Moving Tokens on Graphs

Some Known Results and Open Questions

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Outline

In this talk, I will

1. Briefly introduce reconfiguration problems.
2. Explain some known results on
   - Independent Set Reconfiguration
   - k-Path Vertex Cover Reconfiguration

   as examples about specific reconfiguration problems.
3. Introduce some relevant open questions.

Reconfiguration Problems

Reconfiguration Problems

- **Instance:**
  - Two configurations A, B.
    E.g., two states of Rubik's Cube Puzzle.
  - A reconfiguration rule (defining whether two arbitrary configurations are adjacent).
    E.g., rotating one face of the cube 90, 180, or 270 degrees.
- **Question:** Does there exist a sequence of adjacent configurations that transforms A into B?

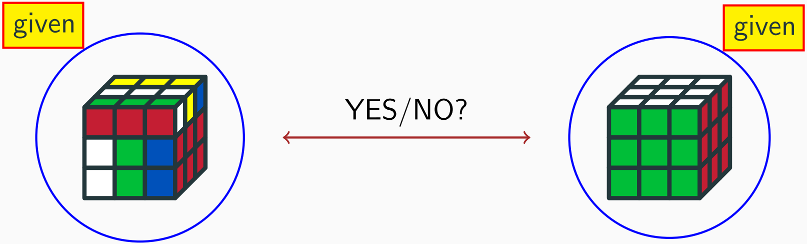

Figure 1: An instance of Rubik's Cube Puzzle.

Reconfiguration Problems

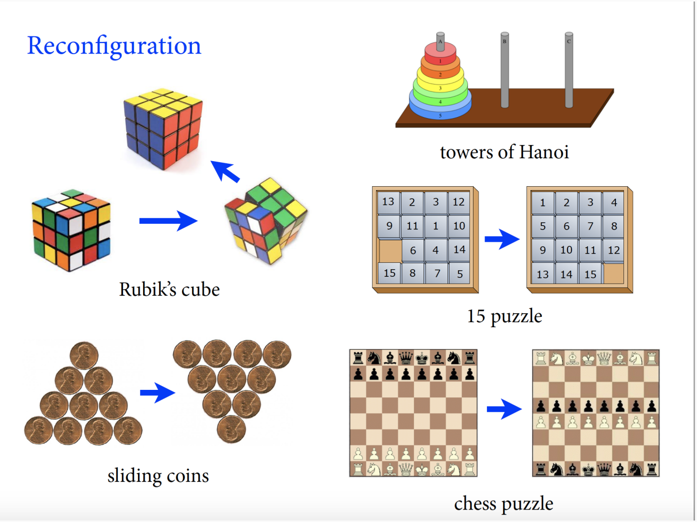

Figure 2: Reconfiguration. (© Anna Lubiw, in her tutorial "Reconfiguration and geometry" at CoRe2019.)

Naturally, we can define the corresponding reconfiguration graph.

- Each configuration is a node.
- Two nodes (configurations) X, Y are adjacent if one can be obtained from the other by applying the given reconfiguration rule exactly once.

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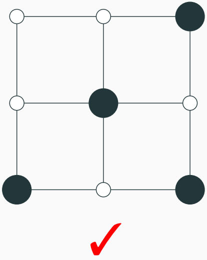
One may ask

- Reachability: a path between two nodes of a reconfiguration graph?
- Shortest Reconfiguration: find a shortest path (if exists) between two nodes of a reconfiguration graph?
- Connectivity: a reconfiguration graph is connected?
- Diameter: the diameter of a reconfiguration graph is bounded?

Independent Set Reconfiguration

Independent Set

An independent set of a graph G is a vertex-subset $I \subseteq V(G)$ such that no two members of I are connected by and edge in G.



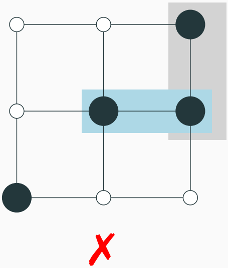


Figure 3: Examples of independent sets.

Token Sliding (TS)

Imagine that each member of an independent set contains a token.

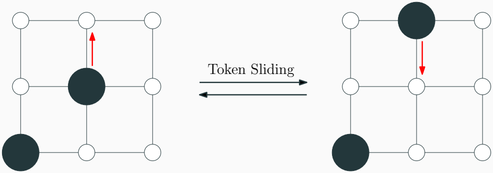


Figure 4: Token Sliding: move a token to one of its neighbors.

