4. Trees

Part 2

Objectives

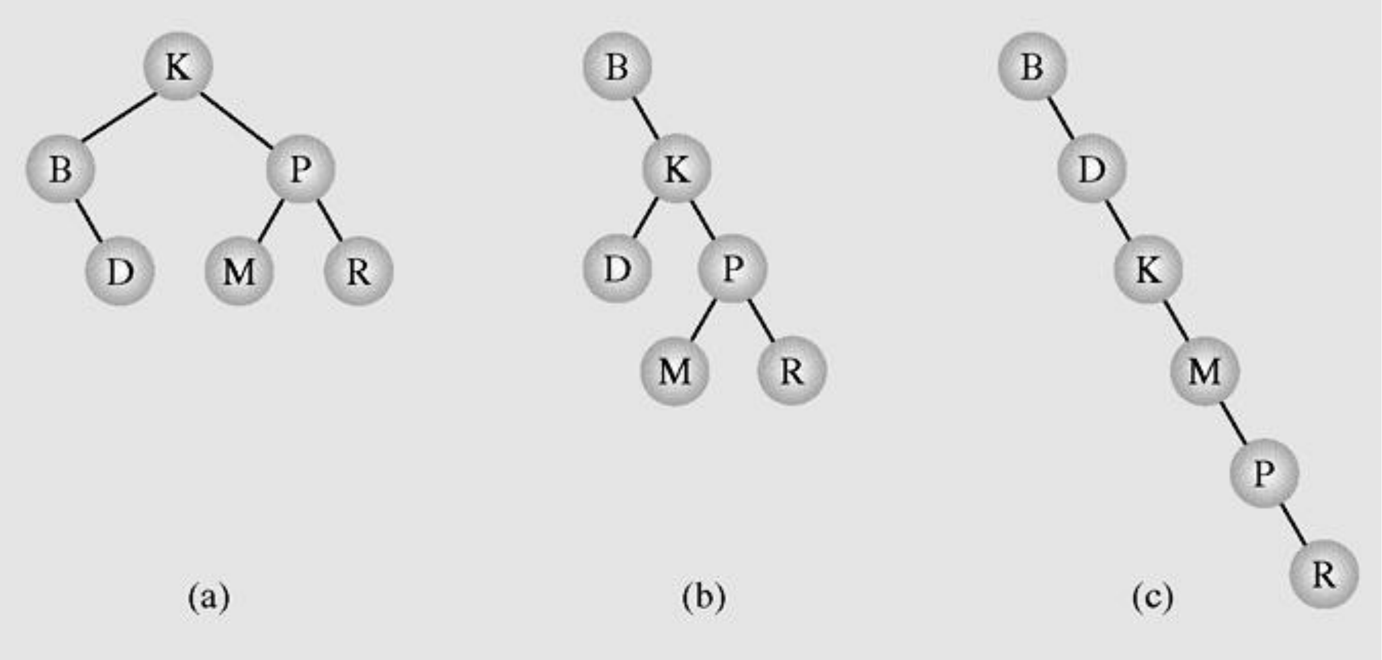
- Balanced Binary Tree Definitions
- Balancing a Tree - Simple Balanced Algorithm
- Rotations on Binary Search Tree
- AVL Tree
- Insertion in an AVL Tree
- Deletion in an AVL Tree
- Heaps
- Heaps as Priority Queue
- Polish Notation and Expression Trees

Balanced Binary Tree Definitions

- A binary tree is **height-balanced** if for every internal node *p* of *T*, the heights of the children of *p* differ by at most 1.
- A tree is considered **perfectly balanced** if it is height-balanced and all leaves are to be found on one level or two levels.

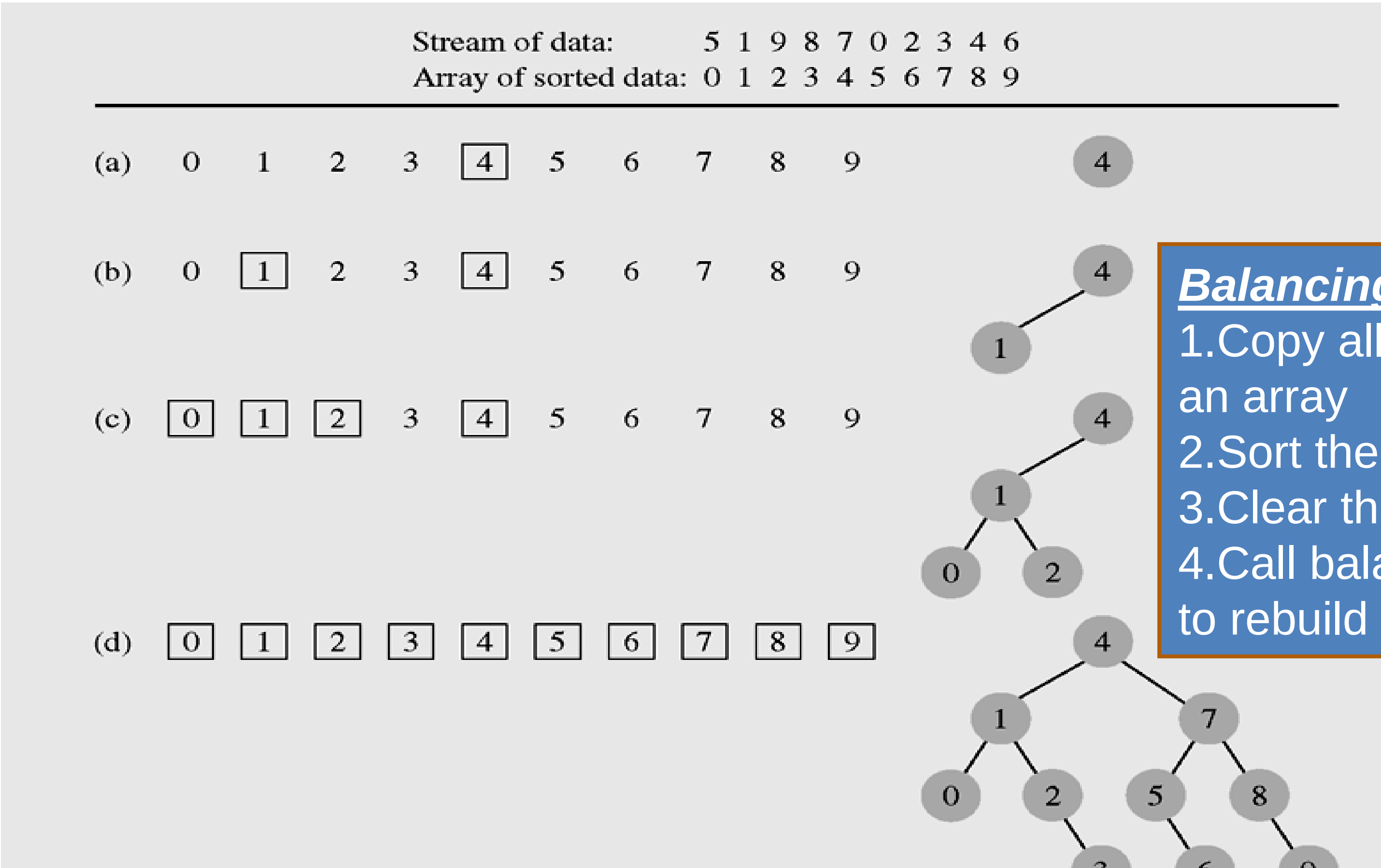
What is a balanced tree in general?
There are several ways to define "Balanced". The main goal is to keep the depths of all nodes to be O(log(n)).

Why need to balance a Binary Search Tree?



