should expect this to be impossible. After all, one could in the worst case just ignore the information and pretend that one has never received it. This is confirmed by the following theorem, which applies to any decision-theoretic agent:

The expected value of information is nonnegative:

$$\forall\, \mathbf{e}, E_j \quad VPI_{\mathbf{e}}(E_j) \geq 0\ .$$

The theorem follows directly from the definition of VPI, and we leave the proof as an exercise (Exercise 16.18). It is, of course, a theorem about *expected* value, not *actual* value. Additional information can easily lead to a plan that *turns out to* be worse than the original plan if the information happens to be misleading. For example, a medical test that gives a false positive result may lead to unnecessary surgery; but that does not mean that the test shouldn't be done.

It is important to remember that VPI depends on the current state of information, which is why it is subscripted. It can change as more information is acquired. For any given piece of evidence E_j, the value of acquiring it can go down (e.g., if another variable strongly constrains the posterior for E_j) or up (e.g., if another variable provides a clue on which E_j builds, enabling a new and better plan to be devised). Thus, VPI is not additive. That is,

$$VPI_{\mathbf{e}}(E_j, E_k) \neq VPI_{\mathbf{e}}(E_j) + VPI_{\mathbf{e}}(E_k) \qquad \text{(in general)} .$$

VPI is, however, order independent. That is,

$$VPI_{\mathbf{e}}(E_j, E_k) = VPI_{\mathbf{e}}(E_j) + VPI_{\mathbf{e},e_j}(E_k) = VPI_{\mathbf{e}}(E_k) + VPI_{\mathbf{e},e_k}(E_j) .$$

Order independence distinguishes sensing actions from ordinary actions and simplifies the problem of calculating the value of a sequence of sensing actions.

16.6.4 Implementation of an information-gathering agent

A sensible agent should ask questions in a reasonable order, should avoid asking questions that are irrelevant, should take into account the importance of each piece of information in relation to its cost, and should stop asking questions when that is appropriate. All of these capabilities can be achieved by using the value of information as a guide.

Figure 16.9 shows the overall design of an agent that can gather information intelligently before acting. For now, we assume that with each observable evidence variable E_j, there is an associated cost, $Cost(E_j)$, which reflects the cost of obtaining the evidence through tests, consultants, questions, or whatever. The agent requests what appears to be the most efficient observation in terms of utility gain per unit cost. We assume that the result of the action $Request(E_j)$ is that the next percept provides the value of E_j. If no observation is worth its cost, the agent selects a "real" action.

MYOPIC

The agent algorithm we have described implements a form of information gathering that is called **myopic**. This is because it uses the VPI formula shortsightedly, calculating the value of information as if only a single evidence variable will be acquired. Myopic control is based on the same heuristic idea as greedy search and often works well in practice. (For example, it has been shown to outperform expert physicians in selecting diagnostic tests.)

```
function INFORMATION-GATHERING-AGENT(percept) returns an action
   persistent: D, a decision network

   integrate percept into D
   j ← the value that maximizes VPI(E_j) / Cost(E_j)
   if VPI(E_j) > Cost(E_j)
      return REQUEST(E_j)
   else return the best action from D
```

Figure 16.9 Design of a simple information-gathering agent. The agent works by repeatedly selecting the observation with the highest information value, until the cost of the next observation is greater than its expected benefit.

However, if there is no single evidence variable that will help a lot, a myopic agent might hastily take an action when it would have been better to request two or more variables first and then take action. A better approach in this situation would be to construct a *conditional plan* (as described in Section 11.3.2) that asks for variable values and takes different next steps depending on the answer.

One final consideration is the effect a series of questions will have on a human respondent. People may respond better to a series of questions if they "make sense," so some expert systems are built to take this into account, asking questions in an order that maximizes the total utility of the system and human rather than an order that maximizes value of information.

16.7 DECISION-THEORETIC EXPERT SYSTEMS

DECISION ANALYSIS

The field of **decision analysis**, which evolved in the 1950s and 1960s, studies the application of decision theory to actual decision problems. It is used to help make rational decisions in important domains where the stakes are high, such as business, government, law, military strategy, medical diagnosis and public health, engineering design, and resource management. The process involves a careful study of the possible actions and outcomes, as well as the preferences placed on each outcome. It is traditional in decision analysis to talk about two roles: the **decision maker** states preferences between outcomes, and the **decision analyst** enumerates the possible actions and outcomes and elicits preferences from the decision maker to determine the best course of action. Until the early 1980s, the main purpose of decision analysis was to help humans make decisions that actually reflect their own preferences. As more and more decision processes become automated, decision analysis is increasingly used to ensure that the automated processes are behaving as desired.

