1a) $n^5 + n - 20 \notin O(n^4)$

Definition of $O(g(n))$

$\exists M > 0, \exists n_0 > 0, \forall n > n_0 \quad (n^5 + n - 20) \leq M n^4$

$\neg\, O(g(n))$

$\forall M > 0, \forall n_0 > 0, \exists n > n_0 \quad (n^5 + n - 20) > M n^4$

① Let $M > 0$ and $n_0 > 0$ be arbitrary and we would prove $(n^5 + n - 20) > M n^4$

② Let $n = \max\{2M, n_0, 40\} + 1$

③ Observe $n > \max\{2M, n_0, 40\}$

④ Since $n > 1$ then $\frac{1}{n^3} < \frac{1}{40}$; and $n > 2M$ then $\frac{n}{2} > M$

$$n^5 + n - 20 > n^5 - 20$$
$$= n^5\left(1 - \frac{20}{n^5}\right) > n^5\left(1 - \tfrac{1}{2}\right)$$
$$= \frac{n}{2} \cdot n^4 > M n^4$$

as needed

b) $2^{n^2} \in \Omega(8^n)$

Definition:

$\exists M > 0, \exists n_0 > 0, \forall n > n_0, \quad 2^{n^2} \geq M 8^n$

① Let $M = 1$ and Let $n_0 = 3$

② Let $n > 3$ be arbitrary, we would prove $\forall n > 3$, $2^{n^2} \geq M8^n$

③ By ②, then $n^2 > 3n$

$2^{n^2} \geq 2^{3n}$

④ By ① and ③

$2^{n^2} \geq 2^{3n} \geq M8^n$

as needed

2a) $\log_3(9^n + 6^n) \in \Theta(n)$

Def:

$\exists M > 0, \exists n_0 > 0, \forall n > n_0 \quad M_1 n \leq \log_3(9^n+6^n) \wedge \log_3(9^n+6^n) \leq M_2 n$

1) Let $M_1 = 2$, $M_2 = 3$ and $n_0 = 1$

2) Let $n > n_0$ be arbitrary

3) Since $M_1 = 2$;

$\log_3(9^n+6^n) > \log_3(9^n) = 2n$

as needed

and since $M_2 = 3$

$\log_3(9^n+6^n) < \log_3(9^n+9^n)$

$= \log_3(2) + 2n < 3n$

as needed

2b) $n^{1/2} - n^{1/3} \notin \Theta(n^{1/6})$

Definition of Θ

$\exists M>0, \exists n_0>0, \forall n>n_0 \quad M_1 n^{1/6} \leq n^{1/2} - n^{1/3} \quad \wedge \quad n^{1/2} - n^{1/3} \leq M_2 n^{1/6}$

$\neg\Theta$

$\forall M>0, \forall n_0>0, \exists n>n_0 \quad M_1 n^{1/6} > n^{1/2} - n^{1/3} \quad \vee \quad n^{1/2} - n^{1/3} > M_2 n^{1/6}$

1. Let $M>0$ and $n_0>0$ be arbitrary we will prove $n^{1/2} - n^{1/3} > M_2 n^{1/6}$

2. Let $n = \max\{n_0, 8M^3, 2\} + 1$

3. Observe $n > \max\{n_0, 8M^3, 2\}$

4. By observation

$$\frac{1}{n^{1/6}} < 1/2 \quad \text{and} \quad n^{1/3} > 2M$$

$$n^{1/2} - n^{1/3} = n^{1/2}\left(1 - \frac{1}{n^{1/6}}\right) > n^{1/2}(1 - 1/2)$$
$$= n^{1/3} \cdot n^{1/6}\left(1 - 1/2\right) > \frac{2M}{2} \cdot n^{1/6}$$
$$= M \cdot n^{1/6} \quad \text{as needed}$$

3a) $n^3 - 5n^2 \notin o(8n^3)$

Definition $o(8n^3)$

$\forall M > 0, \exists n_0 > 0, \forall n > n_0, \quad n^3 - 5n^2 \leq M 8n^3$

$\neg\, o(8n^3)$

$\exists M > 0, \quad \forall n_0 > 0, \quad \exists n > n_0, \quad n^3 - 5n^2 > M 8n^3$