Different binary search trees with the same information

Balancing a Tree - *Simple Balance Algorithm - 1*



Creating a binary search tree from an ordered array

Balancing a tree:

1. Copy all tree nodes to an array
2. Sort the array
3. Clear the tree
4. Call balance() method to rebuild the tree

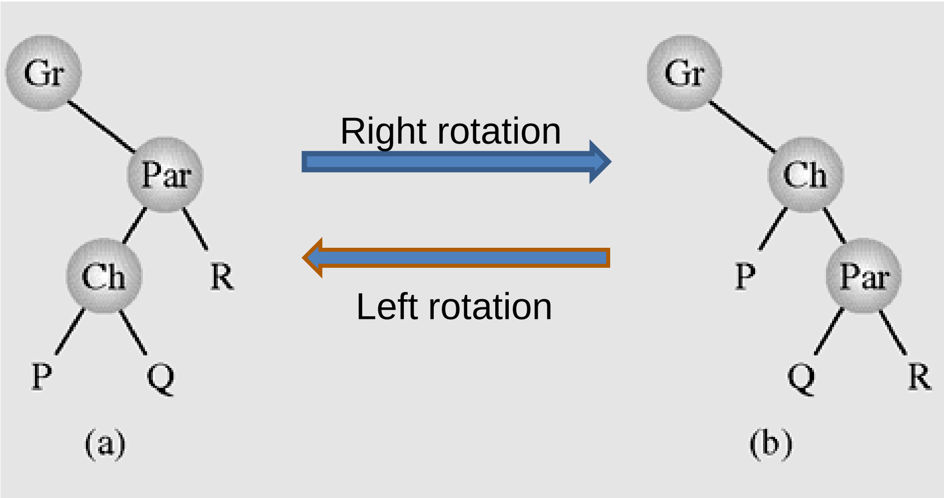
Balancing a Tree - *Simple Balance Algorithm - 2*

```
public void balance(T data[], int first, int last) {
        if (first <= last) {
            int middle = (first + last)/2;
            insert(data[middle]);
            balance(data,first,middle-1);
            balance(data,middle+1,last);
        }
    }
public void balance(T data[]) {
        balance(data,0,data.length-1);
    }
```

Rotations on Binary Search Tree

Right Rotation

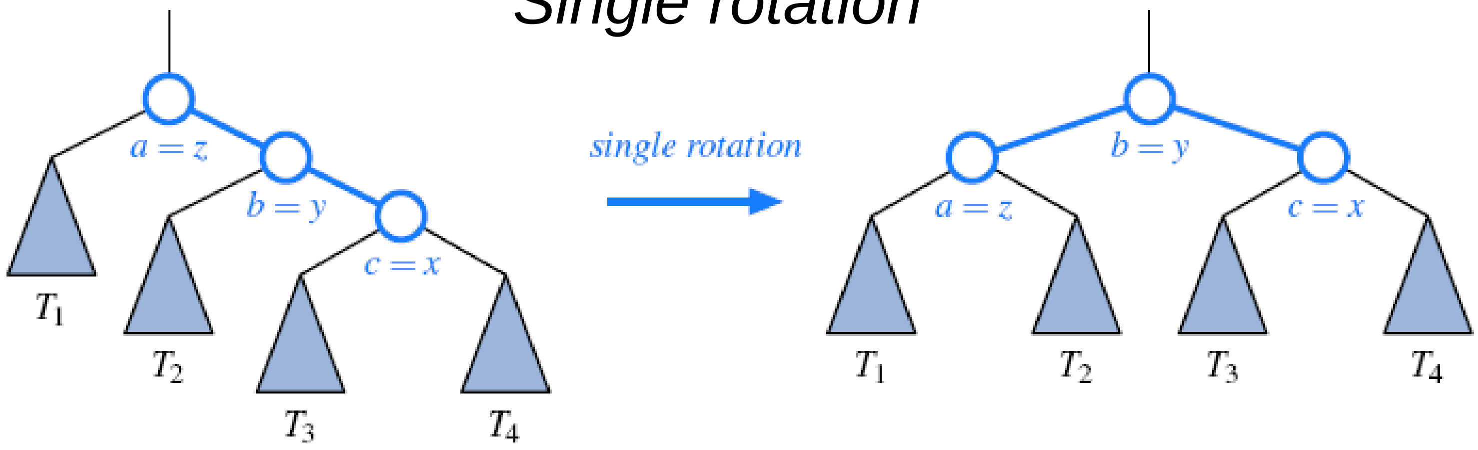
```
rotateRight: rotate the node Par to right about its left
child Ch
     Ch become new root of the subtree
     Right subtree of Ch becomes left subtree of Par
     Par subtree becomes right subtree of Ch
```



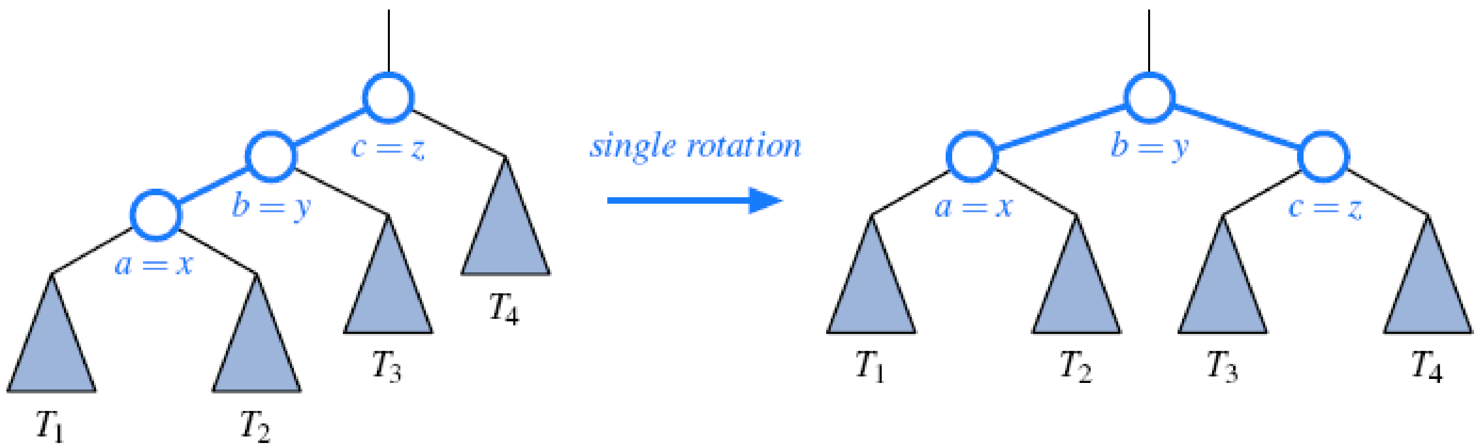
Right rotation of Par about its' left child Ch

Rotations on Binary Search Tree demo - 1

Single rotation



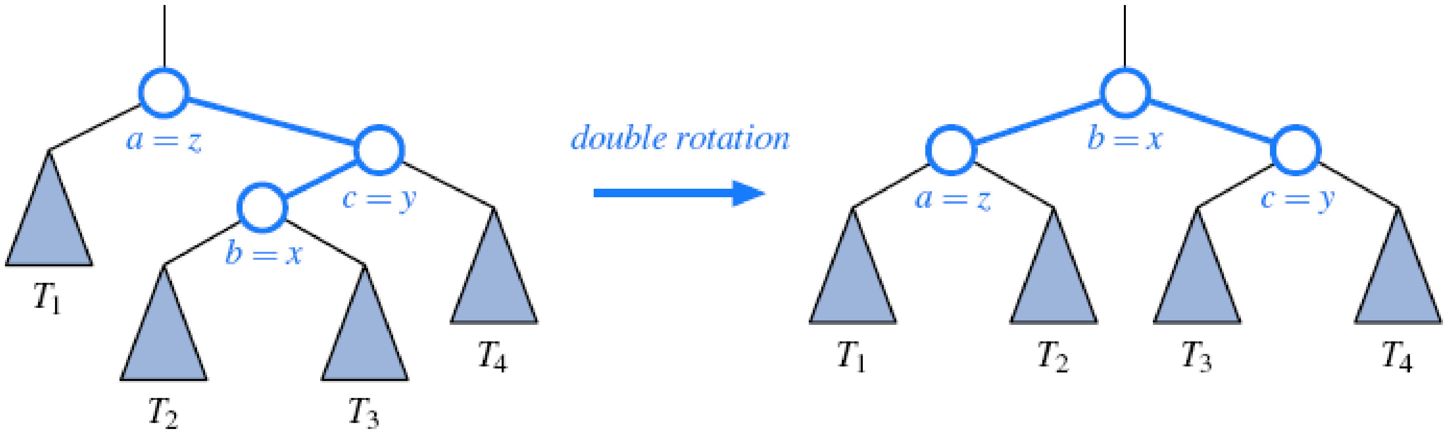
(a)



