**Question 5**

a. Use induction to prove that for any positive integer n, 3 divide $n^3 + 2n$

Base case: 1,

When $n = 1$, $n^3 + 2n$ is 3, which is divisible by 3.

Inductive step:

Suppose that for positive integer k, $k^3 + 2k$, is divisible by 3,

$k^3 + 2k = 3m$, for some positive integer m

then we will show that

$(k+1)^3 + 2(k+1)$ is also divisible by 3

$k^3 = 3m - 2k$

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

Substituting in $k^3$:

$= 3m - 2k + 3k^2 + 3k + 1 + 2k + 2$

$= 3m + 3k^2 + 3k + 3$

$(k+1)^3 + 2(k+1) = 3(m + k^2 + k + 1)$

Since m and k are both positive integers, $(m + k^2 + k + 1)$ must also be an integer and therefore $(k+1)^3 + 2(k+1)$ is divisible by 3.

■

b. Use strong induction to prove that any positive integer n(n >= 2) can be written as a product of primes

Base case:

When n = 2, it is a prime number itself.

Inductive step:

Assume that for all k >= 2, any integer j (between 2 and k) can be expressed as a product of prime numbers. We will prove that k + 1 can also be expressed as a product of primes

If k+1 is prime, then it is a product of one prime number (itself).

If k + 1 is composite, then it can be expressed as a product of 2 integers, a and b, each of which are at least 2.

If (k+ 1) = a * b, then a = (k+1)/b

Since a is strictly less than k + 1, then a is less than k.

We can apply the same argument to show that b = (k + 1)/a and is less than a.

Thus a and b fall into the range from 2 through k and we can apply the inductive hypothesis to them and each a and b can be expressed as a product of primes.

$a = p_1 * p_2 \ldots * p_x$

$b = q_1 * q_2 \ldots * q_y$

(k + 1) can be expressed as a product of primes:

$(k+1) = a * b = (p_1 * p_2 \ldots * p_x) * (q_1 * q_2 \ldots * q)$

■

**Question 6**

Solve the following questions from zybooks:

a. Exercise 7.4.1 sections a, b, c, d, e, f, g

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

a. $\sum_{j=1}^{3} 3^{2} = \frac{3(3+1)(2*3+1)}{6}$

= 14.

b. P(k)

$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

c. P(k+1)

$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

Simplified:

$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

d. The base case would be the smallest value that proves true, in this case it would P(1) would be true.

e. In the inductive step we must show that for all positive integers k, P(k) implies P(k + 1)

f. P(k)

g. For: $\sum_{j=1}^{n} j^{2} = \frac{n(n+1)(2*n+1)}{6}$

Base case: n = 1, $\sum_{j=1}^{1} 1^{2} = \frac{1(1+1)(2*1+1)}{6}$

Inductive step:

For all positive integers where k >= 1, if P(k) is true, then we will show P(k + 1) is true.

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore, P(k + 1) is true.

■

b. Exercise 7.4.3 section c

Prove that for n ≥ 1, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Base case:

n =1

$$\sum_{j=1}^{1} \frac{1}{1^2} \leq 2 - \frac{1}{1}$$

Inductive step: for all positive integers k >=1, if P(k), then P(k + 1) is true

P(k)

$$\sum_{j=1}^{k} \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

P(k+1)

$$\sum_{j=1}^{k+1} \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$$

$$\sum_{j=1}^{k+1} = \sum_{j=1}^{k} \frac{1}{k^2} + \frac{1}{(k+1)^2}$$

$\leq\ 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

$\leq\ 2 - \frac{1}{k} + \frac{1}{k(k+1)}$ (this was the hint given)

$=\ 2 - \frac{1(k+1)}{k(k+1)} + \frac{1}{k(k+1)}$

$=\ 2 - \frac{1}{k+1}$

■

c. Exercise 7.5.1 section a

Prove that for any positive integer n, 4 evenly divides $3^{2n}$ -1

Base case:

n = 1, P(1) = $3^2$ -1 = 8, 8 is divisible by 4.

Inductive step:

For any k greater than or equal to 4, if P(k) is true, then P(k+1) is also true.

P(k) = $3^{2k}$ - 1 = 4m, for some integer m.

P(k +1) = $3^{2(k+1)}$ - 1.

= $3^{2k}$ * $3^2$ -1

Inductive step:

$3^{2k}$ - 1 = 4m

$3^{2k}$ = 4m + 1

P(k + 1) = (4m + 1) *$3^2$ - 1

= 4 * 9m + 9 - 1

= 4(9m + 2)

Since m is an integer, 9m + 2 must also be an integer, and therefore P(k + 1) is divisible by 4.

■