DECISION MAKER

DECISION ANALYST

Early expert system research concentrated on answering questions, rather than on making decisions. Those systems that did recommend actions rather than providing opinions on matters of fact generally did so using condition-action rules, rather than with explicit representations of outcomes and preferences. The emergence of Bayesian networks in the late 1980s made it possible to build large-scale systems that generated sound probabilistic inferences from evidence. The addition of decision networks means that expert systems can be developed that recommend optimal decisions, reflecting the preferences of the agent as well as the available evidence.

A system that incorporates utilities can avoid one of the most common pitfalls associated with the consultation process: confusing likelihood and importance. A common strategy in early medical expert systems, for example, was to rank possible diagnoses in order of likelihood and report the most likely. Unfortunately, this can be disastrous! For the majority of patients in general practice, the two most *likely* diagnoses are usually "There's nothing wrong with you" and "You have a bad cold," but if the third most likely diagnosis for a given patient is lung cancer, that's a serious matter. Obviously, a testing or treatment plan should depend both on probabilities and utilities. Current medical expert systems can take into account the value of information to recommend tests, and then describe a differential diagnosis.

We now describe the knowledge engineering process for decision-theoretic expert systems. As an example we consider the problem of selecting a medical treatment for a kind of congenital heart disease in children (see Lucas, 1996).

AORTIC COARCTATION

About 0.8% of children are born with a heart anomaly, the most common being **aortic coarctation** (a constriction of the aorta). It can be treated with surgery, angioplasty (expanding the aorta with a balloon placed inside the artery), or medication. The problem is to decide what treatment to use and when to do it: the younger the infant, the greater the risks of certain treatments, but one mustn't wait too long. A decision-theoretic expert system for this problem can be created by a team consisting of at least one domain expert (a pediatric cardiologist) and one knowledge engineer. The process can be broken down into the following steps:

Create a causal model. Determine the possible symptoms, disorders, treatments, and outcomes. Then draw arcs between them, indicating what disorders cause what symptoms, and what treatments alleviate what disorders. Some of this will be well known to the domain expert, and some will come from the literature. Often the model will match well with the informal graphical descriptions given in medical textbooks.

Simplify to a qualitative decision model. Since we are using the model to make treatment decisions and not for other purposes (such as determining the joint probability of certain symptom/disorder combinations), we can often simplify by removing variables that are not involved in treatment decisions. Sometimes variables will have to be split or joined to match the expert's intuitions. For example, the original aortic coarctation model had a *Treatment* variable with values *surgery*, *angioplasty*, and *medication*, and a separate variable for *Timing* of the treatment. But the expert had a hard time thinking of these separately, so they were combined, with *Treatment* taking on values such as *surgery in 1 month*. This gives us the model of Figure 16.10.

Assign probabilities. Probabilities can come from patient databases, literature studies, or the expert's subjective assessments. Note that a diagnostic system will reason from symptoms and other observations to the disease or other cause of the problems. Thus, in the early years of building these systems, experts were asked for the probability of a cause given an effect. In general they found this difficult to do, and were better able to assess the probability of an effect given a cause. So modern systems usually assess causal knowledge and encode it directly in the Bayesian network structure of the model, leaving the diagnostic reasoning to the Bayesian network inference algorithms (Shachter and Heckerman, 1987).

Assign utilities. When there are a small number of possible outcomes, they can be enumerated and evaluated individually using the methods of Section 16.3.1. We would create a scale from best to worst outcome and give each a numeric value, for example 0 for death and 1 for complete recovery. We would then place the other outcomes on this scale. This can be done by the expert, but it is better if the patient (or in the case of infants, the patient's parents) can be involved, because different people have different preferences. If there are exponentially many outcomes, we need some way to combine them using multiattribute utility functions. For example, we may say that the costs of various complications are additive.

GOLD STANDARD

Verify and refine the model. To evaluate the system we need a set of correct (input, output) pairs; a so-called **gold standard** to compare against. For medical expert systems this usually means assembling the best available doctors, presenting them with a few cases,

