# Chapter 7

# Projective Geometry and Camera Models

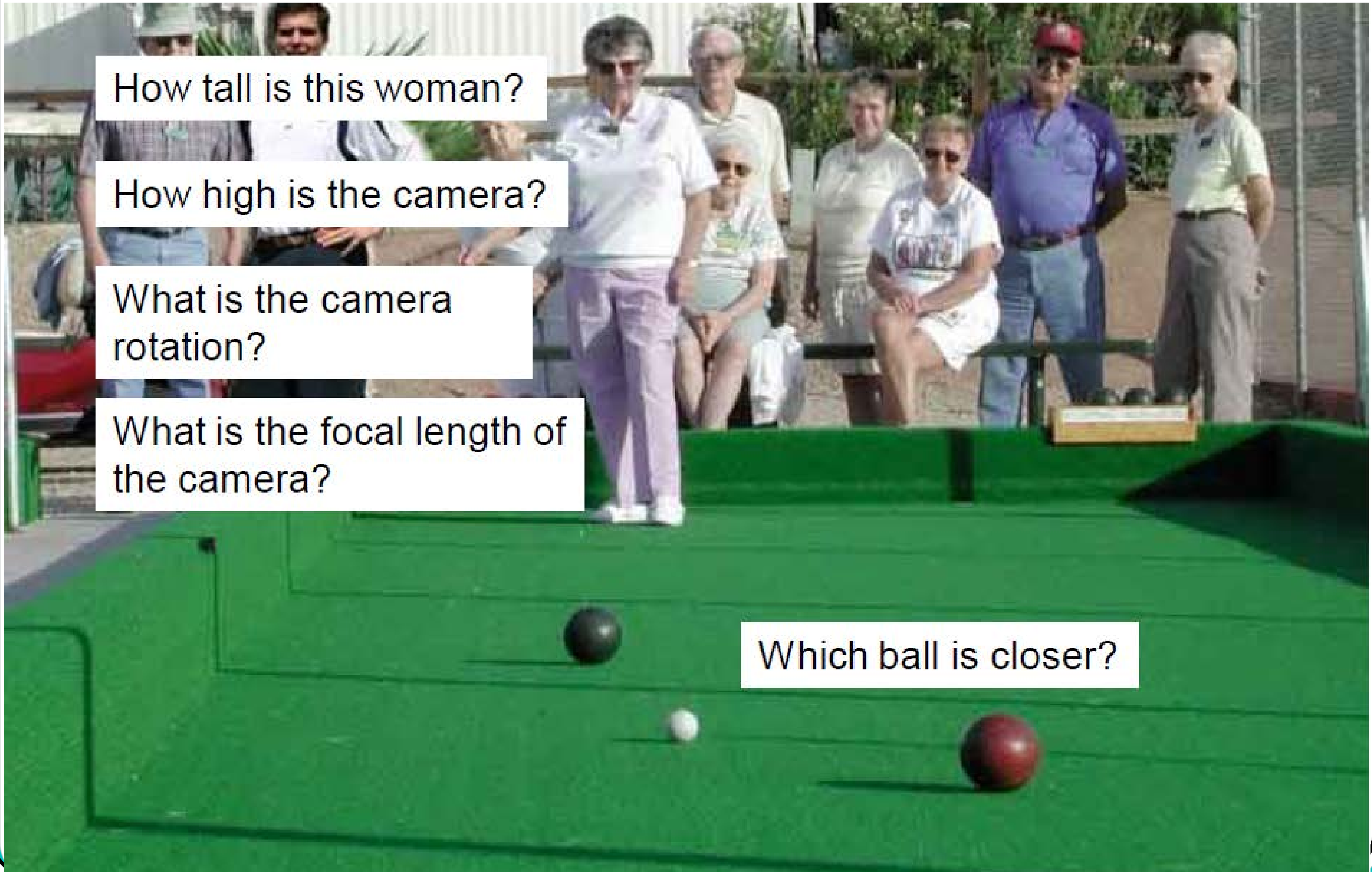
***James Hays, Brown University***

# Contents

Mapping between image and world coordinates

- Projective geometry
  - Vanishing points and lines
- Pinhole camera model
- Cameras & lenses
- Projection matrix

