Simulating Missile Trajectory with DEs

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GitHub

Introduction

In a ballistic missile trajectory simulation, the system of differential equations used to describe the ballistic model is a highly complex system. In particular, the six-degree of freedom model used most frequently, solves for the missile's components of acceleration, velocity, and position at discrete time intervals. The usual approach for simulation is the 4th Order Runge Kutta method. This poster will be diving into a different, and potentially more efficient algorithm, called the Parker-Sochacki Method (PSM for short).

Variables and Assumptions

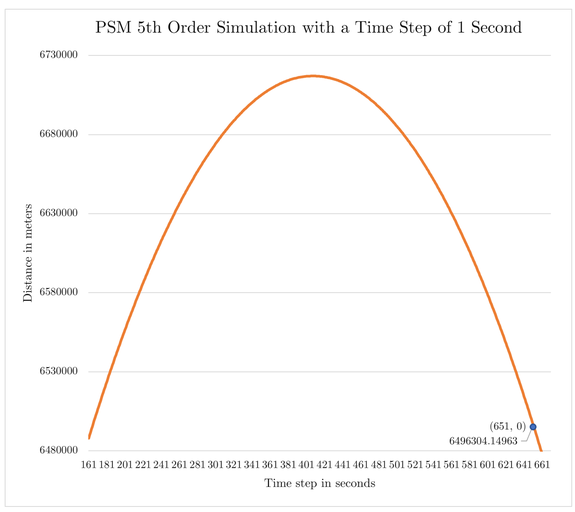
- Initial position coordinates, velocity, and acceleration
- PSM Order, time step, and tolerance
- Gravity coefficients for x, y, and z components
- Assuming that mass and acceleration are constants to be ignored

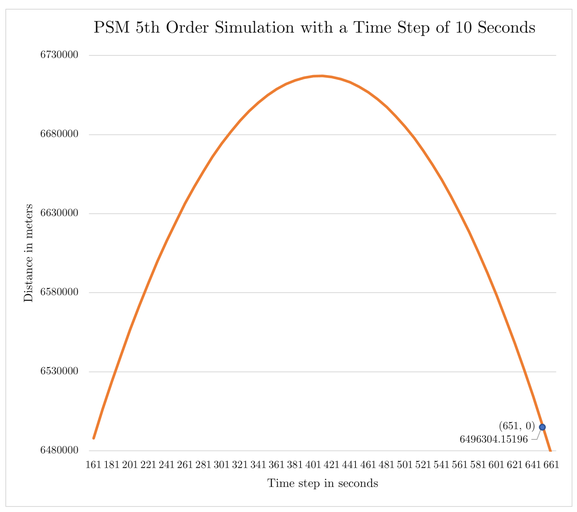
Problem Identification

When solving for missile trajectory, the Parker-Sochacki method needs initial conditions for position coordinates, velocity, and acceleration. It solves the trajectory by taking each of these components and solves them using a power series where the order of accuracy is simulated by the Cauchy product - an example of how this works is shown to the right of this box. This accurately simulates the trajectory by using coefficients corresponding to the x, y, and z coordinates of the missile, as well as a time step that can be defined by the user.

The direct benefit of using PSM however, is the fact that you can take big time steps maintaining accuracy, which is not true for RK4. In RK4, a time step of 1 is typically used, whereas in PSM you can take much bigger time steps, like 10 or even 20, depending on the order you use. Using 5th order PSM (PSM5), for example, means that for each coefficient there are 5 extra values stored which are used to solve for the next coefficient. With PSM5 and a time step of 10, the problem can be computed quicker and more efficiently than using RK4 with a time step of 1. Two plots are shown below, displaying the accuracy of PSM5 when taking a time step of 1 and then 10.

Verification of Model





In the plots above, the one on the left is using a time step of 1 and the one on the right using a time step of 10. Labeled, is the point (651, 0), since it is near the end the of the plot, where the amount of rounding error is going to be the greatest. As can be seen, the distance, which is in meters, is only off by less than .01.

Cauchy Product Example Problem

Use the power series method to solve $y' = y^2 \; ; y(0) = -1$

$$y = \sum_{k=0}^{\infty} c_k t^k \Rightarrow y' = \sum_{k=1}^{\infty} k c_k t^{k-1}$$

Plug into $y' = y^2$

$$y' = \sum_{k=1}^{\infty} k c_k t^{k-1} = \left(\sum_{k=0}^{\infty} c_k t^k \right) \left(\sum_{k=0}^{\infty} c_k t^k \right)$$

Rewrite y^2 to match the form of the cauchy product

$$y' = \sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{j=0}^{k} c_j c_{k-j} \right)}_{\text{Cauchy product}} t^k \tag{1}$$

Set $k = 1$ on both sides of the equation to solve for c_k

$$y' = \sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} c_j c_{k-j-1} \right) t^{k-1}$$

$$c_k = \frac{\sum_{j=0}^{k-1} c_j c_{k-j-1} t^{k-1}}{k} \tag{2}$$

Solving for c_k, we can see each coefficient is a function of previous coefficients

$$\therefore c_1 = c_0 c_0, c_2 = \frac{c_0 c_1 + c_1 c_0}{2}, c_3 = \frac{c_0 c_2 + c_1 c_1 + c_2 c_0}{2}, \ldots$$

Conclusion

PSM is an efficient and highly adaptive way to simulate ballistic missile trajectory. By plugging in initial conditions of position, velocity, and acceleration, it can accurately simulate the trajectory of a missile in various situations. Going more into the method, there are a few things that could not be talked about on this poster, such as the different gravity models that can be used in the simulations, the different ways to make parts of the algorithm run more efficiently, and even how to have the PSM algorithm select the most optimal order and time step given different constraints.

References

All of the source code and references are located on the GitHub repository and can be accessed by scanning the QR code at the top of the poster, or going to the following URL: https://github.com/tierneycolin/ModelingFinalProject