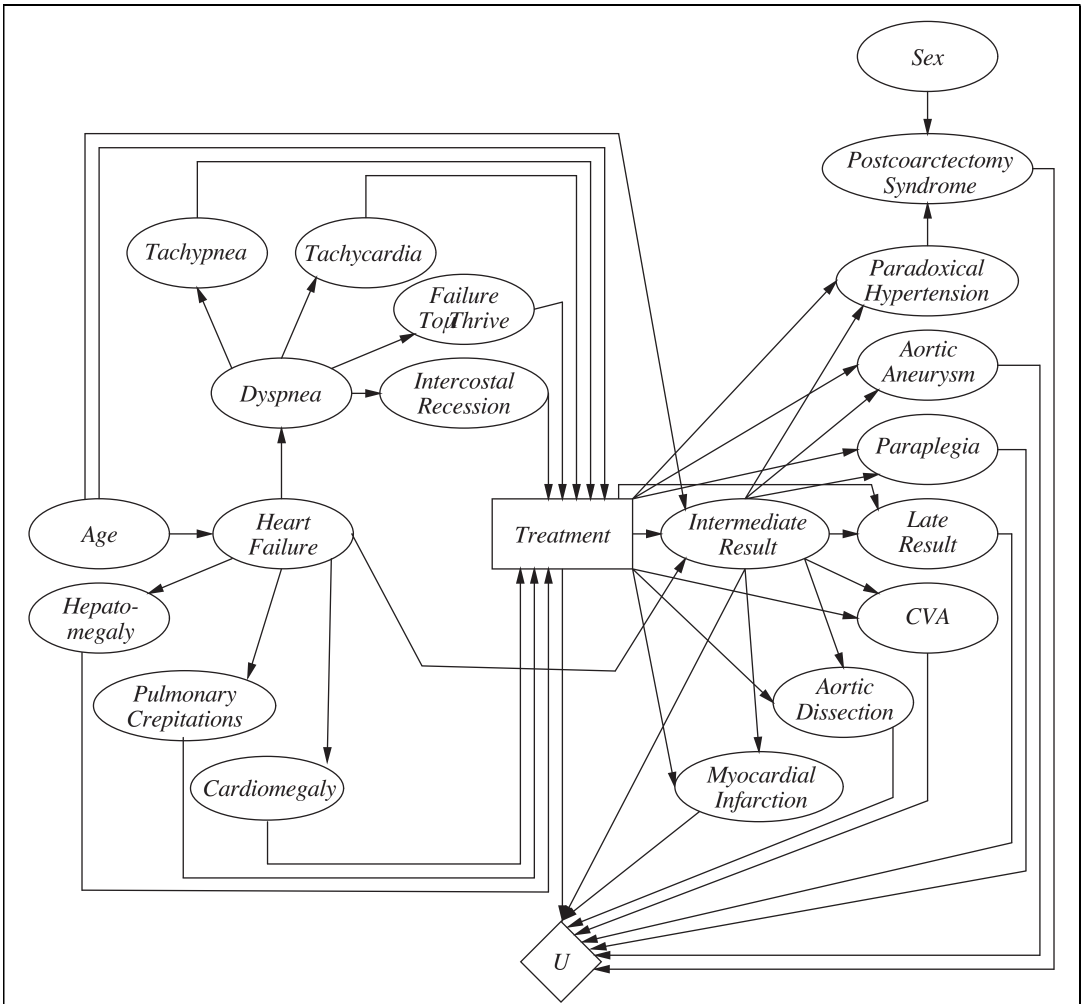


Figure 16.10 Influence diagram for aortic coarctation (courtesy of Peter Lucas).

and asking them for their diagnosis and recommended treatment plan. We then see how well the system matches their recommendations. If it does poorly, we try to isolate the parts that are going wrong and fix them. It can be useful to run the system "backward." Instead of presenting the system with symptoms and asking for a diagnosis, we can present it with a diagnosis such as "heart failure," examine the predicted probability of symptoms such as tachycardia, and compare with the medical literature.

SENSITIVITY ANALYSIS

Perform sensitivity analysis. This important step checks whether the best decision is sensitive to small changes in the assigned probabilities and utilities by systematically varying those parameters and running the evaluation again. If small changes lead to significantly different decisions, then it could be worthwhile to spend more resources to collect better data. If all variations lead to the same decision, then the agent will have more confidence that it is the right decision. Sensitivity analysis is particularly important, because one of the main criticisms of probabilistic approaches to expert systems is that it is too difficult to assess the numerical probabilities required. Sensitivity analysis often reveals that many of the numbers need be specified only very approximately. For example, we might be uncertain about the conditional probability $P(tachycardia \mid dyspnea)$, but if the optimal decision is reasonably robust to small variations in the probability, then our ignorance is less of a concern.

16.8 SUMMARY

This chapter shows how to combine utility theory with probability to enable an agent to select actions that will maximize its expected performance.

