Quantum Entanglement, Fair Sampling, and Reality

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Abstract

Quantum entanglement and its associated effects are among the most interesting topics in quantum mechanics, and provide a fascinating glimpse at the counter-intuitive world of quantum phenomena. This report studies entanglement through a thought experiment known as Mermins device. First, we introduce the entangled Greenberger-Horne-Zeilinger (GHZ) state, and provide a brief explanation of the Einstein-Podolsky-Rosen (EPR) Paradox. We will discuss the normal Mermin's device, a thought experiment done under the assumption that hidden variables exist. We then propose a modified form of Mermin's device that can be reconciled with experimental data, and demonstrate simulation results that verify our theoretical derivations. Finally, we summarize the operation of Mermin's Device and provide a brief description of entanglement phenomena on quantum computing and quantum information theory.

Keywords: Entanglement, Bell's Paradox, EPR Paradox, Mermin's Device

1. Introduction

Consider the following experiment. Three subjects, labeled A, B, and C, are respectively given paper cards K_A, K_B, and K_C, each of which is either labeled X or Y. They have been told beforehand that *either all three of them receive cards labeled X, or only one of them is labeled X*. They are also told that after receiving the cards, they should write down $+1$ or -1 on their card. Let us respectively denote $f_A(X)$, $f_B(X)$, and $f_C(X)$ as the number A, B, or C would write down if an X card is received, and similarly define

$f_A(Y)$, $f_B(Y)$, and $f_C(Y)$. The subjects are allowed to discuss strategies before they are given their paper cards, but not afterwards. *We say that the three subjects have won the game if and only if*:

$$f_A(K_A)\cdot f_B(K_B)\cdot f_C(K_C) = \begin{cases} 1, & K_A = K_B = K_C = X \\ -1, & \text{Exactly one of } K_A, K_B, \text{ or } K_C \text{ is } X \end{cases} \tag{1}$$

An interesting question arises—Is it possible for the three subjects to construct a strategy that guarantees them victory? Below, we will show that without using quantum mechanics, (i.e. in a classical setting), it is impossible for A, B, and C to win the game.

Suppose, in the most general sense, that the subjects devise some extra parameter k, to influence the probabilities with which they choose to write down $+1$ and -1. We suppose by contradiction that this strategy guarantees victory. Then there must exist some (perhaps not-unique) $k = k_0$ such that:

$$\begin{aligned} f_A(X,k_0)\cdot f_B(X,k_0)\cdot f_C(X,k_0) &= +1 \\ f_A(Y,k_0)\cdot f_B(Y,k_0)\cdot f_C(X,k_0) &= -1 \\ f_A(Y,k_0)\cdot f_B(X,k_0)\cdot f_C(Y,k_0) &= -1 \\ f_A(X,k_0)\cdot f_B(Y,k_0)\cdot f_C(Y,k_0) &= -1 \end{aligned}$$

In this case, if we multiply the left-hand and right-hand sides of the above equation, we arrive at the contradiction:

$$[f_A(X,k_0)\cdot f_A(Y,k_0)\cdot f_B(X,k_0)\cdot f_B(Y,k_0)\cdot f_C(X,k_0)\cdot f_C(Y,k_0)]^2 = -1$$

From the above discussion, we conclude that the subjects cannot prevail using any classical scheme.

However, if A, B, and C share a quantum state and plan strategies accordingly, it is possible for them to always win. Suppose the three subjects share the *Greenberger-Horne-Zeilinger* (*GHZ*) state, as proposed and described by quantum physicists Daniel Greenberger, Michael Horne, and Anton Zeilinger in 1989:

$$|\Psi\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

We recall that the Pauli matrices for the X and Y directions are:

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \equiv \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

Note that ± 1 are the eigenvalues of each operator. Let us denote by $|+x\rangle$ and $|-x\rangle$ the eigenstates of σ_x corresponding with $+1$ and -1, respectively, and similarly define $|+y\rangle$ and $|-y\rangle$. Then we have:

$$\begin{aligned} |+x\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-x\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ |+y\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \\ |-y\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \end{aligned}$$

Now, suppose that *each subject measures its state using the* $|+x\rangle, |-x\rangle$ *basis if he/she receives an* X*, and using the* $|+y\rangle, |-y\rangle$ *basis if he/she receives an* Y.