# Camera and World Geometry

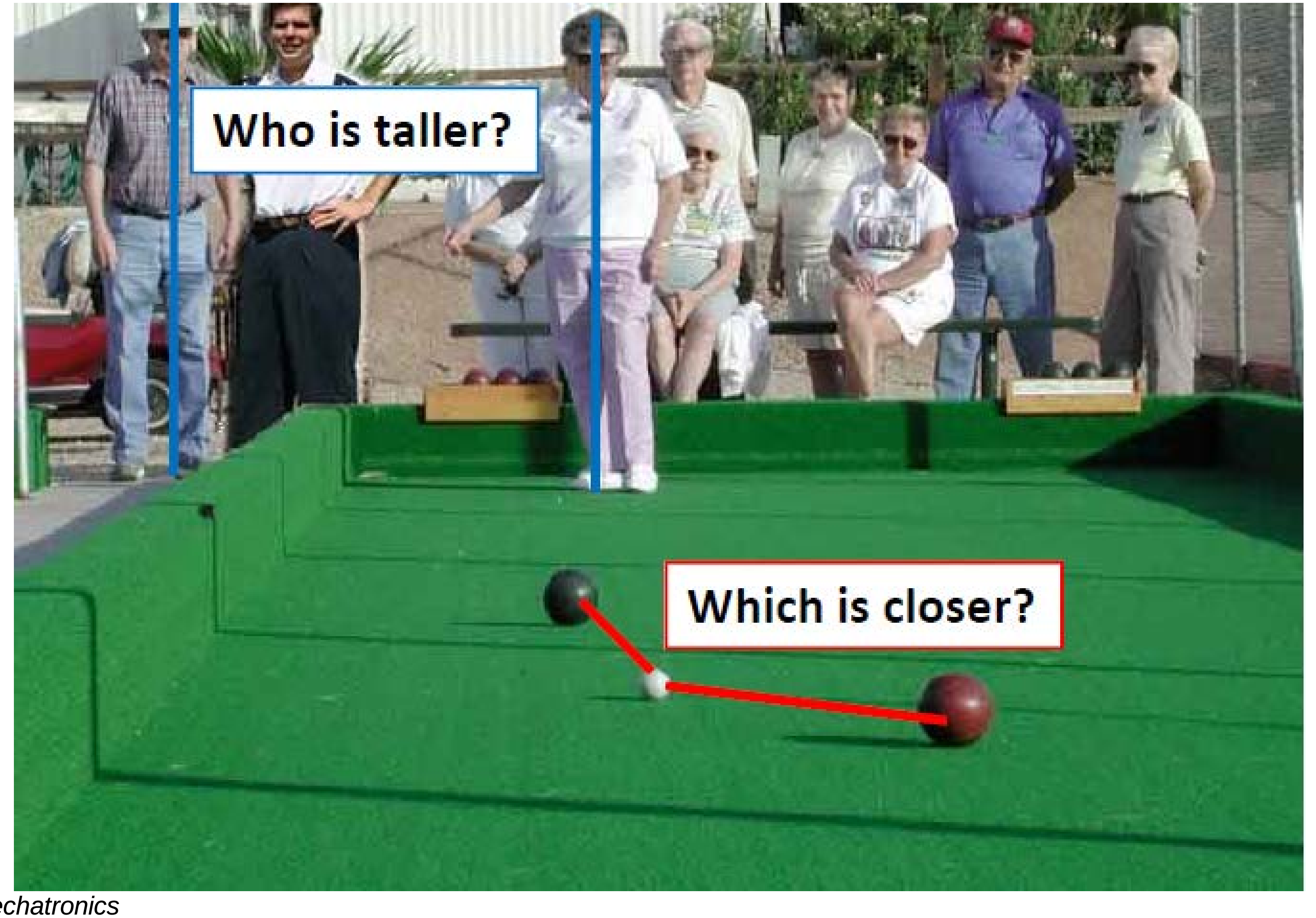


# Projection can be tricky…

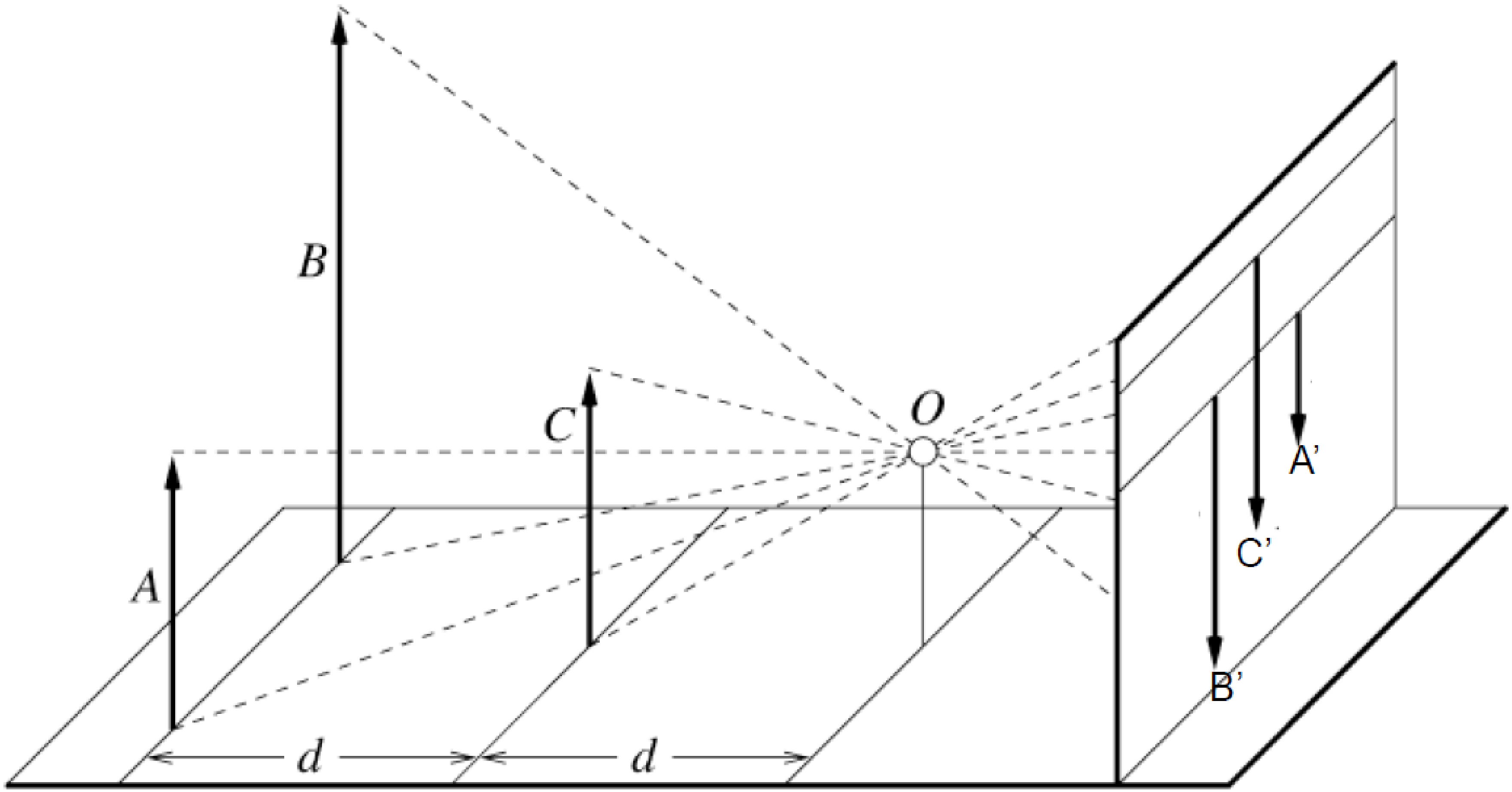
# Projection can be tricky…

# Projective Geometry

- What is lost?
  - Length

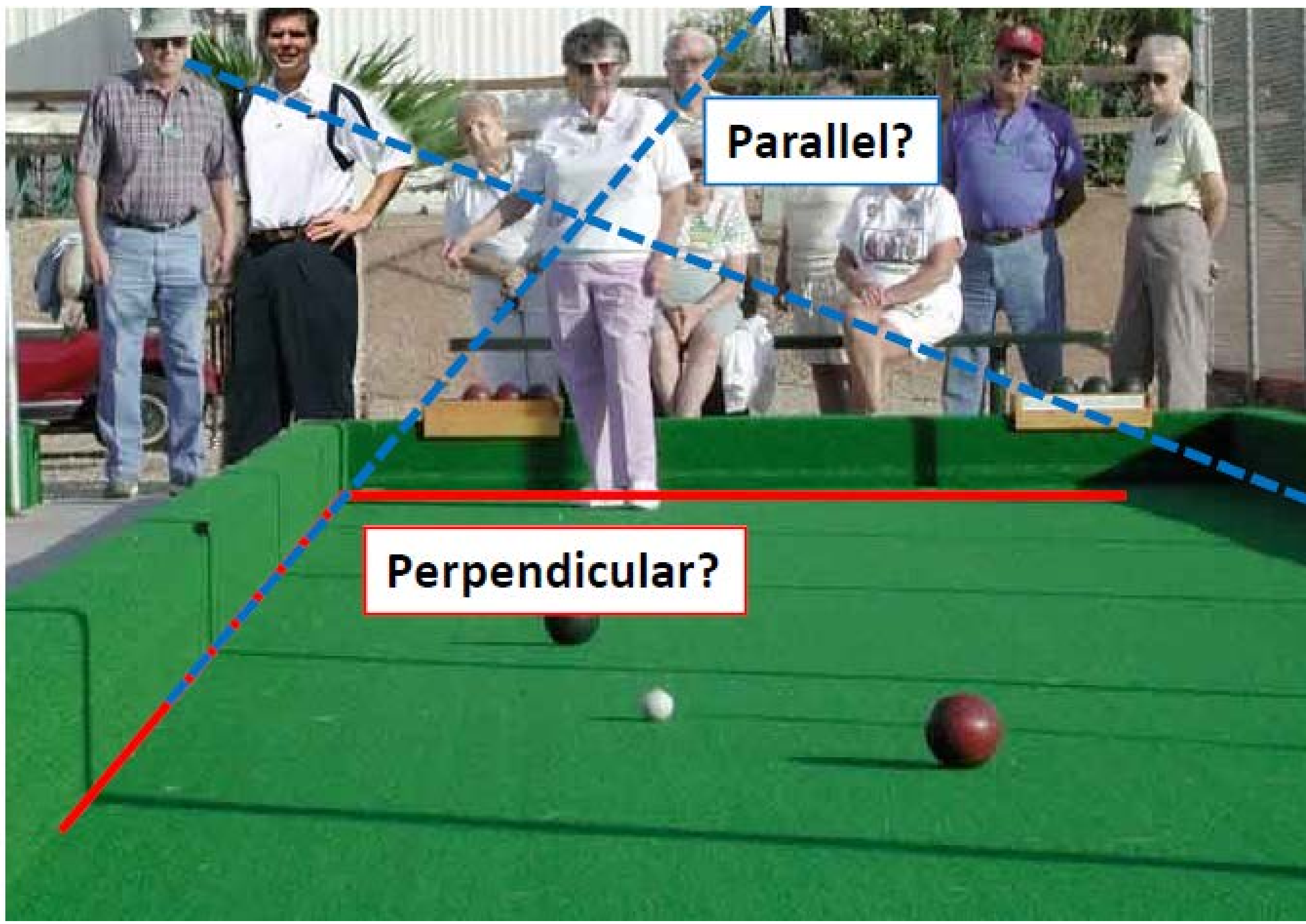


# Length is not preserved



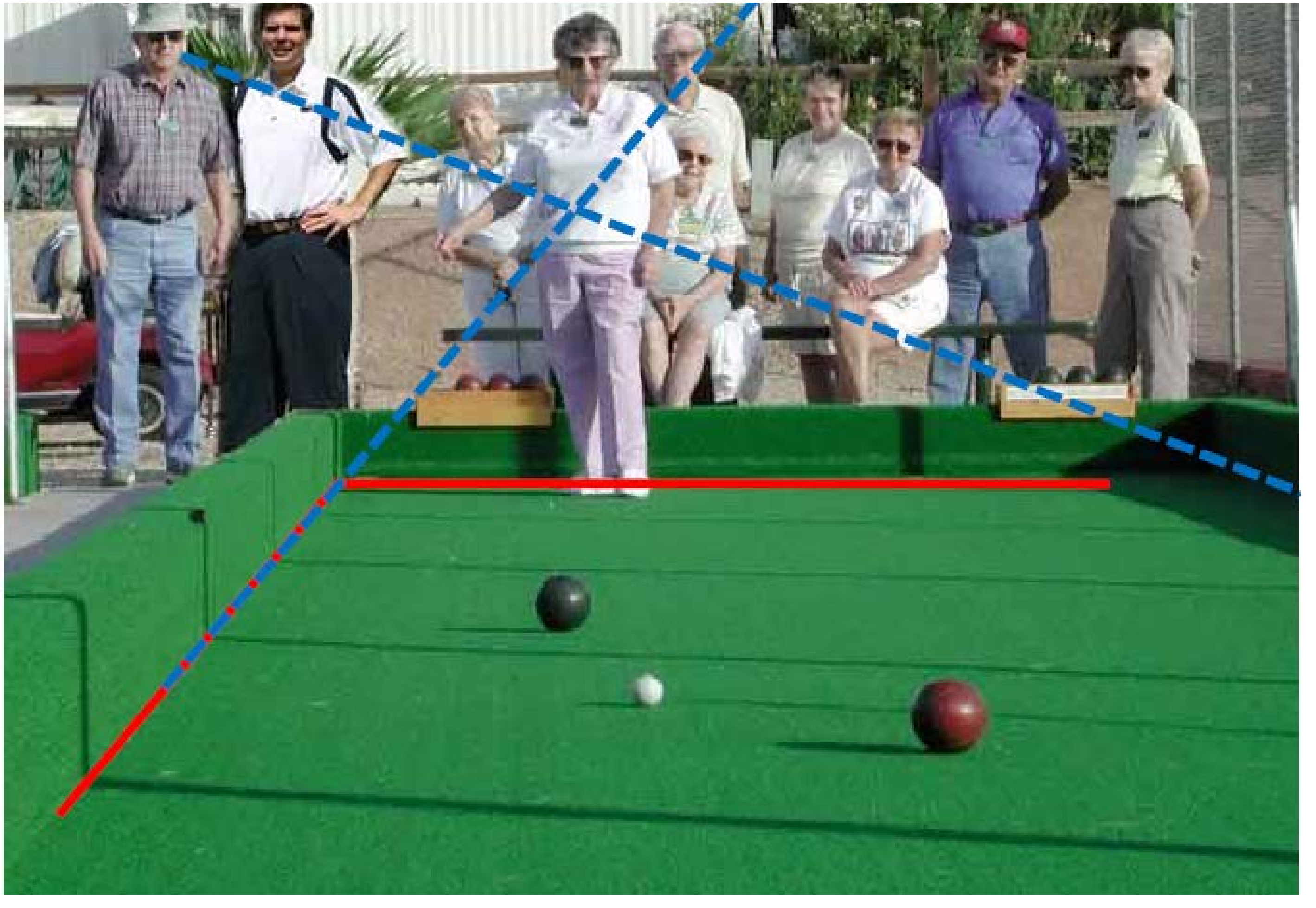
# Projective Geometry

- What is lost?
  - Length
  - Angles



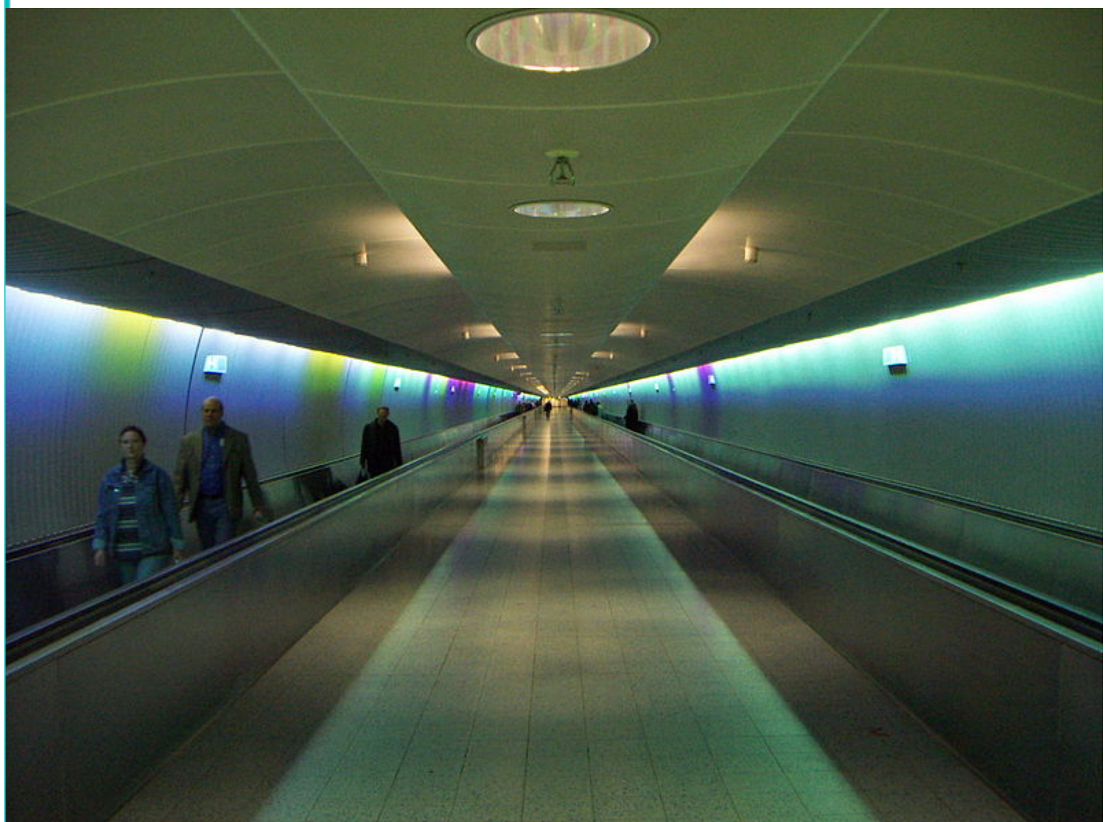
# Projective Geometry

- What is preserved?
  - Straight lines are still straight

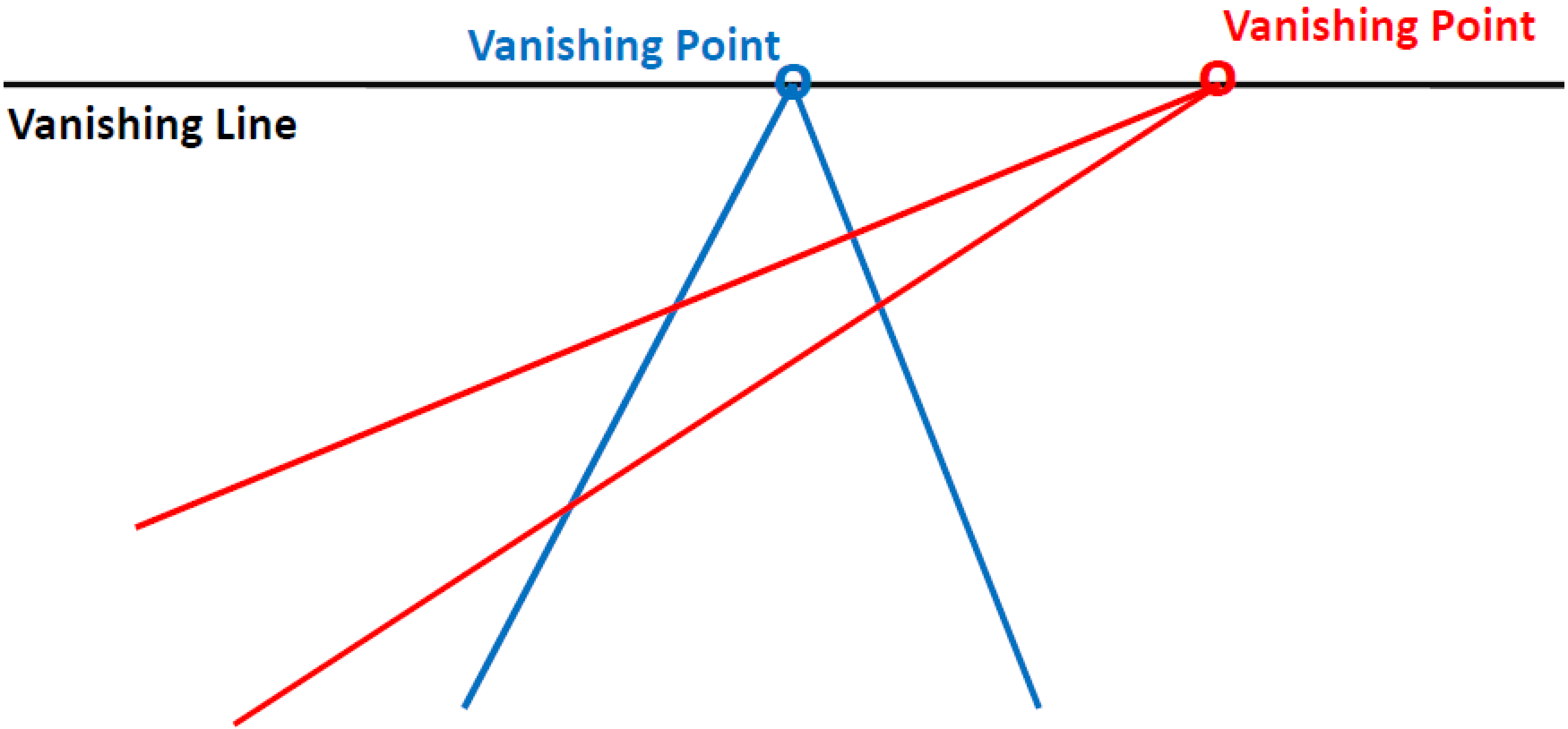


# Vanishing points and lines

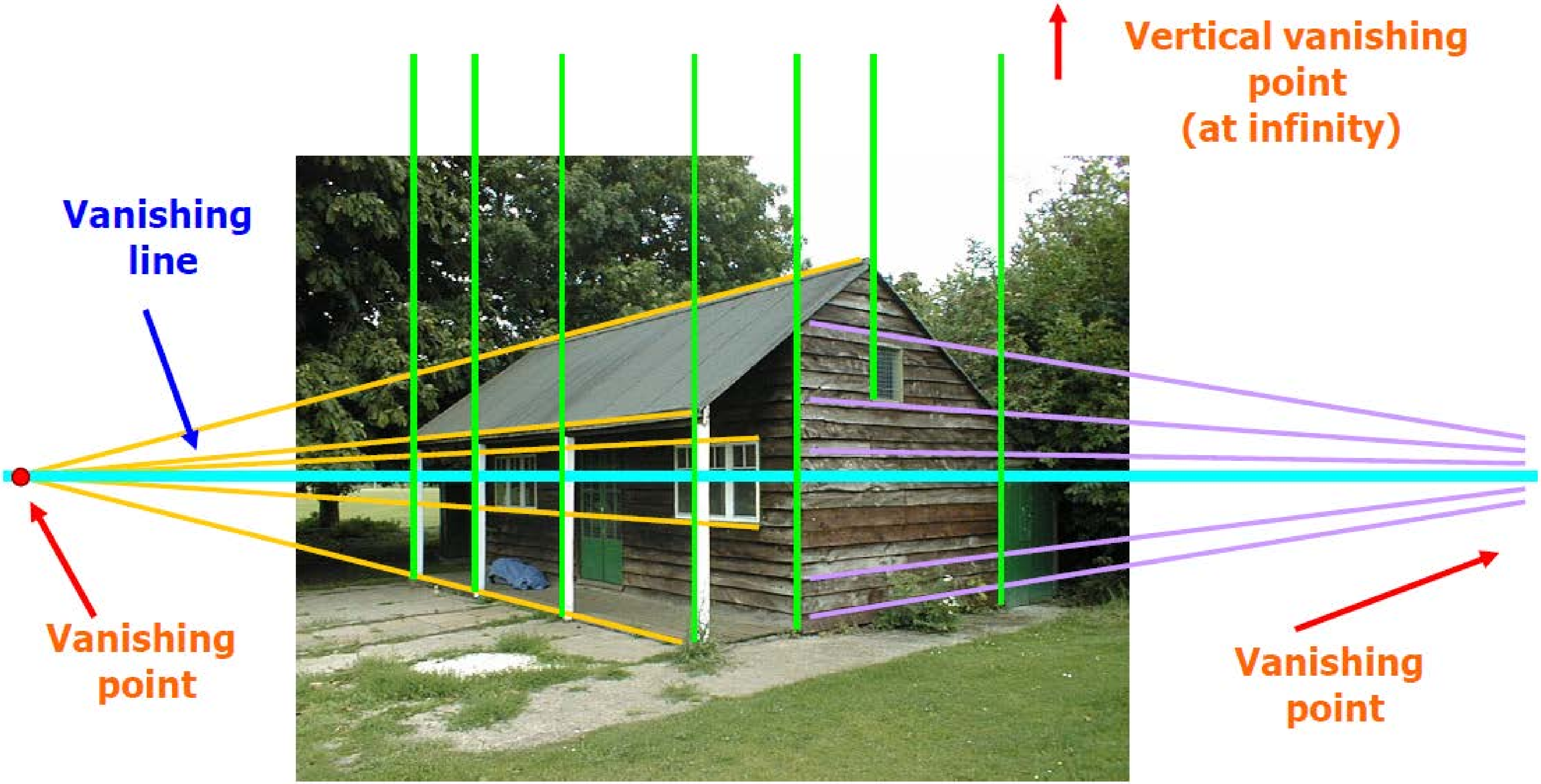
- Parallel lines in the world intersect in the image at a "vanishing point".



