2.1

A symmetric matrix $(n+1) \times (n+1)$ is positive semi-definite if and only if all eigenvalues are non-negative.

Let, v be an arbitrary vector. Using the spectral decompontion we have,

$$v^T R v = (v^T U)\, \mathrm{diag}(\lambda) (U^T v) = \sum^{n} \lambda_l [v^T T]_l)^2$$

where U is a matrix containing the n orthogonal eigenvectors of R. The above expression is non-negative for all v if and only if $\lambda_l \geq 0$ for all $l = [l(0)\ l(1) \dots l(n)]$.

If R is an $(n+1)\times(n+1)$ symmetric matrix with real entries, then it has n orthogonal eigenvectors. We need to show that all the roots of the characteristic polynomial of **R** are real numbers.

Let, $z=a+bi$ is a complex number, it complex conjugate is defined by, $\bar{z}=a-bi$. We have $z\bar{z}=(a+bi)(a-bi)=a^2+b^2$, so $z\bar{z}$ is always a non-negative real number (and equal 0 only when $z=0$).

It is also true that if w, z are complex numbers then $\overline{wz}=\bar{w}\bar{z}$.

Let, v be a vector whose entries are allowed to be complex. It is no longer true that $v.v\geq 0$ with equality only when $v=0$. However, if $\bar{v}$ is the complex conjugate of v, it is true that $\bar{v}.v\geq 0$ with equality only when $v=0$. Indeed,

$$\begin{bmatrix} a_1-b_1i \\ a_2-b_2i \\ \vdots \\ a_n-b_ni \end{bmatrix}\begin{bmatrix} a_1+b_1i \\ a_2+b_2i \\ \vdots \\ a_n+b_ni \end{bmatrix} = (a_1^2+b_1^2)+(a_2^2+b_2^2)+\dots+(a_n^2+b_n^2)$$

which is always non negative and equals zero only when all the entries a_i and b_i are zero.

If λ is a eigenvalue of the real symmetric matrix R and there is a non zero vector v, also with complex entries such that $Rv=\lambda v$. By taking complex conjugate of both sides, noting that $\bar{R}=R$ since R has real entries, we get $\overline{Rv}=\overline{\lambda v} \Rightarrow R\bar{v}=\bar{\lambda}\bar{v}$. Then using that $R^T=R$,

$$\bar{v}^T R v = \bar{v}^T (Rv) = v^T(\lambda v) = \lambda(\bar{v}.v),$$

$$\bar{v}^T R v = (R\bar{v})^T v = (\bar{\lambda}\bar{v})^T v = \bar{\lambda}(\bar{v}.v).$$

since, $v\neq 0$, we have $\bar{v}.v\neq 0$. Thus $\lambda=\bar{\lambda}$, which mean $\lambda\in$ Real