- **Probability theory** describes what an agent should believe on the basis of evidence, **utility theory** describes what an agent wants, and **decision theory** puts the two together to describe what an agent should do.
- We can use decision theory to build a system that makes decisions by considering all possible actions and choosing the one that leads to the best expected outcome. Such a system is known as a **rational agent**.
- Utility theory shows that an agent whose preferences between lotteries are consistent with a set of simple axioms can be described as possessing a utility function; furthermore, the agent selects actions as if maximizing its expected utility.
- **Multiattribute utility theory** deals with utilities that depend on several distinct attributes of states. **Stochastic dominance** is a particularly useful technique for making unambiguous decisions, even without precise utility values for attributes.
- **Decision networks** provide a simple formalism for expressing and solving decision problems. They are a natural extension of Bayesian networks, containing decision and utility nodes in addition to chance nodes.
- Sometimes, solving a problem involves finding more information before making a decision. The **value of information** is defined as the expected improvement in utility compared with making a decision without the information.
- **Expert systems** that incorporate utility information have additional capabilities compared with pure inference systems. In addition to being able to make decisions, they can use the value of information to decide which questions to ask, if any; they can recommend contingency plans; and they can calculate the sensitivity of their decisions to small changes in probability and utility assessments.

BIBLIOGRAPHICAL AND HISTORICAL NOTES

The book *L'art de Penser*, also known as the *Port-Royal Logic* (Arnauld, 1662) states:

> To judge what one must do to obtain a good or avoid an evil, it is necessary to consider not only the good and the evil in itself, but also the probability that it happens or does not happen; and to view geometrically the proportion that all these things have together.

Modern texts talk of *utility* rather than good and evil, but this statement correctly notes that one should multiply utility by probability ("view geometrically") to give expected utility, and maximize that over all outcomes ("all these things") to "judge what one must do." It is remarkable how much this got right, 350 years ago, and only 8 years after Pascal and Fermat showed how to use probability correctly. The Port-Royal Logic also marked the first publication of Pascal's wager.

Daniel Bernoulli (1738), investigating the St. Petersburg paradox (see Exercise 16.3), was the first to realize the importance of preference measurement for lotteries, writing "the *value* of an item must not be based on its *price*, but rather on the *utility* that it yields" (italics his). Utilitarian philosopher Jeremy Bentham (1823) proposed the **hedonic calculus** for weighing "pleasures" and "pains," arguing that all decisions (not just monetary ones) could be reduced to utility comparisons.

The derivation of numerical utilities from preferences was first carried out by Ramsey (1931); the axioms for preference in the present text are closer in form to those rediscovered in *Theory of Games and Economic Behavior* (von Neumann and Morgenstern, 1944). A good presentation of these axioms, in the course of a discussion on risk preference, is given by Howard (1977). Ramsey had derived subjective probabilities (not just utilities) from an agent's preferences; Savage (1954) and Jeffrey (1983) carry out more recent constructions of this kind. Von Winterfeldt and Edwards (1986) provide a modern perspective on decision analysis and its relationship to human preference structures. The micromort utility measure is discussed by Howard (1989). A 1994 survey by the *Economist* set the value of a life at between $750,000 and $2.6 million. However, Richard Thaler (1992) found irrational framing effects on the price one is willing to pay to avoid a risk of death versus the price one is willing to be paid to accept a risk. For a 1/1000 chance, a respondent wouldn't pay more than $200 to remove the risk, but wouldn't accept $50,000 to take on the risk. How much are people willing to pay for a QALY? When it comes down to a specific case of saving oneself or a family member, the number is approximately "whatever I've got." But we can ask at a societal level: suppose there is a vaccine that would yield X QALYs but costs Y dollars; is it worth it? In this case people report a wide range of values from around $10,000 to $150,000 per QALY (Prades *et al.*, 2008). QALYs are much more widely used in medical and social policy decision making than are micromorts; see (Russell, 1990) for a typical example of an argument for a major change in public health policy on grounds of increased expected utility measured in QALYs.

The **optimizer's curse** was brought to the attention of decision analysts in a forceful way by Smith and Winkler (2006), who pointed out that the financial benefits to the client projected by analysts for their proposed course of action almost never materialized. They trace this directly to the bias introduced by selecting an optimal action and show that a more complete Bayesian analysis eliminates the problem. The same underlying concept has been called **post-decision disappointment** by Harrison and March (1984) and was noted in the context of analyzing capital investment projects by Brown (1974). The optimizer's curse is also closely related to the **winner's curse** (Capen *et al.*, 1971; Thaler, 1992), which applies to competitive bidding in auctions: whoever wins the auction is very likely to have overestimated the value of the object in question. Capen *et al.* quote a petroleum engineer on the

POST-DECISION DISAPPOINTMENT

WINNER'S CURSE

topic of bidding for oil-drilling rights: “If one wins a tract against two or three others he may feel fine about his good fortune. But how should he feel if he won against 50 others? Ill.” Finally, behind both curses is the general phenomenon of **regression to the mean**, whereby individuals selected on the basis of exceptional characteristics previously exhibited will, with high probability, become less exceptional in future.