# Vanishing points and lines

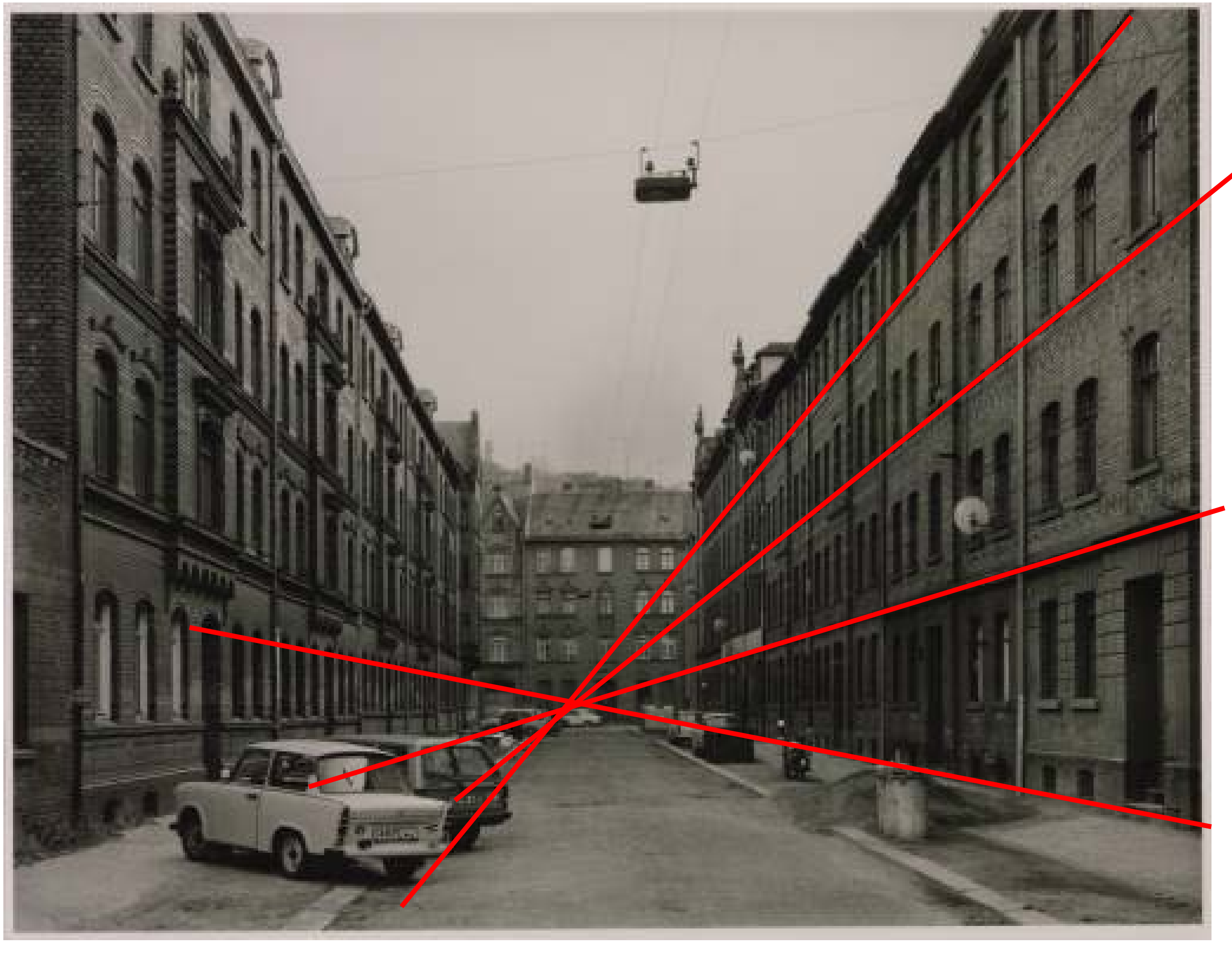


# Vanishing points and lines

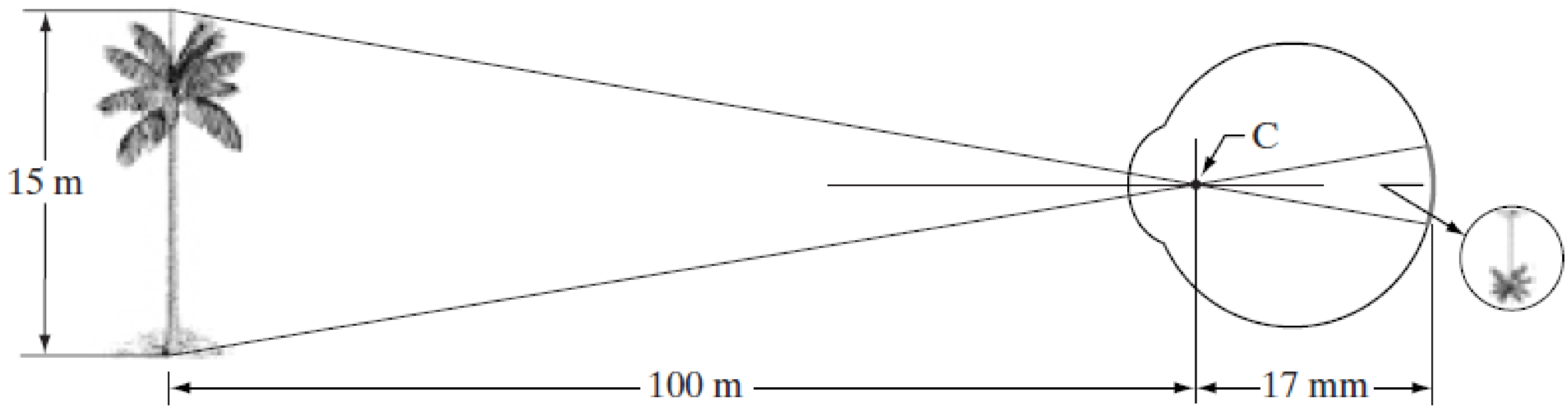


# Vanishing points and lines

# Note on estimating vanishing points

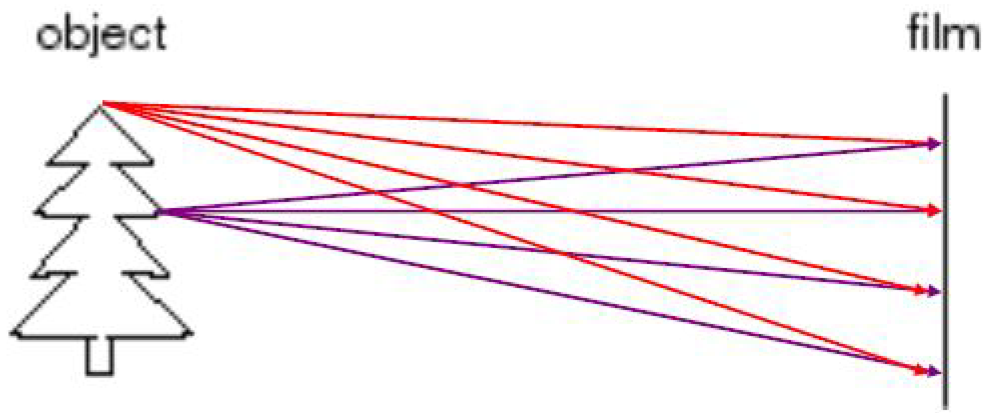


# How do we see the world



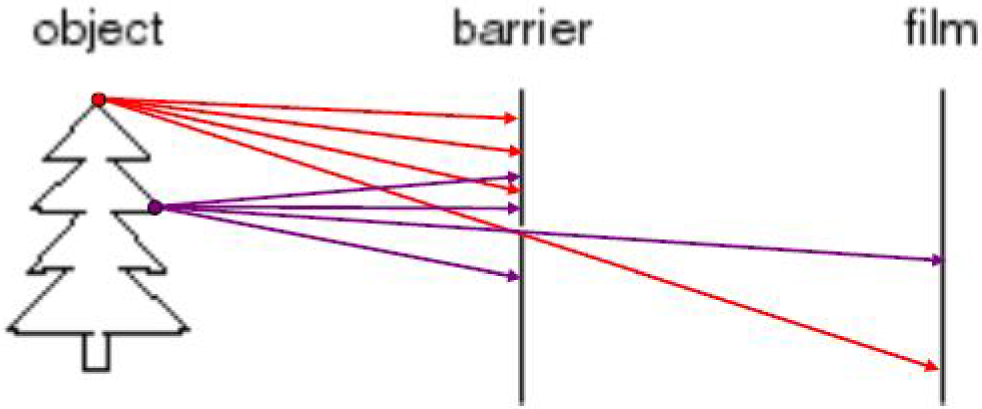

***Graphical representation of the eye looking at a palm tree. Point C is the optical center of the lens.***

# How do we see the world?




- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?

# Pinhole camera




- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture**

# Camera obscura: the pre-camera

- Known during classical period in China and Greece (e.g. Mo-Ti, China, 470BC to 390BC)

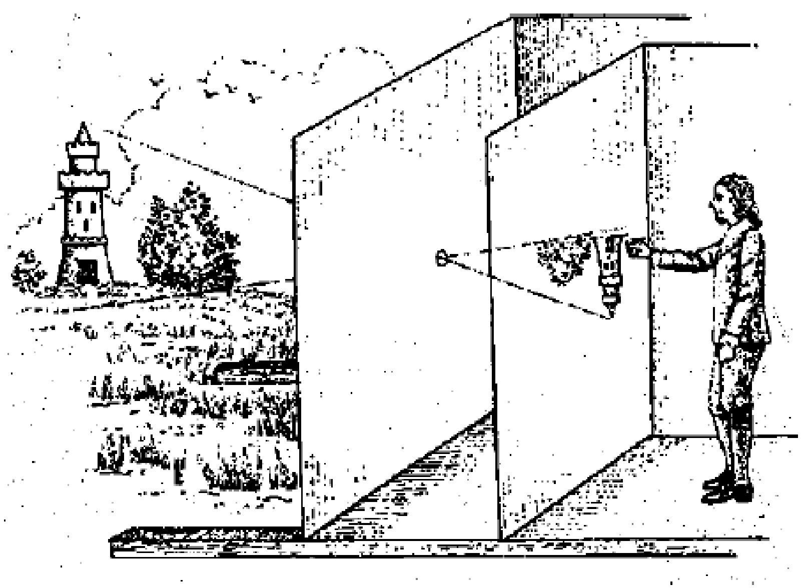


Illustration of Camera Obscura

Freestanding camera obscura at UNC Chapel Hill

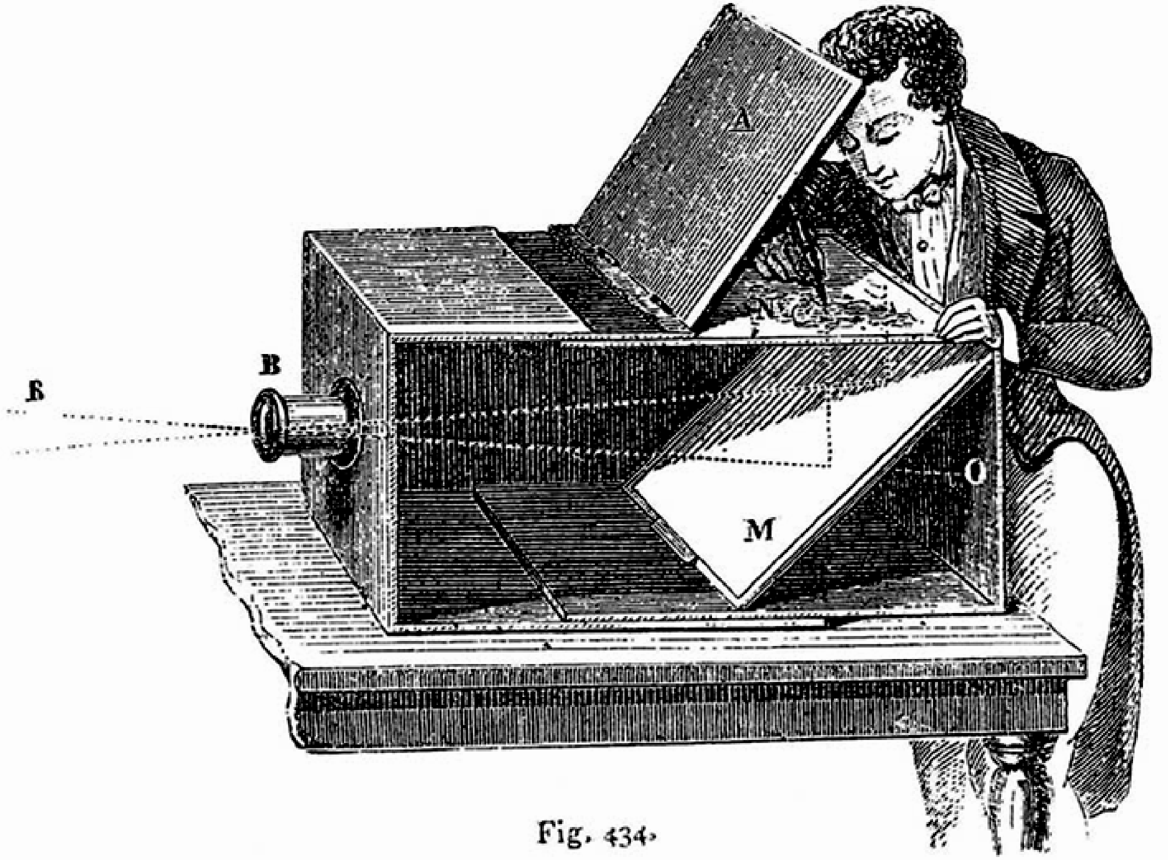
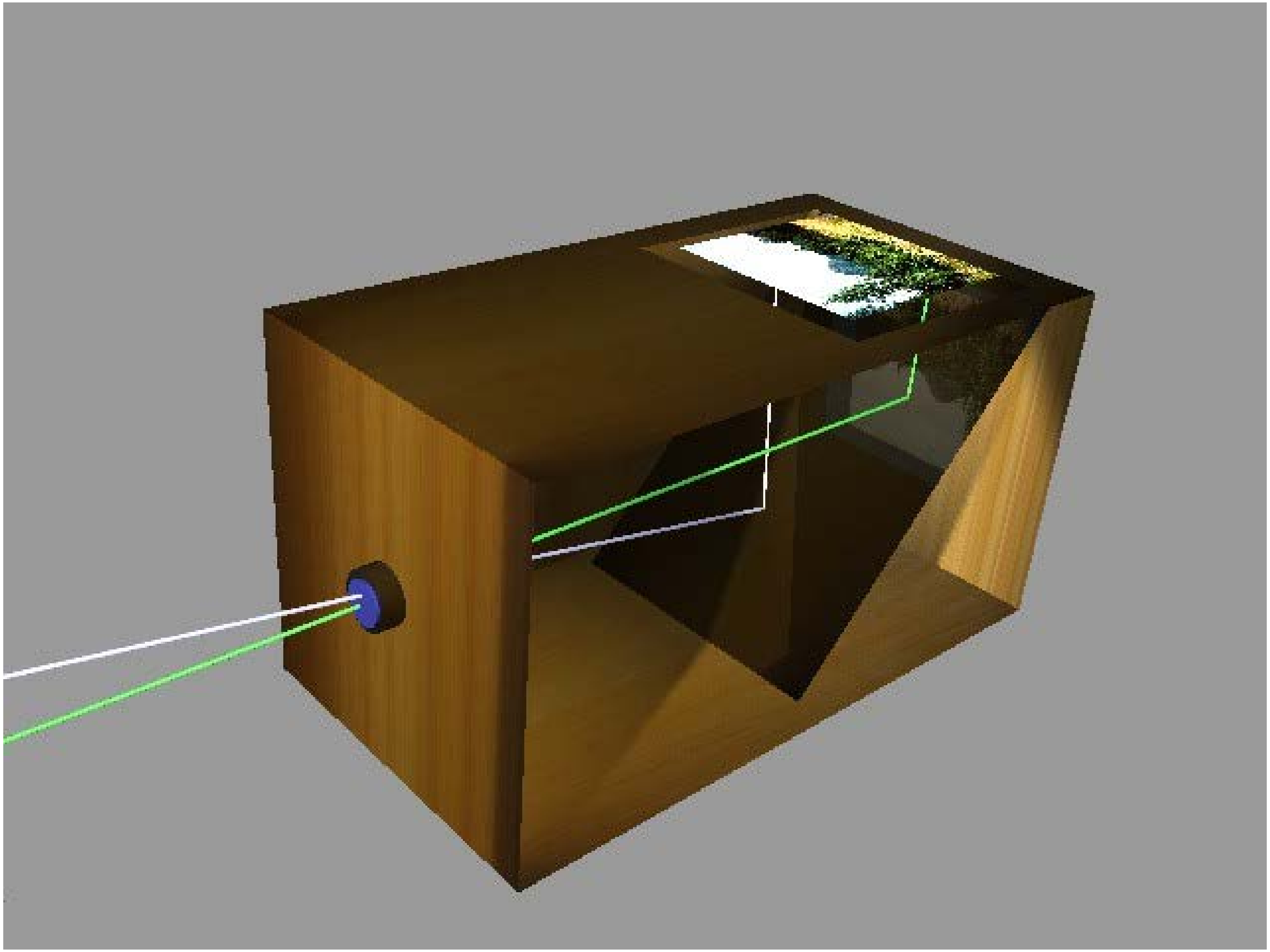
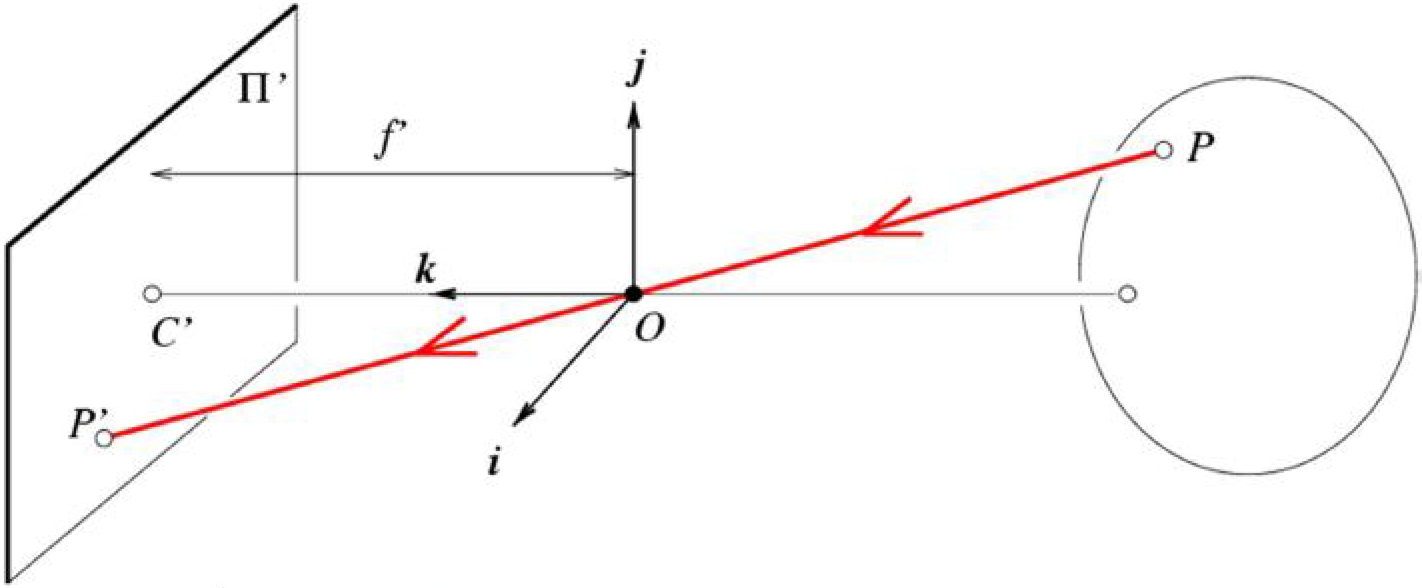


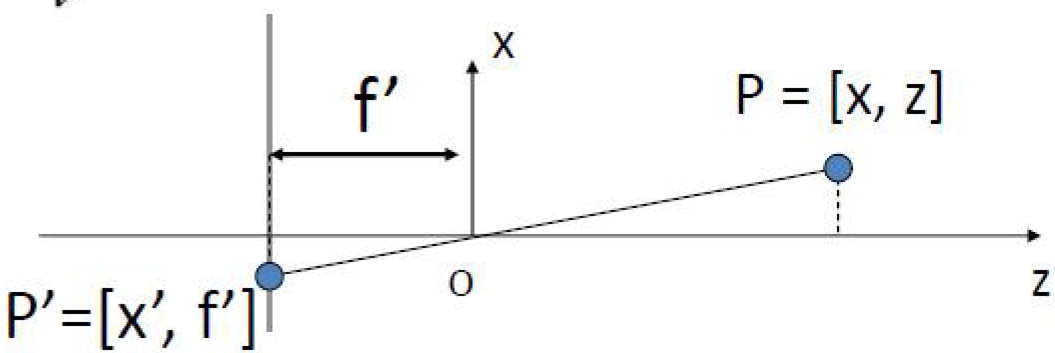

Fig. 434.

