ESTIMATING POWERS AND ROOTS FAST

ESSENTIAL TACTICS TO SPEED UP
MENTAL CALCULATIONS WITH WHOLE NUMBERS

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MULTIPLYING AND DIVIDING FAST. ESSENTIAL TACTICS TO SPEED UP
MENTAL CALCULATIONS WITH WHOLE NUMBERS

This book is dedicated with infinite gratitude to my mother, an exemplary educator and human being, and to the inspiring teachers and professors who instilled in me the love of learning and the passion to teach. I have them to thank for the many ways in which I have improved over the years as a communicator, researcher, and instructor. Their imprint lives in every page of this book.
Of any mistakes or inaccuracies in content, method, or strategy, they are, naturally, completely innocent.

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PREFACE

This book will not teach you how to estimate exponents or roots.

I assume that you know how to do that already…. with a calculator.

The goal of this book is to enable you to calculate these mentally, without a calculator.

Learning to estimate exponents and radicals in-your-feet, fast, will save you time, give your brain a workout, and with a little social finesse, maybe help you impress someone. But that's only at a bare minimum. You'll benefit from the teachings of this book not just the next time you are doing math homework, but also anytime you are trying to figure out the actual dimensions of a space, such as a box, a space in the wall, or the size of a room or a house or lot.

In fact, when working with any unit of distance space or volume we are dealing with squares and cubes. How many meters is that 84 square meters apartment actually? Or what is the volume of that recipient or silo? While we normally express these dimensions in square or cubic meters, inches, kilometers, miles and what not, it is interesting to be able to compare in a different meaningful way, what the space of this is. Even knowing square roots is used in everyday life, when dealing with any sort of triangle, for instance the diagonal that goes from the top left of a wall to the top bottom right. You can estimate that if you know the lengths off the sides and approximate a square root. No calculator needed.

Or if you work with computers, you may have heard of disk space memory units as megabytes, gigabytes, terabytes and so on. Did you know that an approximation of these quantities can be understood with exponents? Exponents also have their use in estimating incredibly large (or small) quantities, and while remembering all those digits may be hard, and of marginal use for mental math, it is important that we can think of them in an abbreviated way…. And we do this by expressing and thinking about these with exponents as well.

Ok. Perhaps you are a student.

Mental arithmetic is not just useful in middle and or high school where the applications may be the most obvious. Even after you've graduated, whether you are going to college or graduate school, whichever your major, fast mental arithmetic can save you precious next time you are taking that time-sensitive standardized test. And especially if you are going into the sciences, you will deal with radicals and exponents, so a firm grasp, and speed calculating them will set you apart the earlier you come to do it.

The truth is, we never stop benefiting from knowing arithmetic. You may find it incredibly useful and time saving advanced college math courses like linear algebra where you still need to do arithmetic to do matrix operations for example. Business, science, social science major? You'll probably use adding and subtracting quite a bit.

Math is everywhere, and you do – or should – use it every day to optimize your time and do things better, faster, and more effectively. Estimating exponents and roots fast without a calculator is a great step to improving your mental speed and accuracy in arithmetic.

Whatever your reason for coming on this journey with me, know that I fully believe it's worth the time and effort.

Welcome aboard.

INTRODUCTION

This book is an effort to combine the various scattered mental arithmetic tricks that have been taught by teachers many generations behind and in different and distant corners of the world, into a consistent and comprehensive system that will enable you to calculate or estimate exponents and radicals.

While much has been written about the four basic arithmetic operations in the context of fast mental calculations, less has been said about calculating exponents and radicals fast. This should come as no surprise, since the applications of exponents and radicals are not as ubiquitous as those of the four basic operations. We use them less often in every-day life and so less emphasis is placed upon them. Few people who have had formal instruction in elementary school, middle or high school will have difficulty reciting the table of two by heart, but we'd be hard pressed to find even a few people who can apply squares to even single digit figures as easily.

Further discouraging such memorization is that numbers, even squares and cubes (let alone exponents of larger numbers) become inordinately large, very quickly. What use is then knowing by heart a 6-digit figure such as $6^{7,}$ 279,936. How useful is this, especially given calculators and multiple faster methods of arriving at this figure.

For this reason, whenever numbers exceed five figures, we consider unnecessary and time wasting to remember them by heart and we choose instead to arrive at an approximation, which we will note in the form of a single digit number with two decimals times 10 elevated to an n^{th} power which will correspond to the number of digits of the actual result, like so.

For 6^7 which is 279,936, we could express it as 2.80×10^5. Note we rounded the hundredth place up and consequently the tenths places as well. This allows us to arrive at an approximation of a number which in most scenarios will be sufficient. 2.80×10^5 is a number in the hundred-thousands, and 10^6 in the millions, 10^7 in the ten-millions and so on and so forth.

Fast mental arithmetic, requires, as does most any other skill, consistent practice, and a little flexibility. Math, like languages, and many other arts, cannot be mastered without investing significant time doing it. Calculating and estimating exponents and radicals, however basic they may appear, are no exception to this rule. The approach we have adopted in this book is that of including both theoretical chapters to present the information, and practice chapters – which are essential to achieve mastery, efficiency in memory and speed, as well as an appendix, to see worked through examples so that you can see one way of thinking through the calculations.

Theoretical chapters are used to present the techniques, explaining the logic behind them, along with one or more examples, while practice chapters are intended to engage you in applying those skills, or at minimum at helping you start to apply them, since the amount of practice required to master fast arithmetic is beyond the scope of the fifty or so pages of this manuscript.

The appendix, at the end of the book is not merely a whimsical addition. It is intended to show you, step by step how the system outlined in the book is applied to the actual examples we ask you to solve in practice chapters. The appendix should be read in tandem with the practice chapters since the exact steps we take in the appendix are outlined there.

It's worth noting, that reading this whole book and doing all the practice exercises several times is no guarantee you will come out calculating and estimating exponents and radicals fast. Beyond the space limitation, success will depend on many other things such as your previous background and the actual attention you are paying to what you are doing (the speed and concentration required for reading a mathematics methods book, even one as simple as this one, is different from that required of a fiction novel or a social science textbook). How much you get out of this book, will be function, if you will, in great measure, of what you put into it, and how seriously you think about these techniques beyond the short time it will take you to read this book.

Think of the schema presented here as a system that you can apply, and that you can personalize and tweak according to your own previous math background, as you go about practicing on your own in everyday situations. The key to mastery, is practice.

Although I show one, and occasionally two ways of going about an operation, as I say perhaps too many times throughout the book, which actual combination of techniques and sequencing of techniques is optimal for improving your mental calculation speed, will largely depend on the background with which you come to this book.

For most people squaring 2 is easier than cubing 2, - but this may not always the case, someone may proceed to solving the problem by multiplying the number in their head, while another may have had more practical use with cubes and may find the latter calculation more readily accessible.