REGRESSION TO THE MEAN

The Allais paradox, due to Nobel Prize-winning economist Maurice Allais (1953) was tested experimentally (Tversky and Kahneman, 1982; Conlisk, 1989) to show that people are consistently inconsistent in their judgments. The Ellsberg paradox on ambiguity aversion was introduced in the Ph.D. thesis of Daniel Ellsberg (Ellsberg, 1962), who went on to become a military analyst at the RAND Corporation and to leak documents known as The Pentagon Papers, which contributed to the end of the Vietnam war and the resignation of President Nixon. Fox and Tversky (1995) describe a further study of ambiguity aversion. Mark Machina (2005) gives an overview of choice under uncertainty and how it can vary from expected utility theory.

There has been a recent outpouring of more-or-less popular books on human irrationality. The best known is *Predictably Irrational* (Ariely, 2009); others include *Sway* (Brafman and Brafman, 2009), *Nudge* (Thaler and Sunstein, 2009), *Kluge* (Marcus, 2009), *How We Decide* (Lehrer, 2009) and *On Being Certain* (Burton, 2009). They complement the classic (Kahneman *et al.*, 1982) and the article that started it all (Kahneman and Tversky, 1979). The field of evolutionary psychology (Buss, 2005), on the other hand, has run counter to this literature, arguing that humans are quite rational in evolutionarily appropriate contexts. Its adherents point out that irrationality is penalized by definition in an evolutionary context and show that in some cases it is an artifact of the experimental setup (Cummins and Allen, 1998). There has been a recent resurgence of interest in Bayesian models of cognition, overturning decades of pessimism (Oaksford and Chater, 1998; Elio, 2002; Chater and Oaksford, 2008).

Keeney and Raiffa (1976) give a thorough introduction to multiattribute utility theory. They describe early computer implementations of methods for eliciting the necessary parameters for a multiattribute utility function and include extensive accounts of real applications of the theory. In AI, the principal reference for MAUT is Wellman’s (1985) paper, which includes a system called URP (Utility Reasoning Package) that can use a collection of statements about preference independence and conditional independence to analyze the structure of decision problems. The use of stochastic dominance together with qualitative probability models was investigated extensively by Wellman (1988, 1990a). Wellman and Doyle (1992) provide a preliminary sketch of how a complex set of utility-independence relationships might be used to provide a structured model of a utility function, in much the same way that Bayesian networks provide a structured model of joint probability distributions. Bacchus and Grove (1995, 1996) and La Mura and Shoham (1999) give further results along these lines.

Decision theory has been a standard tool in economics, finance, and management science since the 1950s. Until the 1980s, decision trees were the main tool used for representing simple decision problems. Smith (1988) gives an overview of the methodology of decision analysis. Influence diagrams were introduced by Howard and Matheson (1984), based on earlier work at SRI (Miller *et al.*, 1976). Howard and Matheson’s method involved the derivation of a decision tree from a decision network, but in general the tree is of exponential size. Shachter (1986) developed a method for making decisions based directly on a decision network, without the creation of an intermediate decision tree. This algorithm was also one of the first to provide complete inference for multiply connected Bayesian networks. Zhang *et al.* (1994) showed how to take advantage of conditional independence of information to reduce the size of trees in practice; they use the term *decision network* for networks that use this approach (although others use it as a synonym for influence diagram). Nilsson and Lauritzen (2000) link algorithms for decision networks to ongoing developments in clustering algorithms for Bayesian networks. Koller and Milch (2003) show how influence diagrams can be used to solve games that involve gathering information by opposing players, and Detwarasiti and Shachter (2005) show how influence diagrams can be used as an aid to decision making for a team that shares goals but is unable to share all information perfectly. The collection by Oliver and Smith (1990) has a number of useful articles on decision networks, as does the 1990 special issue of the journal *Networks*. Papers on decision networks and utility modeling also appear regularly in the journals *Management Science* and *Decision Analysis*.

The theory of information value was explored first in the context of statistical experiments, where a quasi-utility (entropy reduction) was used (Lindley, 1956). The Russian control theorist Ruslan Stratonovich (1965) developed the more general theory presented here, in which information has value by virtue of its ability to affect decisions. Stratonovich's work was not known in the West, where Ron Howard (1966) pioneered the same idea. His paper ends with the remark "If information value theory and associated decision theoretic structures do not in the future occupy a large part of the education of engineers, then the engineering profession will find that its traditional role of managing scientific and economic resources for the benefit of man has been forfeited to another profession." To date, the implied revolution in managerial methods has not occurred.