Recall that under the scheme of this game, either all three subjects receive Xs, or only one of them do. Let us consider the former case, in which all three subject would measure with the $|+x\rangle, |-x\rangle$ basis. Then we would have:

$$
\begin{aligned}
&(\langle +x| \otimes \langle +x| \otimes \langle +x|)|\Psi\rangle \\
=&\frac{1}{\sqrt{8}}(\langle 000| + \langle 001| + \langle 010| + \langle 011| + \langle 100| + \langle 101| + \langle 110| + \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
=&\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&(\langle +x| \otimes \langle -x| \otimes \langle -x|)|\Psi\rangle \\
=&\frac{1}{\sqrt{8}}(\langle 000| - \langle 001| - \langle 010| + \langle 011| + \langle 100| - \langle 101| - \langle 110| + \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
=&\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&(\langle -x| \otimes \langle -x| \otimes \langle -x|)|\Psi\rangle \\
=&\frac{1}{\sqrt{8}}(\langle 000| - \langle 001| - \langle 010| + \langle 011| - \langle 100| + \langle 101| + \langle 110| - \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
=&0
\end{aligned}
$$

$$
\begin{aligned}
&(\langle +x| \otimes \langle -x| \otimes \langle -x|)|\Psi\rangle \\
=&\frac{1}{\sqrt{8}}(\langle 000| - \langle 001| - \langle 010| + \langle 011| + \langle 100| - \langle 101| - \langle 110| + \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
=&0
\end{aligned}
$$

The numbers written down by the three subjects would thus consist of either three $+1$ or one $+1$ and two -1. In either scenario, *the product of the three numbers would be* $+1$.

Similarly, in the case that only one of the three subjects receive an X, only that subject would measure his/her state with the $|+x\rangle, |-x\rangle$ basis, while the other two would measure wth the $|+y\rangle, |-y\rangle$ basis:

$$(\langle +y| \otimes \langle +y| \otimes \langle +x|)|\Psi\rangle$$
$$= \frac{1}{\sqrt{8}}(\langle 000| + \langle 001| + i\langle 010| + i\langle 011| + i\langle 100| + i\langle 101| - \langle 110| - \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$= 0$$

$$(\langle +y| \otimes \langle -y| \otimes \langle +x|)|\Psi\rangle$$
$$= \frac{1}{\sqrt{8}}(\langle 000| + \langle 001| - i\langle 010| - i\langle 011| + i\langle 100| + i\langle 101| + \langle 110| + \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$= \frac{1}{2}$$

$$(\langle +y| \otimes \langle -y| \otimes \langle -x|)|\Psi\rangle$$
$$= \frac{1}{\sqrt{8}}(\langle 000| - \langle 001| - i\langle 010| + i\langle 011| + i\langle 100| - i\langle 101| + \langle 110| - \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$= 0$$

$$(\langle -y| \otimes \langle -y| \otimes \langle -x|)|\Psi\rangle$$
$$= \frac{1}{\sqrt{8}}(\langle 000| - \langle 001| - i\langle 010| + i\langle 011| - i\langle 100| + i\langle 101| - \langle 110| + \langle 111|) \cdot \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$
$$= \frac{1}{2}$$

The numbers written down by the three subjects would thus consist of either three -1 or one -1 and two $+1$. In either scenario, *the product of the three numbers would be* -1.

We thus conclude that under this strategy, by sharing the *GHZ* state, the subjects A, B, C are guaranteed to win.

2. EPR paradox and Bell's theorem

2.1. EPR paradox

As mentioned from the previous section, subjects A, B and C always win the game if we allow them to share quantum entangled states beforehand.

The absurdity of the situation is that it allows physical quantities to exist independently of measurements even though their values can only be defined as members of sets of measurements that are performed together. This implies that quantum theory cannot be equivalent to a certain type of classical theory.

A method of interpreting the GHZ experiment is to simply view it as a Bell theorem experiment. To discuss Bell's theorem, we must first provide a brief introduction of the EPR paradox.

