Chapter 1

Sampling and Reconstruction

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Review of useful equations

- Linear system

$$x(t),\ X(f) \longrightarrow \boxed{\begin{array}{c}\text{Linear system}\\ h(t)\\ H(f)\end{array}} \longrightarrow y(t) = x(t) * h(t),\ Y(f) = X(f)H(f)$$

- Especially, $x(t) = A\cos(2\pi f_0 t + \theta)$

$$y(t) = A\,|H(f_0)|\cos(2\pi f_0 t + \theta + \arg(H(f_0)))$$

- Fourier transform:

$$\cos(2\pi f_0 t) \xleftrightarrow{FT} \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)]$$

$$\sin(2\pi f_0 t) \xleftrightarrow{FT} \frac{1}{2} j[\delta(f + f_0) - \delta(f - f_0)]$$

- Trigonometric formulas: $\cos(a)\cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

$$\sin(a)\sin(b) = -\frac{1}{2}[\cos(a+b) - \cos(a-b)]$$

$$\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

1. Introduction

❖ A typical signal processing system includes 3 stages:

❖ The analog signal is digitalized by an A/D converter

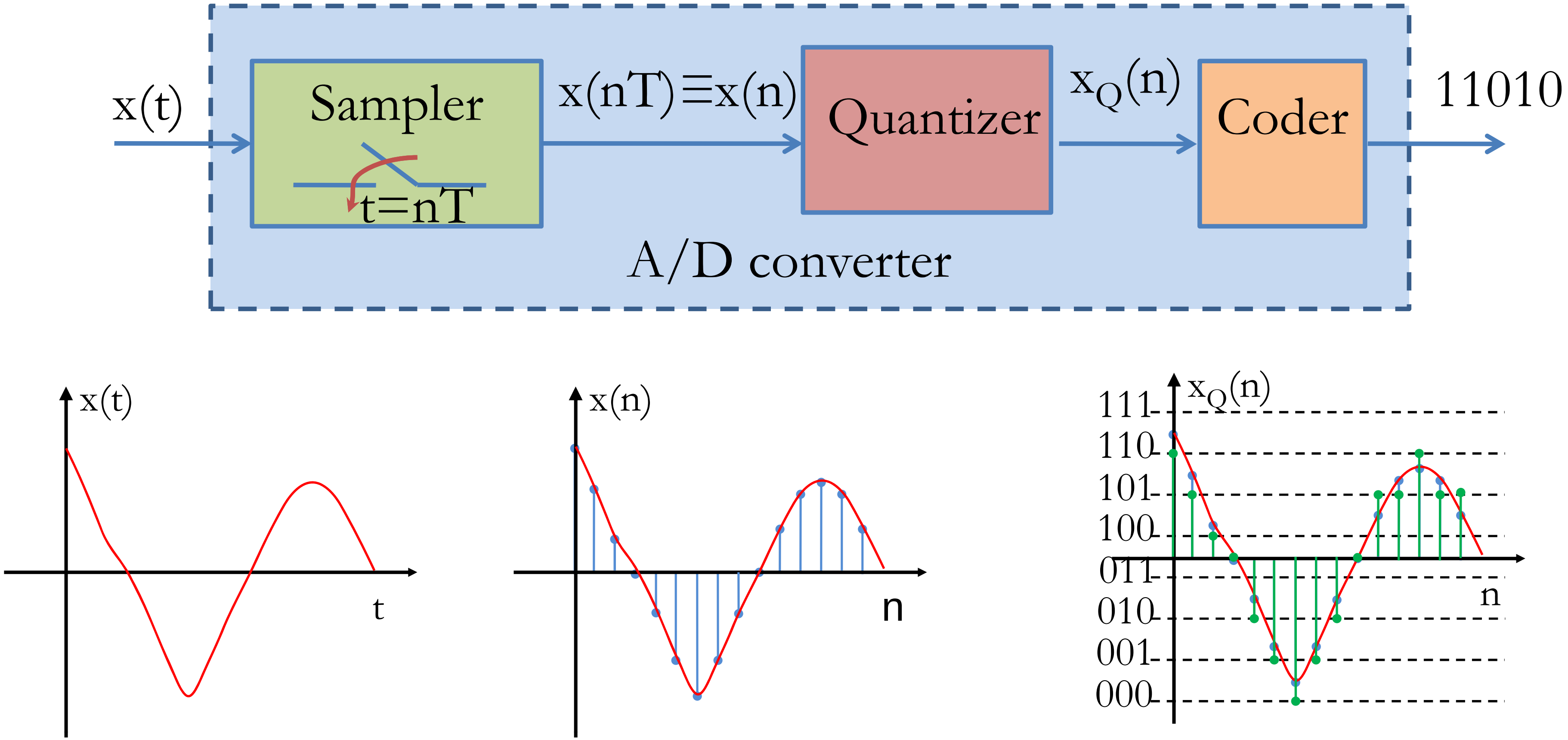
❖ The digitalized samples are processed by a digital signal processor.

- The digital processor can be programmed to perform signal processing operations such as filtering, spectrum estimation. Digital signal processor can be a general purpose computer, DSP chip or other digital hardware.

❖ The resulting output samples are converted back into analog by a D/A converter.

