1 Orthogonal Polynomials with respect to a Discrete Measure

1.1 A Discrete Measure

A discrete N-point measure $d\lambda_N$ supported on a finite number of distinct points can be represented with the set of N support points $\{t_i\}$ and the corresponding weights $\{w_i\}$, $i = 1, 2, \cdots, N$. Given functions u and v, their inner product on $d\lambda_N$ is defined as:

$$\langle u, v\rangle_{d\lambda_N} = \sum_{i=1}^{N} w_i u(t_i)v(t_i). \tag{1}$$

If $\langle u, v\rangle_{d\lambda_N} = 0$, u and v are orthogonal with respect to $d\lambda_N$. The norm of u is defined as:

$$||u||_{d\lambda_N} = \sqrt{\langle u, u\rangle_{d\lambda_N}} = \left(\sum_{i=1}^{N} w_i u(t_i)^2\right)^{1/2}.$$

1.2 Objection

Given the set of N support points $\{t_i\}$ with positive weights $\{w_i\}$, our goal is to establish a set of monic polynomials $\{\pi_k\}$, $k = 0, 1, 2, \cdots, N-1$, that are mutually orthogonal with respect to the discrete measure $d\lambda_N$.

1.3 Orthogonal Polynomials with respect to $d\lambda_N$

The monic orthogonal polynomials with respect to a discrete measure $d\lambda_N$, denoted as $\pi_k(\cdot) = \pi_k(\cdot, d\lambda_N), k = 0, 1, 2, \cdots, N$, satisfy the three-term recurrence relationship:

$$\begin{gathered} \pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t), \;\; k = 0, 1, 2, \cdots, N-2, \\ \pi_{-1}(t) = 0, \;\; \pi_0(t) = 1, \end{gathered} \tag{2}$$

where the recurrence coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ are determined by

$$\begin{aligned} \alpha_k &= \frac{\langle t\pi_k, \pi_k\rangle_{d\lambda_N}}{\langle \pi_k, \pi_k\rangle_{d\lambda_N}}, \;\; k = 0, 1, 2, \cdots, N-1, \\ \beta_k &= \frac{\langle \pi_k, \pi_k\rangle_{d\lambda_N}}{\langle \pi_{k-1}, \pi_{k-1}\rangle_{d\lambda_N}}, \;\; k = 1, 2, \cdots, N-1, \\ \beta_0 &= \langle \pi_0, \pi_0\rangle_{d\lambda_N} = \langle 1, 1\rangle_{d\lambda_N}. \end{aligned} \tag{3}$$

If we normalize every monic polynomial π_k by $\tilde{\pi}_k = \pi_k/||\pi_k||_{d\lambda_N}$, we obtain a set of orthonormal polynomials $\{\tilde{\pi}_k\}$ with a similar three-term recurrence relationship:

$$\begin{gathered} \sqrt{\beta_{k+1}}\tilde{\pi}_{k+1}(t) = (t-\alpha_k)\tilde{\pi}_k(t) - \sqrt{\beta_k}\tilde{\pi}_{k-1}(t), \;\; k = 0, 1, 2, \cdots, N-2, \\ \tilde{\pi}_{-1}(t) = 0, \;\; \tilde{\pi}_0(t) = 1/\sqrt{\beta_0}. \end{gathered} \tag{4}$$

For $n \leq N$, the n^{th} order Jacobi matrix associated with the measure $d\lambda_N$ is a symmetric tridiagonal matrix defined in the form:

$$\mathbf{J_n} = \mathbf{J_n}(d\lambda_N) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ 0 & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}, \tag{5}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are from (3). We write $\mathbf{J_n}(d\lambda_N)$ if we want to exhibit the measure $d\lambda_N$.

Let $\tilde{\boldsymbol{\pi}}(t) = (\tilde{\pi}_0(t), \tilde{\pi}_1(t), \cdots, \tilde{\pi}_{n-1}(t))^T$, the three-term recurrence relationship in (4) can be written in the matrix form:

$$t\tilde{\boldsymbol{\pi}}(t) = \mathbf{J_n}\tilde{\boldsymbol{\pi}}(t) + \sqrt{\beta_n}\tilde{\pi}_n(t)\mathbf{e_n}, \tag{6}$$

where $\mathbf{e_n} = (0, 0, \cdots, 1)^T$ is the last column of the identity matrix I_n.

Denote the zeros of $\tilde{\pi}_n$ (i.e. zeros of π_n) as $\{\tau_v\}$, $v = 1, 2, \cdots, n$. Setting $t = \tau_v$ in (6), we obtain that $\{\tau_v\}$ are eigenvalues of $\mathbf{J_n}$, and $\tilde{\boldsymbol{\pi}}(\tau_v)$ are the corresponding eigenvectors. Thus, we can decompose the Jacobi matrix in the following way:

$$\mathbf{J_n} = \mathbf{V}\mathbf{D}_\tau\mathbf{V^T}, \tag{7}$$

where $\mathbf{V} = (\tilde{\boldsymbol{\pi}}(\tau_1), \tilde{\boldsymbol{\pi}}(\tau_2), \cdots, \tilde{\boldsymbol{\pi}}(\tau_n))$ and $\mathbf{D}_\tau = diag(\tau_1, \tau_2, \cdots, \tau_n)$.

