2.1 Spatial Shift: $F[f(x-a)](\omega) = e^{-i2\pi\omega a} \, . \, F[f](\omega)$

Proof:

$$F[f(x-a)](\omega) = \int_{-\infty}^{+\infty} f(x-a)\, e^{-i2\pi\omega x}\, dx$$

Put, $\tau = x-a$ and $d\tau = dx$

So,
$$F[f(x-a)](\omega) = \int_{-\infty}^{+\infty} f(\tau)\, e^{-i2\pi\omega(\tau+a)}\, d\tau$$

$$= e^{-i2\pi\omega a} \int_{-\infty}^{+\infty} f(\tau)\, e^{-i2\pi\omega\tau}\, d\tau$$

$$= e^{-i2\pi\omega a}\, F[f](\omega).$$

2.2 Convolution: $F[(K * f)(x)](\omega) = F[K](\omega) \, . \, F[f](\omega)$

Proof: The convolution of $K(x)$ and $f(x)$ is defined in the interval $(-\infty, +\infty)$ as $K * f = \int_{-\infty}^{+\infty} K(u)\, f(x-u)\, du$

Now,
$$F\{K*f\} = \int_{-\infty}^{+\infty} e^{-i2\pi\omega x} \left[\int_{-\infty}^{+\infty} K(u)\, f(x-u)\, du \right] dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i2\pi\omega x}\, K(u)\, f(x-u)\, du\, dx$$

changing the order of the integration:

$$= \int_{-\infty}^{+\infty} K(u) \left[\int_{-\infty}^{+\infty} e^{-i2\pi\omega x}\, f(x-u)\, dx \right] du$$

put, $x-u=t \Rightarrow dx = dt$

so,
$$F\{K*f\} = \int_{-\infty}^{+\infty} K(u) \left[\int_{-\infty}^{+\infty} e^{-i2\pi\omega(u+t)}\, f(t)\, dt \right] du$$

$$= \int_{-\infty}^{+\infty} K(u)\, e^{-i2\pi\omega u} \left[\int_{-\infty}^{+\infty} e^{-i2\pi\omega t}\, f(t)\, dt \right] du$$

By the property of definite integral :-

$$F[(K*f)(x)](\omega) = \int_{-\infty}^{+\infty} K(x)\, e^{-i2\pi\omega x}\, dx \int_{-\infty}^{+\infty} e^{-i2\pi\omega x}\, f(x)\, dx$$

$$= F[K](\omega) \, . \, F[f](\omega)$$

2.3 Derivative : $F\left[\frac{\partial f(x)}{\partial x}\right](\omega) = i2\pi\omega \cdot F[f](\omega)$

Proof: we know, $f = F^{-1}\{F\}$

therefore,

$$f(x) = \int_{-\alpha}^{+\alpha} F(\omega)\, e^{i2\pi\omega x}\, d\omega$$

Differentiating both sides with respect to x:

$$f'(x) = \int_{-\alpha}^{+\alpha} i2\pi\omega\, F(\omega)\, e^{i2\pi\omega x}\, d\omega$$

or, $f'(x) = F^{-1}\{i2\pi\omega\, F(\omega)\}$

Taking the Fourier Transform of both sides:

$$F\{f'(x)\} = F\{F^{-1}[i2\pi\omega\, F(\omega)]\}$$

So, $F\left[\frac{\partial f(x)}{\partial x}\right](\omega) = i2\pi\omega.\ F[f](\omega)$

Answer 3.1

It's a high pass filter. We can use this kind of filter for edge detection. The second order derivative in 1-D is:

$$f'[n] = \frac{f[n+1] - 2\,f[n] + f[n-1]}{\tau^2}$$