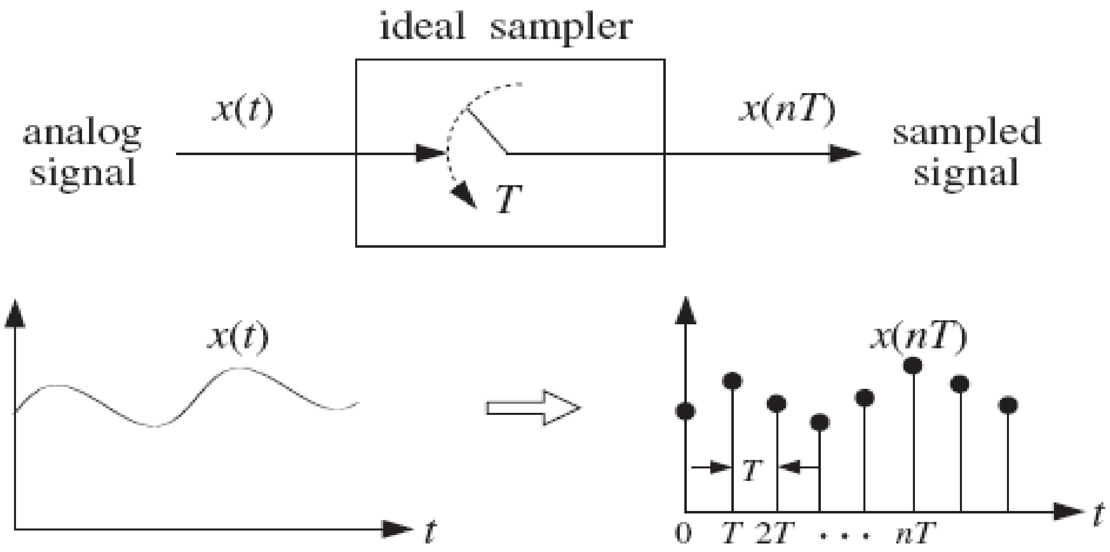
2. Analog to digital conversion

❖ Analog to digital (A/D) conversion is a three-step process.



3. Sampling

- Sampling is to convert a continuous time signal into a discrete time signal. The analog signal is periodically measured at every T seconds



- $x(n) \equiv x(nT) = x(t = nT)$, $n = \ldots.-2, -1, 0, 1, 2, 3\ldots\ldots.$
- T: sampling interval or sampling period (second);
- $f_s = 1/T$: sampling rate or sampling frequency (samples/second or Hz)

3. Sampling-example 1

❖ The analog signal $x(t)=2\cos(2\pi t)$ with t(s) is sampled at the rate $f_s=4$ Hz. Find the discrete-time signal x(n) ?

Solution:

❖ $x(n)\equiv x(nT)=x(n/fs)=2\cos(2\pi n/fs)=2\cos(2\pi n/4)=2\cos(\pi n/2)$

n	0	1	2	3	4
x(n)	2	0	-2	0	2

❖ Plot the signal

3. Sampling-example 2

❖ Consider the two analog sinusoidal signals

$$x_1(t) = 2\cos(2\pi\frac{7}{8}t), \quad x_2(t) = 2\cos(2\pi\frac{1}{8}t); \quad t(s)$$

These signals are sampled at the sampling frequency f_s=1 Hz.

Find the discrete-time signals ?

Solution:

$$x_1(n) \equiv x_1(nT) = x_1(n\frac{1}{f_s}) = 2\cos(2\pi\frac{7}{8}\frac{1}{1}n) = 2\cos(\frac{7}{4}\pi n)$$

$$= 2\cos((2-\frac{1}{4})\pi n) = 2\cos(\frac{\pi}{4}n)$$

$$x_2(n) \equiv x_2(nT) = x_2(n\frac{1}{f_s}) = 2\cos(2\pi\frac{1}{8}\frac{1}{1}n) = 2\cos(\frac{1}{4}\pi n)$$

❖ Observation: $x_1(n)=x_2(n)$ → based on the discrete-time signals, we cannot tell which of two signals are sampled ? These signals are called "alias"