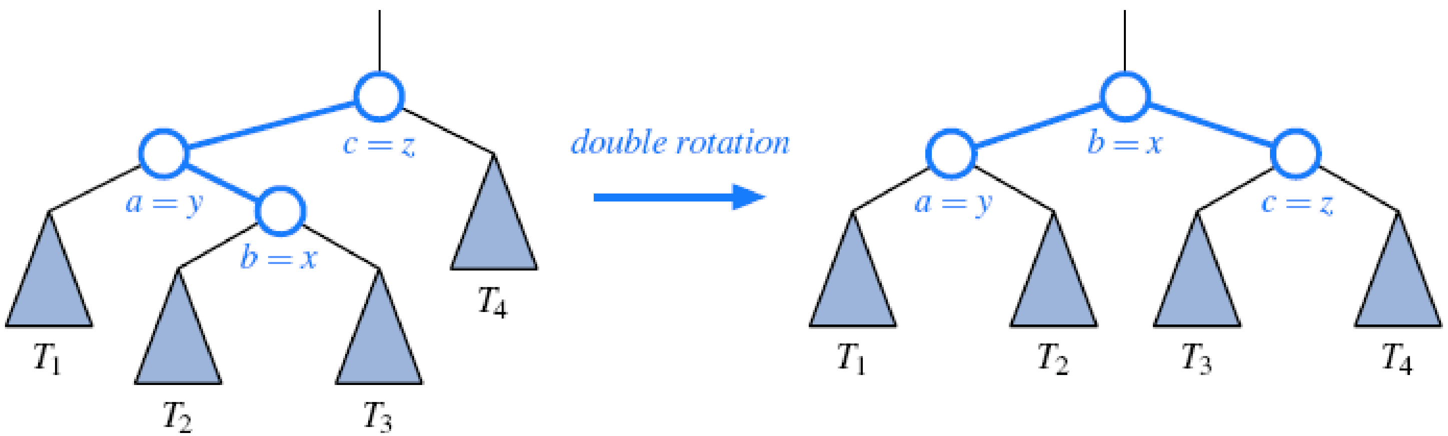
(b)

Rotations on Binary Search Tree demo - 2

Double rotation



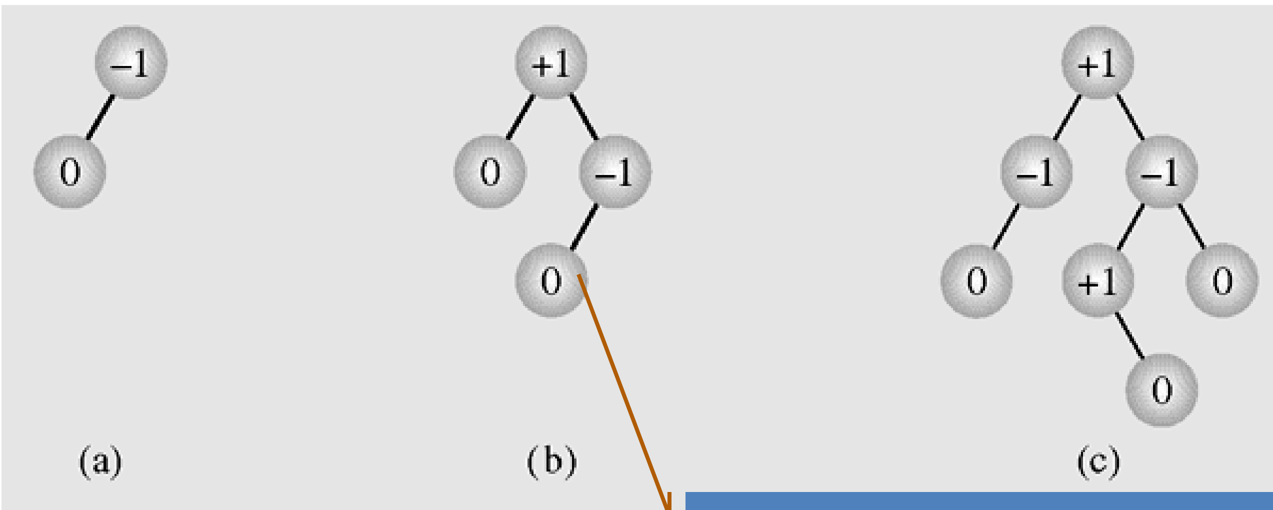
(c)



(d)

AVL Trees

- An **AVL tree** *(by Adelson Velskii, Landis)* is a height-balanced binary search tree.



Examples of AVL trees

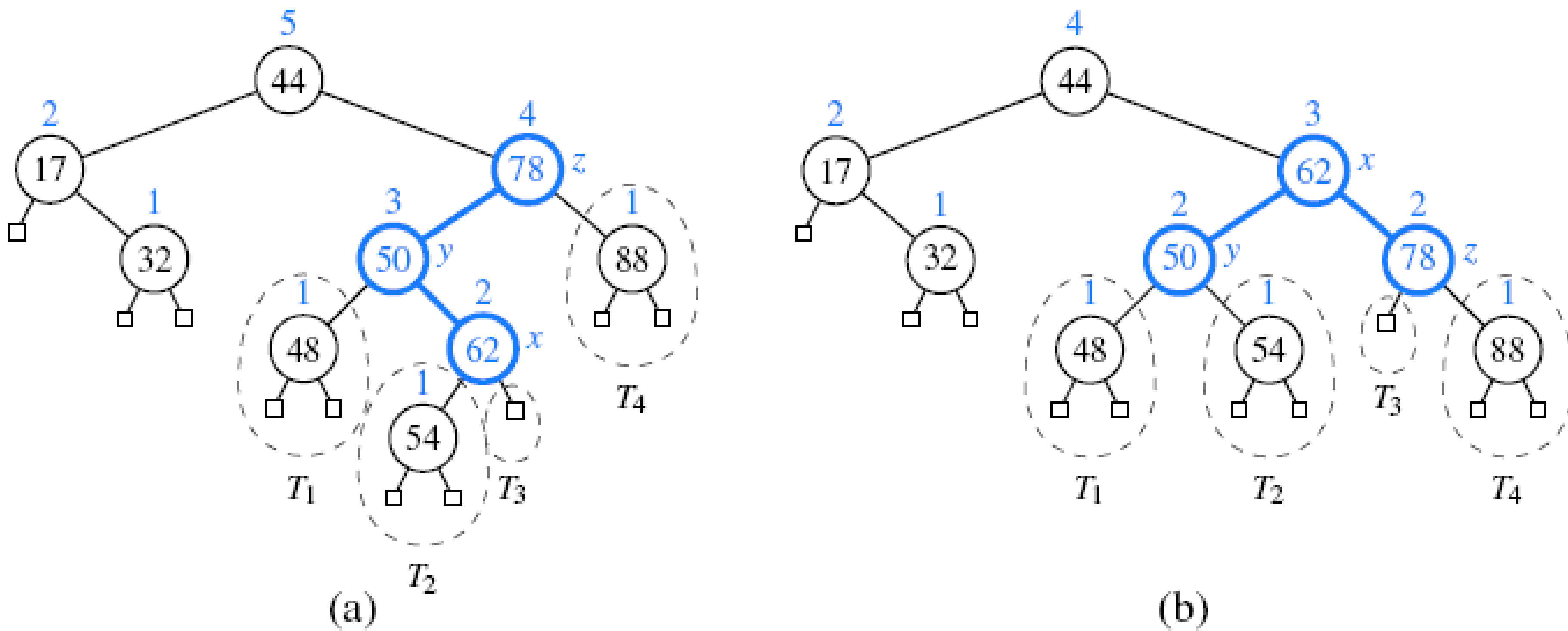
Balance factor = height(right)-height(left)

