Practice Questions for Graph Theory

Representation, Search Algorithms, and Variants for Problem Solving

Raghav B. Venkataramaiyer

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1 Graph Representations

1.1 Vertex Insertion

Question $\overline{ABCD}$ is a closed quadrilateral. A new vertex E is introduced between B and C. Show the adjacency lists before and after the introduction of E. Hence, write an algorithm/ pseudocode in order to introduce a new vertex between an existing edge.

Interpretation Given a closed quadrilateral $\overline{ABCD}$, the adjacency list in 1-indexed format is given as:

```
V = [A, B, C, D]
Adj = [[2, 4],                  # 1 is connected to 2 and 4
       [1, 3],                  # 2 is connected to 1 and 3
       [2, 4],                  # and so forth
       [1, 3]]
```

Here the edge $\overline{BC}$ is defined in two entries of the adjacency list, *i.e.* as vertex `3` in `Adj[2]` and vertex `2` in `Adj[3]`.

Solution In order to introduce a new vertex E between edge $\overline{BC}$,

Step 1 Append vertex E to the vertex list V and get its index.

```
V = [A, B, C, D, E]             # Add E as V[5]
```

Step 2 Remove the edge $\overline{BC}$

```
V = [A, B, C, D, E]
Adj = [[2, 4],
       [1],                    # remove 3 from Adj[2]
       [4],                    # remove 2 from Adj[3]
       [1, 3],
       []]                     # add empty Adj[5]
```

Step 3 Add edges $\overline{BEC}$

```
V = [A, B, C, D, E]
Adj = [[2, 4],
       [1, 5],                 # add 5 to Adj[2]
       [4, 5],                 # add 5 to Adj[3]
       [1, 3],
       [2, 3]]                 # add 2, 3 to Adj[5]
```

Algorithm To introduce a vertex W between an edge (u, v),

```
GRAPH_ADD_VERTEX_BW(G,W,u,v) :
  w = G.V.append(W)            # Insert W into list of
                               # vertices and store the
                               # last appended index.

  G.Adj[u].remove(v)           # Remove v from Adj[u]
  G.Adj[v].remove(u)           # Remove u from Adj[v]

  G.Adj[u].append(w)           # Append w into Adj[u]
  G.Adj[v].append(w)           # Append w into Adj[v]

  G.Adj[w].append(u)           # Append u into Adj[w]
  G.Adj[w].append(v)           # Append v into Adj[w]
```

PS: Here, W in uppercase refers to a variable (*i.e.* vertex information like coordinates of a point etc.) that needs to appended into the list of verts $G.V$. And (u, v) represent the indices of the pair of verts that constitute and edge. We are interested in the Adjacency List (as required by the question,) hence the use of $G.Adj$

1.2 Transpose Graph

Question Given a graph $G(V, M)$, M being the adjacency matrix. A transpose graph would be the one with same set of vertices, but a transposed adjacency matrix, *i.e.* $G^\top(V, M^\top)$.
What does a transpose graph represent? Illustrate with a drawing to support your answer.

Interpretation Recall that,

1. In an adjacency matrix A, the component at i^{th} row, and j^{th} column is given as a_{ij} and it represents whether the edge $v_i \to v_j$ exists.
2. A transpose graph $G^\top(V, M^\top)$ would be any different *iff* $M \neq M^\top$. In other words, if G is a directed graph.
3. The components in the transposed matrix are mirrored across the diagonal. Hence, if $B = A^\top$, then $b_{ij} = a_{ji}$.

Solution Each edge $v_i \to v_j$ in G, transforms to $v_j \to v_i$ in the transpose graph $G^\top$. In other words, the edges are reversed.

This would be any different only in case of a directed graph. Since for an undirected graph $M = M^\top$

Hence, the transpose graph $G^\top(V, M^\top)$ represents $G(V, M)$ with edges reversed.

Illustration [TODO]

1.3 (In/Out)-degree

Question What is the average in-degree of a graph $G(V, E)$, where E is the set of edges in G?

Solution In-degree of a vertex is defined as the number of edges leading onto itself.