The EPR Paradox was devised in 1935 by Albert Einstein, Boris Podolsky, and Nathan Rosen as a thought experiment to demonstrate what they perceived as a lack of completeness in quantum mechanics. A neutral pion, originally imagined to be at rest, decays into a pair of back-to-back photons (photons traveling in opposite directions). The pair of photons is described by a single two-particle wave function. Once separated, the two photons are still described by the same wave function, and a measurement of one observable of the first photon will determine the measurement of the corresponding observable of the second photon, *even though the distance between them is so far that information must travel faster than light in order for this phenomenon to occur*. If photon 1 is found to have spin up along the x-axis, then photon 2 must have spin down along the x-axis, since the total angular momentum of the final-state must be the same as the angular momentum of the initial state, that of a single neutral pion. Likewise, the measurement of another observable of the first system will determine the measurement of the corresponding observable of the second system. Essentially, the authors of the EPR paradox were concerned with the following two propositions:

- Quantum theory is complete.
- Incompatible observables cannot be simultaneous elements of reality.

In order to resolve the above paradox, Einstein, Podolsky, and Rosen postulated the existence of "hidden variables"— unknown properties of the systems that supposedly account for the discrepancies responsible for the occurrence of the EPR Paradox. The three physicists claimed that quantum mechanical theories are incomplete by nature, and cannot completely describe physical reality. Roughly speaking, they questioned the apparent fact that System II already contained all information regarding System I long before the scientist measures any of the observables. Furthermore, they claimed that the hidden variables would be local, so that no instantaneous action at a distance would be necessary. To figure out if local theory can explain and if the hidden variables exist, John Bell formulated an experiment and developed a famous inequality called Bell's inequality.

2.2. Bell's theorem

In 1964 John Bell proposed a mechanism to test for the existence of these hidden variables, and he developed his famous inequality as the basis for such a test. He showed that if the inequality were not satisfied, then it would be impossible to have a local hidden variable theory that accounted for the spin experiment.

To describe the theorem in simple terms, we first imagine a set of devices consisting of three unconnected black boxes. The first two boxes, A and B, are detectors. Each detector has a switch with three possible positions labeled v, w, and z, which respectively represent a measurement direction on z-axis, and two other measuring axes at angles of 120 with respect to each other. The measurement result, involving the spin of the detected particle (up or down), is recorded. The third black box, which serves as the source, sends pairs of particles to detectors A and B. The detectors are arbitrarily far from each other and are not connected in any way, so that the detectors have no influence on one another. The switches on detectors A and B are randomly and independently selected after a pair of particles has left the source and before either particle has arrived at its detector.

We consider a pair of particles produced in the entangled spin state $|S\rangle$, as described below:

$$|S\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2) \tag{2}$$

where the spins are relative to the z-axis. If there exist hidden variables, which can be interpreted as some collaborative strategy between photons, we must measure the expected frequency of different spin (up and down pair) being detected is either 1 or $\frac{5}{9}$. The reason for this is that if the particles contain hidden information all along, then the particles must have decided beforehand the spins for each of the 3 directions. There are 8 possibilities $(\uparrow, \uparrow, \uparrow)$, $(\uparrow, \uparrow, \downarrow)$, $(\uparrow, \downarrow, \downarrow)$, $(\downarrow, \downarrow, \downarrow)$, $(\downarrow, \uparrow, \uparrow)$, $(\uparrow, \downarrow, \uparrow)$ $(\downarrow, \downarrow, \uparrow)$ and $(\downarrow, \uparrow, \downarrow)$ which indicate the direction of spin detected along each axis (v, w, z). Let us briefly discuss the following two cases.

1. $(\uparrow, \uparrow, \uparrow)$

2. $(\uparrow, \downarrow, \uparrow)$

Note that other cases can be obtained by swapping either particles or coordinate axes.