*Lens Based Camera Obscura, 1568*



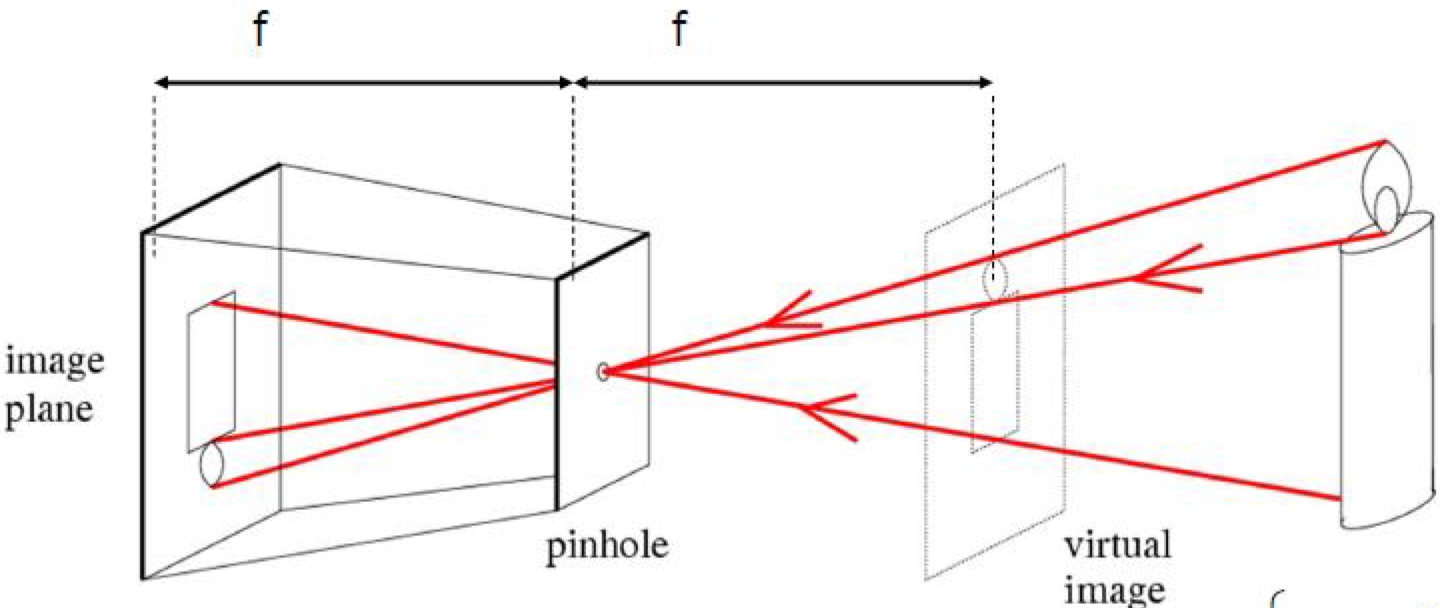
# Pinhole camera




$$\frac{x'}{f'} = \frac{x}{z}$$

# Pinhole camera

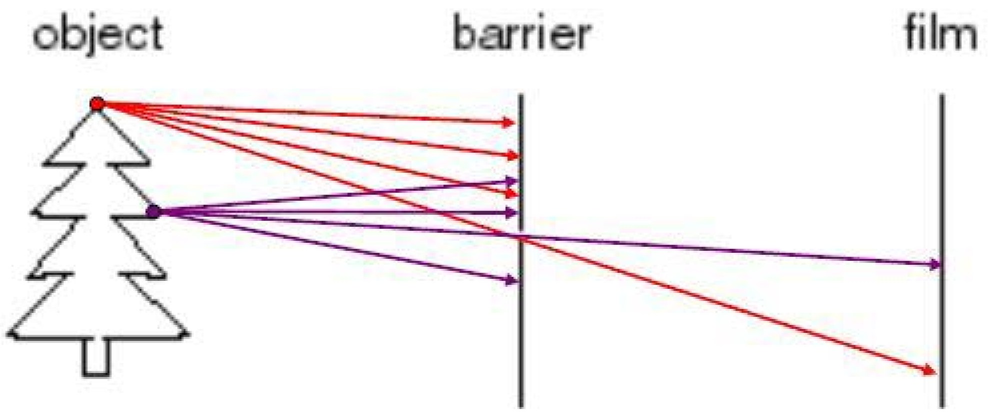



$$\begin{cases} x' = f\,\dfrac{x}{z} \\ y' = f\,\dfrac{y}{z} \end{cases}$$

- Common to draw image plane *in front* of the focal point
- Moving the image plane merely scales the image.

# Pinhole camera

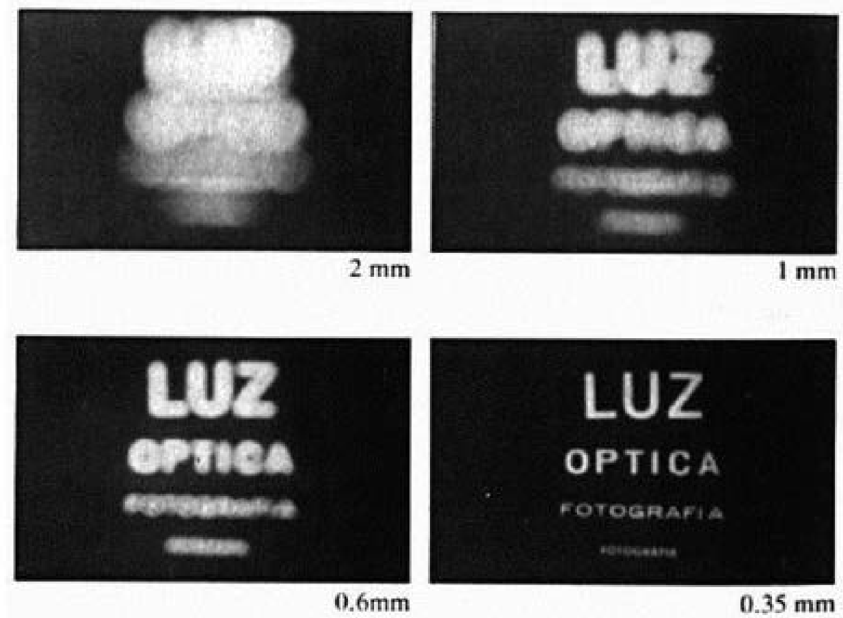
Is the size of the aperture important?




# Cameras & Lenses

Shrinking aperture size

- Rays are mixed up

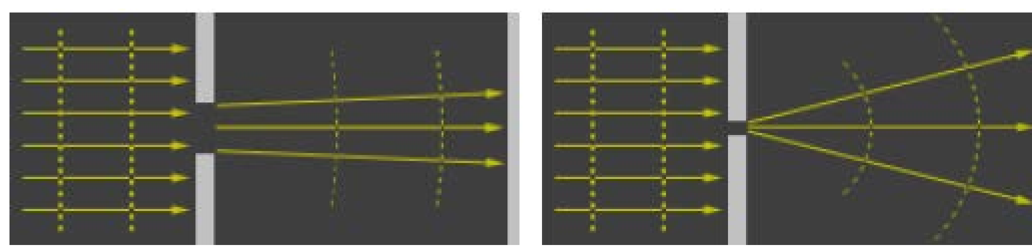



-Why the aperture cannot be too small?

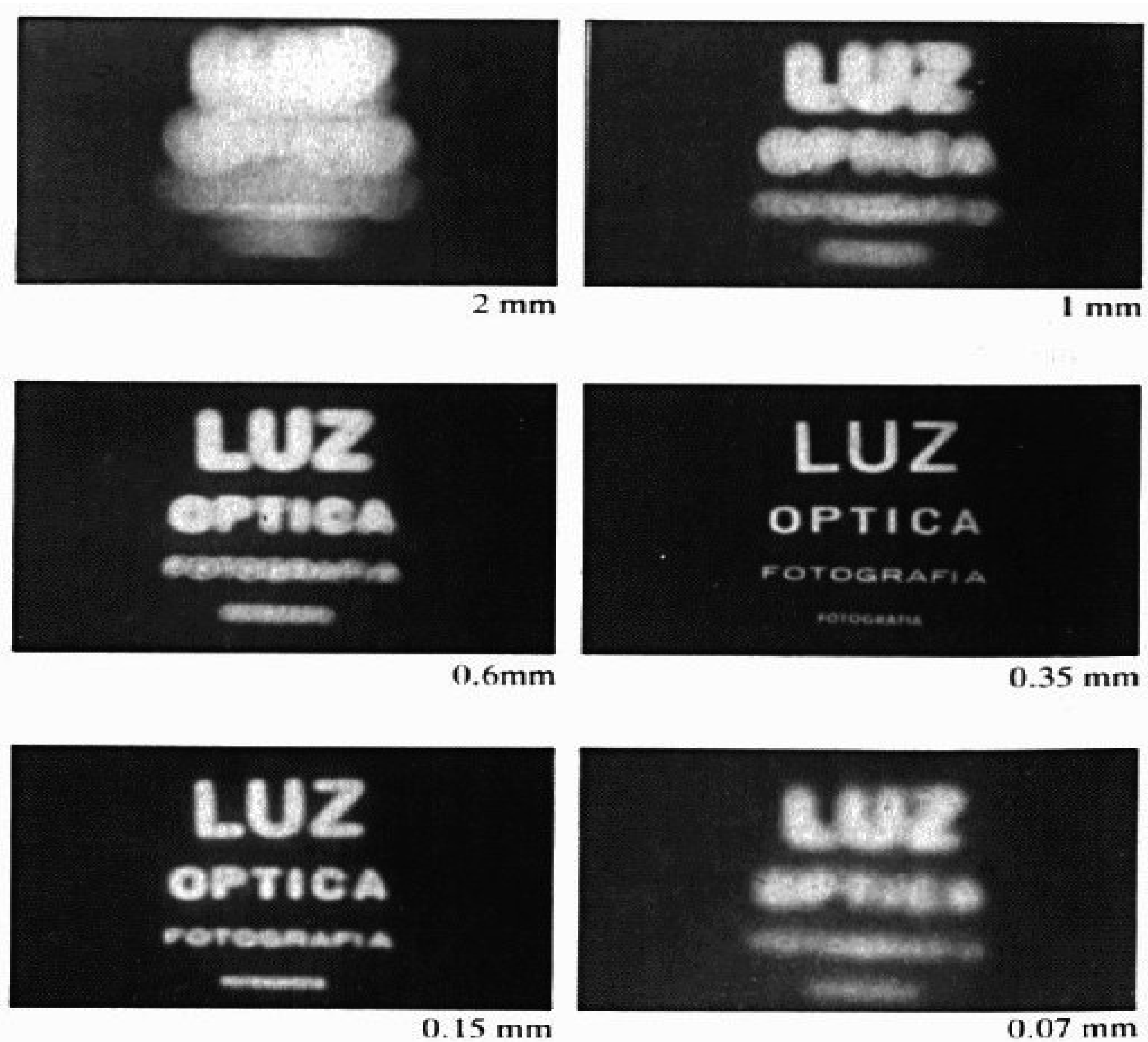
-Less light passes through

-Diffraction effect

*Parallel light rays which pass through a small aperture begin to diverge and interfere with one another. This becomes more significant as the size of the aperture decreases relative to the wavelength of light passing through, but occurs to some extent for any size of aperture or concentrated light source.*



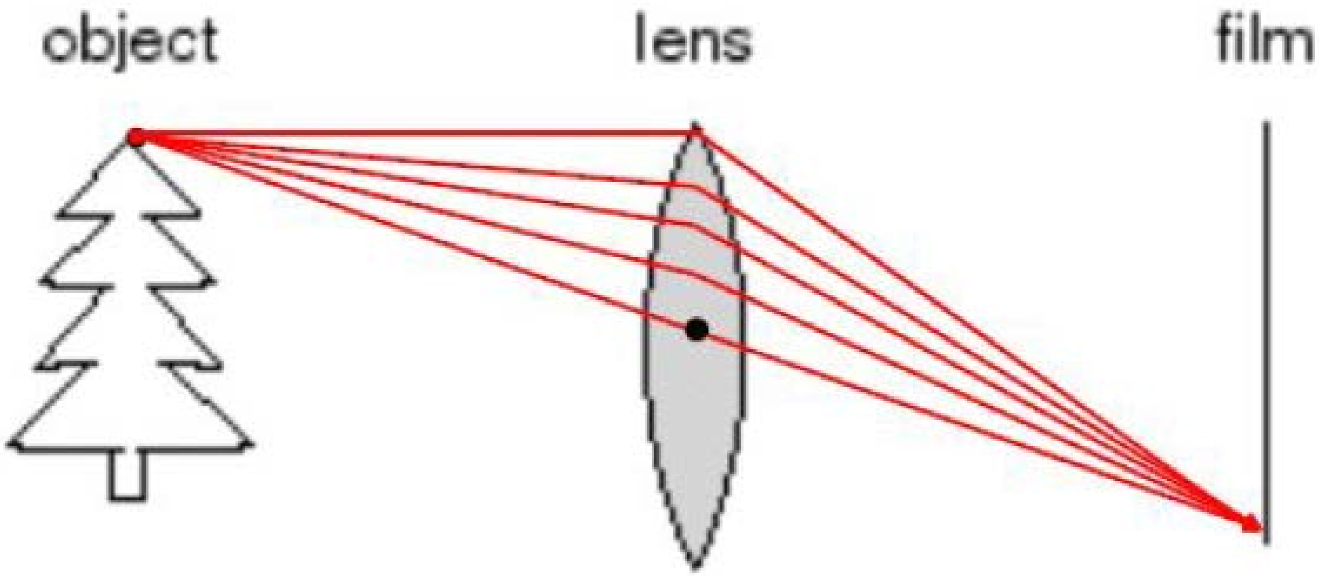
# Cameras and Lenses




**Adding Lenses!**

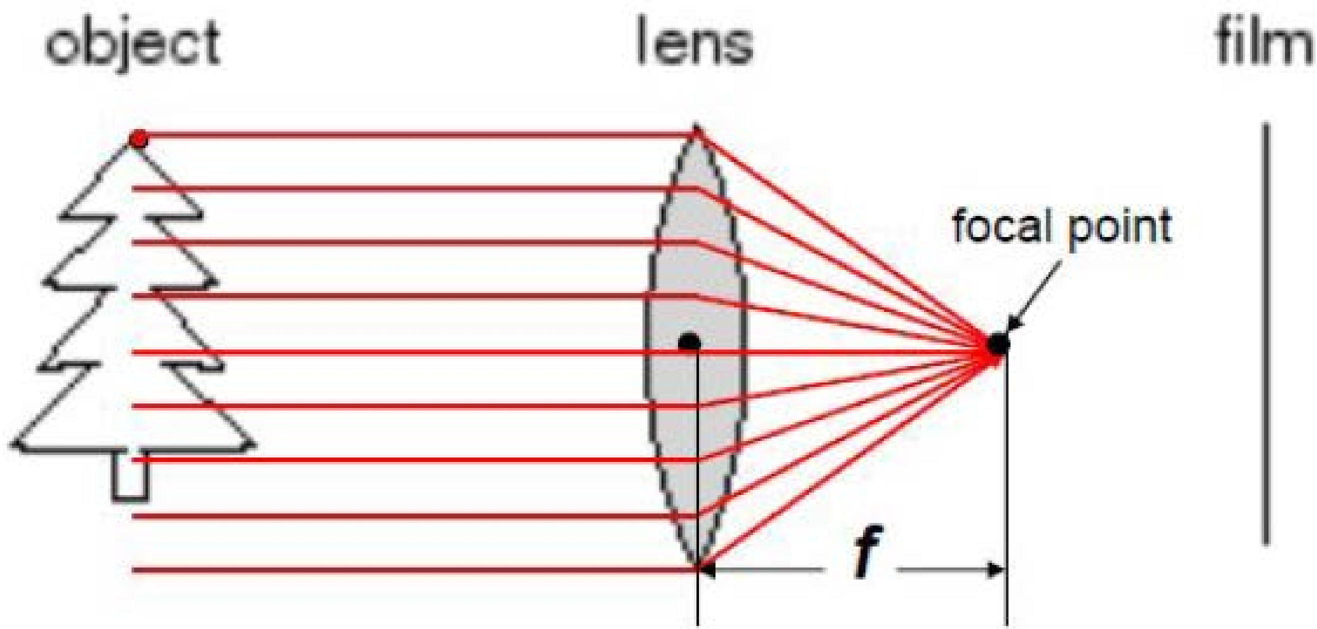
# Cameras and Lenses

- **A lens focuses light onto the film**



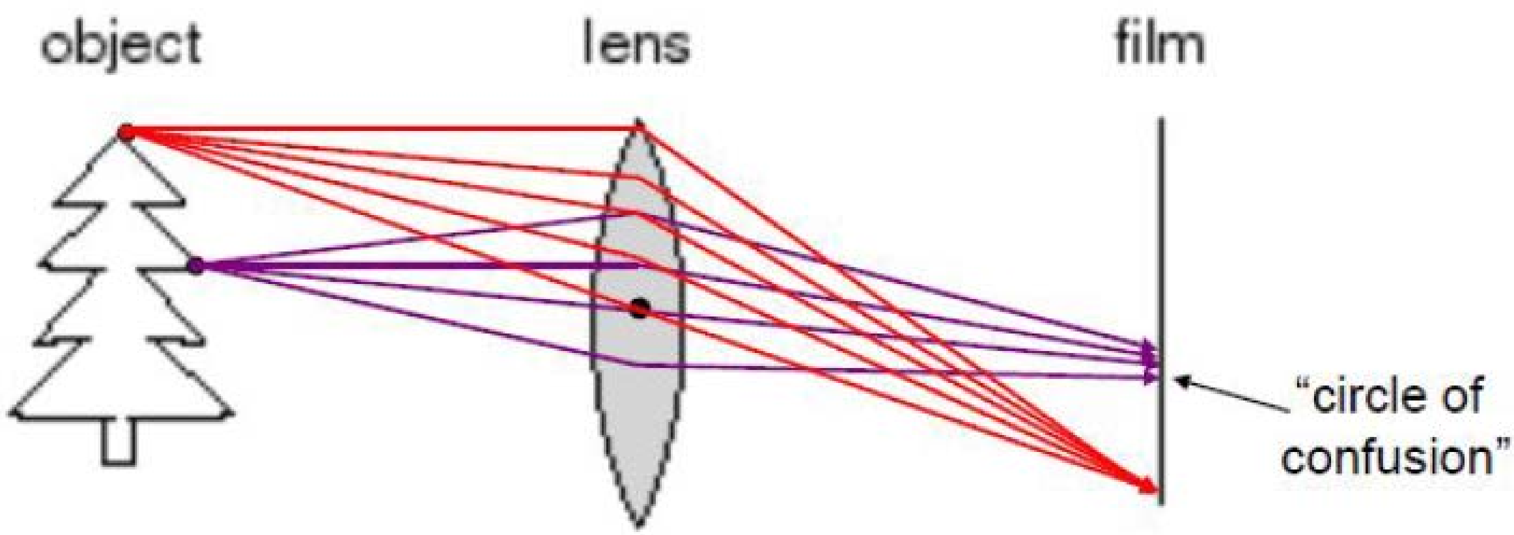
# Cameras and Lenses

- **A lens focuses light onto the film**
  - **Rays passing through the center are not deviated.**
  - **All parallel rays converge to one point on a plane located at the *focal length f*.**



# Cameras and Lenses

- **A lens focuses light onto the film**
  - **There is a specific distance at which objects are "in focus" [other points project to a "circle of confusion" in the image].**



*In optics the refractive index or index of refraction of a substance or medium is a measure of the speed of light in that medium*