Use the material presented here as it best fits your personal skill using your judgement. Do not be a passive observer. Engage.

I would also like to stress before you begin, that it is imperative that you learn by heart a series of operations, most commonly cubes, squares, single digit powers of base 10, and numbers with prime bases and prime exponents. Knowing the basic operations like the back of your hand is essential if you are ever going to achieve proficiency in mental exponents and radicals.

As we move to more complicated calculations, we will start to assume that you have assimilated some of the most basic tactics and concepts, and we will not spell them out thoroughly.

Finally, although I have given the techniques names, these are personal, arbitrary, and functional-intuitive – that is, the names are based on what I thought the technique should be called based on what we are doing with the numbers. Many of these techniques have received different names in different parts of the world by different instructors. The names should help you identify the method and logic to apply, but after you have the techniques down, the idea is that you don't have to say in your head as you are doing the math.

Get familiar with the techniques and apply them, and think as you are doing the mental calculations, what the relationships between numbers is. Think of numbers as abstractions, ideas. Eventually they should just be a fleeting thought in your mind as you do the math.

Are you ready?

Let´s begin.

1. CALCULATING EXPONENTS FAST

In this chapter we'll survey some tactics for calculating exponents fast. We´ll cover first basic ground. This amounts to a fair amount of memorization.

Just as it was useful to memorize multiplication tables in grade school, it is also useful, albeit less so, to know by heart some basic powers. The question is…. How many? It depends on the work you are doing. For example, most people dealing with everyday calculation will typically work with figures below one million; even a figure such as 9^{11} will be pointless to memorize, because it exceeds that amount. On the other hand, crossing the one million barriers with a smaller base like 2, would require an exponent of 30. 2^{30}. For this reason, knowing smaller bases and larger exponents will be a priority.

If you are a scientist, an engineer, or an economist, or any other profession requiring regular knowledge of big numbers, then knowing powers of small bases and two are three-digit exponents, and bases, even two-digit bases with one or more-digit exponents will be useful.

Undoubtedly, you will benefit from memorizing as much as you can, if you have the time. Having said that, on the other hand, for the average person, squares, and cubes, even of large numbers will be more common than powers of 6 or 7, so ceteris paribus, in general it maybe more helpful to know 234^2 than 39^{17}. We'll cover as much ground as possible.

Finally, since elevating a number to a given exponents is a form of multiplication, knowing how to calculate exponents fast can aid you greatly in making fast calculations beyond just exponents. It can aid in multiplication and division, as well as in estimating roots, to a limited extent it may be of use adding and subtracting.

In a way the techniques suggested here are shortcuts, but that does not mean that they won't take work, especially if you have never memorized power tables.

1.1 FOUNDATIONAL POWERS

As a first step. It will be of great help to make note of some foundational powers.

Ideally, you would know up to the 20^{th} power of any number from 0 to 10. But that is a great deal to ask if you are just getting started, and for the higher bases combined with higher exponents, it is likely to be of marginal use. Instead, focus on squares, cubes, fourth powers, and fifth powers. Knowing these will be most useful, as you will see later, even when trying to mentally calculate a two-digit base elevated to a two-digit exponent.

	0	1	2	3	4	5	6	7	8	9	10
x^0	1	1	1	1	1	1	1	1	1	1	1
x^1	0	1	2	3	4	5	6	7	8	9	10
x^2	0	1	4	9	16	25	36	49	64	81	100
x^3	0	1	8	27	64	125	216	343	512	729	1,000
x^4	0	1	16	81	256	625	1,296	2,401	4,096	6,561	10,000
x^5	0	1	32	243	1,024	3,125	7,776	16,807	32,768	59,049	100,000

Notice that even a single digit power with a single digit exponent such as 8^5 or 9^5, is already a five-digit number. If you are mentally computing five or more-digit numbers, and you don't need to be extremely precise, there are two techniques which we'll recommend. The first is to think of 59,049 for example as 59k or 32,768 as 32k+ or 33k if you need to round up. Similarly, if you are dealing with millions say, 59,038,830, (23^5) just think 59m.

If you are dealing with larger figures or you prefer to be more scientific, or you need to express this in written form, know that you can rewrite 59,049 as approximately $59\text{x}10^{3}$. We'll review this further down the road, for now, memorize these basic powers.

Example 1	7^4 2,401

Example 2	4^5 1,024

Example 3	9^3 729

1.2 BREAK DOWN AND MULTIPLY

Knowing the table above by heart and having covered and mastered the multiplication chapters, we can move forward and use one of the laws of exponents to your advantage.

Namely, since $x^m x^n = x^{m+n}$, we need not memorize for example what 9^6 (a number beyond the scope of the table above) is since it is equal 9^3 x 9^3, which we do know. And since we have memorized the table above, we can invoke with ease 729 x 729.

Or 7^8 can be thought of as 7^4 times 7^4, or 2,401 x 2,401. If you extend your knowledge of the powers table…. as you should eventually, you could invoke 7^6 times 7^2 which will be an easier product. Why? Because multiplying a large number by a smaller number generally involves less mental overhead than multiplying two medium sized numbers.

Compare having to mentally compute

2,401 x 2,401 with 117,649 x 49. The second product, has more carries using the traditional method, but if you mastered the concepts in the fast multiplication, even the calculation should be relatively easy.

Similarly, 2^7, is best calculated as 2^5 times 2^2 with the above table than it would be 2^4 x 2^3

Compare having to compute 32 x 4 versus 16 x 8.

This reference might be useful in trying to decide how to break down powers with exponents larger than five but less than 10

Break down

x^9 into x^4 x x^5, or $(x^3)^3$
x^8 into x^4 x x^4, or $(x^4)^2$
x^7 into x^2 x x^5
x^6 into x^3 x x^3, or $(x^3)^2$

But although memorizing the above would be useful. If you want to want to extend this further and aim at complete proficiency in fast powers, you should memorize the table presented below to apply more advanced techniques to multiple digit bases and two-digit exponents. And master the squares and cubes, if those are the ones you plan to use the most.

	0	1	2	3	4	5	6	7	8	9	10
x^0	1	1	**1**	**1**	1	1	1	1	1	1	1
x^1	0	1	**2**	**3**	4	5	6	7	8	9	10
x^2	0	1	**4**	**9**	16	25	36	49	64	81	100
x^3	0	1	**8**	**27**	64	125	216	343	512	729	1,000
x^4	0	1	**16**	**81**	256	625	1,296	2,401	4,096	6,561	10,000
x^5	0	1	**32**	**243**	1,024	3,125	7,776	16,807	32,768	59,049	100,000
x^6	0	1	**64**	**729**							
x^7	0	1	**128**	**2,187**							
x^8	0	1	**256**	**6,561**							
x^9	0	1	**512**	**19,683**							
x^{10}	0	1	**1,024**	**59,049**							

Example 1	3^9 19,683
Example 2	3^7 2,187
Example 3	8^6 262,144

1.3 BEYOND SIMPLE BREAK DOWN AND MULTIPLY

With an extended table, covering x through the 10^{th} power, there's a lot more you can do.