3. Sampling-example 2

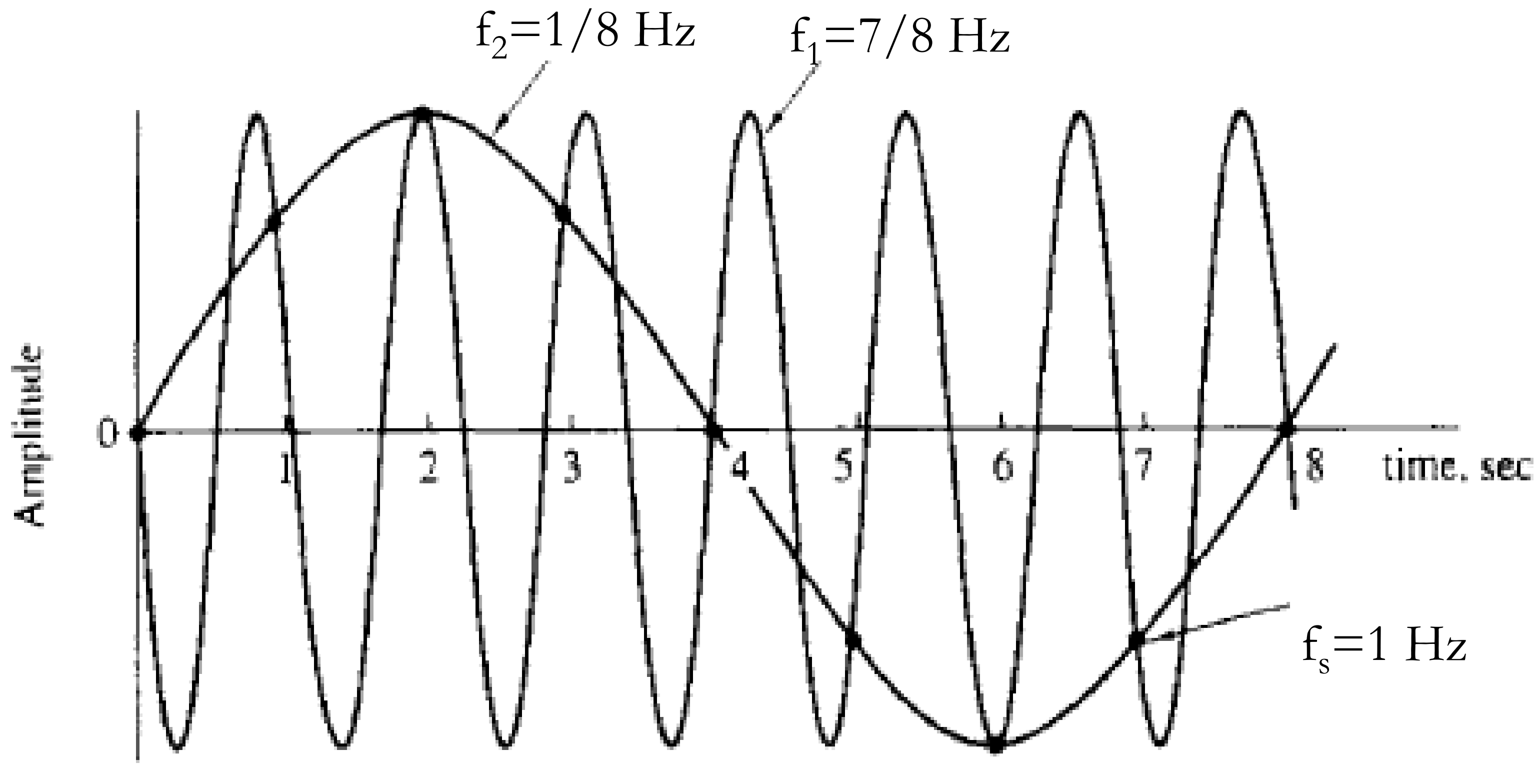


Fig: Illustration of aliasing

3. Sampling-Aliasing of Sinusoids

- In general, the sampling of a continuous-time sinusoidal signal $x(t) = A\cos(2\pi f_0 t + \theta)$ at a sampling rate $f_s = 1/T$ results in a discrete-time signal x(n).

- The sinusoids $x_k(t) = A\cos(2\pi f_k t + \theta)$ is sampled at f_s , resulting in a discrete time signal $x_k(n)$.

- If $f_k = f_0 + kf_s$, $k = 0, \pm 1, \pm 2, \ldots$, then $x(n) = x_k(n)$.

Proof: (in class)

- Remarks: We can that the frequencies $f_k = f_0 + kf_s$ are indistinguishable from the frequency f_0 after sampling and hence they are aliases of f_0

4. Sampling Theorem-Sinusoids

❖ Consider the analog signal $x(t) = A\cos(\Omega t) = A\cos(2\pi ft)$ where Ω is the frequency (rad/s) of the analog signal, and f=Ω/2π is the frequency in cycles/s or Hz. The signal is sampled at the three rate f_s=8f, f_s=4f, and f_s=2f.

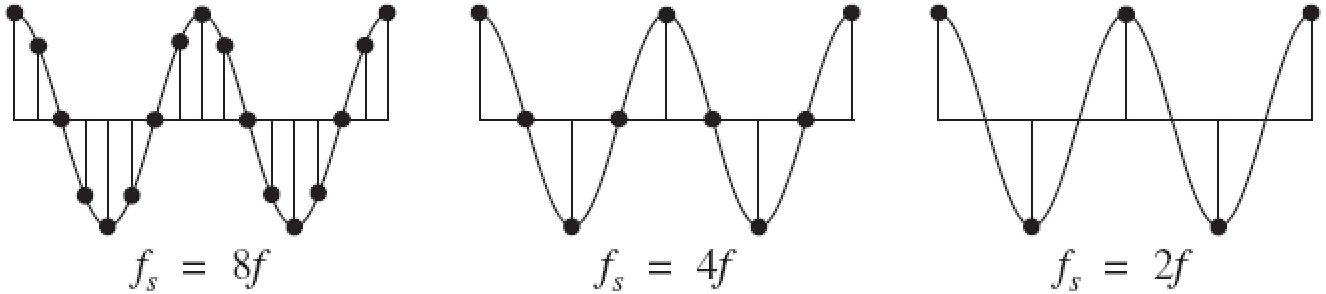


Fig: Sinusoid sampled at different rates

❖ Note that $\dfrac{f_s}{f} = \dfrac{samples/\sec}{cycles/\sec} = \dfrac{samples}{cycle}$

❖ To sample a single sinusoid properly, we must require $\dfrac{f_s}{f} \geq 2\dfrac{samples}{cycle}$

4. Sampling Theorem

- For accurate representation of a signal x(t) by its time samples x(nT), two conditions must be met:

1) The signal x(t) must be bandlimitted, i.e., its frequency spectrum must be limited to f_{max}

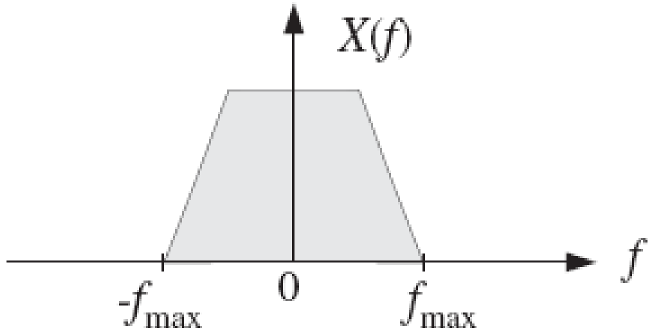


Fig: Typical bandlimited spectrum

2) The sampling rate f_s must be chosen at least twice the maximum frequency f_{max}. $\boxed{f_s \geq 2 f_{max}}$

- $f_s=2f_{max}$ is called Nyquist rate; $f_s/2$ is called Nyquist frequency; $[-f_s/2, f_s/2]$ is Nyquist interval.