Token Jumping (TJ)

Imagine that each member of an independent set contains a token.

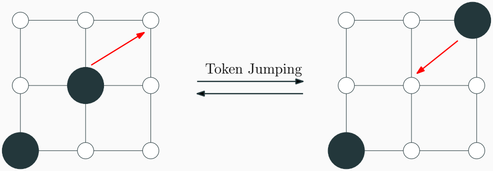


Figure 5: Token Jumping: move a token to one of unoccupied vertices.

Remark: Token Jumping generalizes Token Sliding.

Token Addition and Removal (TAR(u))

Imagine that each member of an independent set contains a token.

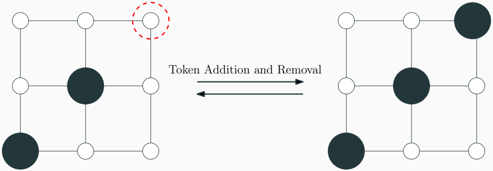


Figure 6: Token Addition and Removal: add or remove a token such that the number of tokens is always at least u.

Remark: Token Addition and Removal generalizes Token Jumping.

Independent Set Reconfiguration (ISR)

- **Instance:**
  - Two independent sets I, J of a graph G.
  - One of the following reconfiguration rules: TS, TJ, TAR(u).
- **Question:** Does there exist a sequence of adjacent independent sets that transforms I into J?

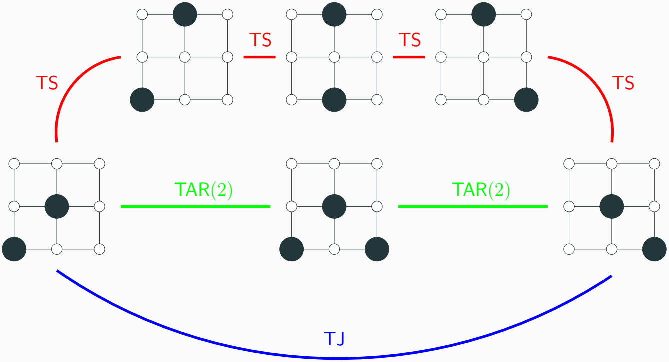


Figure 7: Examples of ISR's yes-instances.

Independent Set Reconfiguration (ISR)

- ISR is $\mathtt{PSPACE}$-complete on several well-known graph classes, including
  - perfect graphs [Kamiński, Medvedev, and Milanič 2012],
  - planar graphs of maximum degree 3 [Hearn and Demaine 2005]
  - bounded treewidth graphs [Wrochna 2018],
  - split graphs [Belmonte et al. 2019] and bipartite graphs [Lokshtanov and Mouawad 2019] under TS.

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- Naturally, one may ask:
  - which graph class makes the problem "easy/hard"?
    Open Question: graphs of treewidth ≤ 2?
  - which parameter makes the problem "easy/hard"?
    A natural parameter: the number of tokens. [this talk]

ISR Parameterized by the Number of Tokens

- **Instance:**
  - Two independent sets I, J of a graph G.
  - One of the following reconfiguration rules: TS, TJ, TAR(u).
- **Parameter:** The number of tokens k.
- **Question:** Does there exist a sequence of adjacent independent sets that transforms I into J?

This problem is W[1]-hard

- under TAR [Mouawad et al. 2017],
- under TJ [Ito et al. 2020]
- under TS [this talk]

and is in FPT

- on planar graphs under TJ [Ito, Kamiński, and Ono 2014].

Remind: How to prove $\mathtt{W}[1]$-hardness?

Parameterized Reduction from problem P to problem Q

is a function ϕ with the following properties:

- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where k is the parameter of x,
- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P.
- If k is the parameter of x and k' is a parameter of $\phi(x)$, then $k' \leq g(k)$ for some function g.

- Transforming an Independent Set instance (G, k) into a Vertex Cover instance $(G, n-k)$ is not a parameterized reduction.
- Transforming an Independent Set instance (G, k) into a Clique instance $(\overline{G}, k)$ is a parameterized reduction.