We won't provide a complete table with such number or recommend that you memorize these. As mentioned in the introduction, the digits can grow quite large and unmanageable. If you really want to memorize 20-digit results, do go ahead and do so… We recommend techniques for dealing with such scenarios.

Specifically, we recommend you develop an extended table covering not only numbers elevated to the 10^{th} power, but if you need to go beyond, only memorize the 20^{th}, 30^{th}, 40^{th}, 50^{th} power and so forth until the 100^{th} power. Why not memorize a number to the 83th power? Because you can handle 83 breaking it downs as x^{80} times x^{3}

With a number such as 5^{14} we can simply think of it as 5^{10} times 5^{4}.

	0	1	2	3
x^{10}	1	1	1,024 or $1.02x10^{3}$	59,049 or $5.90x10^{4}$
x^{20}	0	1	$\approx 1.05x10^{6}$	$\approx 3.49x10^{9}$
x^{30}	0	1	$\approx 1.07x10^{9}$	$\approx 2.06x10^{14}$
x^{40}	0	1	$\approx 1.10x10^{12}$	$\approx 1.22x10^{19}$
x^{50}	0	1	$\approx 1.13x10^{15}$	$\approx 7.18x10^{23}$
x^{60}	0	1	$\approx 1.15x10^{18}$	$\approx 4.24x10^{28}$
x^{70}	0	1	$\approx 1.18x10^{21}$	$\approx 2.50x10^{33}$
x^{80}	0	1	$\approx 1.21x20^{24}$	$\approx 1.48x10^{38}$
x^{90}	0	1	$\approx 1.24x10^{27}$	$\approx 1.73x10^{42}$
x^{100}	0	1	$\approx 1.27x10^{30}$	$\approx 5.15x10^{47}$

For example, solving x^{47}, simply if you know x through the 10^{th} power and then multiple powers as exponents like 20, 30 and 40, in x^{20}, x^{30}, x^{40}, you can compute any two-digit exponent by simply breaking the operation into a product of two parts you´ve committed to memory, always making the multiplicand a power of the highest multiple of 10 the original exponent can reach, and the multiplicator a relatively smaller figure corresponding to x^{1} through $x^{9.}$

Like so,

4^{47}.

And we can break it as 4^{40} x 4^{7}.

And if you didn't memorize 4^7, do 4^{40} x 4^5 x 4^2

Because of this technique we might even suggest that going beyond memorizing any exponent beyond 9 that is not a multiple of 10 is inefficient.

Again, there's no point in memorizing what x^{37}th is since we can simply break it down as x^{30} times x^7.

Then x^{37} is much easier to compute since you have memorized x^{30} and you must multiply it by a relatively small number x^7. Which, in mental math makes things more manageable.

Again, you could save yourself memorizing x^{30}, x^{40}, and so forth and potentially instead do x^{10} x x^{10} x x^{10} x x^7, or even $(x^{10})^4$ x x^7 but these mental setups are inefficient since they create overhead and become more difficult to manage or retain in memory.

So finally, if you have x^{94}, you can just think of it as x^{90} times x^4…. Effectively breaking down the exponent into a multiple of then and then a number between 0 and 10.

Again, the big number is committed to memory, all you need to do is remember it by heart, so you can multiply it by the relatively smaller x^4.

Example 1	2^{42} 2^{40} x 2^2 1.10×10^{12} x 4 $\approx 4.40 \times 10^{12}$

Example 2	3^{35} 3^{30} x 3^5 2.06×10^{14} x 243 $\approx 50 \times 10^{15\,x}$

Example 3	3^{91} $3^{90} \times 3^{1}$ $1.73 \times 10^{42} \times 3$ $\approx 5.19 \times 10^{42}$

1.4 ELEVATING MULTIPLES OF 10

Moving ahead, it is also of great use to know what 10 elevated to various powers is,

Namely,

10^{1} is 10
10^{2} is 100
10^{3} is 1,000
10^{4} is 10,000
10^{5} is 100,000
10^{6} is 1,000,000
10^{7} is 10,000,000
10^{8} is 100,000,000
10^{9} is 1,000,000,000
$10^{10}$is 10,000,000,000

But there is absolutely no need to memorize this, just remember instead that 10 elevated to whatever power will result in a number that has 1 in its left most place value, followed by as many zeros as the result of the exponent.

But why is this important to know? After all, 13^{6} and even 12^{6} are so far apart from 10^{6} that knowing 10^{6} doesn't really help by means of rounding the base to the nearest multiple of 10 to approximate the number.

The use of knowing this is may be less apparent now than it may be when trying to solve a problem.

Suppose you are trying to solve 12^{24}.

Since our base is 2 digits, the techniques we've learned so far may not be of much use.

One way of going about this might be to break down both the base and/or the exponent to arrive at an estimate, like so,

$2^{24} \times 6^{24}$

Which could be expressed as,

$(2^{4})^{6} \times (6^{4})^{6}$

$16^{6} \times 1{,}296^{6}$

Now if you've arrived this far, since 12^{24} is a very large number, chances are, and especially if you are computing this mentally, that you want to reach an approximation of the result.

What you can do at this point is take advantage of the base elevated to an exponent, to rethink the problem as

$16^{6} \times 1{,}296^{6}$
$(16 \times 1{,}296)^{6}$

$(20{,}736)^{6}$

How is this any better?

Knowing the powers of base 10, and scientific short notation we can rethink this as approximately,

$(2.07 \times 10^{4})^{6}$

Approximately,

(79×10^{24})

Which, depending on your purposes and need for precision will at worst partially approximate of the actual result of 12^{24} at best give you a very clear idea of the figure you are dealing with.