① Let $M = 1/16$

② Let $n_0 > 0$ be arbitrary and $n = \max\{n_0, 10\} + 1$, we will prove $n^3 - 5n^2 > M 8n^3$

③ Observe $n > n_0$ and $n > 10$

$1/n < 1/10$

$$n^3 - 5n^2 = n^3\left(1 - \frac{5}{n}\right) > n^3\left(1 - \frac{5}{10}\right)$$
$$= n^3/2$$ as needed.

b) $\sqrt{n^3 - 4n} \in \omega(n)$

$\forall M > 0, \exists n_0 > 0, \forall n > n_0, \sqrt{n^3 - 4n} \geq Mn$

① Let $M > 0$ be arbitrary

② Let $n_0 = \max\{8, 2M^2\}$

③ Let $n > n_0$ be arbitrary

④ Observe $n > \max\{8, 2M^2\}$

⑤ $n > 8$, then $\frac{1}{n^2} < \frac{1}{8}$ and $n > 2M^2$ then $\sqrt{\frac{n}{2}} > M$

$$\sqrt{n^3 - 4n} = \sqrt{n^3(1 - 4/n^2)} \geq \sqrt{n^3(1 - 4/8)}$$

$$= \sqrt{\frac{n^3}{2}}$$

$$= n \cdot \sqrt{\frac{n}{2}} > Mn$$

as needed

4) To prove $n \log_a(n) \in o(n^b)$

a) For all $b > 1$, by (1.2), $\log_a(n) \in o(n^{b-1})$

b) By (5.1), $n \in \cancel{\Theta}(n)$ and By (3.1) $n \in O(n)$

c) By (5.4) $n \log_a(n) \in o(n^b)$

4b) To prove $n^a \cdot b^n \in o(n!)$

1. By (1.4), $n^a \in o(b^n)$
2. By (5.1), $b^n \in \Theta(b^n)$ and by 3.1 $b^n \in O(b^n)$
3. By (5.4), 1 and 2 $n^a \cdot b^n \in o(b^{2n})$.
4. Since $b > 1$, then $b^2 > 1$, by 1.6 $(b^2)^n \in o(n!)$
5. By (2.3), 3 and 4;

$$n^a b^n \in o(b^{2n}) \text{ and } (b^2)^n \in o(n!), \text{ then}$$

$$n^a \cdot b^n \in o(n!)$$

4c) $a + bn \in \Theta(n)$

① By (5.1), $a \in \Theta(1)$ and by (3.1) $a \in O(1)$

② By (1.1), $1 \in o(\log_a(n))$ and by (1.2) $\log_{(a)} n \in o(n)$,

③ By (2.3) and ②, $1 \in o(n)$ and by (3.4), $1 \in O(n)$

④ Since $a \in O(1)$ and $1 \in O(n)$, by (2.1) $a \in O(n)$

⑤ By (5.1) $bn \in \Theta(n)$ and by (3.1) $bn \in O(n)$

⑥ Since $a \in O(n)$ and $bn \in O(n)$, by (4.1) $a + bn \in O(n)$

5a) If $fg \in \omega(g)$, $f \notin \Theta(1)$

① Suppose $fg \in \omega(g)$, we would prove $f \notin \Theta(1)$

② By (5.1) $1/g \in \Theta(1/g)$ and by 3.1 $1/g \in \Omega(1/g)$

③ Since $fg \in \omega(g)$ and $1/g \in \Omega(1/g)$ by (5.5)
$f \in \omega(1)$

④ By (3.7) since $f \in \omega(1)$, then $f \notin O(1)$

⑤ By (3.1) $f \in \Theta(1)$ if and only if $f \in O(1)$ and $f \in \Omega(1)$. But by ④ $f \notin O(1)$. Therefore $f \notin \Theta(1)$

5b) If $g \in o(f)$, then $(f+g) \in \Theta(f)$

① Suppose $g \in o(f)$, We claim $(f+g) \in \Theta(f)$ it suffices to show that by 3.1, $(f+g) \in O(f)$ and $(f+g) \in \Omega(f)$

(2) By ① and 3.4 $g \in O(f)$

(3) By 5.1 $f \in \Theta(f)$ and 3.1 $f \in O(f)$ and $f \, \Omega(f)$

(4) By 4.2 and ③ $f+g \in \Omega(f)$ and by 4.1, ② and ③, $f+g \in O(f)$ which concludes the proof.