Insertion algorithm in an AVL Tree

- **Insert a node as in a binary search tree.**
- **Recalculate balance fator of nodes from the inserted node back to the root**.
  - If there is no unbalanced node (which means that no balance factors -2 or 2) in the tree then stop.
  - If we found the first node p whose balance factor is 2 or -2 then we should rotate p about his son. If p's and his son's balance factors have the same sign, then only single rotation should be done. If the signs of p and q are different then double rotation should be done: at first q is rotated, and then p.

    Rotation rule: for left unbalane the right rotation should be used and vice versa.
  - It can be proved that for insertion, at most one rotation (single or double) should be done. That is because the height of the node to be rotated after rotation is exactly the same as its' height before inserting a new key, thus the heights of all **ancestors of p are unchanged.**

Insertion in an AVL Tree demo



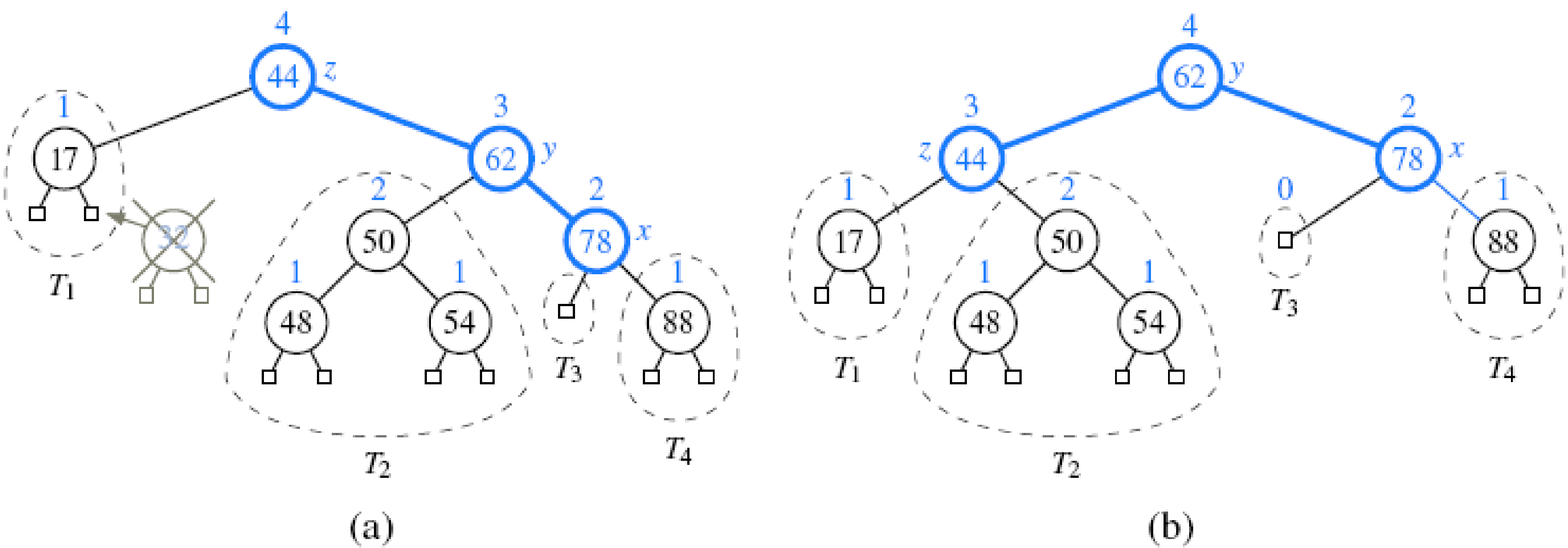
Balancing a tree after insertion the key 54

Deletion algorithm in an AVL Tree

- **Delete a node as in a binary search tree.**
- **Recalculate balance fator of nodes from the deleted node back to the root.**
  - If there is no unbalanced node (which means that no balance factors -2 or 2) in the tree then stop.
  - If we found the first node p whose balance factor is 2 or -2 then we should rotate p about his son. If p's and his son's balance factors have the same sign, then only single rotation should be done. If the signs of p and q are different then double rotation should be done: at first q is rotated, and then p.

    Rotation rule: for left unbalane the right rotation should be used and vice versa.
  - It can be proved that not the same as the case of insertion, in deletion the height of the rotated node p may be shorter by 1, thus it may causes the heights of p's ancestors shorter. Thus we should continue to balance until the height of rotated node is unchanged, and sometimes we do this to the root.

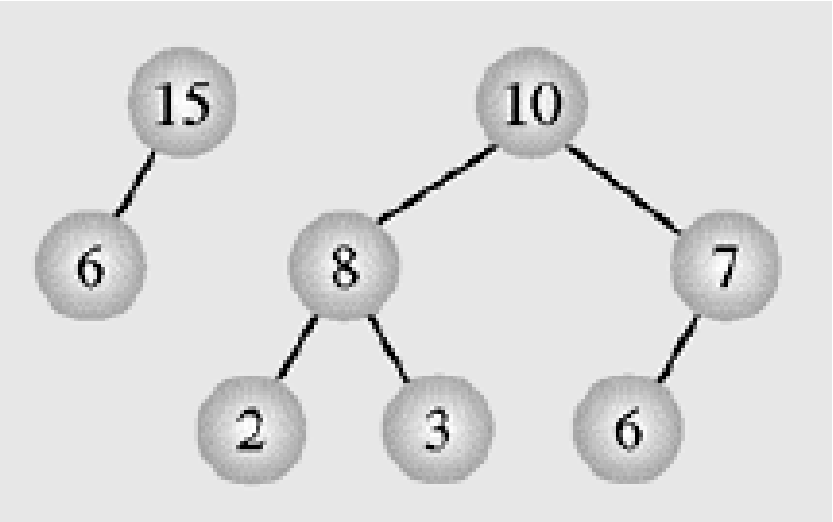
Deletion in an AVL Tree demo



Rebalancing an AVL tree after deleting the key 32

Heaps - 1

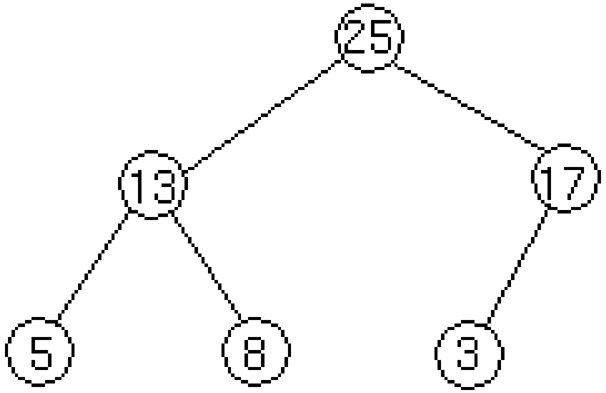
- A particular kind of binary tree, called a **heap**, has two properties:
  - The value of each node is greater than or equal to the values stored in each of its children
  - The tree is **nearly complete**, i.e. it is perfectly balanced, and the leaves in the last level are all in the leftmost positions