- Swapping photons: $(\uparrow, \uparrow, \uparrow) \rightarrow (\downarrow, \downarrow, \downarrow)$
- Swapping axes: $(\uparrow, \downarrow, \uparrow) \rightarrow (\uparrow, \uparrow, \downarrow)$

First, let us consider the case that one of the photon is in the spin state $(\uparrow, \uparrow, \uparrow)$; then, no matter which axes get picked, there must always exist another photon in the spin state $(\downarrow, \downarrow, \downarrow)$. Next, if one of the photons is in the spin state $(\uparrow, \downarrow, \uparrow)$, then another photon must be in the spin state $(\downarrow, \uparrow, \downarrow)$. When measurements begin, switches are randomly chosen, which means that there are nine equally probable pairs of positions for the two detectors: $(v, v), (v, w), (v, z), (w, v), (w, w), (w, z), (z, z), (z, v)$,and (z, w). Of these 9 positions, 5 are opposite spin pairs. Therefore, if photons picked one of these two plans (hidden variables) they will always be more likely to have the opposite spin. This contradicts the result that the measurement result for the $|S\rangle$ state is always $\frac{1}{2}$, as described below.

If A detects a photon with spin $|\uparrow\rangle$ along the z-axis, the spin of another photon along z-axis must be $\downarrow$; since the switched axis is chosen randomly, it means we choose the z-axis $\frac{1}{3}$ of the time. However, if the particle is measured in one of the other two directions, it makes an angle of 60 degrees with these measurement directions. Thus, the result should be "$|\downarrow\rangle$" one-fourth of the time since the probability depend on the square of the cosine of half the angle.

$$P(S) = \cos^2(\frac{\theta}{2}) \tag{3}$$

Therefore, the probability that B still detects $|\downarrow\rangle$ is

$$P_B(\downarrow) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2} \tag{4}$$

which corresponds perfectly to the experiment result. We conclude that no hidden variables exist among the particles.

In the next section, we are going to simulate the device discussed in Bell's theorem, also known as Mermin's devices.

3. Mechanism of Mermin's Device

3.1. Basic Rules

1. Three Devices:

   A source is placed in the middle and two detectors, A and B, and on both sides of the particle source. These three devices operate independently.

2. Source:

   In any round, the source can simultaneously sent a pair of particles in opposite directions. One particle goes to detector A, and another goes to detector B. In addition, particles in a pair carry the same information, and detectors A and B receive these two particles at the same time.

3. Detectors:

   There are three switches (1, 2, and 3) and two lamps (with red and green) on each of the detectors. We use "R" to indicate red lamps, and "G" for green lamps.

4. Each Round:

   In the beginning, switches will randomly point to a number (that is, 1, 2, or 3) with equal probability for each round. Switches do not necessarily point to the same number. As detectors A and B receive the particles, one of the corresponding lamps for each detector will fire according to the instruction sets. The counter records whether the colors of two lighted lamps are the same or not.

5. Instruction Sets:

   Instruction sets contain information describing a particle. Since two particles in one pair carry the same information, we can define an "identical" instruction set for both of the detectors in one round.
   The instruction sets involve selecting $\{R\}$ or $\{G\}$ three times, corresponding to the three switches. That is, there are 8 possible instruction sets—$\{RRR\}$, $\{RRG\}$, $\{RGR\}$, $\{GRR\}$, $\{RGG\}$, $\{GRG\}$, $\{GGR\}$, and $\{GGG\}$.

   For example, suppose the instruction set is $\{RRG\}$. In this case, if the switch points to 1 or 2, the red lamps will fire when the detector receives a particle. If the switch points to 3, the green lamps will fire when the detector receive a particle. Similarly, if the instruction set is defined to $\{RRR\}$, then the detector always fire the red lights regardless of the position of the switch.

6. The Entire Process:

   During the entire process, the source sends an arbitrary large number of pairs of particles, and the counter counts for each round. At the end, the ratios of the same color or different colors detected by detectors A and B are recorded.

7. Sample Space for Detectors:

   In conclusion, there are three switches on each of the detectors. The possible positions of these detectors thus form the sample space $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$, where the first and second numbers indicate the location of the switch for detector A and detector B, respectively. Each of these nine items has a probability of $\frac{1}{9}$.
   Also, there are two lamps on detectors, which can be labeled as "RR", "RG", "GR", and "GG" similar to the switches labels. The corresponding probabilities of these possibilities are not uniform, and are dependent to the instruction sets. Therefore, one can use the label "12RG" to indicate the situation in which the switch points to number 1 and the red lamp fires on detector A, while the switch point to number 2 and the green lamp fires on detector B.