4. Sampling Theorem

❖ The values of f_{max} and f_s depend on the application

Application	fmax	fs
Biomedical	1 KHz	2 KHz
Speech	4 KHz	8 KHz
Audio	20 KHz	40 KHz
Video	4 MHz	8 MHz

4. Sampling Theorem-Spectrum Replication

- Let $x(nT) = \hat{x}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = x(t)s(t)$ where $s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$
- s(t) is periodic, thus, its Fourier series are given by

$$s(t) = \sum_{n=-\infty}^{\infty} S_n e^{j2\pi f_s nt} \quad \text{where} \quad S_n = \frac{1}{T}\int_T \delta(t)e^{-j2\pi f_s nt}dt = \frac{1}{T}\int_T \delta(t)dt = \frac{1}{T}$$

Thus, $s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi f_s nt}$

which results in $\hat{x}(t) = x(t)s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)e^{j2\pi n f_s t}$

- Taking the Fourier transform of $\hat{x}(t)$ yields $\boxed{\hat{X}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(f-nf_s)}$

- Observation: The spectrum of discrete-time signal is a sum of the original spectrum of analog signal and its periodic replication at the interval f_s.

4. Sampling Theorem-Spectrum Replication

❖ $f_s/2 \geq f_{max}$

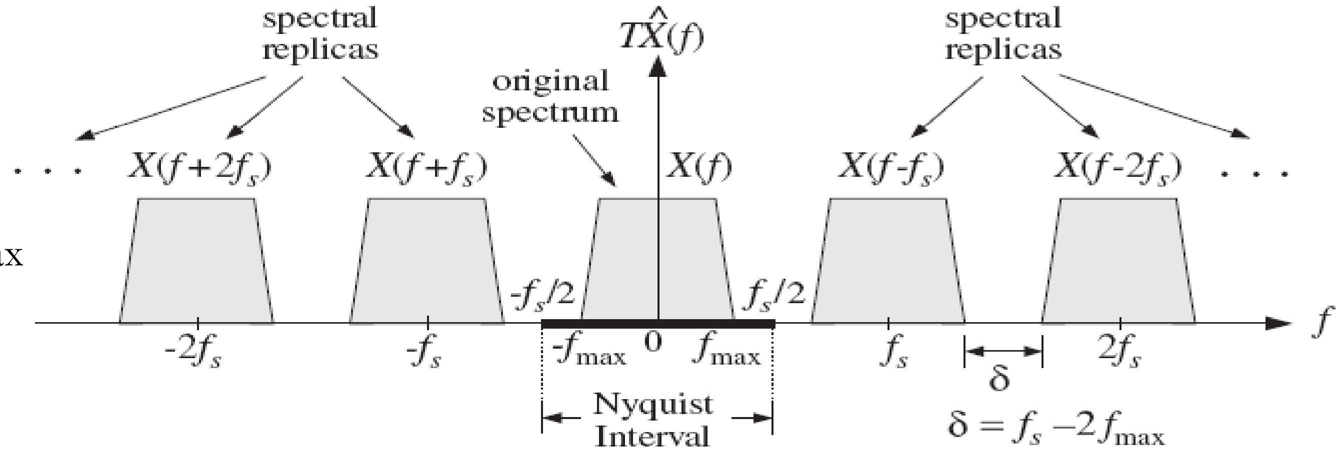


Fig: Spectrum replication caused by sampling

❖ $f_s/2 < f_{max}$

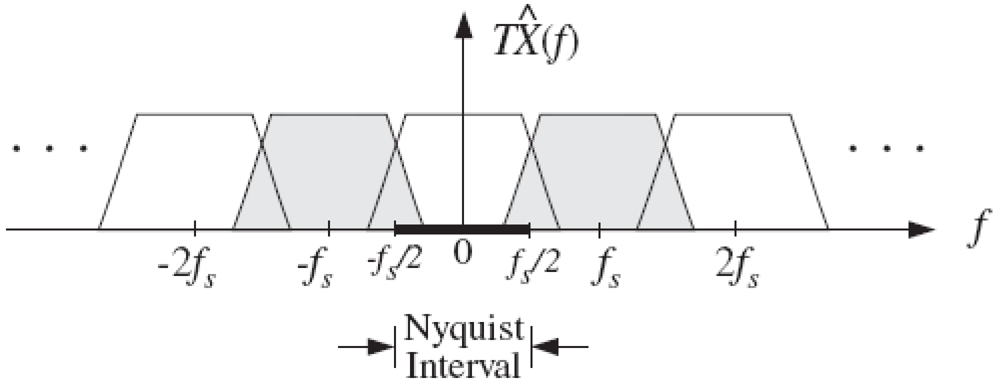


Fig: Aliasing caused by overlapping spectral replicas

5. Ideal Analog reconstruction

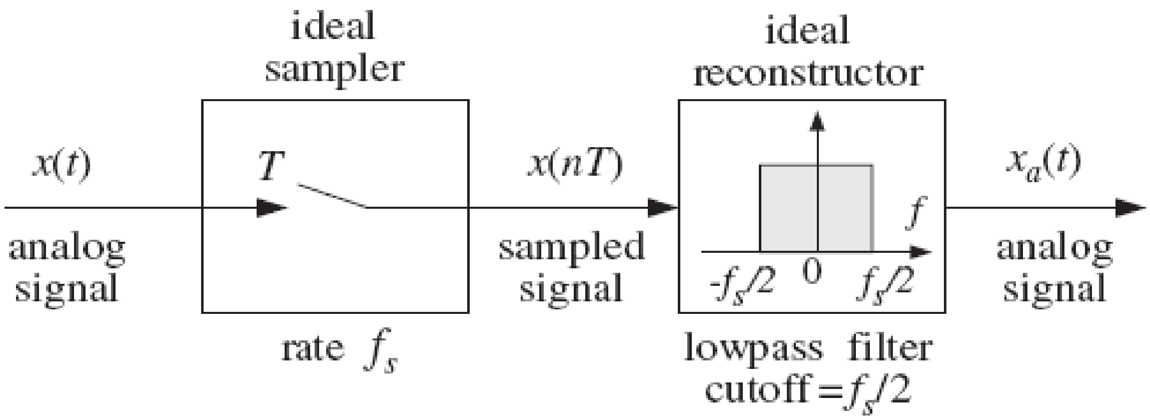


Fig: Ideal reconstructor as a lowpass filter

- An ideal reconstructor acts as a lowpass filter with cutoff frequency equal to the Nyquist frequency fs/2.
- An ideal reconstructor (lowpass filter) $$\boxed{H(f) = \begin{cases} T & f \in [-f_s/2, f_s/2] \\ 0 & otherwise \end{cases}}$$

Then

$$\boxed{\hat{X}_a(f) = \hat{X}(f)H(f) = X(f)}$$

5. Analog reconstruction-Example 1

- The analog signal $x(t)=\cos(20\pi t)$ is sampled at the sampling frequency fs=40 Hz.

  a) Plot the spectrum of signal $x(t)$?

  b) Find the discrete time signal $x(n)$?

  c) Plot the spectrum of signal $x(n)$?

  d) The signal $x(n)$ is an input of the ideal reconstructor, find the reconstructed signal $x_a(t)$?