ISR Parameterized by the Number of Tokens under TS

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INDEPENDENT SET (G', k') $\Longrightarrow$ INDEPENDENT SET RECONFIGURATION under TS (G, I, J, k)

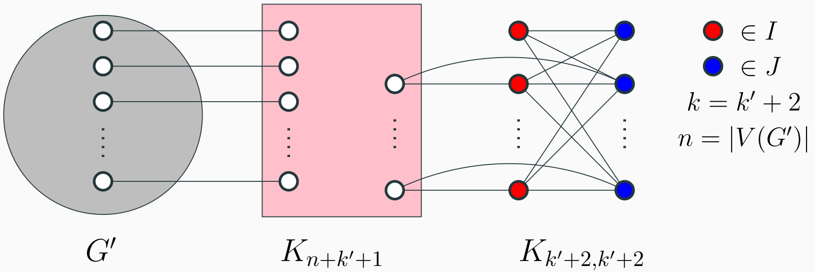

Figure 8: A parameterized reduction from INDEPENDENT SET parameterized by solution size to ISR under TS parameterized by the number of tokens.

Claim: G' has an independent set of size $\geq k'$ if and only if I can be reconfigured into J under TS in G.

Open Question: FPT algorithm on planar graphs?

k-Path Vertex Cover Reconfiguration

k-Path Vertex Cover (k-PVC)

A k-path vertex cover of a graph G is a vertex-subset $I \subseteq V(G)$ such that each path on k vertices contains a member of I.

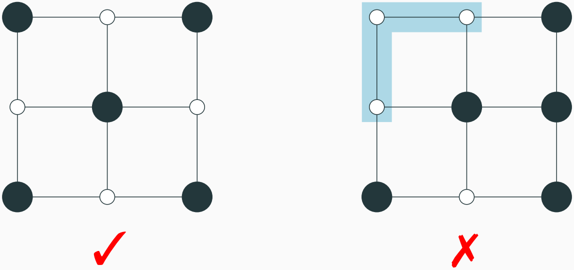

Figure 9: Examples of 3-path vertex covers.

Remind: When $k = 2$, it is called a vertex cover.

k-Path Vertex Cover Reconfiguration (k-PVCR)

- **Instance:**
  - Two k-path vertex covers I, J of a graph G.
  - One of the following reconfiguration rules: TS, TJ, TAR(u).
- **Question:** Does there exist a sequence of adjacent k-path vertex covers that transforms I into J?

Remark: Unlike in ISR, the TAR(u) rule requires that the number of tokens must be at most u.

Theorem (Hoang, Suzuki, and Yagita 2020)

k-PVCR is PSPACE*-complete on planar graphs of maximum degree 3, bounded treewidth graphs (under all rules), and bipartite graphs and chordal graphs (under* TS*). On the other hand, it can be solved in polynomial time on trees under* TJ *and* TAR.

Open Question: k-PVCR under TS on trees?

k-PVCR Parameterized by The Number of Tokens

- **Instance:**
  - Two k-path vertex covers I, J of a graph G.
  - One of the following reconfiguration rules: TS, TJ, TAR(u).
- **Parameter:** The number of tokens t.
- **Question:** Does there exist a sequence of adjacent k-path vertex covers that transforms I into J?

k-PVCR Parameterized by The Number of Tokens

- **Instance:**
  - Two k-path vertex covers I, J of a graph G.
  - One of the following reconfiguration rules: TS, TJ, TAR(u).
- **Parameter:** The number of tokens t.
- **Question:** Does there exist a sequence of adjacent k-path vertex covers that transforms I into J?

Observation: From this meta-theorem, k-PVCR parameterized by the number of tokens is in FPT.

Theorem (Mouawad et al. 2017)

If a t-subset problem Q is superset-closed and has an FPT *algorithm to enumerate all its minimal solutions of cardinality at most t, the number of which is bounded by a function of t, then Q* Reconfiguration *parameterized by t is in* FPT*, as well as the search and shortest path variants.*

Open Question: More efficient FPT algorithms where k is small?

Resources and References

Resources and References

Surveys on Reconfiguration

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C.M. Mynhardt and S. Nasserasr (2019). "Reconfiguration of colourings and dominating sets in graphs". In: *50 years of Combinatorics, Graph Theory, and Computing*. Ed. by Fan Chung et al. 1st. CRC Press, pp. 171–191. DOI: 10.1201/9780429280092-10

Open Problems from CoRe 2019

https://pagesperso.g-scop.grenoble-inp.fr/~bousquen/CoRe_2019/CoRe_2019_Open_Problems.pdf

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