Which is actually

7.95×10^{25} or 80×10^{24}

Example 1	
	15^{12} $3^{12} \text{ x } 5^{12}$ $3^{10} \text{ x } 3^{2} \text{ x } 5^{5} \text{ x } 5^{4} \text{ x } 5^{3}$ $\approx 5.90\text{x}10^{4} \text{ x } 9 \text{ x } 3{,}125 \text{ x } 625 \text{ x } 125$ $\approx 5.90\text{x}10^{4} \text{ x } 9 \text{ x } 3{,}125 \text{ x } 78{,}125$ $\approx 5.90\text{x}10^{4} \text{ x } 9 \text{ x } 2.44\text{x}10^{8}$ $\approx 5.90\text{x}10^{4} \text{ x } 22\text{x}10^{8}$ $\approx 5.90\text{x}10^{4} \text{ x } 22\text{x}10^{8}$ $\approx 5.90 \text{ x } 22 \text{ x } 10^{12}$ $\approx 129.8 \text{ x } 10^{12}$ $\approx 1.3\text{x}10^{14}$

Example 2

16^{12}

$2^{12} \times 8^{12}$

$2^{10} \times 2^{2} \times 8^{12}$

$1{,}024 \times 4 \times 8^{12}$

$4{,}096 \times 2^{12} \times 4^{12}$

$4{,}096 \times 2^{10} \times 2^{2} \times 4^{12}$

$4{,}096 \times 1{,}024 \times 4 \times 4^{12}$

$4{,}096 \times 4{,}096 \times 2^{12} \times 2^{12}$

$4{,}096^{4}$

$\approx (4.10 \times 10^{3})^{4}$

$\approx 4.10^{4} \times 10^{12}$

$\approx 283 \times 10^{12}$

Example 3

72^{42}

$9^{42} \times 8^{42}$

$9^{40} \times 9^{2} \times 8^{40} \times 8^{2}$

$3^{40} \times 3^{40} \times 3^{2} \times 3^{2} \times 2^{40} \times 2^{40} \times 2^{40} \times 2^{2} \times 2^{2} \times 2^{2}$

$3^{80} \times 3^{4} \times 2^{100} \times 2^{20} \times 2^{6}$

$\approx 1.48 \times 10^{38} \times 81 \times 1.27 \times 10^{30} \times 1.05 \times 10^{6} \times 64$

$\approx 1.48 \times 1.27 \times 1.05 \times 10^{74} \times 5{,}184$

$\approx 1.88 \times 1.05 \times 10^{74} \times 5{,}184$

$\approx 1.97 \times 10^{74} \times 5{,}184$

$\approx 10{,}212 \times 10^{74}$

1.5 LARGER BASES

While in general the preferred method for solving two-digits bases is the one presented previously, if you prefer you can use an alternative method, available by the law of exponents.

Whether you want to use this to practice mental multiplication or because you find it easier than the method we proposed previously, we present it here.

Because of the law of exponents that states that

$(x \times y)^n = x^n \times y^n$,

when dealing with large bases and (hopefully, but not necessarily) small exponents, we can factor the base as a product of two or more numbers, and find take the power of the product of each of those numbers, like so,

$48^3 = 12^3 \times 4^3$ which we could also factor and express as $3^3 \times 4^3 \times 4^3$ or 27 x 64 x 64… approximately one hundred and ten thousand.

$21^4 = 3^4 \times 7^4 = 81 \times 2{,}401 = 194{,}481$

$45^6 = 5^6 \times 9^6 = 5^5 \times 5^1 \times 9^5 \times 9^1 = 3{,}125 \times 5 \times 59{,}049 \times 9 = 15{,}625 \times 531{,}441 \approx 8.30 \times 10^9$

or

$30^2 = 5^2 \times 6^2$ which you could calculate… or simply observe that for any multiple of 10 squared in terms of the rightmost digits, and then append however many zeros are needed, as in 30^2 is nothing but 30 x 30 is 900…

And similarly,

$40^2 = 40 \times 40 = 1600$

$70^3 = 70 \times 70 \times 70 = 343{,}000$

Notice also that you could calculate the latter 2, knowing what we know as

$40^2 = 10^2 \times 4^2 = 100 \times 16$

$70^3 = 10^3 \times 7^3 = 1{,}000 \times 343$

And so on and so forth…

Let's look at another example with a complication.

$111^3 = 3^3 \times 37^3$

Here we are stuck in a rut. 37 is a prime so it can't be further broken down.

But a table of the first few prime numbers can be of value.

	11	13	17	19	23	29	31	37	41	43	47
x^0	1	1	1	1	1	1	1	1	1	1	1
x^2	121	169	289	361	529	841	961	1,369	1681	1849	2209
x^3	1,331	2,197	4,913	6,859	12,167	24,389	29,791	50,653	68,921	79,507	103,823
x^4	14,641	28,561	83,521	130,321	279,841	707,281	923,521	1,874,161	2,825,761	3,418,801	489,681
x^5	161,051	371,293	1,419,857	2,476,099	6,436,343	20,511,149	28,629,151	69,343,957	115,856,201	147,008,443	229,345,007

So,

$111^3 = 3^3 \times 37^3 = 27 \times 50{,}653 = 1{,}367{,}631$

Other examples,

$39^2 = 13^2 \times 3^2 = 169 \times 9 = 1{,}521$

$205^3 = 41^3 \times 5^3 = 68{,}921 \times 125 = 8{,}615{,}125$

$68^4 = 17^4 \times 4^4 = 83{,}521 \times 256 = 21{,}381{,}376$

Example 1

93^4

$31^4 \times 3^4$

$923{,}521 \times 81$

$\approx 9.23 \times 10^5 \times 81$

$\approx 7.47 \times 10^7$

Example 2	58^2 29^2 x 2^3 841 x 4 3,364

Example 3	141^3 47^3 x 3^3 103,823 x 27 ≈ 1.04 x 10^5 x 27 2,803,221

2. PROGRESSIVE PRACTICE: POWERS

2.1 EXPONENTS TO MEMORIZE

9^3 (single digit base and exponent)

1^0 (single digit base and exponent)

8^5 (single digit base and exponent)

5^2 (single digit base and exponent)

2^1 (single digit base and exponent)

3^4 (single digit base and exponent)

4^8 (single digit base and exponent)

6^9 (single digit base and exponent)

7^7 (single digit base and exponent)

2.2 BASE TEN

10^2 (exponents indicate appended zeros)

10^5 (exponents indicate appended zeros)

10^7 (exponents indicate appended zeros)

10^{20} (exponents indicate appended zeros)

10^{40} (exponents indicate appended zeros)

10^{80} (exponents indicate appended zeros)

2.3 POWER OF TEN EXPONENTS

0^{10} (base 0-3, exponents that are multiples of 10)

1^{20} (base 0-3, exponents that are multiples of 10)

2^{30} (base 0-3, exponents that are multiples of 10)

3^{40} (base 0-3, exponents that are multiples of 10) (estimate approximation)

2^{50} (base 0-3, exponents that are multiples of 10) (estimate approximation)

2.4 BREAK DOWN BASE OR EXPONENT

34^2 (break down the base and multiply) (memorize prime squares)

27^2 (break down the base and multiply)

56^2 (break down the base and multiply)

66^2 (break down the base and multiply)

72^2 (break down the base and multiply)

116^3 (break down the base and multiply)

84^{21} (break down the exponent and multiply) (estimate approximation)

25^{25} (break down the exponent and multiply) (estimate approximation)