5. Analog reconstruction-Example 2

❖ The analog signal $x(t)=\cos(100\pi t)$ is sampled at the sampling frequency fs=40 Hz.

a) Plot the spectrum of signal x(t) ?

b) Find the discrete time signal x(n) ?

c) Plot the spectrum of signal x(n) ?

d) The signal x(n) is an input of the ideal reconstructor, find the reconstructed signal $x_a(t)$?

5. Analog reconstruction

- Remarks: $x_a(t)$ contains only the frequency components that lie in the Nyquist interval (NI) $[-f_s//2, f_s/2]$.

- $x(t),\ f_0 \in \mathrm{NI} \xrightarrow{\text{sampling at } f_s} x(n) \xrightarrow{\text{ideal reconstructor}} x_a(t),\ f_a = f_0$

- $x_k(t),\ f_k = f_0 + kf_s \xrightarrow{\text{sampling at } f_s} x(n) \xrightarrow{\text{ideal reconstructor}} x_a(t),\ f_a = f_0$

- The frequency f_a of reconstructed signal $x_a(t)$ is obtained by adding to or substracting from f_0 (f_k) enough multiples of fs until it lies within the Nyquist interval $[-f_s//2, f_s/2]$.. That is

$$\boxed{f_a = f \bmod (f_s)}$$

5. Analog reconstruction-Example 3

- The analog signal $x(t)=10\sin(4\pi t)+6\sin(16\pi t)$ is sampled at the rate 20 Hz. Find the reconstructed signal $x_a(t)$?

5. Analog reconstruction-Example 4

❖ Let $x(t)$ be the sum of sinusoidal signals
$x(t)=4+3\cos(\pi t)+2\cos(2\pi t)+\cos(3\pi t)$ where t is in milliseconds.

a) Determine the minimum sampling rate that will not cause any aliasing effects ?
b) To observe aliasing effects, suppose this signal is sampled at half its Nyquist rate. Determine the signal $x_a(t)$ that would be aliased with $x(t)$? Plot the spectrum of signal $x(n)$ for this sampling rate?

6. Ideal antialiasing prefilter

❖ The signals in practice may not bandlimitted, thus they must be filtered by a lowpass filter

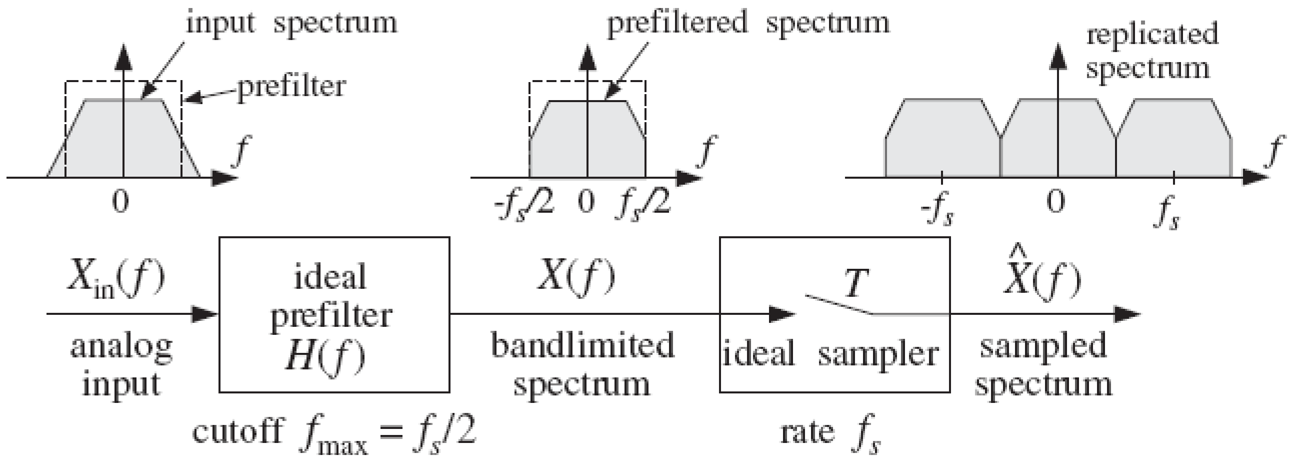


Fig: Ideal antialiasing prefilter

6. Practical antialiasing prefilter

- A lowpass filter: [$-f_{pass}$, f_{pass}] is the frequency range of interest for the application ($f_{max}=f_{pass}$)
- The Nyquist frequency fs/2 is in the middle of transition region.
- The stopband frequency f_{stop} and the minimum stopband attenuation A_{stop} dB must be chosen appropriately to minimize the aliasing effects.