Recent work by Krause and Guestrin (2009) shows that computing the exact non-myopic value of information is intractable even in polytree networks. There are other cases—more restricted than general value of information—in which the myopic algorithm does provide a provably good approximation to the optimal sequence of observations (Krause *et al.*, 2008). In some cases—for example, looking for treasure buried in one of n places—ranking experiments in order of success probability divided by cost gives an optimal solution (Kadane and Simon, 1977).

Surprisingly few early AI researchers adopted decision-theoretic tools after the early applications in medical decision making described in Chapter 13. One of the few exceptions was Jerry Feldman, who applied decision theory to problems in vision (Feldman and Yakimovsky, 1974) and planning (Feldman and Sproull, 1977). After the resurgence of interest in probabilistic methods in AI in the 1980s, decision-theoretic expert systems gained widespread acceptance (Horvitz *et al.*, 1988; Cowell *et al.*, 2002). In fact, from 1991 onward, the cover design of the journal *Artificial Intelligence* has depicted a decision network, although some artistic license appears to have been taken with the direction of the arrows.

EXERCISES

16.1 (Adapted from David Heckerman.) This exercise concerns the **Almanac Game**, which is used by decision analysts to calibrate numeric estimation. For each of the questions that follow, give your best guess of the answer, that is, a number that you think is as likely to be too high as it is to be too low. Also give your guess at a 25th percentile estimate, that is, a number that you think has a 25% chance of being too high, and a 75% chance of being too low. Do the same for the 75th percentile. (Thus, you should give three estimates in all—low, median, and high—for each question.)

a. Number of passengers who flew between New York and Los Angeles in 1989.
b. Population of Warsaw in 1992.
c. Year in which Coronado discovered the Mississippi River.
d. Number of votes received by Jimmy Carter in the 1976 presidential election.
e. Age of the oldest living tree, as of 2002.
f. Height of the Hoover Dam in feet.
g. Number of eggs produced in Oregon in 1985.
h. Number of Buddhists in the world in 1992.
i. Number of deaths due to AIDS in the United States in 1981.
j. Number of U.S. patents granted in 1901.

The correct answers appear after the last exercise of this chapter. From the point of view of decision analysis, the interesting thing is not how close your median guesses came to the real answers, but rather how often the real answer came within your 25% and 75% bounds. If it was about half the time, then your bounds are accurate. But if you're like most people, you will be more sure of yourself than you should be, and fewer than half the answers will fall within the bounds. With practice, you can calibrate yourself to give realistic bounds, and thus be more useful in supplying information for decision making. Try this second set of questions and see if there is any improvement:

a. Year of birth of Zsa Zsa Gabor.
b. Maximum distance from Mars to the sun in miles.
c. Value in dollars of exports of wheat from the United States in 1992.
d. Tons handled by the port of Honolulu in 1991.
e. Annual salary in dollars of the governor of California in 1993.
f. Population of San Diego in 1990.
g. Year in which Roger Williams founded Providence, Rhode Island.
h. Height of Mt. Kilimanjaro in feet.
i. Length of the Brooklyn Bridge in feet.
j. Number of deaths due to automobile accidents in the United States in 1992.

16.2 Chris considers four used cars before buying the one with maximum expected utility. Pat considers ten cars and does the same. All other things being equal, which one is more likely to have the better car? Which is more likely to be disappointed with their car's quality? By how much (in terms of standard deviations of expected quality)?

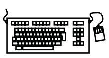
16.3 In 1713, Nicolas Bernoulli stated a puzzle, now called the St. Petersburg paradox, which works as follows. You have the opportunity to play a game in which a fair coin is tossed repeatedly until it comes up heads. If the first heads appears on the nth toss, you win 2^n dollars.

a. Show that the expected monetary value of this game is infinite.

b. How much would you, personally, pay to play the game?

c. Nicolas's cousin Daniel Bernoulli resolved the apparent paradox in 1738 by suggesting that the utility of money is measured on a logarithmic scale (i.e., $U(S_n) = a \log_2 n + b$, where S_n is the state of having $\$n$). What is the expected utility of the game under this assumption?

d. What is the maximum amount that it would be rational to pay to play the game, assuming that one's initial wealth is $\$k$?



16.4 Write a computer program to automate the process in Exercise 16.9. Try your program out on several people of different net worth and political outlook. Comment on the consistency of your results, both for an individual and across individuals.