8. Two Possible Results:

   (a) **Case 1**—Both of the switches point to the same number. Situations can be labeled as 11, 22, 33.

   (b) **Case 2**—Two switches point to different number. Situations can be labeled as 12, 21, 13, 31, 23, 32.

At this stage, one focuses on the probability for whether the two colors of the firing lamps are identical or not, so one can summarize the instruction sets into $\{RRR\}$ and $\{RRG\}$. The result for $\{GGG\}$ is as same as $\{RRR\}$, and results for the 5 remaining sets are as same as $\{RRG\}$.

3.2. Result of Original Mermin's device

1. **Case 1**—For each of the eight instruction sets, since the switches point to the same number, the lights always fire the same color. That is

$$P(RR \text{ or } GG|\text{case } 1) = 1$$

2. **Case 2**—For instruction sets $\{RRR\}$ or $\{GGG\}$, the two lamps always fire the same color, regardless of the location of the switches. Thus, the probability of detecting the same color from each lamp is 1:‘

$$P(RR|\text{ case 2 and } \{RRR\}) = P(GG|\text{ case 2 and } \{GGG\}) = 1$$

   For the six remaining instruction sets, such as $\{RRG\}$, one can analyze the results using the following steps. Consider the set $\{RRG\}$, which as previously discussed indicates that a particle fires "R" when the location of the switch is 1 or 2, and fires "G" when the location of the switch is 3. The sample space of results is$\{12RR, 21RR, 13RG, 31GR, 23RG,$ $32GR\}$. Among these outcomes, "12RR" and "21RR" clearly indicate same color, and therefore:

$$\begin{cases} P(\text{same color }, RR) = \dfrac{1}{3} \\ P(\text{different colors}) = \dfrac{2}{3} \end{cases}$$

Finally, we summarize all results from original Mermin's device into the following table: (Same color indicates "RR" or "GG")

Instruction Sets	RRR	RRG
Case 1	1	1
Case 2	1	$\frac{1}{3}$

Table 1: Probability of same color under each condition

4. Mechanism of Extended Mermin's Device

4.1. Conundrum of Normal Mermin's device

The same color probability $\frac{1}{3}$ in normal Mermin's devices do not agree with the theoretical probability $\frac{1}{4}$. This conundrum can be resolved by introducing a new instruction element **N** to normal Mermin's devices, corresponding to the event in which **none** of the lamps are fired in the detector, thus creating *extended Mermins devices.*

4.2. New Instruction Set

N indicates that no lamps are fired in the detector. There are two possible reasons why a detectors lamp may fail to fire. First, **N** could be caused by the detector itself (1^{st} interpretation), or N could carried by the incident particles (2^{nd} interpretation). Below, we will discuss the physics behind each interpretation.

- 1^{st} Interpretation:

  In the first interpretation, the absence of a flash is caused by the inefficiency of the detector. This indicates that **N** occurs independently of the incident particles. For example, if the detector "misses" incident particles, no lamp will be fired. In accordance with the first interpretation, we may postulate the existence of an additional switch **0** in the detector. The switch points to **0** when event **N** occurs. However, when we calculate the probability that the two detectors fire different colors, event **N** is disregarded. That is to say, we can ignore switch **0** when calculating the probability, which implies that the device exhibits the same behavior as the normal Mermin's device. The conundrum thus cannot be solved using the first interpretation.

- 2^{nd} Interpretation:

  In the second interpretation, the event **N** is caused by incident particles. In this interpretation, we designate one of the switches in the detector to correspond to the event **N**, to create the *Extended Mermins device.* In the the Extended Mermin's device there are two constraints on the event **N**. First, **N** can only replace a color that appears twice in the instruction set. Second, **N** can only appear in one of the instruction sets (for one of the detectors), meaning that the instruction sets for the two detectors no longer remain the same.

Finally, we prove in Section 4.3 that *under certain constraints, the second interpretation can reconcile the behavior of the extended Mermins device with recorded experimental results* (recall that the behavior of the regular Mermins device contradicts experimental results.) For example, if the original instruction set is "GGR", we may replace switch 2 with **N**, and the new instruction set becomes "GNR". The probabilities of the switch landing on positions G, N, or R are still identical and independent ($= \frac{1}{3}$).