- These two properties define a **max heap**
- If "greater" in the first property is replaced with "less," then the definition specifies a **min heap**
- **Heap is used as Priority Queue and for heap-sorting**

Heaps - 2

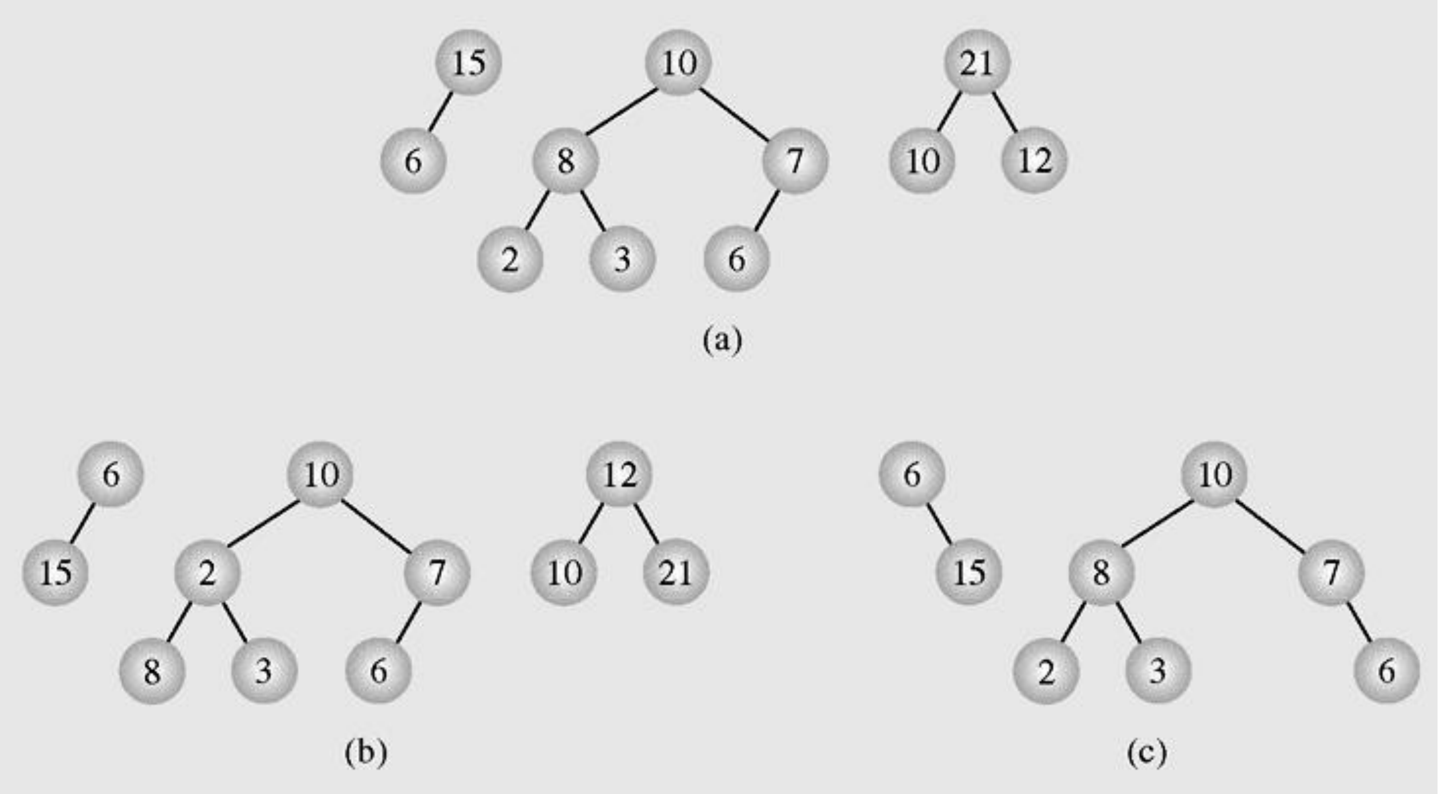
We can represent heap by array in level order, going from left to right. The array corresponding to the heap above is [25, 13, 17, 5, 8, 3].



The root of the tree A[0] and given index *i* of a node, the indices of its parent, left child and right child can be computed:

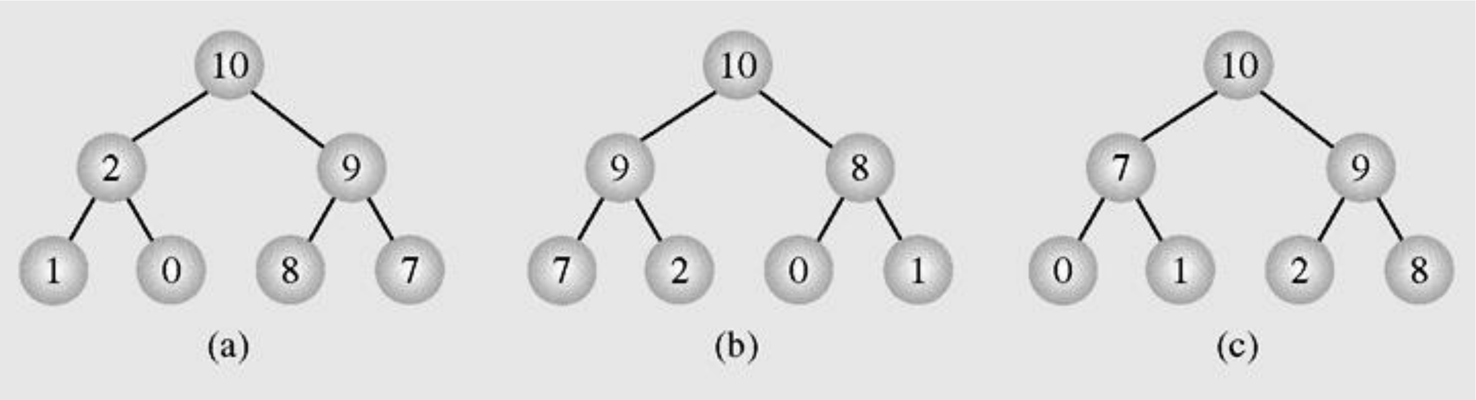
PARENT (*i*)
    return floor((*i-1)*/2)
LEFT (*i*)
    return 2*i*+*1*
RIGHT (*i*)
    return 2(*i* + 1)