16.5 The Surprise Candy Company makes candy in two flavors: 70% are strawberry flavor and 30% are anchovy flavor. Each new piece of candy starts out with a round shape; as it moves along the production line, a machine randomly selects a certain percentage to be trimmed into a square; then, each piece is wrapped in a wrapper whose color is chosen randomly to be red or brown. 80% of the strawberry candies are round and 80% have a red wrapper, while 90% of the anchovy candies are square and 90% have a brown wrapper. All candies are sold individually in sealed, identical, black boxes.

Now you, the customer, have just bought a Surprise candy at the store but have not yet opened the box. Consider the three Bayes nets in Figure 16.11.

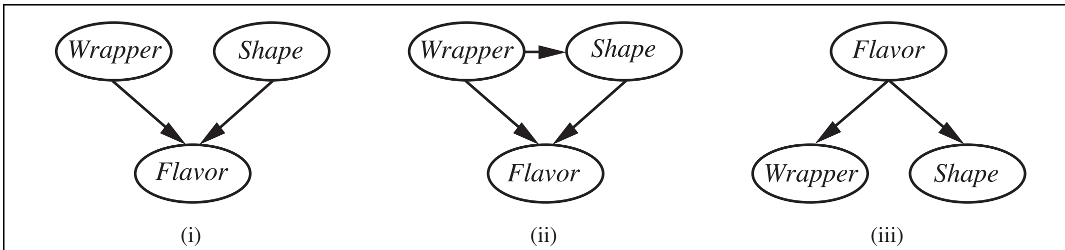


Figure 16.11 Three proposed Bayes nets for the Surprise Candy problem, Exercise 16.5.

a. Which network(s) can correctly represent $\mathbf{P}(Flavor, Wrapper, Shape)$?

b. Which network is the best representation for this problem?

c. Does network (i) assert that $\mathbf{P}(Wrapper|Shape) = \mathbf{P}(Wrapper)$?

d. What is the probability that your candy has a red wrapper?

e. In the box is a round candy with a red wrapper. What is the probability that its flavor is strawberry?

f. A unwrapped strawberry candy is worth s on the open market and an unwrapped anchovy candy is worth a. Write an expression for the value of an unopened candy box.

g. A new law prohibits trading of unwrapped candies, but it is still legal to trade wrapped candies (out of the box). Is an unopened candy box now worth more than less than, or the same as before?

16.6 Prove that the judgments $B \succ A$ and $C \succ D$ in the Allais paradox (page 620) violate the axiom of substitutability.

16.7 Consider the Allais paradox described on page 620: an agent who prefers B over A (taking the sure thing), and C over D (taking the higher EMV) is not acting rationally, according to utility theory. Do you think this indicates a problem for the agent, a problem for the theory, or no problem at all? Explain.

16.8 Tickets to a lottery cost \$1. There are two possible prizes: a \$10 payoff with probability 1/50, and a \$1,000,000 payoff with probability 1/2,000,000. What is the expected monetary value of a lottery ticket? When (if ever) is it rational to buy a ticket? Be precise—show an equation involving utilities. You may assume current wealth of \$$k$ and that $U(S_k) = 0$. You may also assume that $U(S_{k+10}) = 10 \times U(S_{k+1})$, but you may not make any assumptions about $U(S_{k+1,000,000})$. Sociological studies show that people with lower income buy a disproportionate number of lottery tickets. Do you think this is because they are worse decision makers or because they have a different utility function? Consider the value of contemplating the possibility of winning the lottery versus the value of contemplating becoming an action hero while watching an adventure movie.

16.9 Assess your own utility for different incremental amounts of money by running a series of preference tests between some definite amount M_1 and a lottery $[p, M_2; (1-p), 0]$. Choose different values of M_1 and M_2, and vary p until you are indifferent between the two choices. Plot the resulting utility function.

16.10 How much is a micromort worth to you? Devise a protocol to determine this. Ask questions based both on paying to avoid risk and being paid to accept risk.

16.11 Let continuous variables $X_1, \ldots, X_k$ be independently distributed according to the same probability density function $f(x)$. Prove that the density function for $\max\{X_1, \ldots, X_k\}$ is given by $kf(x)(F(x))^{k-1}$, where F is the cumulative distribution for f.