It can be shown that there exists an orthogonal similarity transformation from $\{\tau_v\}$ in $\mathbf{D}_\tau$ to the recurrence coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ in $\mathbf{J_n}$:

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{\boldsymbol{\lambda}}^T \\ \sqrt{\boldsymbol{\lambda}} & \mathbf{D}_\tau \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{V}^T \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{\beta_0}\mathbf{e_1}^T \\ \sqrt{\beta_0}\mathbf{e_1} & \mathbf{J_n} \end{bmatrix}, \tag{8}$$

where $\mathbf{J_n} = \mathbf{J_n}(d\lambda_N)$ is from (5), $\mathbf{V}$, $\mathbf{D}_\tau$ are from (7), and

$$\sqrt{\boldsymbol{\lambda}} = (\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n})^T,$$

with λ_v denoting the weight of the point τ_v in the n-point Gauss quadrature rule.

1.4 A Stable Method to Compute Recurrence Coefficients

In order to evaluate the discrete orthogonal polynomials $\{\pi_k\}$, we have to compute the recurrence coefficients $\{\alpha_k\}$ and $\{\beta_k\}$. An intuitive method, known as the Stieltjes precedure, focuses on the three-term recurrence relationship in (2) and computes $\{\pi_k\}$ and $\{\alpha_k\}$, $\{\beta_k\}$ alternatively. However, this procedure is numerically unstable as k increases. Below we introduce a stable method to obtain $\{\alpha_k\}$ and $\{\beta_k\}$ through the Lanczos process.

1.4.1 The Lanczos Process

The k^{th} Krylov subspace of a matrix $\mathbf{A} \in \mathbb{C}^{N\times N}$ and a nonzero vector $\mathbf{b} \in \mathbb{C}^N$ is defined as

$$K_k(\mathbf{A}, \mathbf{b}) = span\{\mathbf{b}, \mathbf{A}\mathbf{b}, \cdots, \mathbf{A}^{k-1}\mathbf{b}\}, \ \ 1 \le k \le N.$$

Given a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{N\times N}$ and a vector $\mathbf{b} \in \mathbb{C}^N$, the Lanczos process computes the Hessenberg reduction of $\mathbf{A}$, defined as $\mathbf{H} = \mathbf{Q}^\star \mathbf{A}\mathbf{Q}$, where $\mathbf{H} \in \mathbb{C}^{N\times N}$ is symmetric tridiagonal, and $\mathbf{Q} \in \mathbb{C}^{N\times N}$ is unitary. The columns of $\mathbf{Q}$, denoted as $\{\mathbf{q_1}, \mathbf{q_2}, \cdots, \mathbf{q_N}\}$, where $\mathbf{q_1} = \mathbf{b}/||\mathbf{b}||_2$, form a orthonormal basis of $K_N(\mathbf{A}, \mathbf{b})$.

1.4.2 The Stable Method (?)

Starting with N support points $\{t_i\}$ and their weights $\{w_i\}$, we define vector $\sqrt{\mathbf{w}}$ and diagonal matrix $\mathbf{D_t}$ as follows:

$$\sqrt{\mathbf{w}} = (\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_N})^T,$$
$$\mathbf{D_t} = diag(t_1, t_2, \cdots, t_N).$$

Similar to (8), we have

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q_1}^T \end{bmatrix} \begin{bmatrix} 1 & \sqrt{\mathbf{w}}^T \\ \sqrt{\mathbf{w}} & \mathbf{D_t} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q_1} \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{\beta_0}\mathbf{e_1}^T \\ \sqrt{\beta_0}\mathbf{e_1} & \mathbf{J_N} \end{bmatrix}. \tag{9}$$

Accordingly, we construct a Hermitian matrix $\mathbf{A} = \begin{bmatrix} 1 & \sqrt{\mathbf{w}}^T \\ \sqrt{\mathbf{w}} & \mathbf{D_t} \end{bmatrix}$ and take $\mathbf{b} = \mathbf{e_1}$. The Lanczos process leads to a symmetric tridiagonal matrix as on the right of (9), the major and minor diagonal elements of which are the recurrence coefficients of interest.

For any given order $K \le N - 1$, , we tridiagonalize $\mathbf{A}$ iteratively till the result contains $\mathbf{J_K}$. With the recurrence coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ from $\mathbf{J_K}$, and the three-term recurrence relationship in (2), we can evaluate $\{\pi_k\}, k = 0, 1, \cdots, K$ at all support points $\{t_i\}, i = 1, 2, \cdots, N$.