Heaps - 3



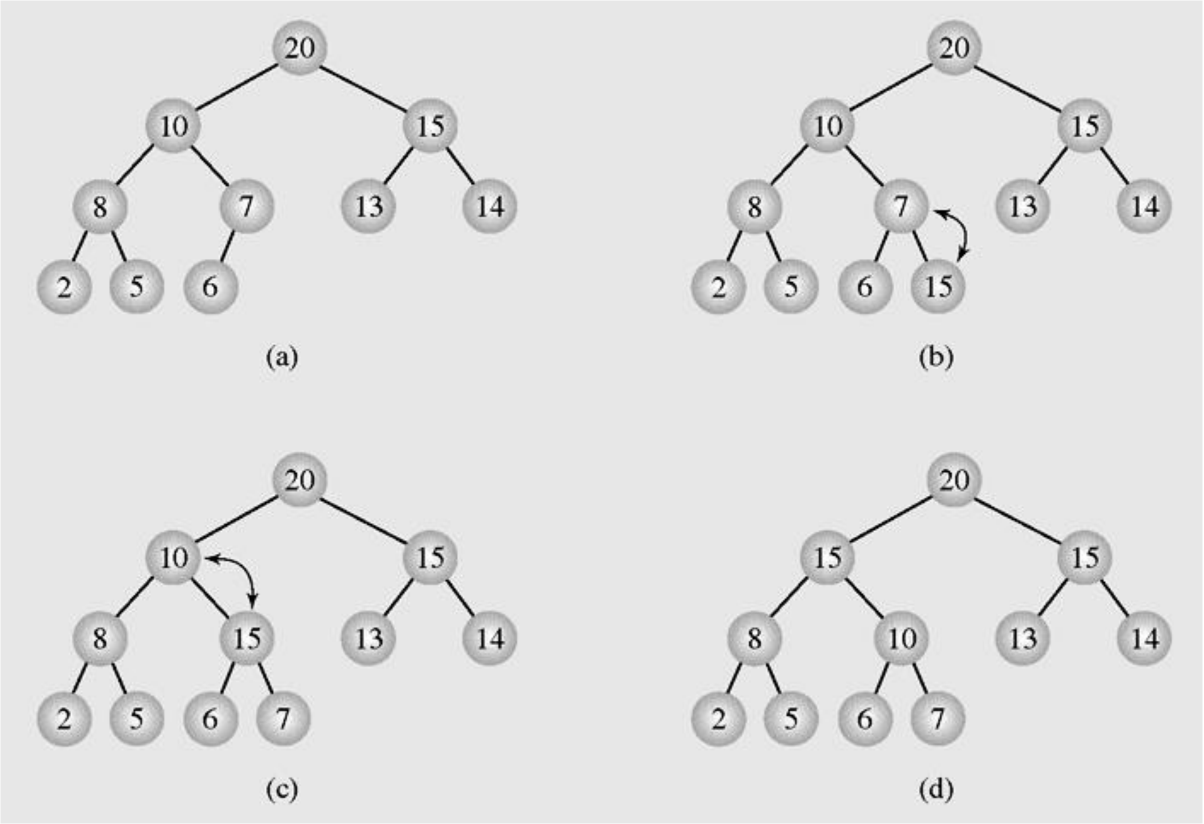
Examples of (a) heaps and (b–c) non-heaps

Heaps - 4



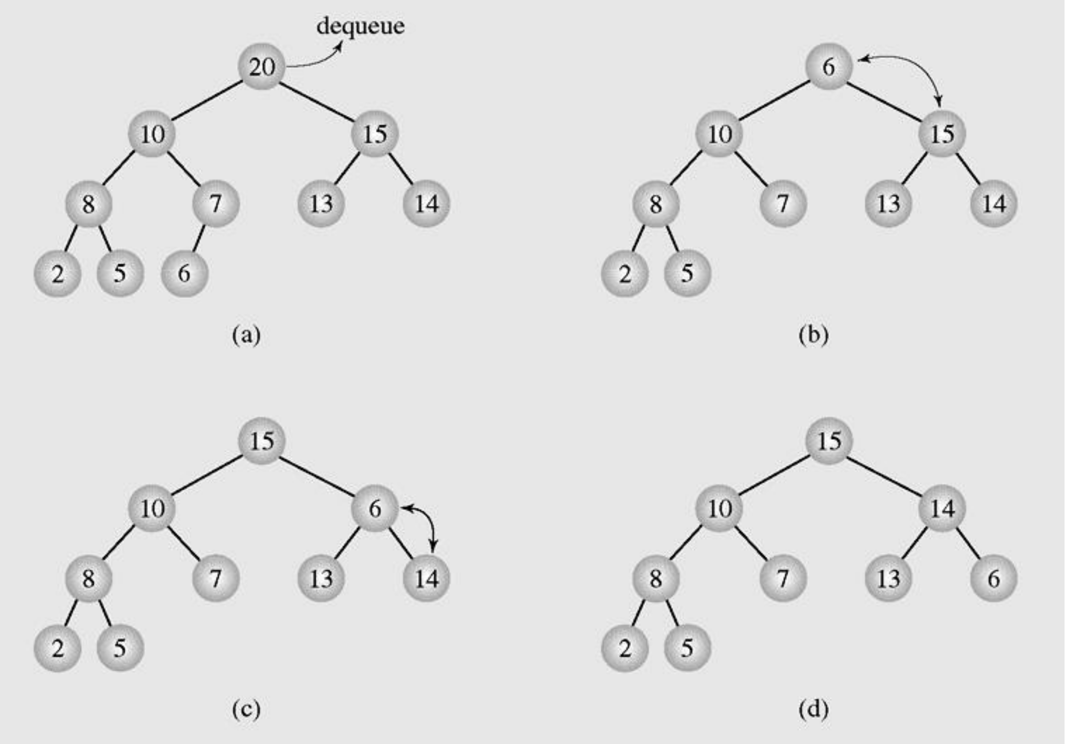
Different heaps constructed with the same elements