16.12 Economists often make use of an exponential utility function for money: $U(x) = -e^{x/R}$, where R is a positive constant representing an individual's risk tolerance. Risk tolerance reflects how likely an individual is to accept a lottery with a particular expected monetary value (EMV) versus some certain payoff. As R (which is measured in the same units as x) becomes larger, the individual becomes less risk-averse.

a. Assume Mary has an exponential utility function with $R = \$500$. Mary is given the choice between receiving \$500 with certainty (probability 1) or participating in a lottery which has a 60% probability of winning \$5000 and a 40% probability of winning nothing. Assuming Marry acts rationally, which option would she choose? Show how you derived your answer.

b. Consider the choice between receiving \$100 with certainty (probability 1) or participating in a lottery which has a 50% probability of winning \$500 and a 50% probability of winning nothing. Approximate the value of R (to 3 significant digits) in an exponential utility function that would cause an individual to be indifferent to these two alternatives. (You might find it helpful to write a short program to help you solve this problem.)

16.13 Repeat Exercise 16.16, using the action-utility representation shown in Figure 16.7.

16.14 For either of the airport-siting diagrams from Exercises 16.16 and 16.13, to which conditional probability table entry is the utility most sensitive, given the available evidence?

16.15 Consider a student who has the choice to buy or not buy a textbook for a course. We'll model this as a decision problem with one Boolean decision node, B, indicating whether the agent chooses to buy the book, and two Boolean chance nodes, M, indicating whether the student has mastered the material in the book, and P, indicating whether the student passes the course. Of course, there is also a utility node, U. A certain student, Sam, has an additive utility function: 0 for not buying the book and -\$100 for buying it; and \$2000 for passing the course and 0 for not passing. Sam's conditional probability estimates are as follows:

$$
\begin{array}{ll}
P(p|b,m) = 0.9 & P(m|b) = 0.9 \\
P(p|b,\neg m) = 0.5 & P(m|\neg b) = 0.7 \\
P(p|\neg b,m) = 0.8 & \\
P(p|\neg b,\neg m) = 0.3 &
\end{array}
$$

You might think that P would be independent of B given M, But this course has an open-book final—so having the book helps.

a. Draw the decision network for this problem.

b. Compute the expected utility of buying the book and of not buying it.

c. What should Sam do?

16.16 This exercise completes the analysis of the airport-siting problem in Figure 16.6.

a. Provide reasonable variable domains, probabilities, and utilities for the network, assuming that there are three possible sites.

b. Solve the decision problem.

c. What happens if changes in technology mean that each aircraft generates half the noise?

d. What if noise avoidance becomes three times more important?

e. Calculate the VPI for *AirTraffic*, *Litigation*, and *Construction* in your model.

16.17 (Adapted from Pearl (1988).) A used-car buyer can decide to carry out various tests with various costs (e.g., kick the tires, take the car to a qualified mechanic) and then, depending on the outcome of the tests, decide which car to buy. We will assume that the buyer is deciding whether to buy car c_1, that there is time to carry out at most one test, and that t_1 is the test of c_1 and costs \$50.

A car can be in good shape (quality q^+) or bad shape (quality q^-), and the tests might help indicate what shape the car is in. Car c_1 costs \$1,500, and its market value is \$2,000 if it is in good shape; if not, \$700 in repairs will be needed to make it in good shape. The buyer's estimate is that c_1 has a 70% chance of being in good shape.

a. Draw the decision network that represents this problem.

b. Calculate the expected net gain from buying c_1, given no test.

c. Tests can be described by the probability that the car will pass or fail the test given that the car is in good or bad shape. We have the following information:
$P(pass(c_1, t_1)|q^+(c_1)) = 0.8$
$P(pass(c_1, t_1)|q^-(c_1)) = 0.35$
Use Bayes' theorem to calculate the probability that the car will pass (or fail) its test and hence the probability that it is in good (or bad) shape given each possible test outcome.

d. Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

e. Calculate the value of information of the test, and derive an optimal conditional plan for the buyer.

16.18 Recall the definition of *value of information* in Section 16.6.

a. Prove that the value of information is nonnegative and order independent.

b. Explain why it is that some people would prefer not to get some information—for example, not wanting to know the sex of their baby when an ultrasound is done.

SUBMODULARITY

c. A function f on sets is **submodular** if, for any element x and any sets A and B such that $A \subseteq B$, adding x to A gives a greater increase in f than adding x to B:

$$A \subseteq B \Rightarrow (f(A \cup \{x\}) - f(A)) \geq (f(B \cup \{x\}) - f(B)) .$$

Submodularity captures the intuitive notion of *diminishing returns*. Is the value of information, viewed as a function f on sets of possible observations, submodular? Prove this or find a counterexample.

The answers to Exercise 16.1 (where M stands for million): First set: 3M, 1.6M, 1541, 41M, 4768, 221, 649M, 295M, 132, 25,546. Second set: 1917, 155M, 4,500M, 11M, 120,000, 1.1M, 1636, 19,340, 1,595, 41,710.