Let $d_{\text{in}}(v)$ represent the in-degree of vertex v. Then the average in-degree is given as the sum of in-degrees divided by the size of number of verts,

$$\mathbb{E}[d_{\text{in}}(v)] = \frac{\sum_{v \in V} d_{\text{in}}(v)}{|V|}$$

Intuitively speaking, the sum of all in-degrees is the same as the number of edges. Hence,

$$\mathbb{E}[d_{\text{in}}(v)] = \frac{|E|}{|V|}$$

In further detail In-degree of a vertex is the same as counting the non-zeros in one (specific) column of an adjacency matrix representation M for the set of edges E.

Similarly, the sum $\sum_{v \in V} d_{\text{in}}(v)$ is equivalent to

- Counting the non-zeros for every the column of M,
- *i.e.* Counting all the non-zeros in M,
- *i.e.* The number of edges.

Hence,

$$\sum_{v \in V} d_{\text{in}}(v) = |E|$$

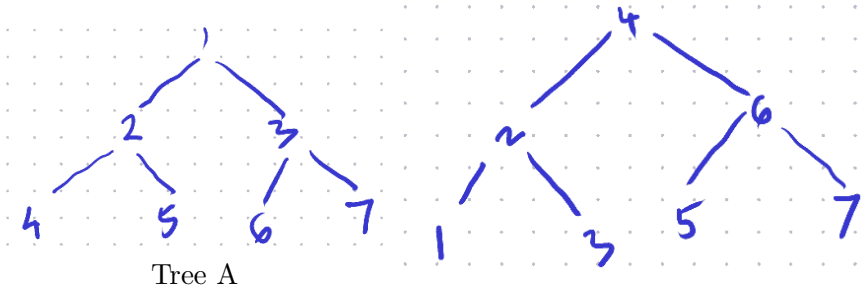
1.4 Representation

Question Provide an adjacency list as well as the adjacency matrix representation for trees A and B in the following figure.

Solution

Tree A

```
Adj = [[2 3]
       [1 4 5]
       [1 6 7]
       [2]
```



Tree A Tree B

```
       [2]
       [3]
       [3]]

M = [[0 1 1 0 0 0 0]
     [1 0 1 1 0 0 0]
     [1 0 0 0 1 1 0]
     [0 1 0 0 0 0 0]
     [0 1 0 0 0 0 0]
     [0 0 1 0 0 0 0]
     [0 0 1 0 0 0 0]]
```

Tree B

```
Adj = [[2]
       [1 3 4]
       [2]
       [2 6]
       [6]
       [4 5 7]
       [6]]
M = [[0 1 0 0 0 0 0]
     [1 0 1 1 0 0 0]
     [0 1 0 0 0 0 0]
     [0 1 0 0 0 1 0]
     [0 0 0 0 0 1 0]
     [0 0 0 1 1 0 1]
     [0 0 0 0 0 1 0]]
```

PS The Adjacency matrix of Tree B is bi-symmetric.

2 Elementary Algorithms

2.1 BFS

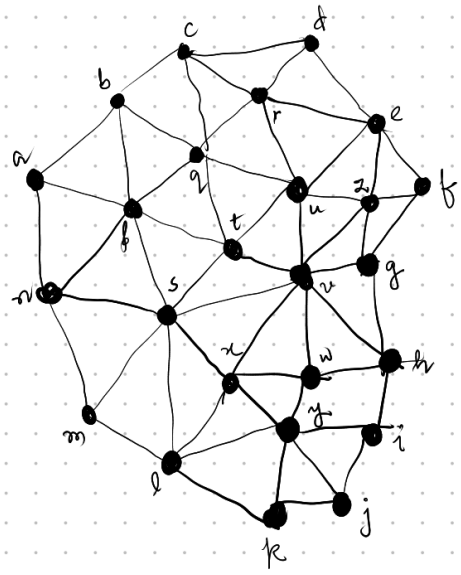


Figure 1: Graph A

Question With reference to Graph A (see Fig 1) **Determine algorithmically**,

1. The shortest path weight $\delta(u, j)$ for the pair (u, j) of vertices.
2. A shortest path between the pair (u, j) of vertices.
3. All shortest-paths originating from vertex u.