12^{12} (break down the exponent and multiply) (estimate approximation)

2.5 PRIME BASE

17^{19} (a prime base with a prime exponent – memorize)

11^{0} (exponent 0 - memorize)

19^{2} (a prime base, squared, multiply)

31^{5} (prime base, low exponent – memorize)

43^{7}(prime base, prime exponent – memorize)

37^{8}(prime base, prime exponent – memorize)

2.6 MULTIPLE OF BASE TEN

200^{2} (treat 0s as numbers to be appended and treat non -zero leftmost characters as base)

350^{2} (treat 0s as numbers to be appended and treat non -zero leftmost characters as base)

470^{2} (treat 0s as numbers to be appended and treat non -zero leftmost characters as base)

$7{,}000^{2}$ (treat 0s as numbers to be appended and treat non -zero leftmost characters as base)

$14{,}000^{2}$(treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime-base and prime exponent - memorize)

$38{,}000^{2}$ (treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime- base and prime exponent - memorize)

$47{,}000^{3}$ (treat 0s as numbers to be appended and treat non-zero leftmost characters as base)

$9{,}000^{3}$ (treat 0s as numbers to be appended and treat non- zero leftmost characters as base)

$4{,}000^{3}$(treat 0s as numbers to be appended and treat non -zero leftmost characters as base)

$4{,}000^{4}$(treat 0s as numbers to be appended and treat non - zero leftmost characters as base)

$66{,}000^{4}$(treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime base and prime exponent - memorize)

$72{,}000^{5}$(treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime base and prime exponent - memorize)

$27{,}000^{6}$ (treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime- base and prime exponent - memorize)

$11{,}000^{7}$ (treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime- base and prime exponent - memorize)

$21{,}000^{8}$(treat 0s as numbers to be appended and treat non - zero leftmost characters as base, prime base and prime exponent - memorize)

3. CALCULATING PERFECT SQUARES AND CUBE ROOTS FAST

Memorizing key exponents and bases in the previous chapter serves immensely in calculating square and cube roots, and in fact, depending on what you have become familiar with and memorized, even higher-level exponents.

This chapter will explore how to calculate and estimate square cube and higher exponent roots fast. The concepts in this chapter build on the concepts presented previously. In fact, we'll review some of the numbers presented previously

3.1 PERFECT SQUARE ROOTS

We can easily calculate square roots for numbers from 1 and 10,000 respectively.

Let´s briefly review basic squares and cubes

x	x^2	x^3
1	**1**	**1**
2	**4**	**8**
3	**9**	**27**
4	**16**	**64**
5	**25**	**125**
6	**36**	**216**
7	**49**	**343**
8	**64**	**512**
9	**81**	**729**
10	**100**	**1,000**
11	**121**	**1,331**
12	**144**	**1,728**
13	**169**	**2,197**
14	**196**	**2,744**
15	**225**	**3,375**
16	**256**	**4,096**
17	**289**	**4,913**
18	**324**	**5,832**
19	**361**	**6,859**
20	**400**	**8,000**

We know that the root of any whole number up to 10,000 will have two digits because is it equal to 100^2. And we know any whole number less than 100^2 need be less than 10,000.

Suppose you have to find the square root of 1,369 (*which is 37*)

First, separate the number in two parts at the hundreds place value.

In this case because we are dealing with a four-digit number, we pick the first two digits, 13 (corresponding to the thousands place value, and the hundreds).

In the previous table we saw:

x	x^2
3	**9**
4	**16**

13 falls in the table in the range with floor above **9** (which is 3^2) and below **16** (which is 4^2).

So, the first digit of the answer then corresponds to the *base of floor of this range*… in other words, 3.

Now shift your focus to 69 (the tens and units). Which number, squared can result in a 9 in the ones place value? From the table above, we see… only 7, as in

x	x^2
7	**49**

Sure, other squares also end in 9,

x	x^2
3	**9**
7	**49**
13	**169**
17	**289**

Knowing the ranges in the tables in the appendix, will save you a lot of time in choosing the right candidate for the ones place value, not only for this technique but for most all techniques presented in this chapter. Familiarize yourself with it.

Without it, we at least can determine that number can be 33 or 37, because it is apparent that both 313 and 317 are too large to be the answer.

7 must be then the ones place value digit, and 37 the root.

What about the square of a smaller, say three-digit number like 225.

We can use the same method. To determine the tens, separate at the hundreds place value, this time the split numbers we'll work with are 2 and 25. 2 corresponds to the range

x	x^2
1	**1**
2	**4**

Where **1** is the floor between **1** and **4** and therefore 1 the tens digit of the result.

And in the case of 25, the only reasonable candidate for a square that can result in a 5 units place value of is 5

x	x^2
5	**25**

We have our result, 15

There's a simple way of systematizing dealing with the square roots units place value:

Memorizing the whole table above is not necessary if you know you are dealing with squares of bases of up to 2 digits, or square root bases of up to four digits.

It's sufficient to know that there is a simple and straightforward correspondence between the ones place value in the base of the square root (*1sBase*) and the units place value of the result (*1sResult*).

Specifically,

1sBase	*1sResult*
1	1 or 9
4	2 or 8
5	5
6	4 or 6
9	7 or 3

Just look at the figures in the table presented at the beginning of the chapter and you'll see these figures match.

The question is then, what happens if you have a ones place value square root base that ends in **1** (***1sBase***). How can you determine if the results one place value (*1sResult)* should be 1 or 9?

For instance, what is the square root of 6,2**41**.

We know that 62 is in the range of 7^2 and 8^2,

x	x^2
7	**49**
8	**64**

We know the result's tens place value is 7, which again is the *base floor of the range*.

For the ones' result place value (*1sResult*) we must again, examine the square roots' base (***1sBase***)

1sResult = 1

According to the table above,

1sBase	*1sResult*
1	1 or 9

We have two possibilities; the answer is either 71 or 79.

We know that 71^2 is closer to 70^2, and 79^2 to 80^2.

Using the tens digit, approximate the answer and guess… 70^2 is 4900 (think 7^2 is 49), while 80^2 is 6400 (think 8^2 is 64).

Since 6,241 is much closer to 80^2 the only reasonable guess is that *1sResult* is 9, and 79 the answer.

