Orbit-Stablizer Theorem

Introduction

Setting

- Let S be a set, $s \in S$.
- Let G be a *permutaion group* on S, (or we say G *acts* on S). That means G is isomorphic to a subgroup of permutations on S, $G \approx H \leq Perm(S)$. We can see every $x \in G$ as a bijection from S to S, so the symbol $x(s)$ means the image of $s \in S$ under function x. Often we also write xs to denote $x(s)$.

Definitions

- The *stablizer* of s in G, denoted by $Stab_G(s)$, is

  $$Stab_G(s) = \{x \in G : xs = s\}$$

  In words, they are all the permutations x in G that leaves s as it is (so they all 'stablize' s).
- The *orbit* of s under G is,

  $$Orb_G(s) = \{xs : x \in G\}$$

  That is, orbit of s is the set of all the destinations it can go.

Theorem

- $Stab_G(s) \leq G$, and

  $$|G| = |Orb_G(s)| \cdot |Stab_G(s)|$$

  note that $Stab_G(s)$ is a subset (or further, subgroup) of G, while $Orb_G(s)$ is subset of S, sort of nothing to do with G.

Interpretation

The theorem suggests that there might be a one-to-one mapping between pairs of indices (i, j), **I'd** call them

Orb_Index and *Stab_Index*, and elements in G. We can visualize them in a table, where the columns correspond to *Stab_Index* and rows being *Orb_Index*:

	1	**2**	**...**	**j**	**...**	$\lvert Stab_G(s)\rvert$
1	1					
2						
...						
i				x		
...						
$\lvert Orb_G(s)\rvert$						

Any $x \in G$ can be identified by a cell (i, j) in this table. Let $t = t_i \in S$ denote the destination of s under $x = x_{i,j}$ and let v denote the pre-image of s under x:

$$t = x(s)$$

$$v = x^{-1}(s)$$

and let h_i denote the mapping in G that only swap s and v, leaving others as it is. i.e.:

$$h_i(s) = v$$

$$h_i(v) = s$$

$$h_i(u) = u \, for \; u \neq s \; or \; v$$

and let $g_j \in Stab_G(s)$ represents mapping that stablizes s (as it is in $Stab_G(s)$), maps v to t, while preserve the other mappings of x except s and v:

$$g_j(s) = s$$

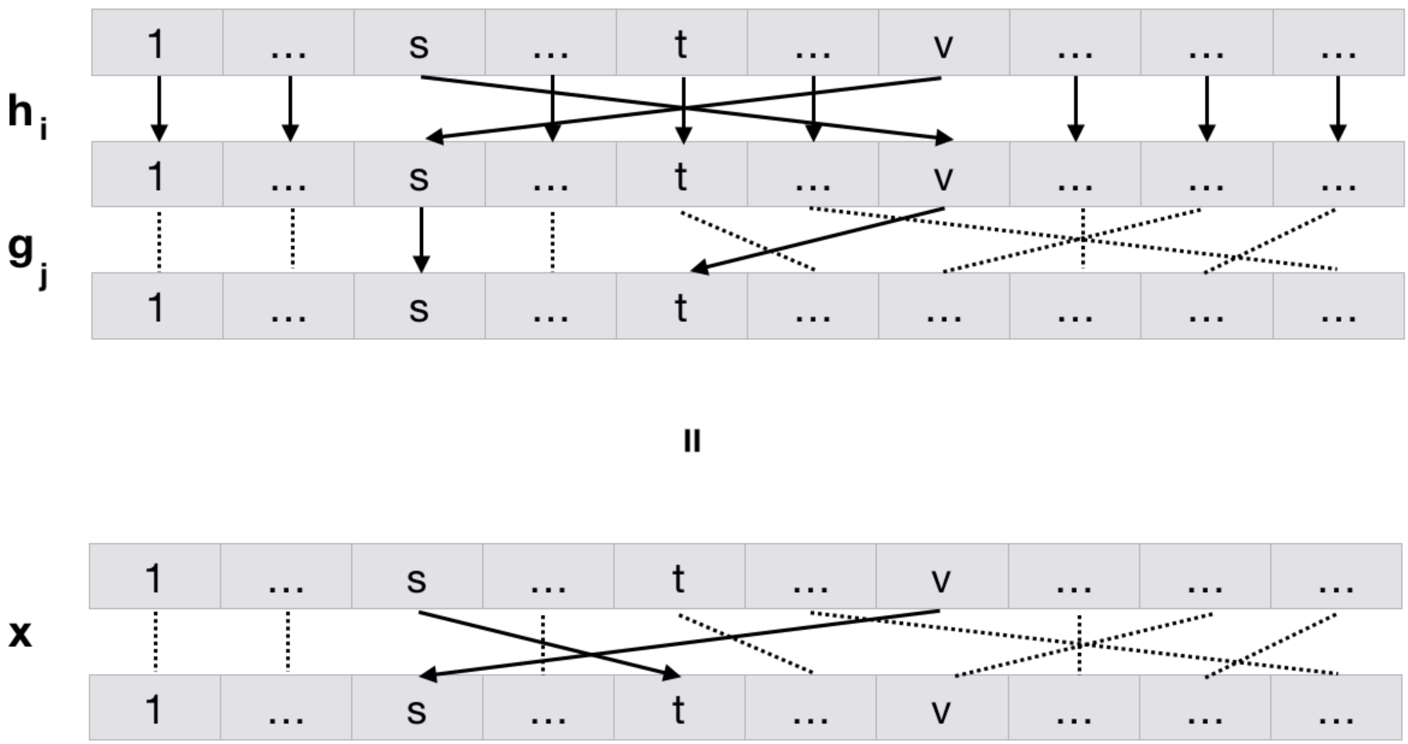
$$g_j(v) = t$$

$$g_j(u) = x(u) \; otherwise$$

Under this setting, x is a composition of h_i and g_j:

$$x = g_j \circ h_i$$

This concept may be better illustrated by:


h_i
g_j
x
1 … s … t … v … … …
1 … s … t … … … … …
1 … s … t … v … … …