$$\boxed{f_s = f_{pass} + f_{stop}}$$

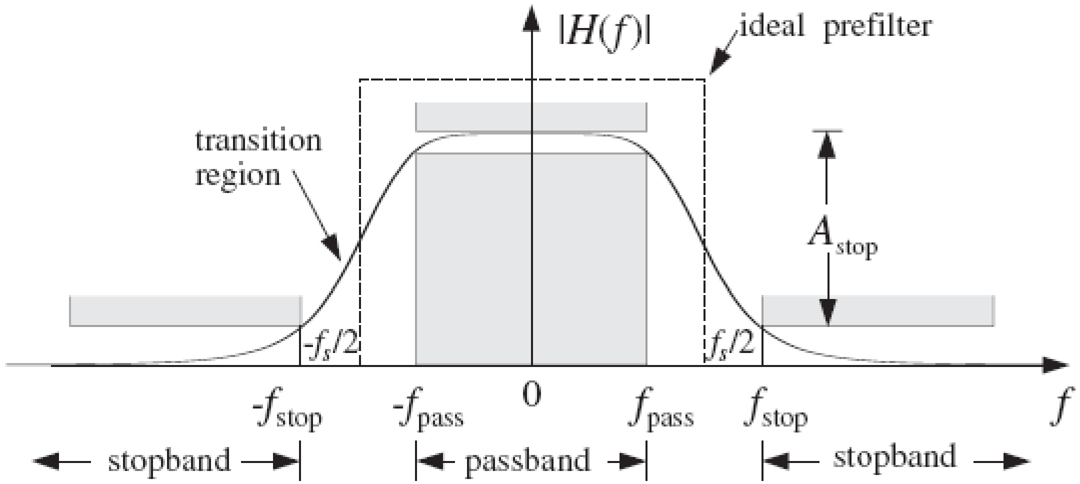


Fig: Practical antialiasing lowpass prefilter

6. Practical antialiasing prefilter

- The attenuation of the filter in decibels is defined as

$$\boxed{A(f) = -20\log_{10}\left|\frac{H(f)}{H(f_0)}\right| (dB)}$$

where f_0 is a convenient reference frequency, typically taken to be at DC for a lowpass filter.

- α_{10} =A(10f)-A(f) (dB/decade): the increase in attenuation when f is changed by a factor of ten.
- α_2 =A(2f)-A(f) (dB/octave): the increase in attenuation when f is changed by a factor of two.
- Analog filter with order N, $|H(f)| \sim 1/f^N$ for large f, thus α_{10} =20N (dB/decade) and α_{10} =6N (dB/octave)

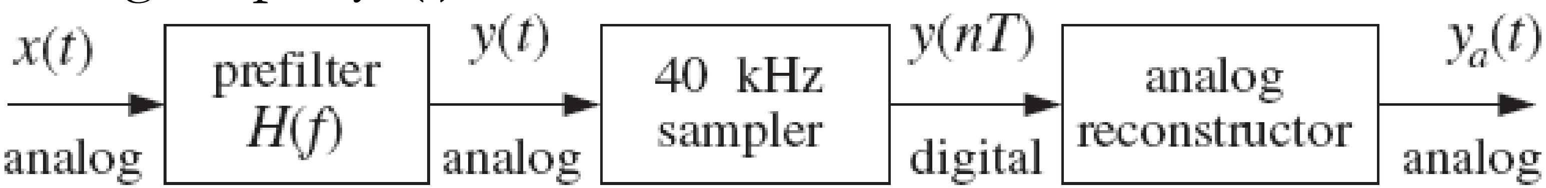
6. Antialiasing prefilter-Example

❖ A sound wave has the form

$$x(t) = 2A\cos(10\pi t) + 2B\cos(30\pi t) + 2C\cos(50\pi t) + 2D\cos(60\pi t) + 2E\cos(90\pi t) + 2F\cos(125\pi t)$$

where t is in milliseconds. What is the frequency content of this signal ? Which parts of it are audible and why ?

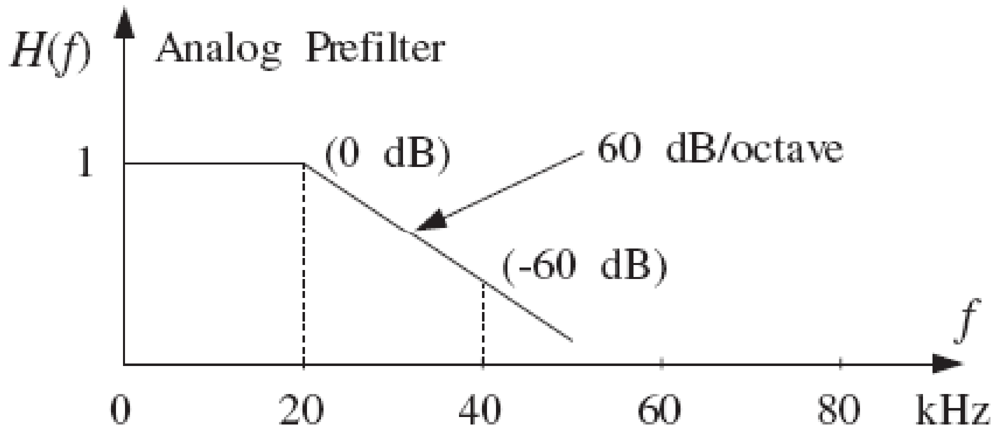
This signal is prefilter by an anlog prefilter H(f). Then, the output y(t) of the prefilter is sampled at a rate of 40KHz and immediately reconstructed by an ideal analog reconstructor, resulting into the final analog output ya(t), as shown below:



6. Antialiasing prefilter-Example

Determine the output signal y(t) and ya(t) in the following cases:

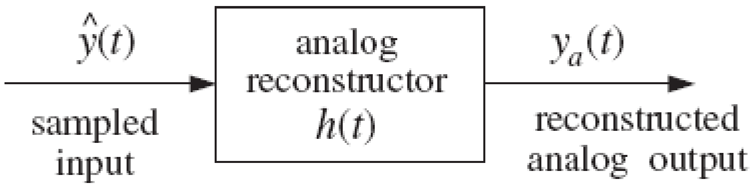
a)When there is no prefilter, that is, H(f)=1 for all f.
b)When H(f) is the ideal prefilter with cutoff fs/2=20 KHz.
c)When H(f) is a practical prefilter with specifications as shown below:

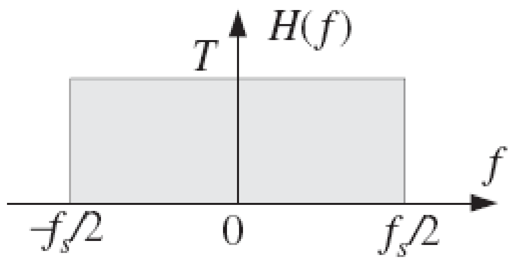


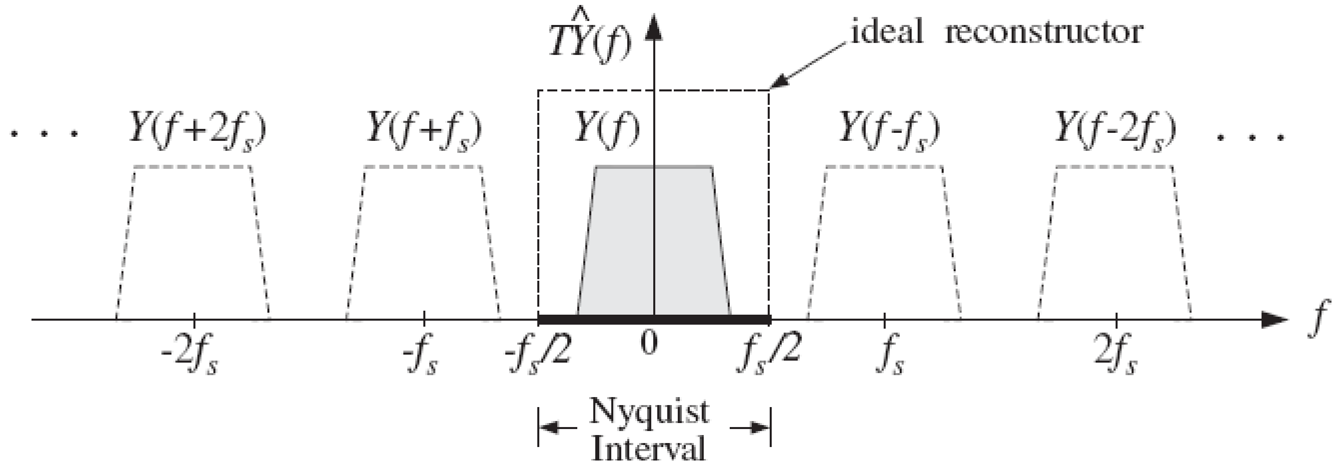
The filter's phase response is assumed to be ignored in this example.

7. Ideal and practical analog reconstructors

❖ An ideal reconstructor is an ideal lowpass filter with cutoff Nyquist frequency fs/2.



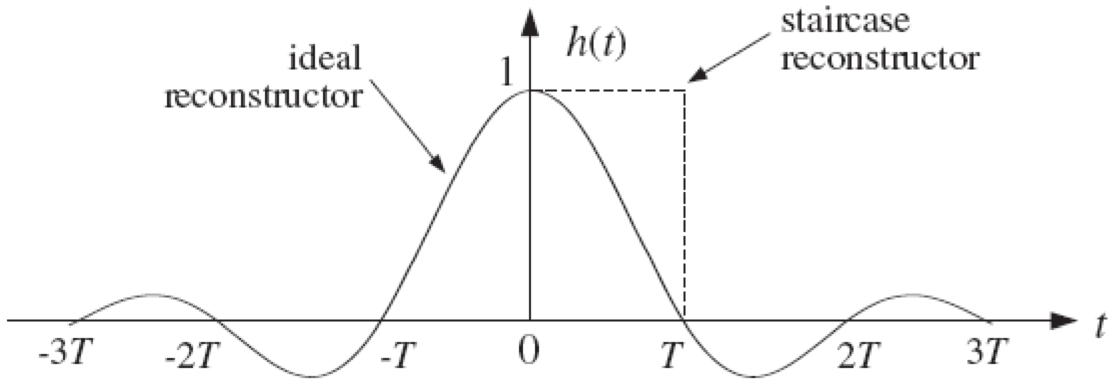


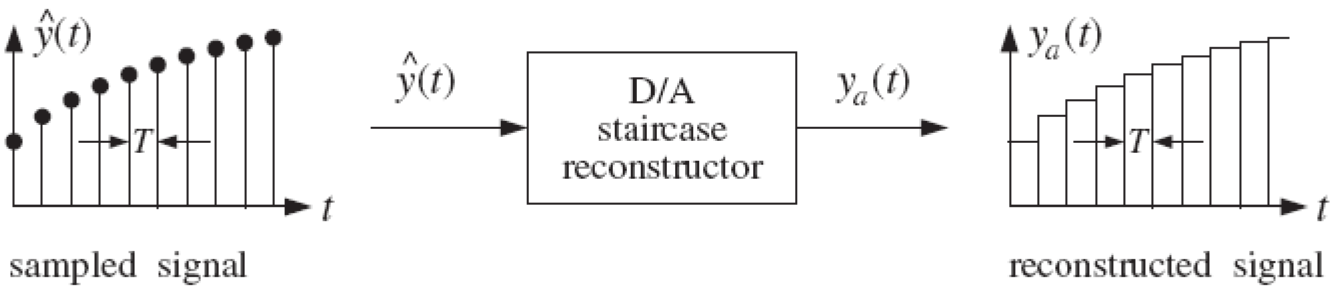


7. Ideal and practical analog reconstructors

- ❖ The ideal reconstructor has the impulse response: $h(t) = \dfrac{\sin(\pi f_s t)}{\pi f_s t}$ which is not realizable since its impulse response is not casual