Example 1	2,916 29 and 16 $5^2 < 26 < 6^2$, so 5? 1sBase 1sResult 6 4 or 6 54 or 56, 54 is closer to 50^2 (2,500) and 56 to 60^2 (3,600) Since 2,916 is closer to 2,500 than to 3,600, then the result must be 54

Example 2	8,464 84 and 64 $9^2 < 84 < 10^2$, so 9? 1sBase 1sResult 4 2 or 8 92 or 98, 92 is closer to 90^2 (8,100) and 98 to 100^2 (10,000) Since 8,464 is closer to 8,100 than to 10,000, then the result must be 92

Example 3	289 2 and 89 $1^2 < 2 < 2^2$, so 1? 1sBase 1sResult 9 7 or 3 13 or 17, 13 is closer to 10^2 (100) and 17 to 20^2 (400) Since 289 is closer to 400 than to 100, then the result must be 17

3.2 PERFECT SQUARE ROOTS OF NUMBERS WITH MORE THAN FOUR DIGITS

To arrive quickly at the square root of a five or six-digit number, we need be acquainted with the concept of a ***digital sum***.

The ***digital sum*** of a number is simply the result of adding all single digits place value figures so as to arrive to a one single figure number, like so.

Find the *digital sum* of

104,976… 1+0+4+9+7+6 = 27… 2+7 = 9

So, the *digital sum* of 104,976 is 9.

Knowing how to calculate the *digital sum* of a number and memorizing the digital sum table below, will help us quickly determine a three-digit result of a five or six-digit square root.

The digital sum of the base of the square root (*DSBase*) corresponds with the digital sum of the result (*DSResult*) of estimating that square root.

DSBase	*DSResult*
1	1 or 8
4	2 or 7
9 or **0**	3, 6 or 9
7	4 or 5

Let´s see these concepts at work with a concrete example.

Find the square root of 104,976.

First, we separate the number at the hundreds, 1,049 on the one hand and 76 on the other.

x	x^2
30	**9**00
40	**16**00

1,049 is between 30^2 and 40^2 or **900** and **1600**.

So, we know the results hundreds place value to be 3.

The units 6, (of 76) corresponds with 4 or 6 as per the table in the previous section.

But 76 is only divisible by 4, and not by 6, so 4 is the *1sResult* figure.

1sBase	*1sResult*
6	4 or 6

3?4 is the partial result, where ? stands for any one-digit figure.

What about the ? in the tens. What figure is that?

Well, we know that the *digital sum* of 104,976 is 9.

And according to the table presented early in this section, that 9 corresponds with a digital sum of 3, 6 or 9 in the result.

DSBase	*DSResult*
9 or **0**	3, 6 or 9

In calculating the ? figure in 3?4, since 3+4 is 7, and we need DSResult to be 3, 6 or 9,
5 is a good candidate for ? since 3+5+4 = 12... 1+2 = 3,
as are 5 and 8 since they also add to the required digital sum,

3+5+4 = 3
3+8+4 = 6
3+2+4 = 9

So which is it?

How do we weed out the incorrect options?

(*Note that we have purposeful jumped into the most complicated scenario in this example, such that solving cases with less complications is easier later*)

Let's consider the left most digits of the possible answer 35, 38, and 32.

Now look at the base of the base square roots leftmost digits: 1,049 (from 104,976).

Is the square of 1,049, 38, 35 or 32?

Note that the square of a number ending in 5 can never end in 9, so 35 is discarded.

And recall,

x	x^2
30	900
40	1600

32 is closer to 30^2
38 is closer to 40^2

Setting it up visually…
32 is closer to 30^2 or 900
38 is closer to 40^2 or 1600

1,049 is considerably closer to 900 than 1600, so the correct answer is indeed 2 for the tens digit.

The result being 324.

Example 1	108,241 1,082 and 41 $30^2 < 1{,}082 < 40^2$, so 3?? 1sBase 1sResult 1 1 or 9 So 3?1 or 3?9 1+0+8+2+4+1 = 1+6 = 7 DSBase DSResult 7 4 or 5 3+x+1 or 3+x+9 = 4 or 5, so the possibilities are: 301, 391, 311, or 319, 329 Ordered in the range of 300^2 (90,000) and 400^2 (160,000) Since 108,241 is closer to 90,000 than to 160,000, but falls toward the middle of the range, we discard 391, and then 301, 311, and 319. 329 is the result.

Example 2

235,225
2,352 and 25
$40^2 < 2{,}352 < 50^2$, so 4??

1sBase	1sResult
5	5

So 3?5
2+3+5+2+2+5 = 1+9 = 1

DSBase	DSResult
1	1 or 8

3+x+5 = 1 or 8, so the possibilities are:
325, 305, 385

Ordered in the range of 400^2 (160,000) and 500^2 (250,000)
Since 235,225 is closer to 250,000 than to 160,000, the result, must be:
385

Example 3

12,544
125 and 44
$10^2 < 125 < 20^2$, so 1??

1sBase	1sResult
4	2 or 8

So 1?2 or 1?8
1+2+5+4+4 = 1+6 = 7

DSBase	DSResult
7	4 or 5

1+x+2 or 1+x+8 = 4 or 5, so the possibilities are:
112, 148, 122, 158.

Ordered in the range of 100^2 (10,000) and 200^2 (40,000)
Since 12,544 is closer to 10,000 than to 40,000, the result cannot be 148 or 158. Furthermore, because it is much closer to 10,000 than to close to the upper limit of the first quartile between 10,000 and 40,000 (roughly 17,500), the result cannot be 122 (which is in the vicinity of the corresponding range between 100 and 200), and must be
112

3.3 EXTENDING THE METHOD TO PERFECT CUBES OF UP TO SIX DIGITS

When trying to extend the method to cube roots with bases of at most six digits, we should note that the result will always be a two-digit number. Why? Because **100^3**, which is the first whole positive number above **99** for which we can find a cube, is 1,000,000, and any number below it must be at most 6 digits, like 999,999.

On the other hand, **10^3**, where the base is the smallest two-digit number, is equal to 1,000, a four-digit figure. Any positive whole number above it in the number line, like **11^3** will necessarily be larger and therefore be of equal or greater number of digits.

We use the following correspondence table to determine the ones place value in the result,

1sBase	*1sResult*
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

This is easier to remember than might appear at first glance. Note that 1, 4, 5, 6, and 9 correspond to, well themselves, while 2 and 3, correspond with 8 and 7, and 8 and 7 with 2 and 3.

A useful extension of the original table of squares and cubes will come in handy.

x	**x^2**	**x^3**
10	100	1,000
20	400	8,000
30	900	27,000
40	1,600	64,000
50	2,500	125,000
60	3,600	216,000
70	4,900	343,000
80	6,400	512,000

90	8,100	729,000
100	10,000	1,000,000

By no means is it necessary to memorize this table.

Why?

Because we can figure it out if we know the cubes of the single digit numbers.

Note,

Knowing for example that 5^3 is 125, we can easily infer that 50^3 is 125,000.

Remember that the cube will have as many additional zeros to match the exponent.

x	**x^2**	**x^3**
10	100	1,000
20	400	8,000
30	900	27,000
40	1,600	64,000
50	2,500	125,000
60	3,600	216,000
70	4,900	343,000
80	6,400	512,000
90	8,100	729,000
100	10,000	1,000,000

What is the cube root of 658,503?