4.3. Result of extended Mermin's device

Since the instruction sets are no longer identical for a pair of particles headed towards the two detectors, we can no longer use certain sets (for example, $\{RRR\}$) to represent the entire system. Instead, one needs to extend the set of possible instructions. For instance, the new instruction $\{RRN$ - $RRR\}$ can be employed to represent the fact that the instruction of detector A is $\{RRN\}$, which turns to N (null) when the switch points to number 3, while the instruction of detector B is $\{RRR\}$.

Again, at this stage, our focus remains on the probability that the two lamps display the same color, so one can summarize the instruction sets into $\{RRN$ - $RRR\}$, $\{RRN$ - $RRG\}$ and $\{RNG$ - $RRG\}$. We focus on these three sets below, as they can be used to represent all other cases.

1. **Case 1**— For the instruction set $\{RRN$ - $RRR\}$, we disregard experimental results in which detector As switch is on position 3, since no lamp on the detectors fires. This occurs with probability:

$$P(\text{Discarded}|\text{case 1}, \{RRN \text{ - } RRR\}) = \frac{1}{3}$$
$$P(\text{ Effective}|\text{case 1}, \{RRN \text{ - } RRR\}) = \frac{2}{3}$$

Similarly, for instruction sets $\{RRN$ - $RRG\}$ and $\{RNG$ - $RRG\}$, discarded events each occur with probability of $\frac{1}{3}$.

Given an effective (no N detected) situation, since the switches are on the same position, the lights always fire the same color. That is,

$$P(RR \text{ or } GG|\text{case 1, Effective}) = 1$$

2. **Case 2**—

For instruction sets $\{RRN$ - $RRR\}$, when detector A's switch is on position 3, the results are discarded, since no lamp on detectors fire:

$$P(RR|\text{ case 2 and }(\{RRN\text{ - }RRR\}) = \frac{2}{3}$$

$$P(\text{ Discarded }|\text{ case 2 and }(\{RRN\text{ - }RRR\}) = \frac{1}{3}$$

For all other situations, the two lamps always fire the same color, regardless of the location of the switches:

$$P(RR|\text{ case 2 and Effective and }\{RRN\text{ - }RRR\}) = 1$$

For instruction sets $\{RRN$ - $RRG\}$, the sample space of possible results is $\{12RR, 21RR, 13RG, 31NR, 23RG, \text{ and } 32NR\}$.

However, the results are discarded if detector A's switch points to number 3. Thus, the effected sample space is $\{12RR, 21RR, 13RG, \text{ and } 23RG\}$:

$$P(\text{ Discarded }|\text{ case 2 and }\{RRN\text{ - }RRG\}) = \frac{1}{3}$$

$$P(\text{ Effective }|\text{ case 2 and }\{RRN\text{ - }RRG\}) = \frac{2}{3}$$

Among the effective outcomes, "12RR" and "21RR" indicate the same color, while "13RG" and "23RG" indicate different colors:

$$P(RR = \text{ Same color }|\text{ case 2 and Effective and }\{RRN\text{ - }RRG\}) = \frac{1}{2}$$

For instruction sets $\{RNG$ - $RRG\}$, the sample space for the result is $\{12RR, 21NR, 13RG, 31GR, 23NG, 32GR\}$.

Similarly, for $\{RRN\text{-}RRG\}$, the results are discarded if the switch of detector A points to number 2, while the affected sample space is $\{12RR, 13RG, 31GR, 32RG\}$:

$$P(\text{ Discarded }|\text{ case 2 and }\{RNG\text{ - }RRG\}) = \frac{1}{3}$$

$$P(\text{ Effective }|\text{ case 2 and }\{RNG\text{ - }RRG\}) = \frac{2}{3}$$

Among effective (no N detected) outcomes, "12RR" indicate that the same color is observed, while "13RG", "31GR" and "23GR" indicate that different colors are observed. Therefore:

$$P(RR = \text{ Same color} \mid \text{case 2, Effective,}\{RNG\text{ - }RRG\}) = \frac{1}{4}$$