Heaps as Priority Queues - 1



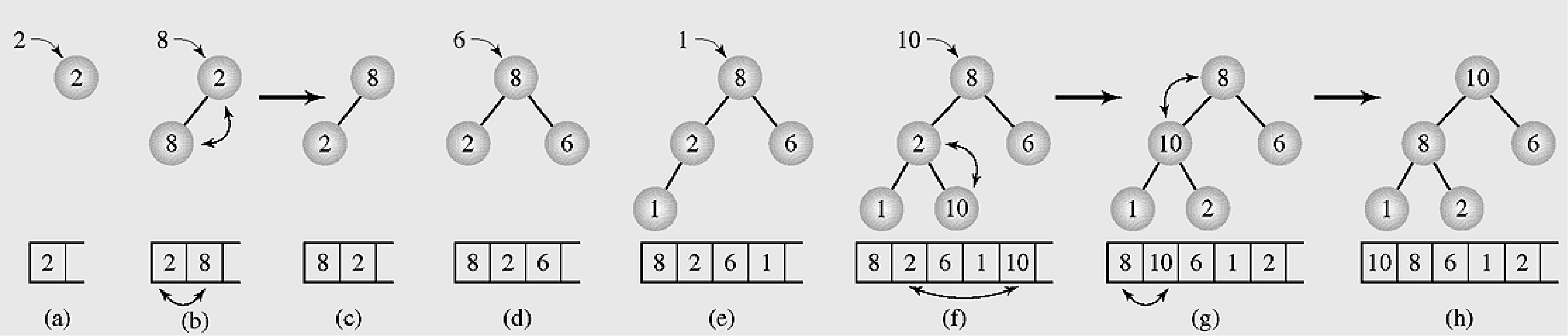
Enqueuing an element to a heap

Heaps as Priority Queues - 2



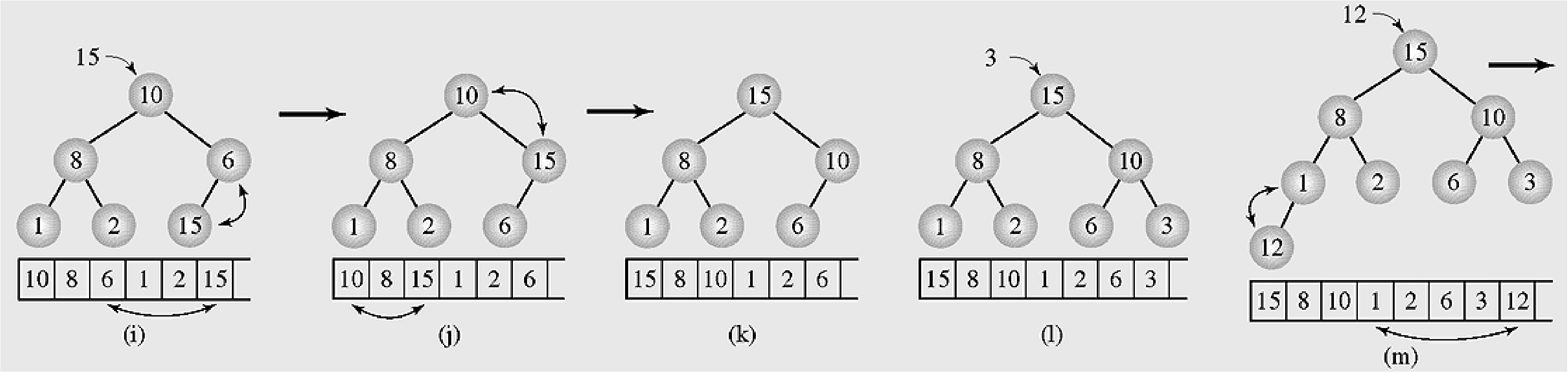
Dequeuing an element from a heap

Organizing Arrays as Heaps - 1



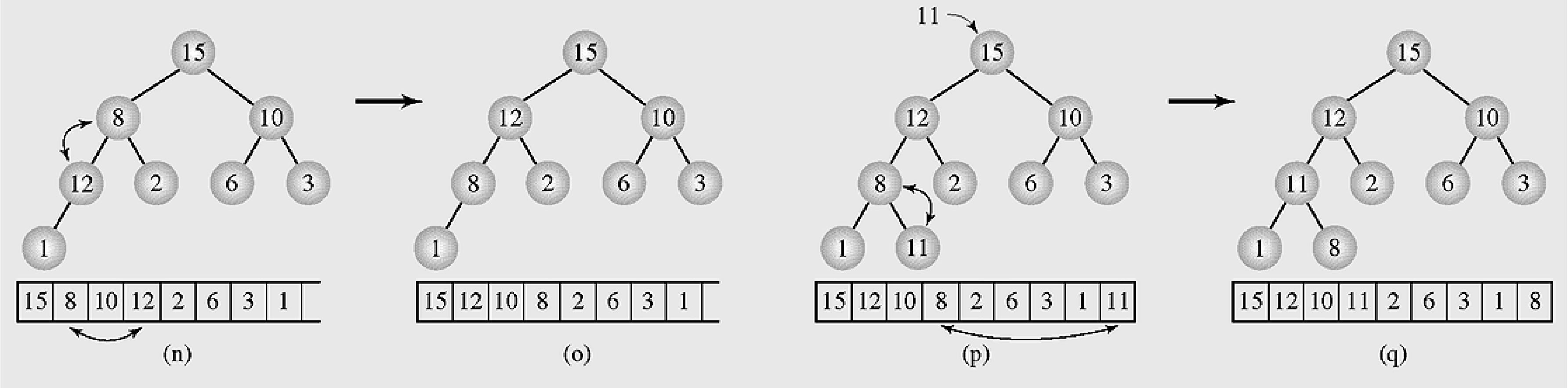
Organizing an array as a heap with a top-down method

Organizing Arrays as Heaps - 2



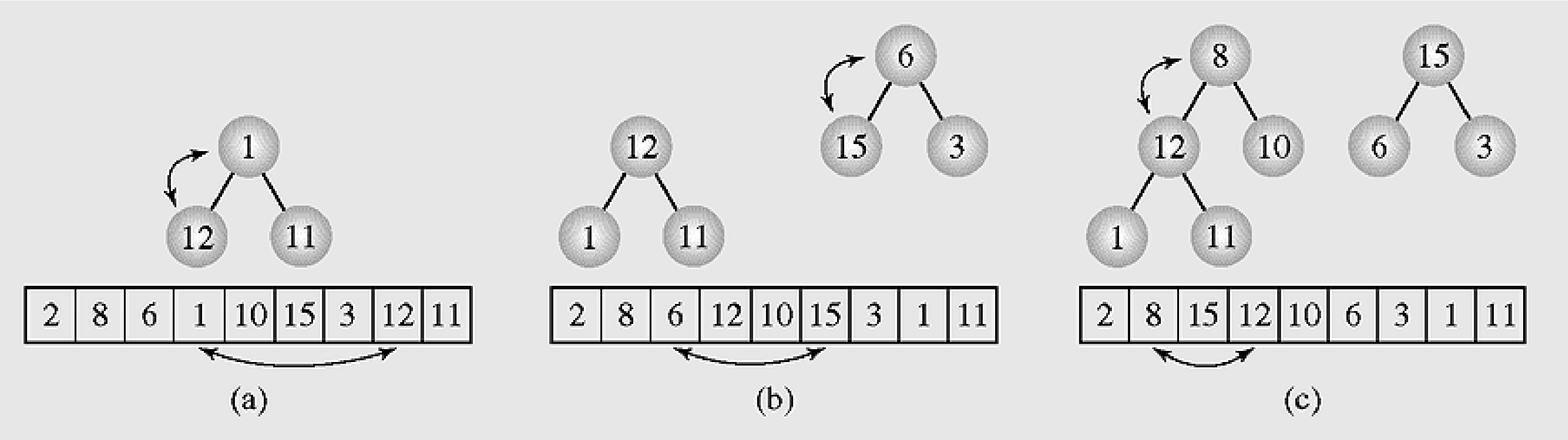
Organizing an array as a heap with a top-down method (continued)

Organizing Arrays as Heaps - 3



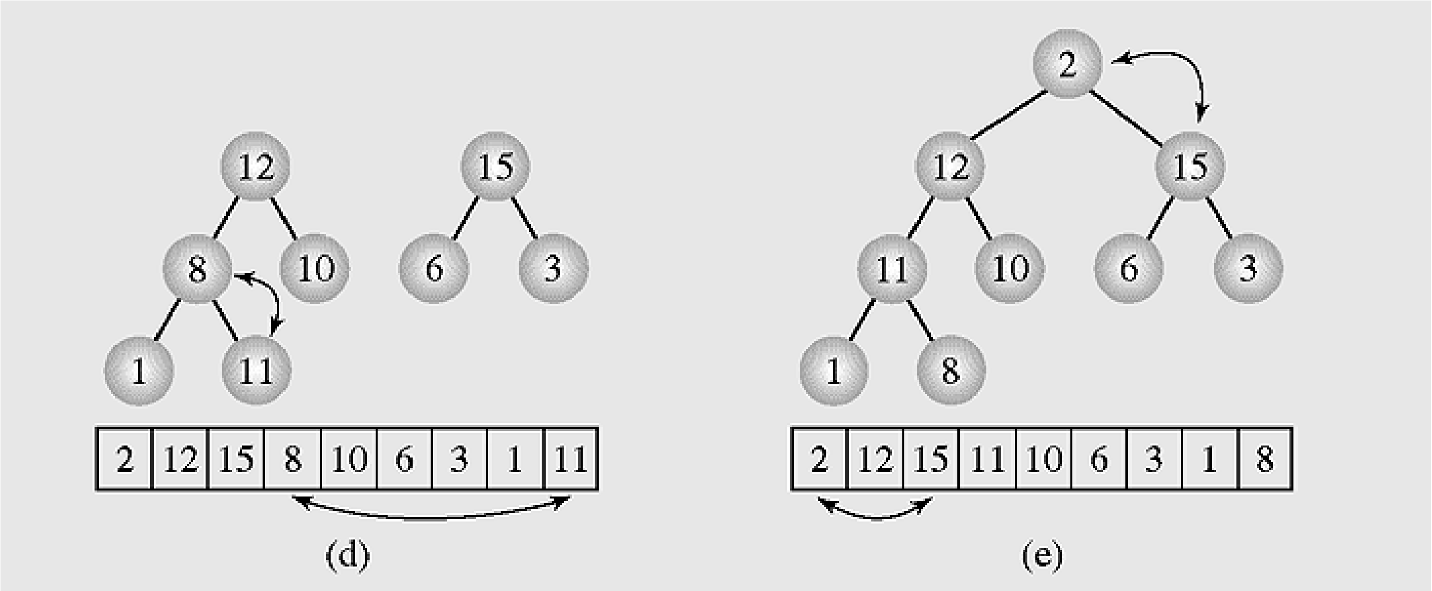
Organizing an array as a heap with a top-down method (continued)