As usual we start breaking the digits at the hundreds into 6,585 and 03.

We know that **80^3** is 512,000 and **90^3** is 729,000.
So, we take the floor figure base (80) tens' place value to be the tens digit of the result, in this case: 8.
Such that the result of the cube root will be 8?.

We know that the units must be 7, because the units 3, corresponds to 7 (as per the first table we presented in this section.

1sBase	*1sResult*
7	3

So, 87 is the result.

What is the cube root of 970,299?

Split at the hundreds into 9,702 and 99.

9 in the tens, because 970,299 is greater than **90^3** and less than **100^3**.

In fact, we can infer that it is a high 90s value because it is so much closer to **100^3** than to **90^3**.

The units digit of the cube root **9**, corresponds with 9,

1sBase	*1sResult*
9	9

so 99 is the perfect cube root of 970,299.

Example 1	$\sqrt[3]{614,125}$ $80^3 < 614,125 < 90^3$, so 8? 1sBase 1sResult 5 5 85
Example 2	$\sqrt[3]{132,651}$ $50^3 < 132,651 < 60^3$, so 5? 1sBase 1sResult 1 1 51
Example 3	$\sqrt[3]{6,859}$ $10^3 < 6,859 < 20^3$, so 1? 1sBase 1sResult 9 9 19

3.4 FINDING CUBE ROOTS OF OF 7, 8 OR 9 DIGITS

So, what if you are trying to find the perfect root of an eight-digit number such as 13,312,053?

We know that for seven to nine-digit numbers for which we try to find a cube, the result will be of 3 digits.

So, we start with the hundreds' place value figure in the result:

To find the hundreds place value result, let's stop for a brief second to consider the intervals of cube powers of multiples of 100.

x	x^2	x^3
100	10,000	1,000,000
200	40,000	8,000,000
300	90,000	27,000,000
400	160,000	64,000,000
500	250,000	125,000,000
600	360,000	216,000,000
700	490,000	343,000,000
800	640,000	512,000,000
900	810,000	729,000,000
1000	1,000,000	1,000,000,000

So, since 3^3 is 27. Then 300^3 is 27,000,000 (effectively appending the six zeros).

The number we are trying to find a root for 13,312,053 will fall in between 200^3 and 300^3 that is, between 8 million and 27 million. So, we quickly find the hundreds place value as the hundreds of the base of the floor of this range, 2 (from 200^3).

The result's ones' place value is easily determined, with the table from the previous section.

The ones digit of the base of the root is 3, which matches with 7.

1sBase	*1sResult*
3	7

So, 2?7.

Finally, just as we did for three-digit square roots, the middle digit is related to the *digital sum*.

Since the *digital sum* of 13,312,053 is 1+3+3+1+2+0+5+3 = 18, 1+8 = 9.

Now the *digital sum* correspondence table relating the digital sum of the cube root with the digital sum of the result is as follows,

DSBase	*DSResult*
1	1 or 4 or 7
8	2 or 5 or 8
9	3 or 6 or 9

Since the *digital sum* of the cube root base is **9**, then the sum of the result has to be either 3, 6 or 9.

With 2?7, we can arrive at such digital sums with 237 (digital sum 3), 267 (digital sum 6), 207 (digital sum 9), or 297 (digital sum 9).

2+3+7 = 3
2+6+7 = 6
2+0+7 = 9
2+9+7 = 9

We proceed in the final step in a similar fashion as we did with three-digit squares.

We take the hundreds and thousands and consider them separately, in ascending order if you will, 20, 23, 26, or 29.

20^3 is 8,000, **30^3** is 27,000.

Our target number 13,000 seems to fall below the middle figure between 8,000 and 27,000,

The result is then 237, because 0 is in the floor, and 6 and 9 are closer to the ceiling of 27,000.

Example 1

478,211,768
$700^2 < 1{,}082 < 800^3$, so 7??

1sBase	1sResult
8	2

So 7?2
4+7+8+2+1+1+7+6+8 = 4+4 = 8

DSBase	DSResult
8	2 or 5 or 8

7+x+2 = 2, 5 or 8, so the possibilities are:
722, 752, or 782
Ordered in the range of 700^2 (343M) and 800^2 (512M)
Since 478M is closer to 512M than to 343M the result must be
782

Example 2

83,453,453
$400^2 <$ 83M $< 500^3$, so 4??

1sBase	1sResult
3	7

So 4?7
8+3+4+5+3+4+5+3 = 3+5 = 8

DSBase	DSResult
8	2 or 5 or 8

4+x+7 = 2, 5 or 8, so the possibilities are:
417, 437, or 467
Ordered in the range of 400^2 (64M) and 500^2 (125M)
Since 83M is closer to 64M than to 125M the result must be either 417 or 437. But 83M is closer to mid-point between 64M and 125M, so the result cannot be 417, and must be
437

Example 3	28,791,000 $300^2 < 28M < 400^3$, so 3?? 1sBase 1sResult 0 0 So 3?0 2+8+7+9+1+0+0+0 = 2+8 = 1+0 = 1 DSBase DSResult 1 1 or 4 or 7 3+x+0 = 1, 4 or 7, so the possibilities are: 370, 310, or 340 Ordered in the range of 300^2 (27M) and 400^2 (64M) Since 28M is closer to 27M than to 64M the result must be 310

3.5 FINDING 4TH, 5TH, ROOTS

The process is similar for 4th and 5th roots.

The problem here is that the root bases of are so inordinately large that they will be very hard to remember when taking digital sums.

For the purposes of mental math, an approximation, that is estimating the leftmost figure (thousands, ten-thousands and so on) may suffice.

Remember the following

Calculating 4th roots:

1. **10^4** is 10,000… Four appended zeros.
   So, the *fourth root of a **five-digit number** will be two digits.*
   -
2. **100^4** is 100,000,000… Eight appended zeros.
   So, the fourth root of a ***seven-digit number*** will be *three digits.*

3. **1000^4** is 1,000,000,000,000… Twelve appended zeros.
   So, the fourth root of a ***thirteen-digit number*** will be *four digits*.

Similarly, for fifth and sixth roots,

Where the power is… then the base of the nth root will have this number of digits

10^5	(*six digits*)	$1.0x10^5$
100^5	(*eleven digits*)	$1.0x10^{10}$
1000^5	(*sixteen digits*)	$1.0x10^{15}$
10^6	(*seven digits*)	$1.0x10^6$
100^6	(*thirteen digits*)	$1.0x10^{12}$
1000^6	(*nineteen digits*)	$1.0x10^{18}$
10^7	(*eight digits*)	$1.0x10^7$
100^7	(*fifteen digits*)	$1.0x10^{14}$
1000^7	(*twenty-two digits*)	$1.0x10^{21}$

And so on, and so forth.