Instruction Sets	Prob in Case 1	Prob in Case 2	Total Prob
RRN-RRR	1	$\frac{1}{3}$	$\frac{5}{9}$
RRN-RRG	1	$\frac{1}{3}$	$\frac{5}{9}$
RNG - RRG	1	$\frac{1}{4}$	$\frac{1}{2}$

Table 2: Probability of same color under each Instruction set

5. Simulation

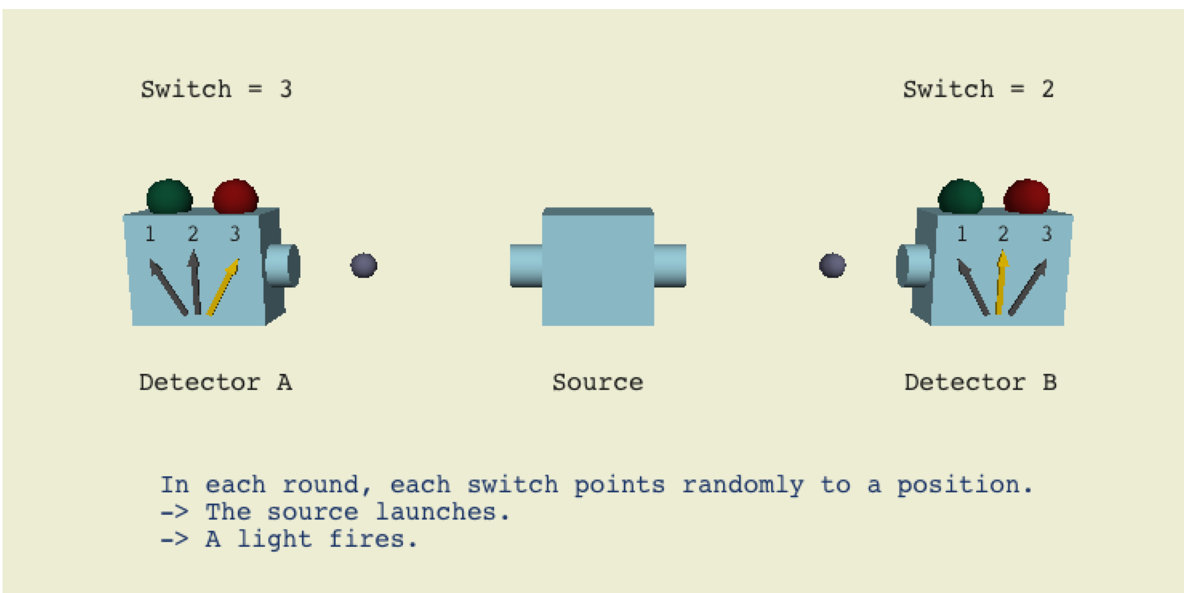

Figure 1: Simulation of Mermin Device

Shown below is an organized table containing our simulation results for the Normal Mermin's device and the Extended Mermin's device. Our source code is on githubblablabla.

Device	Instruction Sets	1 million pairs	10 millions pairs	Theoretical value
Normal	$\{RRR$ - $RRR\}$	1.00000	1.00000	1
Normal	$\{RRG - RRG\}$	0.33398	0.33319	$\frac{1}{3}$
Extended	$\{RRN$ - $RRR\}$	1.00000	1.00000	1
Extended	$\{RNG$ - $RRG\}$	0.25133	0.25025	$\frac{1}{4}$
Extended	$\{RRN$ - $RRG\}$	0.49049	0.49999	$\frac{1}{2}$

Table 3: Probability of same color in our simulation

6. Conclusion

The most fascinating features of quantum mechanics often revolve around the fact that quantum phenomena are often counterintuitive. As we have seen, the probabilistic results derived from Mermins device shed light on seemingly absurd aspects of quantum entanglement. Aside from their value to theoretical physics, concepts illuminated from Mermins device can also be applied to quantum information theory and quantum computation. For instance, we have shown in the first section that certain results, such as an algorithm guaranteed to win the GHZ game, cannot be obtained from classical distribution theories, but can be derived by assigning events to quantum states. Quantum mechanics, in effect, opens the door to entirely new designs of encryption and data processing algorithms. Although Mermin's device may appear to be a mere example of entanglement at first glance, its implications have the power to revolutionize human technology.

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