***n*** **= speed of light in a vacuum / speed of light in medium**

***http://en.wikipedia.org/wiki/Refractive_index#Typical_values***

# Cameras and Lenses

- **Laws of geometric optics:**
  - **Light travels in straight lines in homogeneous medium.**
  - **Reflection upon a surface: incoming ray, surface normal, and reflection are co-planar.**
  - **Refraction: when a ray passes from one medium to another.**

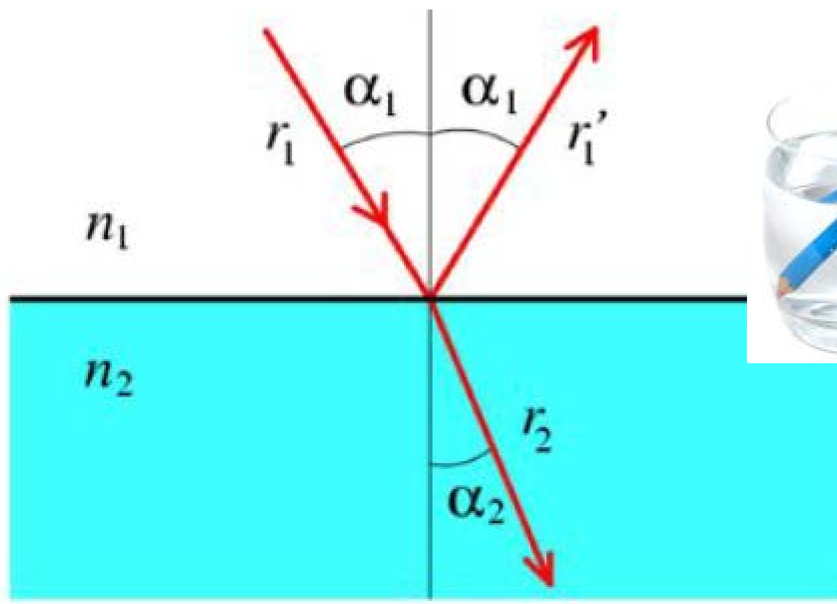
Snell's law

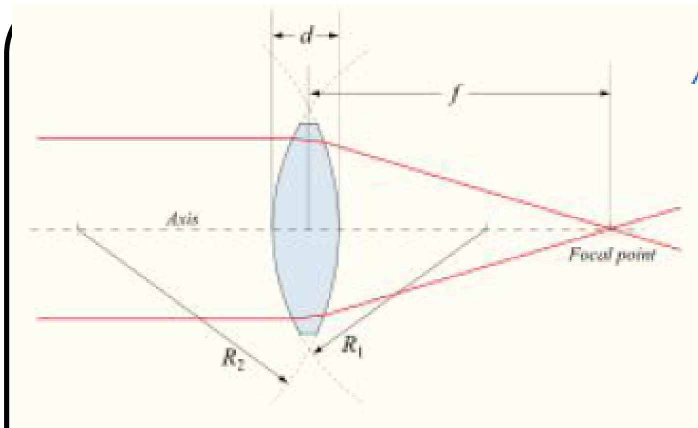
$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

$\alpha_1$ = incident angle

$\alpha_2$ = refraction angle

$n_i$ = index of refraction



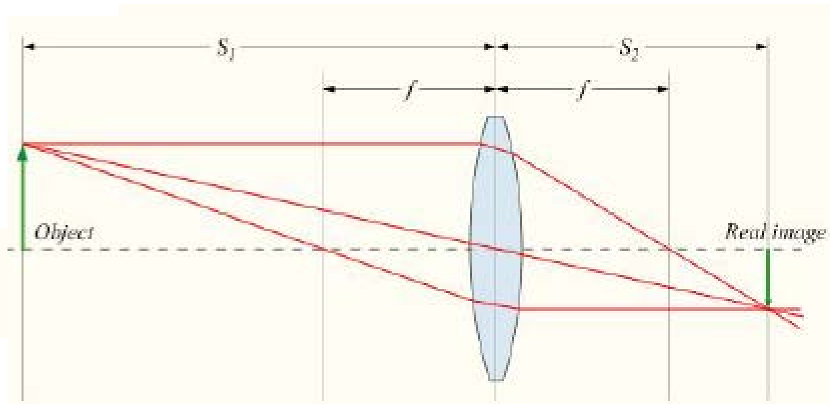

*Positive (converging) lens*

*A lens can be considered a **thin lens** if d << f.*

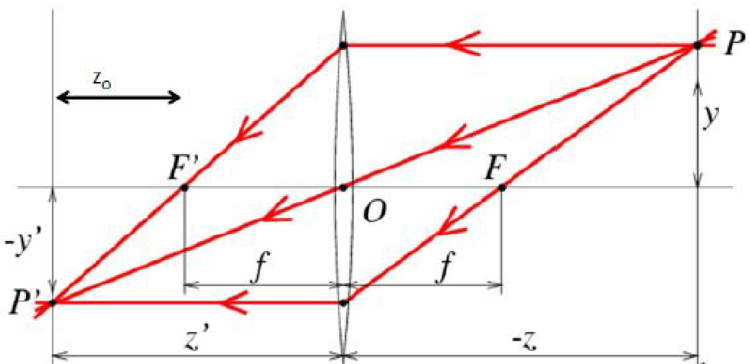
# Thin lens equation

***If d is small compared to $R_1$ and $R_2$, then the thin lens approximation can be made. For a lens in air, f is then given by***

$$\frac{1}{f} \approx (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$




# Thin Lenses




$$z' = f + z_o$$

$$f = \frac{R}{2(n-1)}$$

Snell's law:

$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$ ⇒

Small angles:

$n_1 \alpha_1 \approx n_2 \alpha_2$

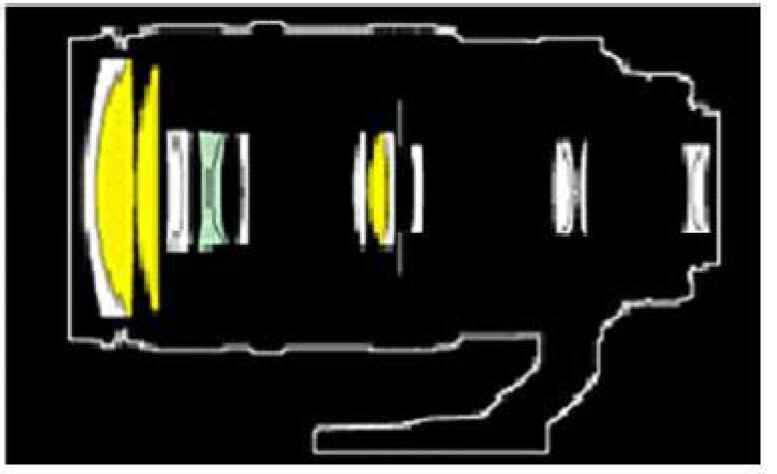
$n_1 = \text{n (lens)}$

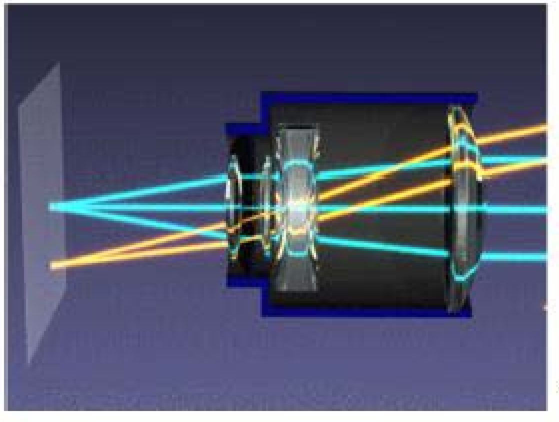
$n_1 = 1 \text{ (air)}$

⇒

$$\begin{cases} x' = z' \dfrac{x}{z} \\ y' = z' \dfrac{y}{z} \end{cases}$$

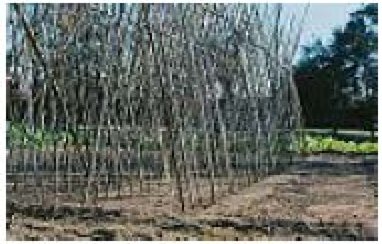
# Cameras & Lenses





Source wikipedia

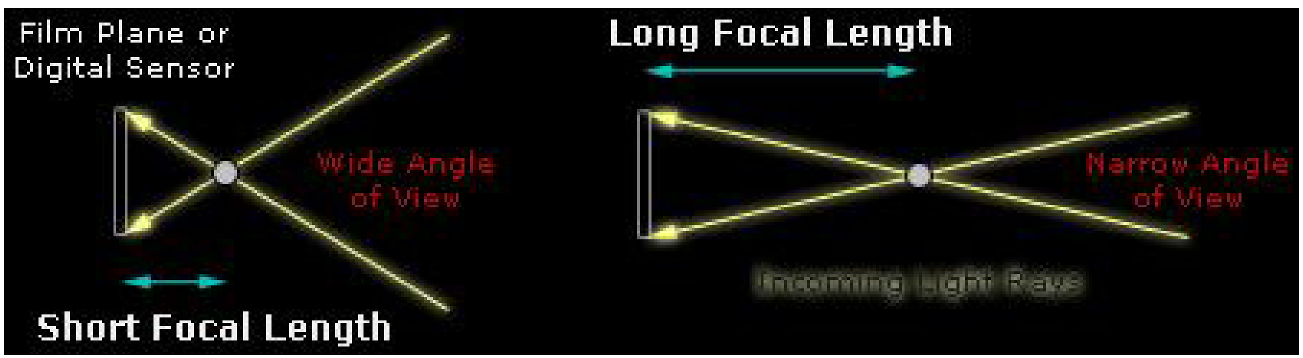
28 mm lens



50 mm lens

70 mm lens

210 mm lens

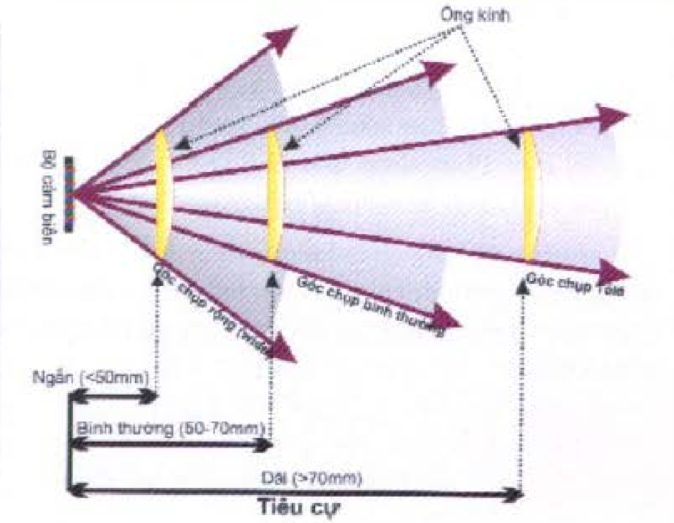



***Lens is described by its focal length, which is the distance in millimeters (mm) between the lens and the image it forms on the sensor or film*** http://photography-cameras.org/camera-lenses/focal-length-of-camera-lens

*Tiêu cự của máy ảnh là thông số cho biết góc nhìn của máy ảnh, nghĩa là khoảng phạm vi mà máy ảnh có thể "thâu tóm" được*

http://www.chuphinh.vn/may-chup-hinh-ky-thuat-so/cac-yeu-to-ky-thuat/75-tieu-cu.html

Hình 13: Minh họa phạm vi thu hình của các góc chụp

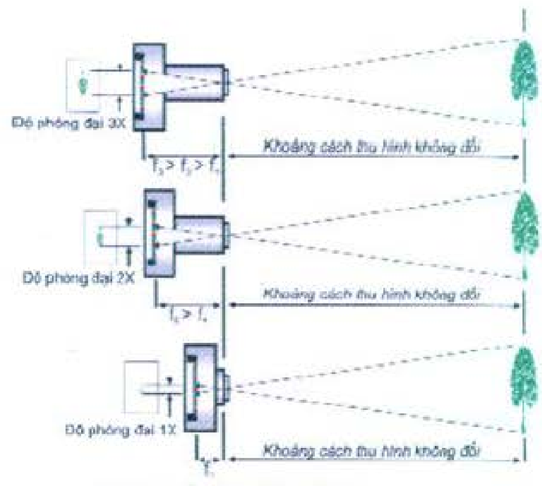



Hình 14: Máy ảnh Sony DSC-S750

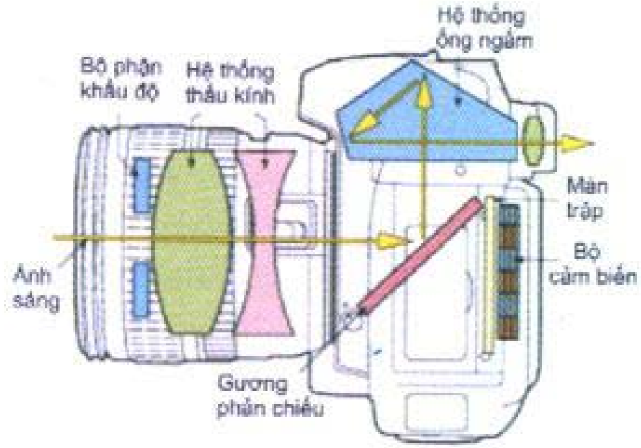

*f = 5.8 – 17.4 mm Với thông số này cho biết máy ảnh có khả năng thay đổi tiêu cự từ 35 mm (góc chụp rộng – wide) đến 105 mm (góc chụp – tele).*

*Optical 3X: Lấy 105/35 (hay lấy 17.4/5.8) ta được kết quả là 3 (thông số zoom )*

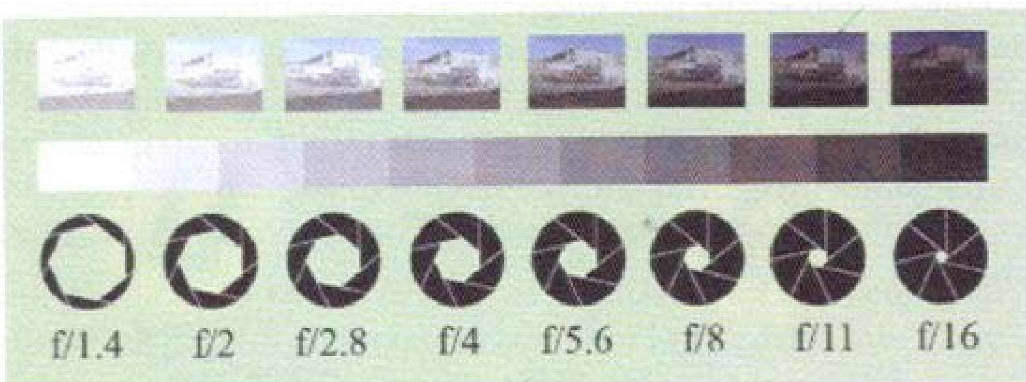
Hình 16: Ảnh hưởng của tiêu cự trên độ lớn của ảnh chụp



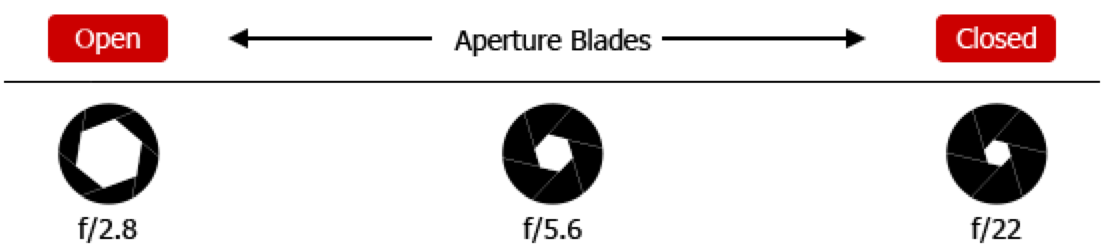
Bộ phận khẩu độ điều tiết lượng ánh sáng đi qua hệ thống thấu kính.

