Coupled Harmonic Oscillators

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(Dated: November 17, 2022)

I. INTRODUCTION

Coupled harmonic oscillators represent many common physical systems and provide a more advanced example of harmonic motion than individual oscillators while still remaining relatively simple to solve mathematically. [1]

For our exploration of this phenomena, we used strings and masses to record frequencies of individual oscillators, modes, and beat phenomena experimentally and make comparisons with the theoretical predictions.

A. Theoretical Background

Coupled harmonic oscillators are systems which experience a restoring force that is proportional to their displacement and are linked in such a way to allow energy to transfer between one another. [1] From Newton's second law and the harmonic oscillator equation, we see that one mode of a coupled oscillator with identical masses and spring constants, is characterized by

$$x_1(t) = cos(\omega_+ t) \tag{1}$$

$$x_2(t) = -cos(\omega_+ t), \tag{2}$$

with a frequency of

$$\omega_+ = \frac{k + 2\kappa}{m}. \tag{3}$$

The other is given by

$$x_1(t) = cos(\omega_- t) \tag{4}$$

$$x_2(t) = cos(\omega_- t), \tag{5}$$

with frequency

$$\omega_- = \sqrt{\frac{k}{m}}. \tag{6}$$

Beats occur when one mass is displaced. Energy is then transferred from the first oscillator to the other until only one is moving, and then the process repeats. [1] There are two frequencies occurring in this process. The "fast" frequency is the frequency of oscillatory motion experienced by one mass. The "slow" frequency is the time it takes for the system's energy to transfer from one oscillator to the other and then back to the first oscillator again. The fast oscillation frequency is given by

$$\frac{(\omega_+ + \omega_-)}{2}, \tag{7}$$

while the slow frequency is represented by

$$\frac{(\omega_+ - \omega_-)}{2}. \tag{8}$$

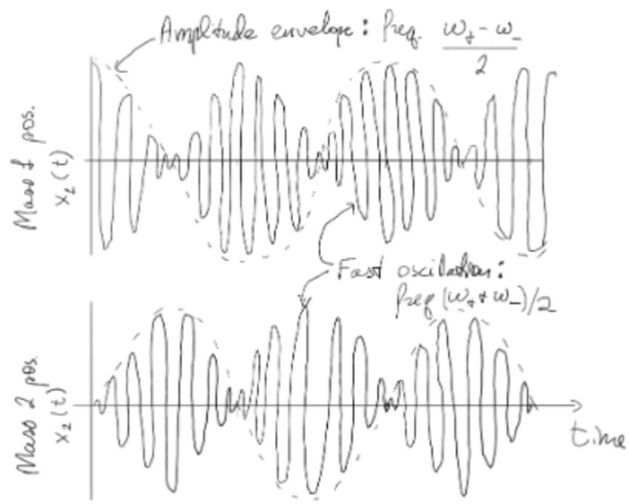

FIG. 1. Beat Phenomena Positions

II. EXPERIMENTAL PROCEDURE

First, we placed our board on a thick cloth to reduce the amount of interference from environmental vibrations as much as possible. Then we inserted two thumbtacks into a board at a distance of 8.400 ± 0.002 cm apart. We then tied two strings to both of the thumbtacks so that there was significant slack. At two points on each of the strings, we then used two more pieces of string to tie two nearly identical masses to create our oscillator.

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After setting up the two oscillators, we manually displaced each separately to measure the individual frequencies and check that they were identical. Then we displaced the oscillators at increasing angles to determine where the system was no longer linear. We then coupled the oscillators by tying a string at the point of intersection. Next, we excited the parallel mode and recorded the frequency. We then produced the symmetric mode by displacing the two masses opposite of one another and again recording the frequency. Finally we induced beats by displacing only one of the masses and recording the fast and slow frequencies.