- ❖ It is practical to use a staircase reconstructor





7. Ideal and practical analog reconstructors

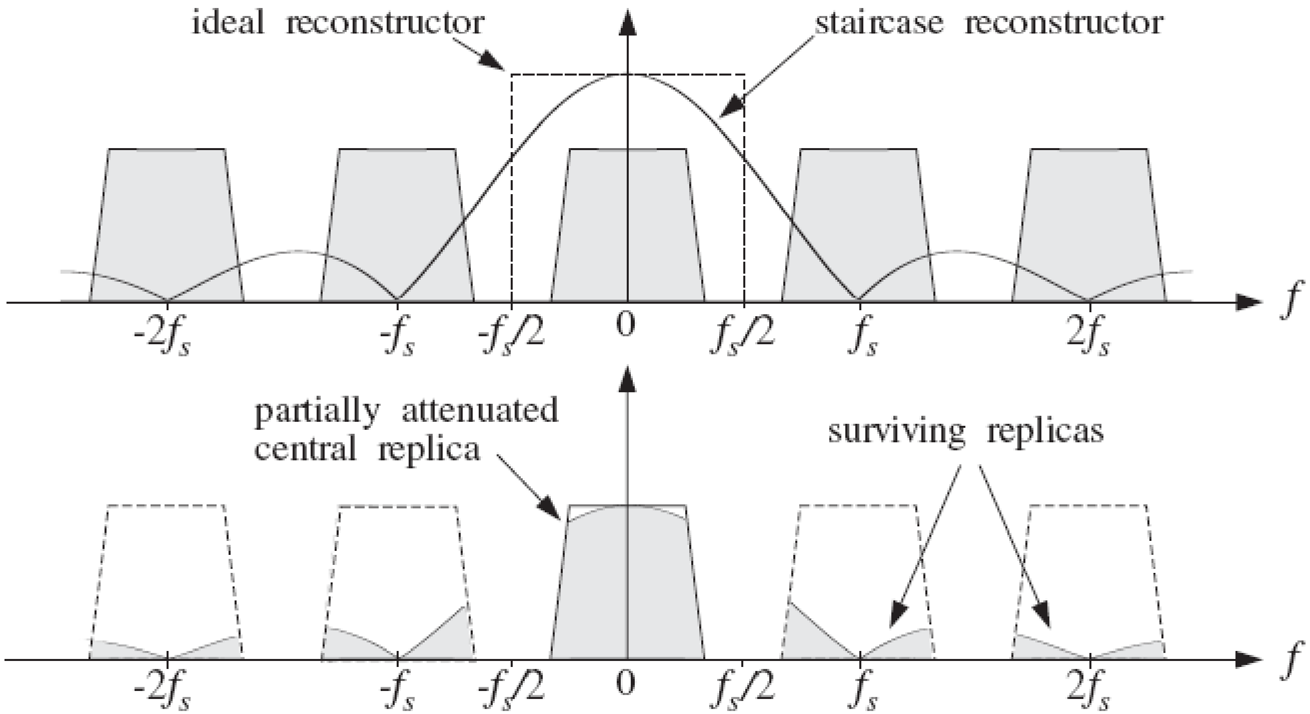


Fig: Frequency response of staircase recontructor

7. Practical reconstructors-antiimage postfilter

❖ An analog lowpass postfilter whose cutoff is Nyquist frequency fs/2 is used to remove the surviving spectral replicas.

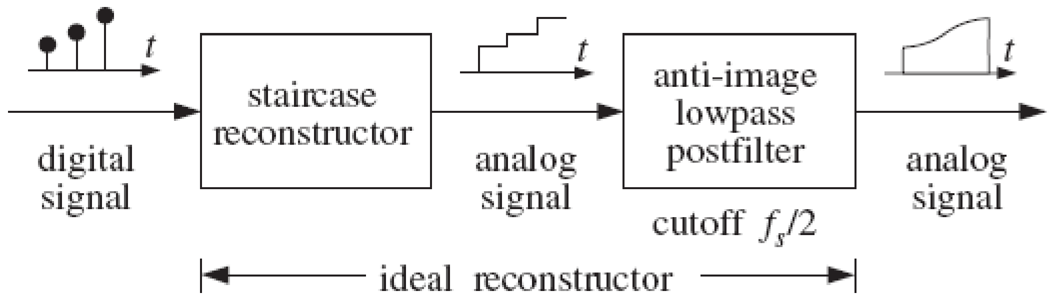


Fig: Analog anti-image postfilter

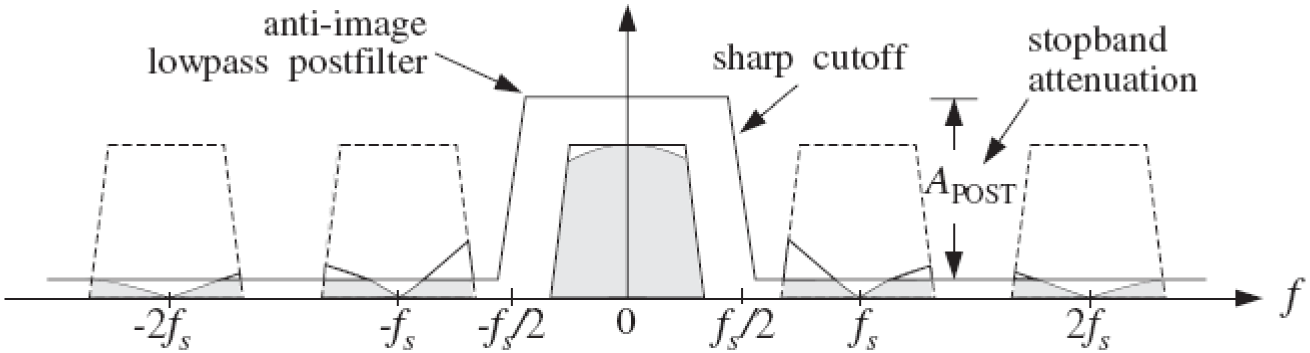


Fig: Spectrum after postfilter

8. Homework

❖ Problems: 1.2, 1.3, 1.4, 1.5, 1.9