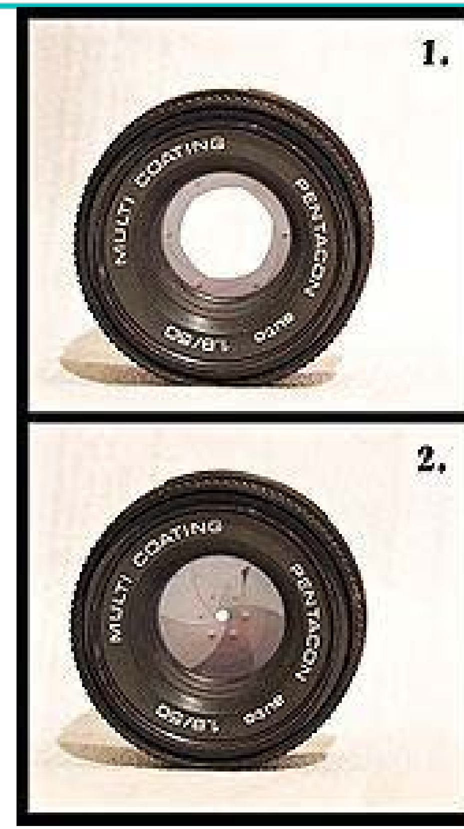



Hình 17: Khẩu độ (Aperture)




Hình 18: Khẩu độ với cường độ ánh sáng tương ứng

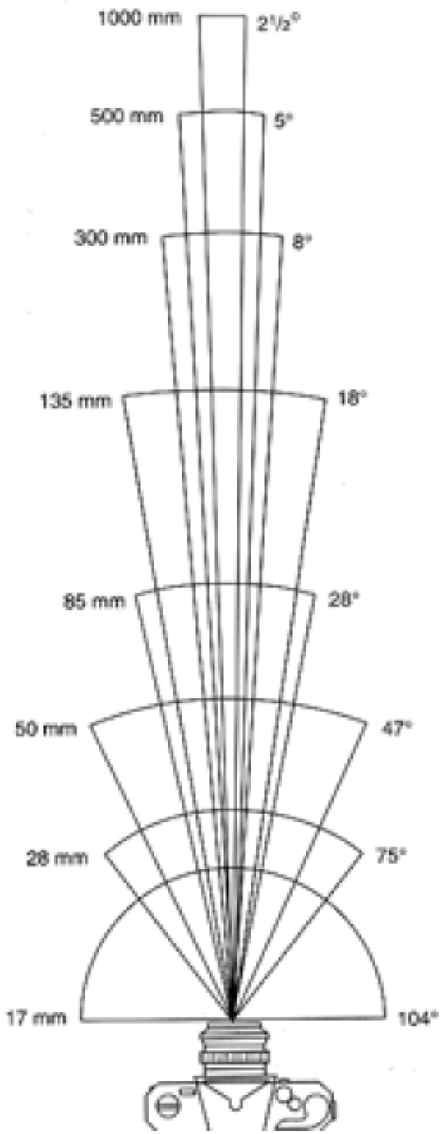



**Large (top) and small (bottom) apertures**

***The aperture is not independent, it must be closely matched to the focal length to get the best lighting effect.***

# Field of View (Zoom, focal length)



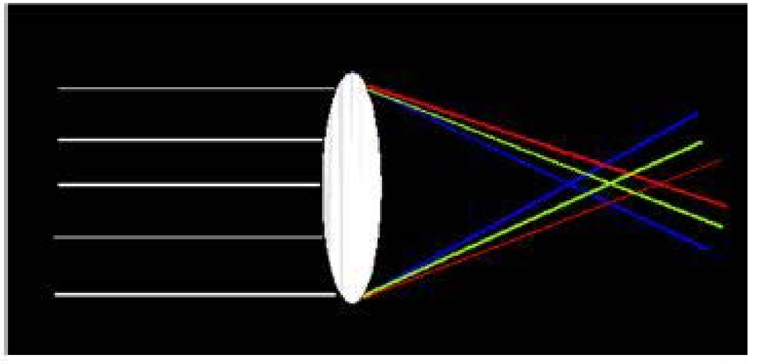


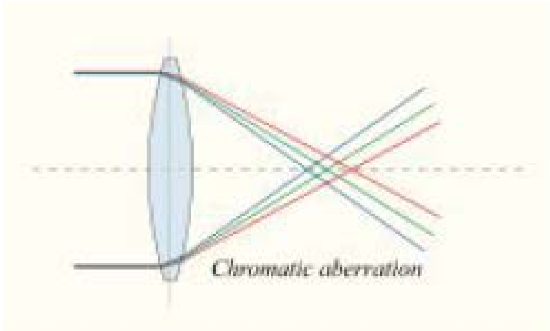
From London and Upton

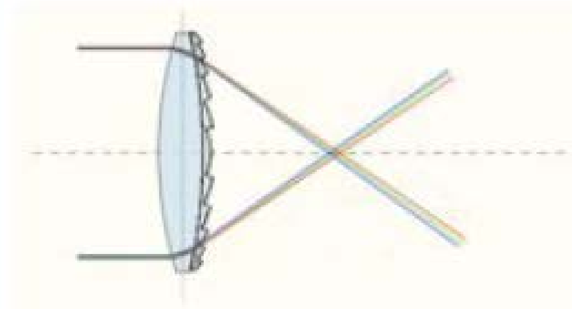
# Issues with lenses: Chromatic Aberration

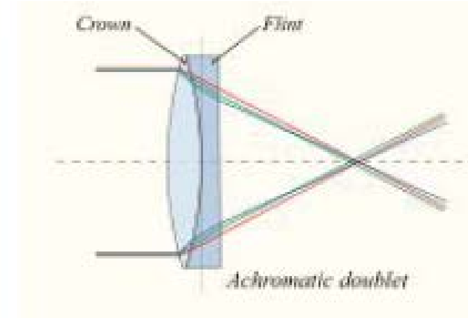
- Lens has different refractive indices for different wavelengths: causes color fringing

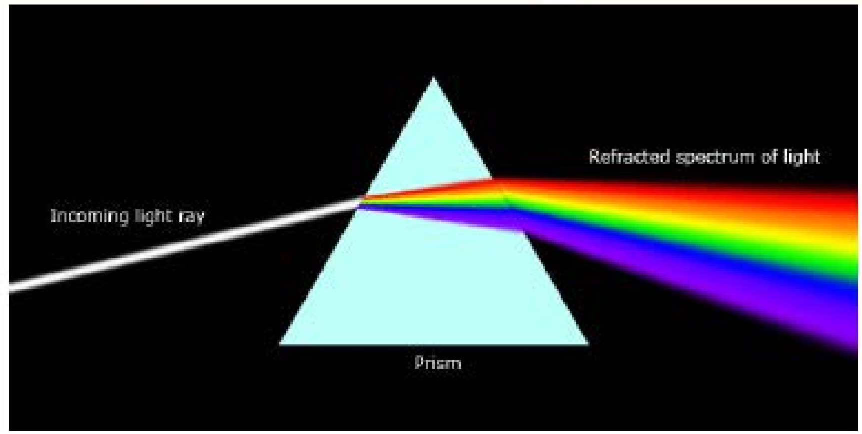
$$f = \frac{R}{2(n-1)}$$

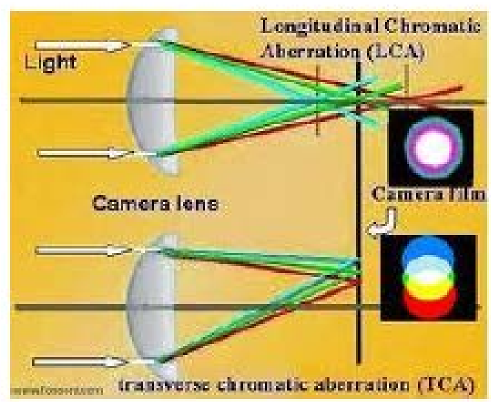



Chromatic aberration




Crown
Flint
Achromatic doublet

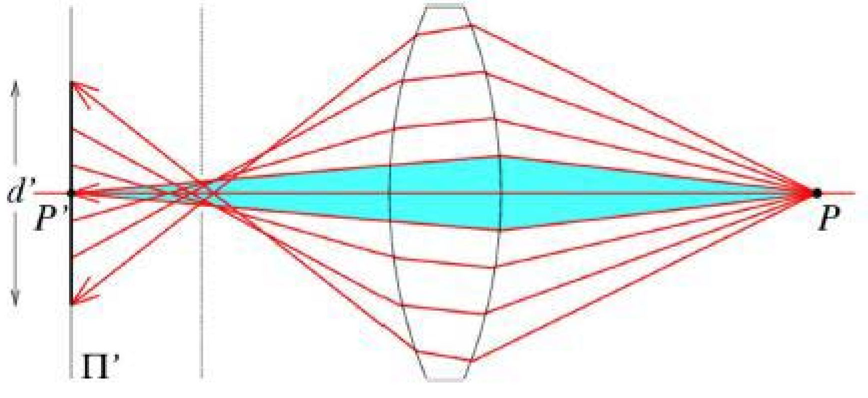

Refracted spectrum of light
Incoming light ray
Prism

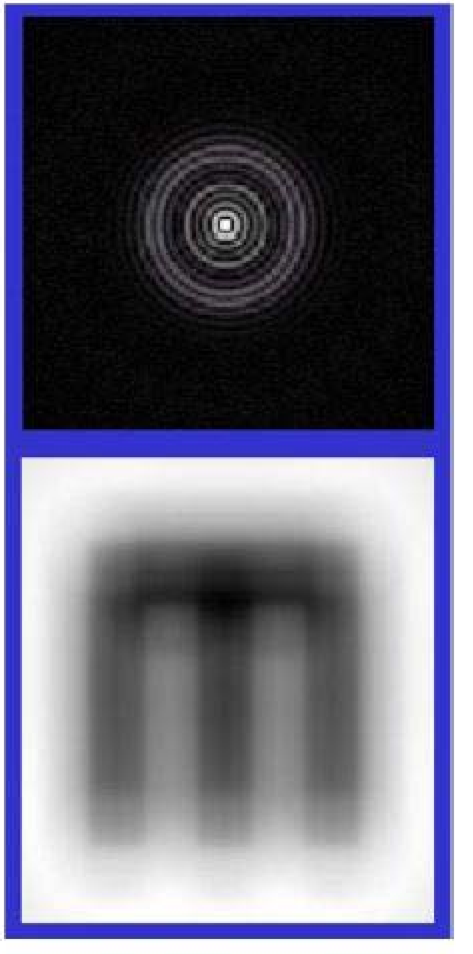

Longitudinal Chromatic
Aberration (LCA)
Light
Camera lens
Camera film
transverse chromatic aberration (TCA)

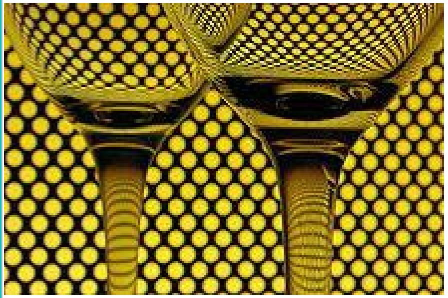

ANIMATING

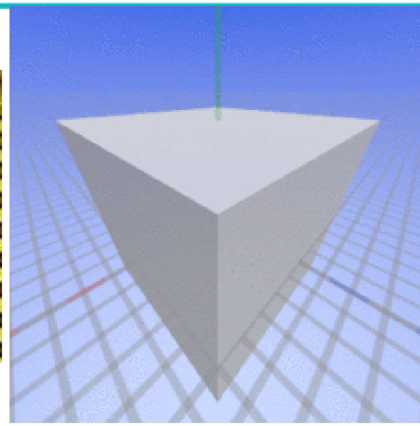
# Issues with lenses: Chromatic Aberration

- Rays farther from the optical axis focus closer



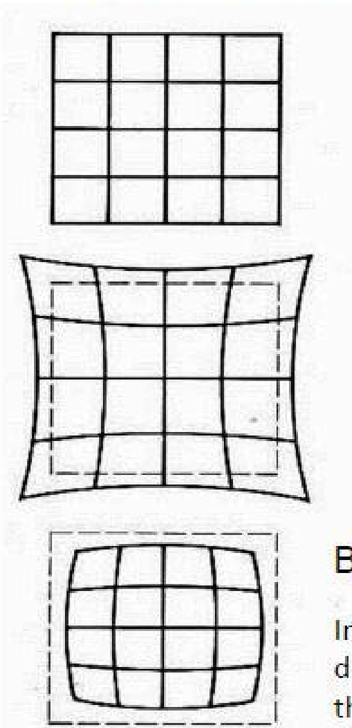




***This simulation shows how adjusting the angle of view of a camera, while varying the camera distance, keeping the object in frame, results in vastly differing images. At narrow angles, large distances, light rays are nearly parallel, resulting in a "flattened" image. At wide angles, short distances, the object appears distorted.***

# Issues with lenses: Chromatic Aberration

– Deviations are most noticeable for rays that pass through the edge of the lens



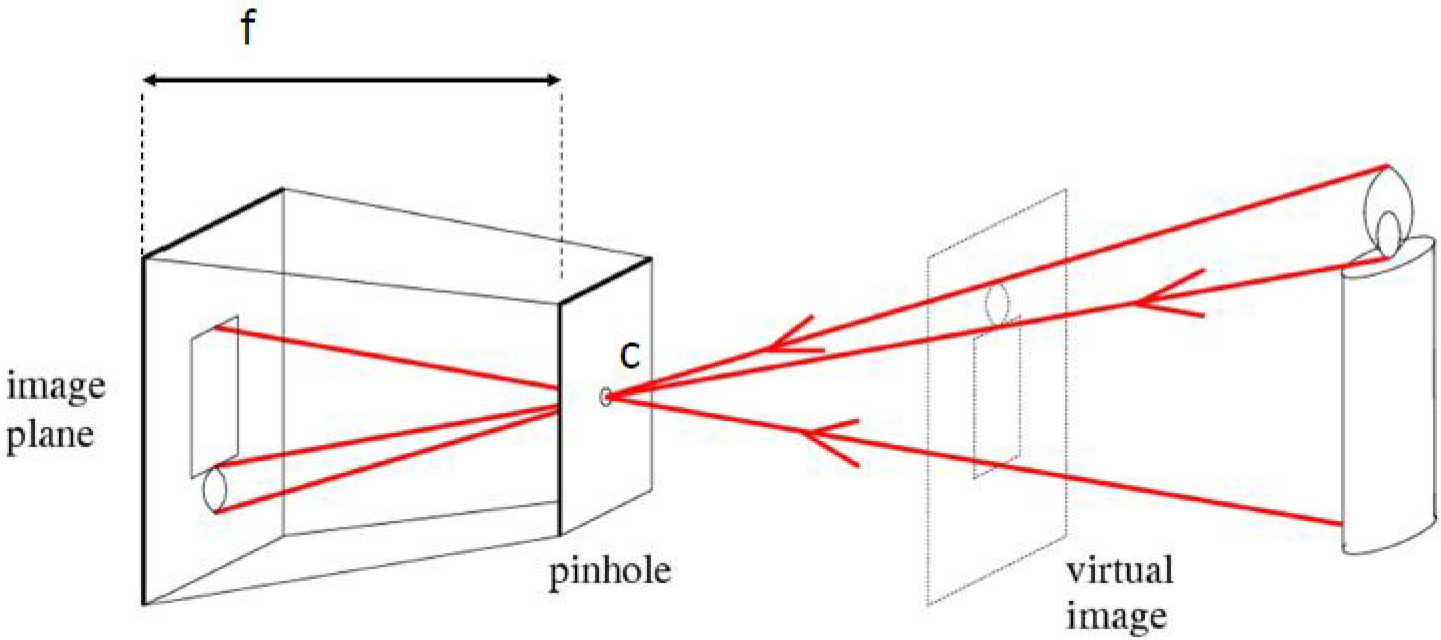
No distortion

Pin cushion

Barrel (fisheye lens)

Image magnification decreases with distance from the optical axis

# Pinhole camera




$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

$$\Re^3 \xrightarrow{E} \Re^2$$

f = focal length

c = center of the camera

# Pinhole camera

Is this a linear transformation?

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

No — division by z is nonlinear!

How to make it linear?

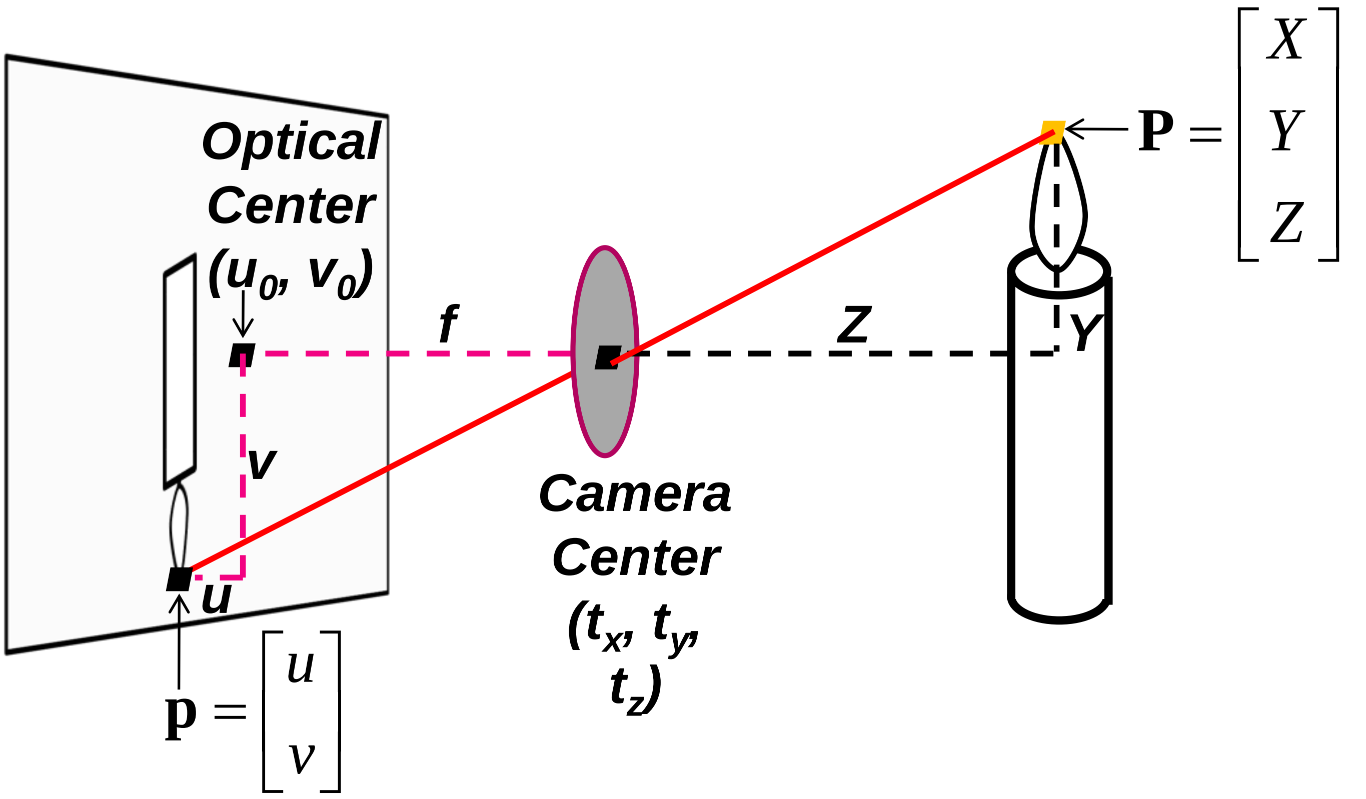
***Mathematically, for a linear system, F, defined by F(x) = y, where x is some sort of stimulus (input) and y is some sort of response (output), the superposition (i.e., sum) of stimuli yields a superposition of the respective responses:***

$$F(x_1 + x_2 + \ldots) = F(x_1) + F(x_2) + \ldots$$

***In the field of electrical engineering, where the x and y signals are allowed to be complex-valued (as is common in signal processing), a linear system must satisfy the superposition property, which requires the system to be additive and homogeneous***

$$F(x_1 + x_2) = F(x_1) + F(x_2) \qquad F(ax) = aF(x)$$

# Projection
## World coordinates ➡ Image coordinates



# Homogeneous coordinates

## *Conversion*

### *Converting to homogeneous coordinates*

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

***homogeneous image coordinates***

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

***homogeneous scene coordinates***

### *Converting from homogeneous coordinates*

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

# Homogeneous coordinates

Perspective Projection Transformation:

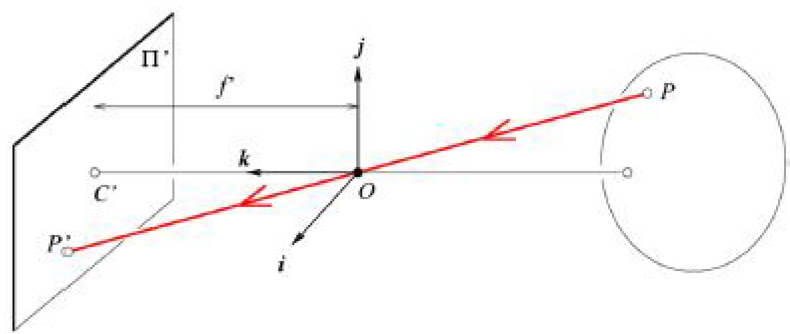
$$P' = \begin{bmatrix} f\,x \\ f\,y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{M}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

**"Projection matrix"**

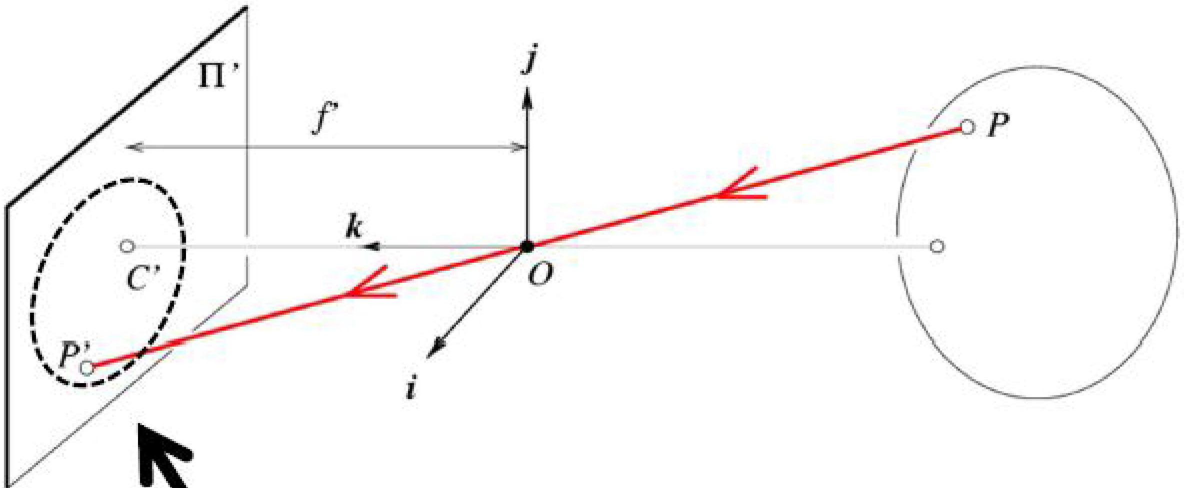
$$P' = M\;P$$

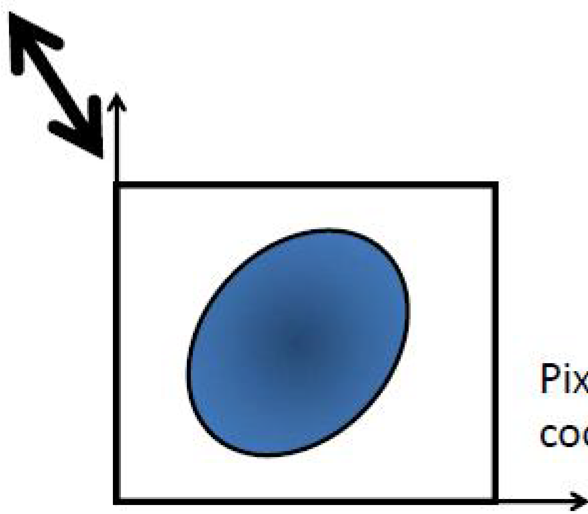
$$\Re^4 \xrightarrow{\text{H}} \Re^3$$

$$P_i' = \begin{bmatrix} f\dfrac{x}{z} \\ f\dfrac{y}{z} \end{bmatrix}$$



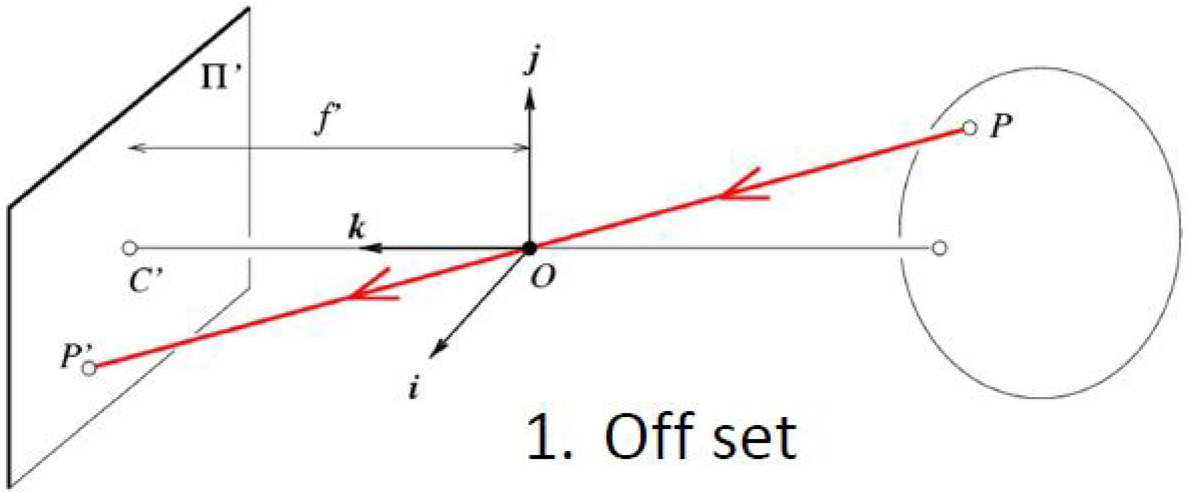
# From retina plane to images



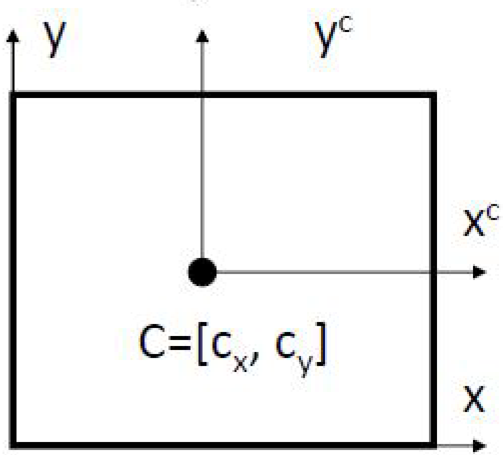



Pixels, bottom-left coordinate systems

# From retina plane to images

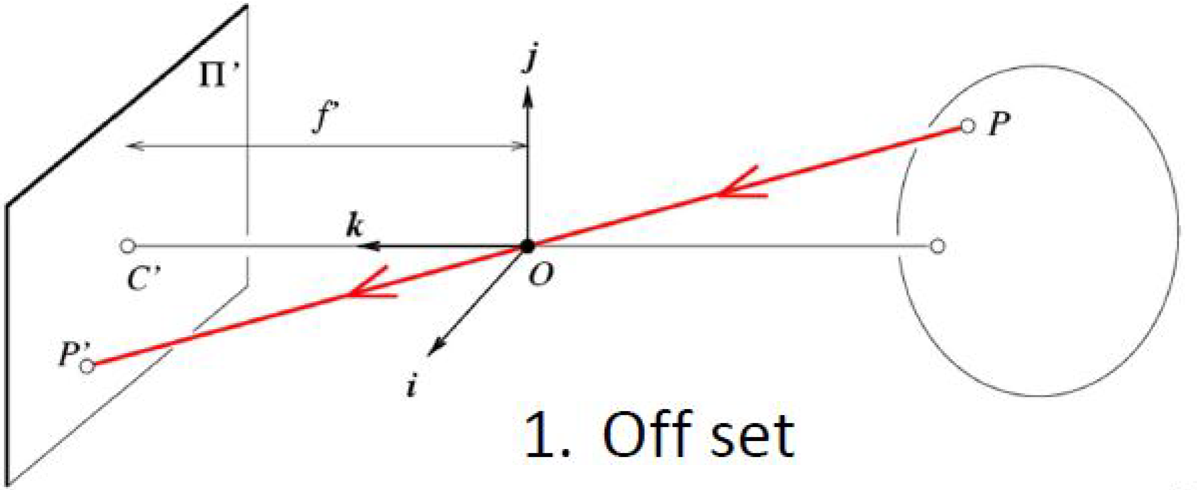


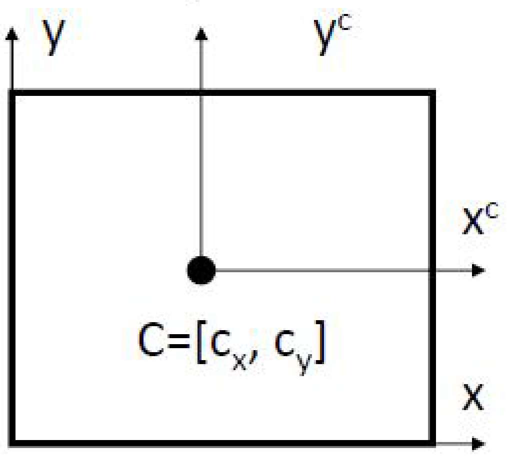

1. Off set




$$(x, y, z) \rightarrow \left(f\frac{x}{z} + c_x, f\frac{y}{z} + c_y\right)$$

# From retina plane to images




1. Off set
2. From metric to pixels

$$(x, y, z) \rightarrow (\underbrace{\boxed{f\ k}}_{\alpha}\frac{x}{z} + c_x, \underbrace{\boxed{f\ l}}_{\beta}\frac{y}{z} + c_y)$$

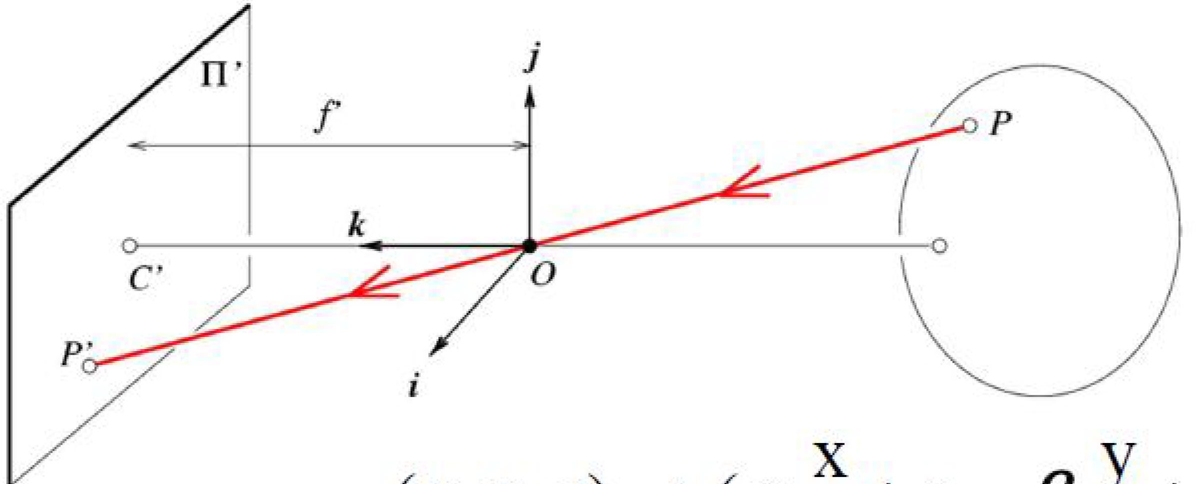
Units: k,l : pixel/m

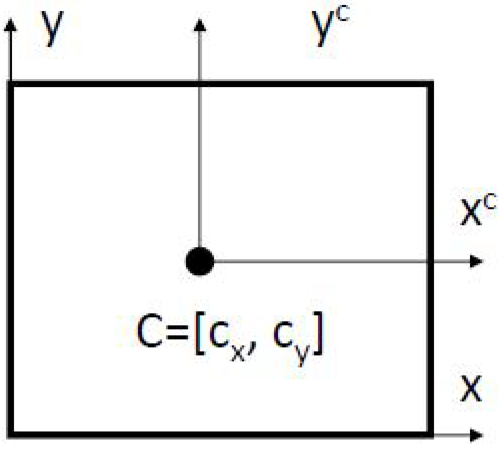
f : m

Non-square pixels

$\alpha$, $\beta$ : pixel

# From retina plane to images

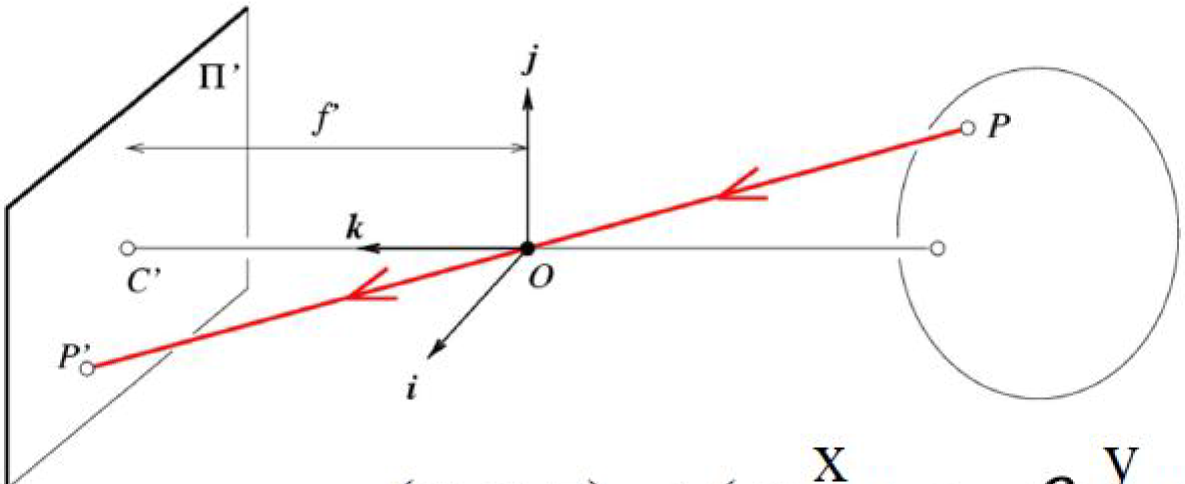


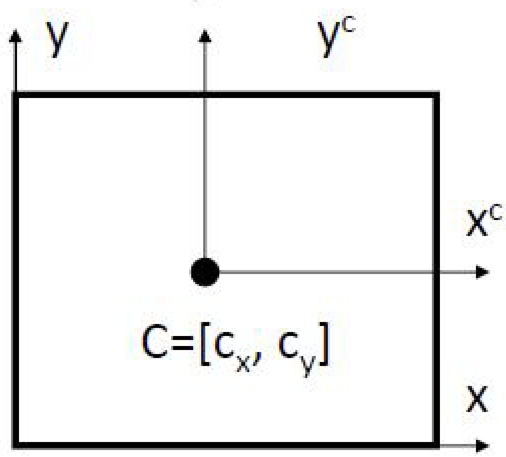




$$(x, y, z) \rightarrow \left(\alpha \frac{x}{z} + c_x, \; \beta \frac{y}{z} + c_y\right)$$

Matrix form?