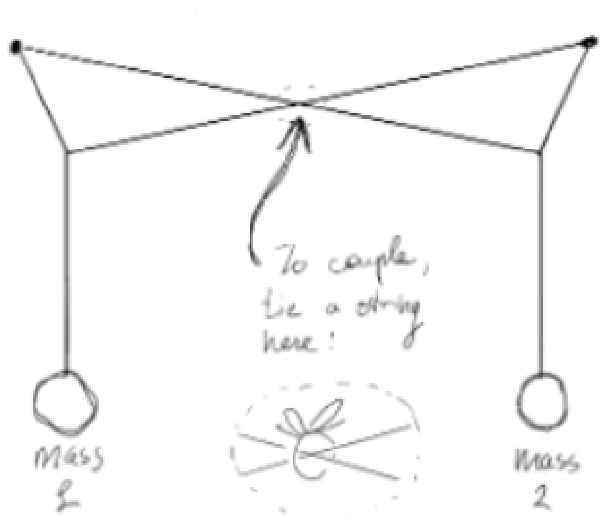

FIG. 2. Coupled Oscillators

III. DATA

TABLE I. Oscillation Frequency of Individual Oscillators

Run	m_1 Frequency [Hz]	m_2 Frequency [Hz]
1	0.63 ± 0.01	0.59 ± 0.01
2	0.62 ± 0.01	0.65 ± 0.01
3	0.59 ± 0.01	0.69 ± 0.01
4	0.65 ± 0.01	0.63 ± 0.01
5	0.62 ± 0.01	0.66 ± 0.01
6	0.65 ± 0.01	0.63 ± 0.01
7	0.56 ± 0.01	0.63 ± 0.01
8	0.55 ± 0.01	0.65 ± 0.01
9	0.62 ± 0.01	0.60 ± 0.01
10	0.65 ± 0.01	0.62 ± 0.01

TABLE II. Beat Phenomena Fast and Slow Frequency

Run	Fast Frequency [Hz]	Slow Frequency [Hz]
1	0.59 ± 0.01	3.82 ± 0.01
2	0.53 ± 0.01	3.97 ± 0.01
3	0.55 ± 0.01	3.96 ± 0.01
4	0.55 ± 0.01	3.46 ± 0.01
5	0.65 ± 0.01	3.80 ± 0.01
6	0.52 ± 0.01	3.93 ± 0.01
7	0.59 ± 0.01	3.82 ± 0.01
8	0.63 ± 0.01	3.73 ± 0.01
9	0.50 ± 0.01	3.95 ± 0.01
10	0.49 ± 0.01	3.85 ± 0.01

TABLE III. Oscillation Frequency of Parallel Mode

Run	Frequency [Hz]
1	0.66 ± 0.01
2	0.65 ± 0.01
3	0.68 ± 0.01
4	0.60 ± 0.01
5	0.70 ± 0.01
6	0.62 ± 0.01
7	0.66 ± 0.01
8	0.60 ± 0.01
9	0.67 ± 0.01
10	0.73 ± 0.01

TABLE IV. Oscillation Frequency of Symmetric Mode

Run	Frequency [Hz]
1	0.55 ± 0.01
2	0.53 ± 0.01
3	0.53 ± 0.01
4	0.51 ± 0.01
5	0.55 ± 0.01
6	0.56 ± 0.01
7	0.55 ± 0.01
8	0.55 ± 0.01
9	0.55 ± 0.01
10	0.55 ± 0.01

IV. ANALYSIS

We found that at an angle of 80 degrees the oscillator no longer behaved linearly. This follows from the small angle approximation which states that for small angles, $sin\theta \approx tan\theta \approx \theta$.

The values and standard deviations were found using Mathematica. The frequency of the m_1 oscillator was 0.61 ± 0.01 [Hz] and m_2 was 0.63 ± 0.01 Hz, which corresponds to a 3% difference. The frequency of the fast beat was 0.57 ± 0.02 Hz and the slow was 3.83 ± 0.04 Hz. The frequency of the symmetric mode was 0.543 ± 0.005 Hz and the parallel mode was 0.66 ± 0.01 Hz.

V. CONCLUSION

We found an equivalent spring constant of $k/m = 0.40 \pm 0.01 s^{-2}$

Unfortunately due to the masses that we used in this experiment, there was a large amount of energy lost in the system to rotational energy of the masses. Although we tried to reduce the interference as much as possible, there was always at least some rotation of the two masses. [2] Although this motion could be analyzed mathematically, the focus of this lab was on one type of motion, so this rotational energy loss was accounted for in the error values.

Other energy was lost in the system from air resistance, although this effect was minimal. Energy was also lost by the system to the environment in the form of heat energy. [2] Noise occurred from air flow although this effect was negligible. However, a non-negligible amount of noise occurred from vibrations from the ground which could couple with our experimental setup. [1] Although we tried to reduce this interference, it was impossible to reduce the effect completely.

Additionally, experimental error occurred when timing the frequency of oscillation. This was done using a digital timer on a smart phone. The stopping and starting time was not done as accurately as if a laser was used to record the mass returning to exactly the same position it started at, and that would make for less error in future experiments.[2] Length and displacement measurements were also taken using a ruler which was only marked to a sixteenth of an inch. Also, there was no way to measure angles in this experiment. Experimental error also occurred due to the string used since it was slightly elastic. Finally, the two masses used were not identical and their masses are unknown, so the experiment could be improved by using identically known masses.

[1] R. Fitzpatrick, *Oscillations and Waves: An Introduction*, 1st ed. (CRC Press, 2017).

[2] P. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for Physical Sciences*, 3rd ed., Vol. 2 (McGraw-Hill Education, 2002).

[3] R. P. Feynman, Phys. Rev. **94**, 262 (1954).

[4] G. C. King, *Vibrations and Waves*, 1st ed. (Wiley, 2009).