Key Insight All the three questions here speak about a shortest path originating from vertex u. This is a uniformly weighted undirected graph, *i.e.* all edges are equally weighted. The solution for shortest path will follow a BFS in such a case.

Solution

1. Running a BFS on the graph gives us the figure, BFS on Graph A (Fig 2) upon termination.

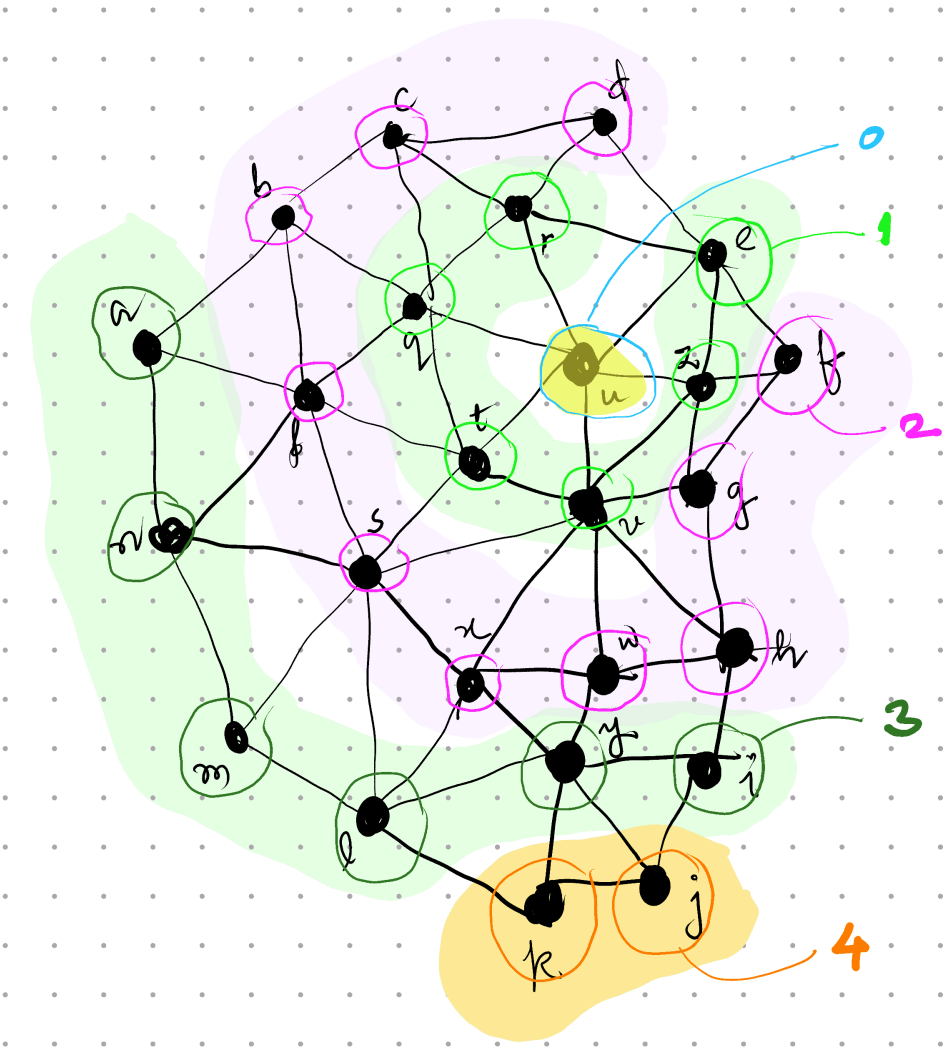


Figure 2: BFS on Graph A

2. The numbers marked are discovery times of the nodes $v \cdot d \; \forall v \in V$.

3. For part (1) the shortest path weight is given as $\delta(u, j) = j \cdot d - u \cdot d$. Computing from the figure, $\delta(u, j) = 4 - 0 = 4$.

4. For part (2) we may pick any one path such that each successive node is from successive level. *i.e.* one of,

   (a) $\langle u, v, h, i, j \rangle$,

   (b) $\langle u, v, w, y, j \rangle$, or

   (c) $\langle u, v, x, y, j \rangle$.

   Recall, that only one of these is, and not all of them are, the required shortest path (*i.e.* discovered in one run).