# Camera matrix

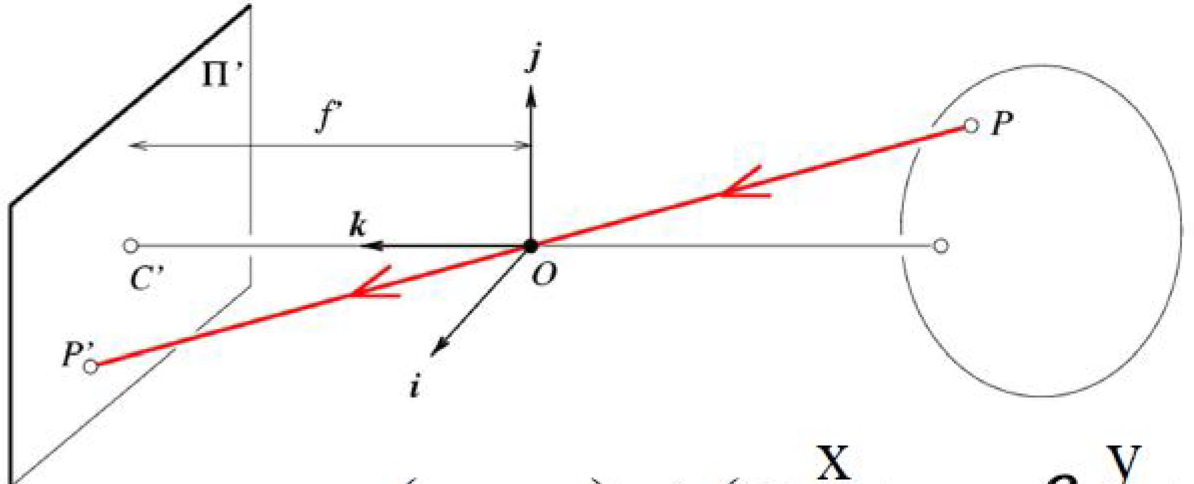


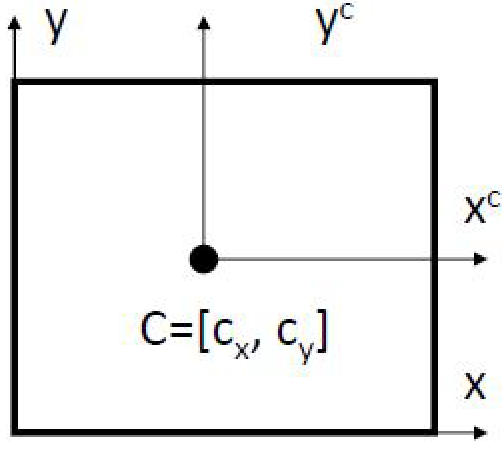

$$(\mathrm{x}, \mathrm{y}, \mathrm{z}) \to \left(\boldsymbol{\alpha}\frac{\mathrm{x}}{\mathrm{z}} + \mathrm{c}_{\mathrm{x}}, \boldsymbol{\beta}\frac{\mathrm{y}}{\mathrm{z}} + \mathrm{c}_{\mathrm{y}}\right)$$

$$P' = \begin{bmatrix} \alpha\, x + c_x z \\ \beta\, y + c_y z \\ z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# Camera matrix

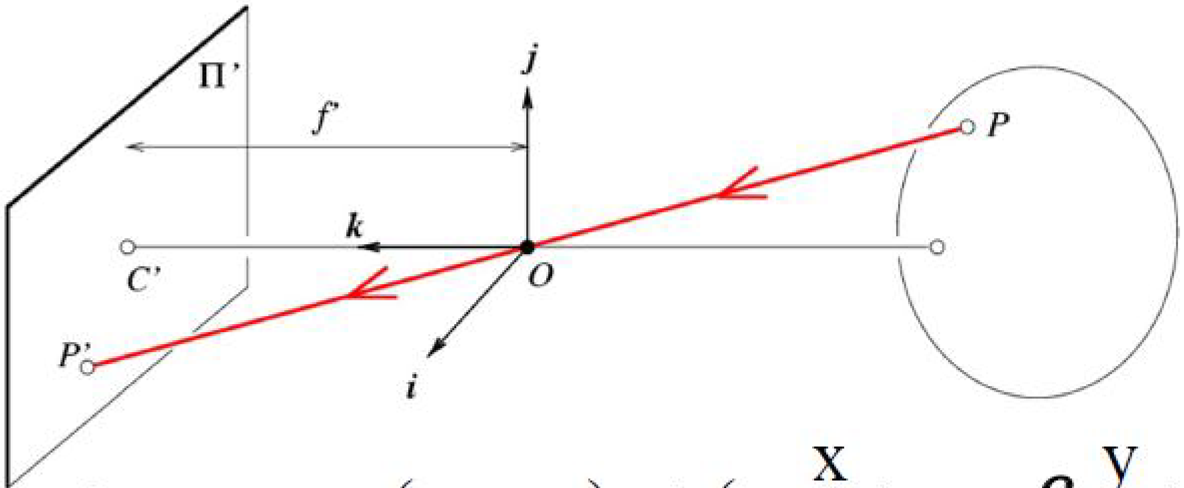


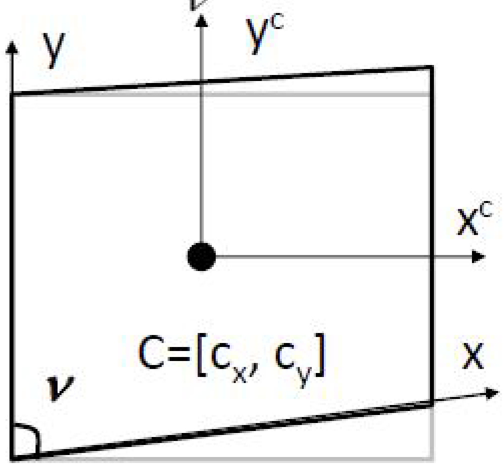

$$(x, y, z) \rightarrow \left(\alpha \frac{x}{z} + c_x, \beta \frac{y}{z} + c_y\right)$$




$$P' = \begin{bmatrix} \alpha & 0 & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# Camera matrix



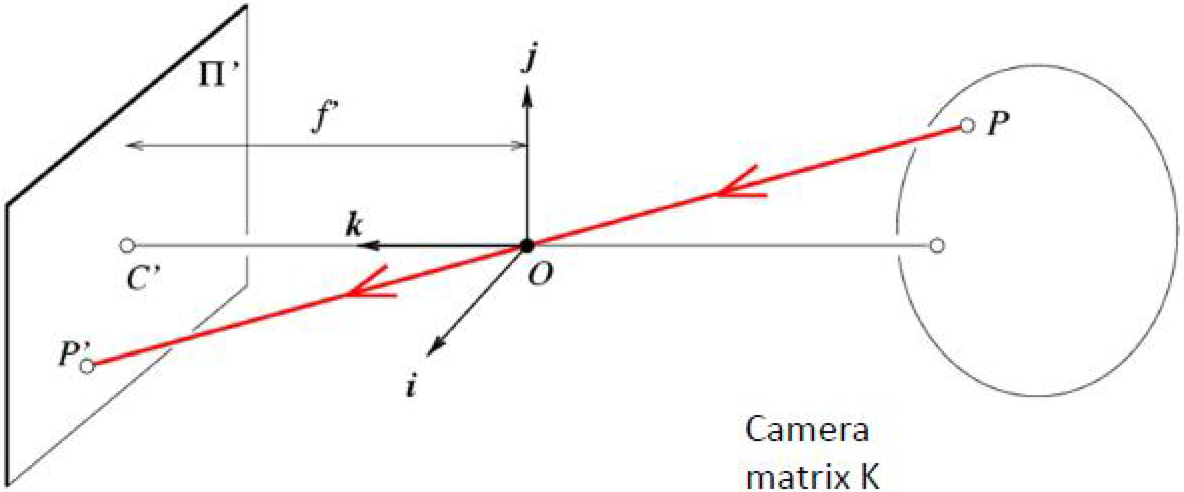

$$(x, y, z) \rightarrow (\boldsymbol{\alpha} \frac{x}{z} + c_x, \boldsymbol{\beta} \frac{y}{z} + c_y)$$

Skew parameter

$$P' = \begin{bmatrix} \alpha & \boxed{s} & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

*Finally, the camera coordinate system may be skewed due to manufacturing error, so that angle* $\theta$ *between two image axes is not equal to 90º.*

# Camera matrix




Camera matrix K

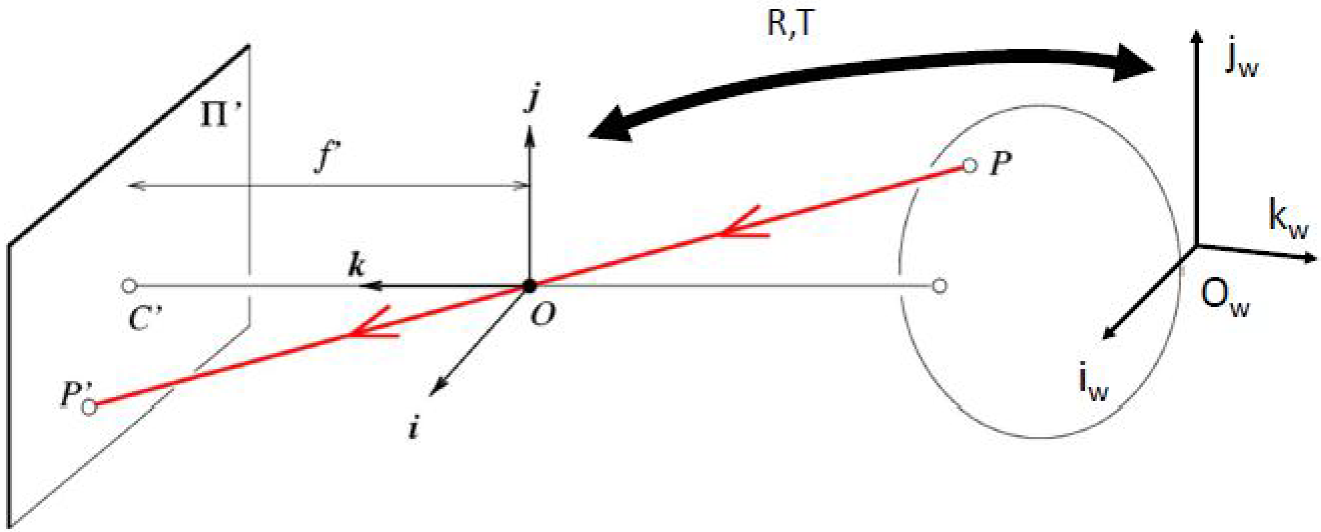
$$P' = M\ P$$

$$= K\begin{bmatrix} I & 0 \end{bmatrix} P$$

$$P' = \begin{bmatrix} \alpha & s & c_x & 0 \\ 0 & \beta & c_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

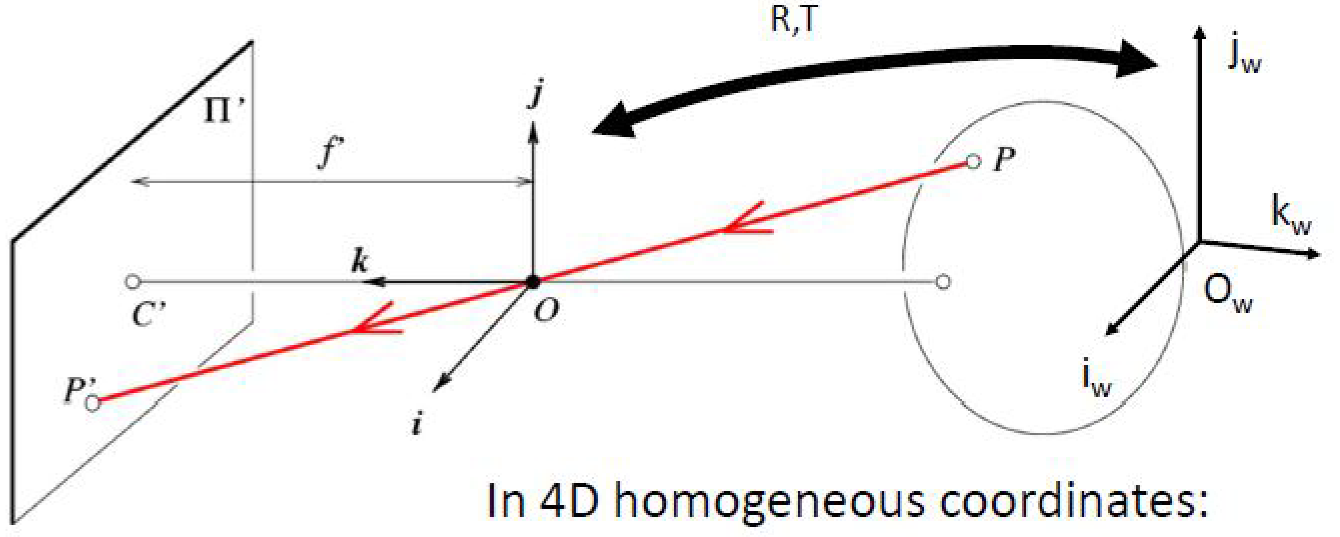
K has 5 degrees of freedom!

# Camera & world reference system




- The mapping is defined within the camera reference system
- What if an object is represented in the world reference system?

# Camera & world reference system




In 4D homogeneous coordinates:

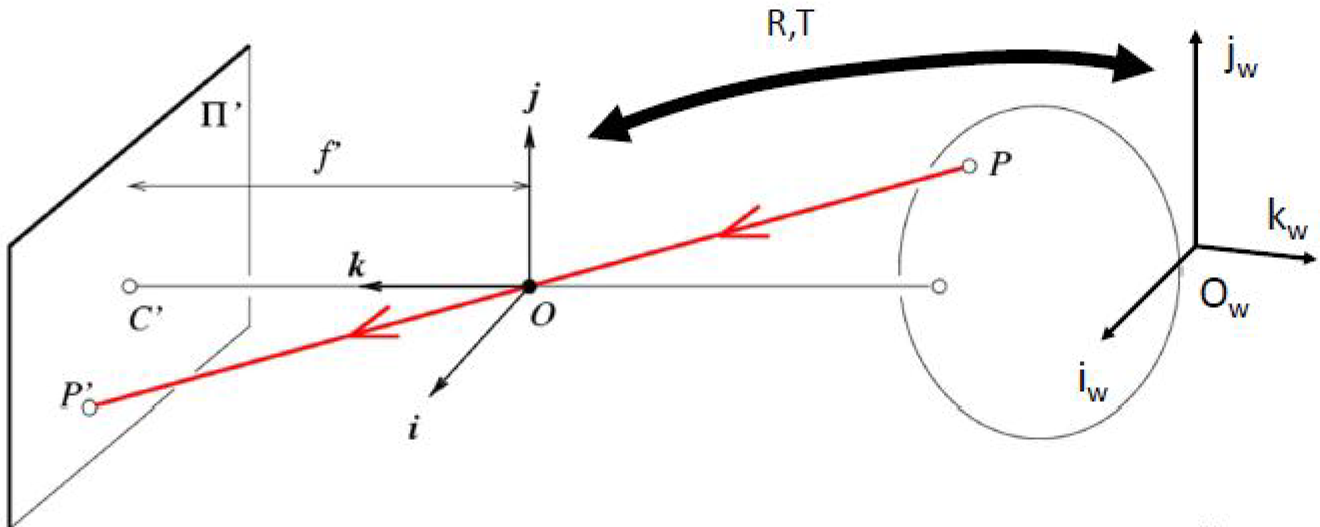
$$P = \begin{bmatrix} R & T \end{bmatrix} P_w$$

$$P' = M\, P_w = \boxed{K} \boxed{\begin{bmatrix} R & T \end{bmatrix}} P_w$$

Internal parameters

External parameters

# Projective cameras



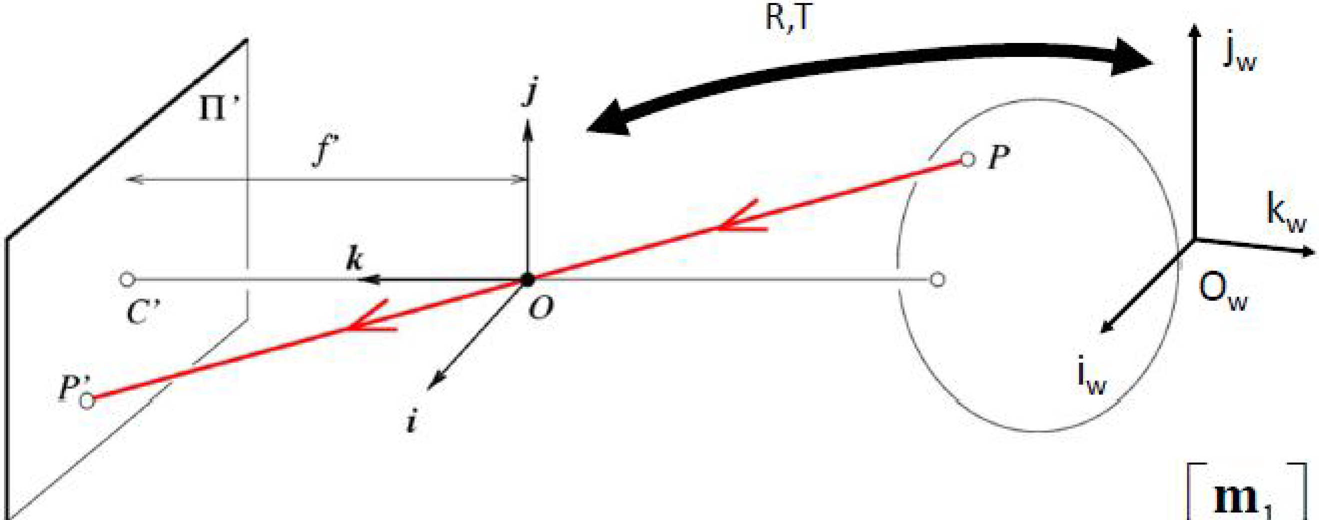

$$P'_{3\times 1} = M\ P_w = K_{3\times 3}\begin{bmatrix} R & T \end{bmatrix}_{3\times 4} P_{w\,4\times 1}$$

$$K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

5 + 3 + 3 =11!

# Projective cameras




$$P'_{3\times1} = M\ P_w = K_{3\times3}\begin{bmatrix} R & T \end{bmatrix}_{3\times4} P_{w4\times1} \qquad M = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{bmatrix}$$

$$(x, y, z)_w \rightarrow \left(\frac{\mathbf{m}_1 P_w}{\mathbf{m}_3 P_w}, \frac{\mathbf{m}_2 P_w}{\mathbf{m}_3 P_w}\right)$$

M is defined up to scale!

Multiplying M by a scalar won't change the image

# Theorem (Faugeras, 1993)

$$M = K\begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} KR & KT \end{bmatrix} = \begin{bmatrix} A & b \end{bmatrix}$$

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha & s & c_x \\ 0 & \beta & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{\alpha} = \mathrm{f\ k};$

$\boldsymbol{\beta} = \mathrm{f\ l}$

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a $3 \times 4$ matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$