Organizing Arrays as Heaps - 4



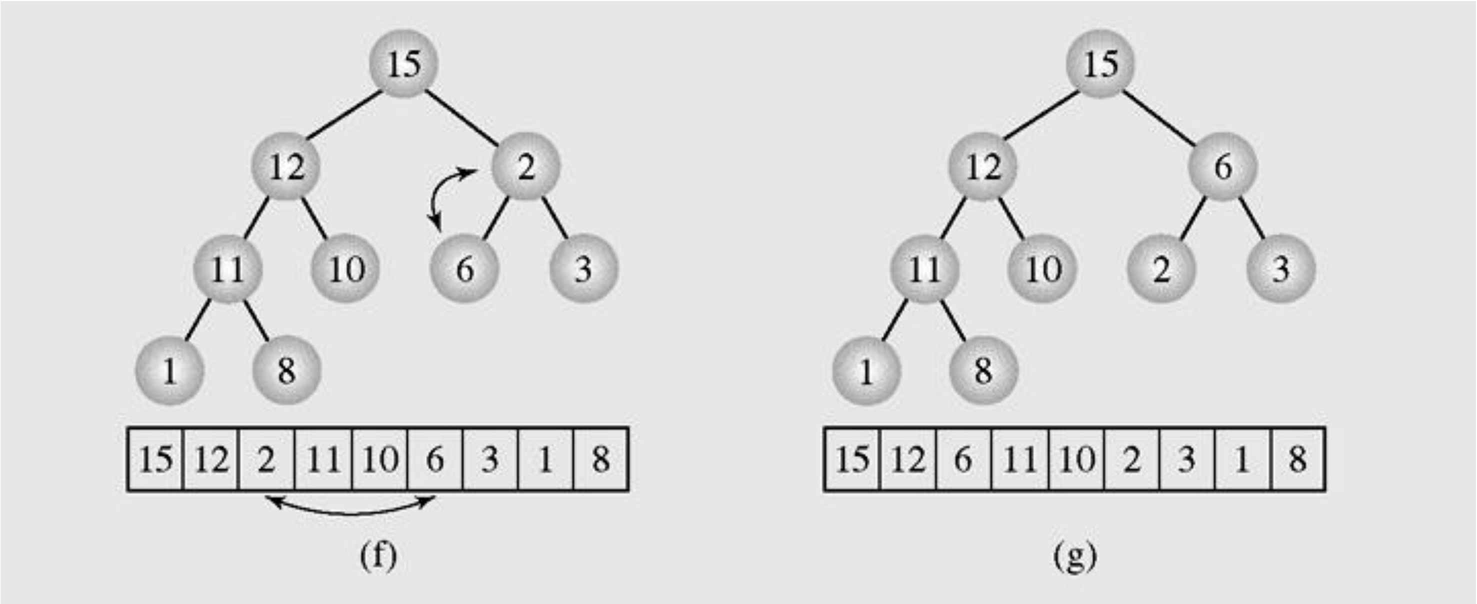
Transforming the array [2 8 6 1 10 15 3 12 11] into a heap with a bottom-up method

Organizing Arrays as Heaps - 5



Transforming the array [2 8 6 1 10 15 3 12 11] into a heap with a bottom-up method (continued)

Organizing Arrays as Heaps - 6



Transforming the array [2 8 6 1 10 15 3 12 11] into a heap with a bottom-up method (continued)

Organizing Arrays as Heaps - 7

```
//Transform the array to HEAP
 int i,s,f;int x;
 for(i=1;i<n;i++)
   { x=a[i]; s=i; //s  is a son, f=(s-1)/2 is father
     while(s>0 && x>a[(s-1)/2])
       { a[s]=a[(s-1)/2];   s=(s-1)/2;  };
         a[s]=x;
       }
   }
```

Organizing Arrays as Heaps - 8

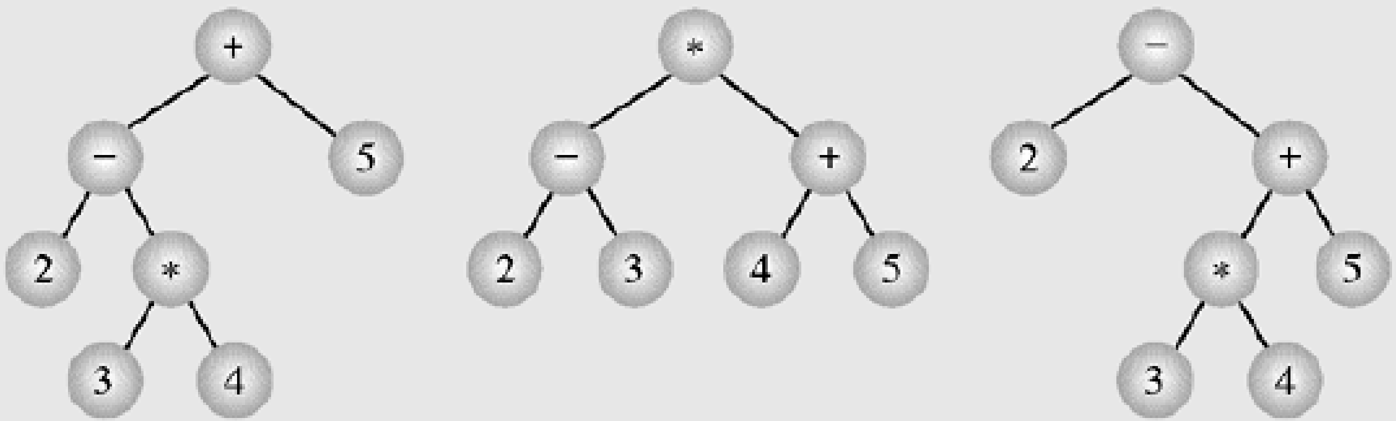
```
// Transform heap to sorted array
for(i=n-1;i>0;i--)
  { x=a[i];a[i]=a[0];
    f=0; //f is father
    s=2*f+1; //s is a left son
   // if the right son is larger then it is selected
    if(s+1<i && a[s]<a[s+1]) s=s+1;
    while(s<i && x<a[s])
       { a[f]=a[s]; f=s; s=2*f+1;
      if(s+1<i && a[s]<a[s+1]) s=s+1;
       };
    a[f]=x;
  };
```

Polish Notation and Expression Trees - 1

- Polish notation is a special notation for propositional logic that eliminates all parentheses from formulas:

  (5 – 6) * 7 = * – 5 6 7 (prefix notation)
- The compiler rejects everything that is not essential to retrieve the proper meaning of formulas.

Polish Notation and Expression Trees - 2



	(a)	(b)	(c)
	2 − 3 * 4 + 5	(2 − 3) * (4 + 5)	2 − (3 * 4 + 5)
Preorder	+ − 2 * 3 4 5	* − 2 3 + 4 5	− 2 + * 3 4 5
Inorder	2 − 3 * 4 + 5	2 − 3 * 4 + 5	2 − 3 * 4 + 5
Postorder	2 3 4 * − 5 +	2 3 − 4 5 + *	2 3 4 * 5 + −

Examples of three expression trees and results of their traversals

Summary - 1

- A tree is a data type that consists of nodes and arcs.
- The root is a node that has no parent; it can have only child nodes.
- Each node has to be reachable from the root through a unique sequence of arcs, called a path.
- An orderly tree is where all elements are stored according to some predetermined criterion of ordering.
- A binary tree is a tree whose nodes have two children (possibly empty), and each child is designated as either a left child or a right child.

Summary

- Balanced Binary Tree
- Balancing a Tree - Simple Balanced Algorithm
- Rotations on Binary Search Tree
- AVL Tree
- Insertion in an AVL Tree
- Deletion in an AVL Tree
- Heaps
- Heaps as Priority Queue
- Polish Notation and Expression Trees

Reading at home

Text book: Data Structures and Algorithms in Java

- 11.2 Balanced Search Trees - 472
- 11.3 AVL Trees - 479
- 9.3 Heaps - 370
- 9.3.1 The Heap Data Structure - 370
- 9.3.2 Implementing a Priority Queue with a Heap - 372