5. For part (3), a BFS tree is required. Its been left that upon the reader to exercise and present as necessary. An easy way out would be to use the adjoining graph (Fig 2) and additionally mark each connection from parent to child as descended during the BFS. Note that the arrow would be a manifestation of line `v.PI = u` in the algorithm (link to the slide). Recall that there may be only one parent to a child, not many, and that the discovery time of the parent is always less than that of the child.

2.2 DFS

Question Given that there are 10 courses in a programme, and corresponding pre-requisites are listed as under, **determine algorithmically** If the programme may be completed successfully by a candidate?

4. depends upon 1 and 2;
5. depends upon 1, 3 and 10;
6. depends upon 2 and 3;
7. depends upon 2, 4 and 5;
8. depends upon 1 and 4;
9. depends upon 5 and 6; and
10. depends upon 3 and 7.

Key Insight We define a relationship $u \to v$ if course u depends upon v. Then we get a dependency graph (*i.e.* a directed graph where relationship is defined when the parent is dependent upon the child).

A topological order $T \equiv \langle v_1, \ldots, v_k \rangle$ of such a graph means that all ancestors of v_i have been listed before v_i itself $\forall v_i \in V$. In simple words, the topological order is one possible order of courses to complete the programme.

However, the topological order is not always possible. From our slides, we know that topological order is only defined for directed acyclic graph (DAG). **Hence, one may complete the programme if the dependency graph is acyclic.**

And **a graph is acyclic if and only if there are no back edges.**

Solution

1. Run a DFS on Dependency Graph;
2. Maintain a list T for Topological Order;
3. Upon finishing the visit to a node, insert the node to the front of the list;
4. Exit abnormally, if encountered a back edge.

If exited abnormally, the graph is acyclic; and the programme can not be completed successfully.

Otherwise, T contains an order of courses that successfully completes the programme.

In figure DFS on Dependency Graph, (Fig 3) nodes have been mentioned with discovery and finish times; and edges has been labelled as B,C,F,T for back edges, cross edges, forward edges and tree edges respectively.

The algorithm terminated upon visiting the edge $7 \to 5$ which is a back edge (labelled B).

Hence the programme can not be completed.

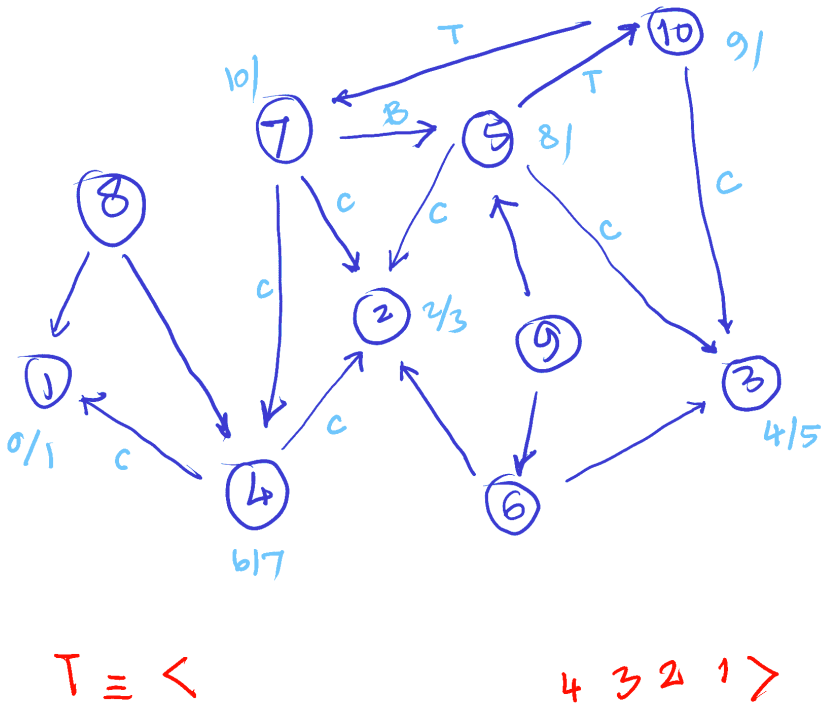


Figure 3: DFS on Dependency Graph