Knowing this has a less apparent use,

To estimate an approximate value for whatever power, remember to take the leftmost digits, and determine the magnitude of the quantity (billion, quadrillion, and so on and so forth, using the table above).

For instance, if you take the 8th root of a number close to $1.0x10^{24}$ you can confidently say that the result will likely be *four-digits*, because we need to divide 24 (the number of appended zeros) to 3 to get 8.

And three is the number of appended zeros of the base 8th power result of the 8th root of the 24 trailing spaces.

Example 1	$\sqrt[5]{(1 \text{ x } 10^5)}$ 10^1 10

Example 2	$\sqrt[7]{(1 \text{ x } 10^{21})}$ 10^3 1,000

Example 3	$\sqrt[6]{(1 \times 10^{12})}$ 10^2 100

4. PROGRESSIVE PRACTICE: ROOTS

We have estimated the basics of roots. Let's move on to some practice exercises so that we can put to use what we've seen.

4.1 SQUARE AND CUBE ROOTS TO MEMORIZE

$^{2}\sqrt{4}$ (memorize)

$^{2}\sqrt{49}$ (memorize)

$^{2}\sqrt{196}$ (memorize)

$^{2}\sqrt{289}$ (memorize)

$^{2}\sqrt{361}$ (memorize)

$^{3}\sqrt{1}$ (memorize)

$^{3}\sqrt{512}$ (memorize)

$^{3}\sqrt{4,913}$ (memorize)

4.2 SQUARE ROOTS OF THREE TO FOUR DIGIT BASES

$^{2}\sqrt{484}$(determine units place value, determine tens place value)

$^{2}\sqrt{1,369}$ (determine units place value, determine tens place value)

$^{2}\sqrt{4,624}$ (determine units place value, determine tens place value)

$^{2}\sqrt{9,801}$ (determine units place value, determine tens place value)

$^{2}\sqrt{3,025}$ (determine units place value, determine tens place value)

$^{2}\sqrt{6,561}$ (determine units place value, determine tens place value)

$^{2}\sqrt{8,649}$ (determine units place value, determine tens place value)

4.3 CUBE ROOTS OF THREE TO SIX DIGIT BASES

$^{3}\sqrt{9,261}$ (determine units place value, determine tens place value)

$^{3}\sqrt{21,952}$ (determine units place value, determine tens place value)

$^{3}\sqrt{46,656}$ (determine units place value, determine tens place value)

$^{3}\sqrt{405,224}$ (determine units place value, determine tens place value)

$^{3}\sqrt{117,649}$ (determine units place value, determine tens place value)

$^{3}\sqrt{373,248}$ (determine units place value, determine tens place value)

$^{3}\sqrt{857,375}$ (determine units place value, determine tens place value)

$^{3}\sqrt{571,787}$ (determine units place value, determine tens place value)

4.4 SQUARE ROOTS OF MORE THAN FOUR DIGIT BASES

$\sqrt[2]{110,889}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{200,704}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{405,769}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{664,225}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{855,625}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{474,721}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[2]{528,529}$ (determine units place value, determine hundreds place value, determine tens place value)

4.5 CUBE ROOTS OF MORE THAN SIX DIGIT BASES

$\sqrt[3]{6,539,203}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{63,044,792}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{296,740,963}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{149,721,291}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{125,000,000}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{334,255,384}$ (determine units place value, determine hundreds place value, determine tens place value)

$\sqrt[3]{359,425,431}$ (determine units place value, determine hundreds place value, determine tens place value

4.6 SIMPLIFY AND ESTIMATE ROOTS

$\sqrt[5]{1.0x10^5}$ (determine the multiplier and simplify the expression)

$\sqrt[19]{1.0x10^{19}}$ (determine the multiplier and simplify the expression)

$\sqrt[9]{1.0x10^9}$ (determine the multiplier and simplify the expression)

$\sqrt[5]{1.0x10^{10}}$ (determine the multiplier and simplify the expression)

$\sqrt[5]{1.0x10^{20}}$ (determine the multiplier and simplify the expression)

$\sqrt[5]{1.0x10^{40}}$ (determine the multiplier and simplify the expression)

$\sqrt[7]{1.0\text{x}10^{21}}$ (determine the multiplier and simplify the expression)

$\sqrt[7]{1.0\text{x}10^{49}}$ (determine the multiplier and simplify the expression)

$\sqrt[9]{1.0\text{x}10^{81}}$ (determine the multiplier and simplify the expression)

$\sqrt[11]{1.0\text{x}10^{121}}$ (determine the multiplier and simplify the expression)

$\sqrt[8]{1.0\text{x}10^{48}}$ (determine the multiplier and simplify the expression)

5. PARTING WORDS

Thank you for reading this book. I really do hope you found it useful and enjoyable. Math is challenging, that's what makes it fun.

If you review and re-do the exercises in the practice chapters, make sure you also review the appendix.
Did you find a better way to arrive at the result? Congratulations!
Success is all about taking matters into your hands!

Did you like the book? Let us know, we love to hear what we did well. It encourages us to pursue greater challenges.

Did you find things you didn't like? Do let us know also. It's only human to make mistakes, however painstakingly we may have reviewed this book before it was published.

In any case, I wish you the best in your journey to achieving mastery in this or any other areas of your life.

And I hope that you will join us as we explore the world of mathematics in future publications.

With warm regards, all the best!

The Sigmacasts Team

APPENDIX: WORKED OUT ANSWERS TO PROBLEMS CHAPTERS 2 AND 4

EXPONENTS TO MEMORIZE

9^3 729	**5^2** 25	**4^8** 65,536
1^0 1	**2^1** 2	**6^9** 10,077,696 or $\approx 10x10^6$
8^5 32,768	**3^4** 81	**7^7** 823,543

BASE TEN

10^2 100	**10^{40}** $1x10^{40}$
10^5 100,000	**10^{80}** $1x10^{80}$ 100
10^{20} $1x10^{20}$	**0^{10}** 0

POWER OF TEN EXPONENTS

0^{10}	10^2	3^{40}
1	**100**	**≈ 1.22x10^{19}**

1^{20}	2^{30}	2^{50}
1	**≈ 1.07x10^{9}**	**≈ 1.13x10^{15}**

BREAK DOWN BASE OR EXPONENT

34^2
$2^2 \times 17^2$
4×289
1,156

66^2
$6^2 \times 11^2$
36×121
4,356

84^{21}
$7^{20} \times 12^{20} \times 84$
$(7^4 \times 12^4)^5 \times 84$
$(7^4 \times 2^4 \times 6^4)^5 \times 84$
$(7^4 \times 2^4 \times 6^4)^5 \times 84$
$(2401 \times 16 \times 1296)^5 \times 84$
$\approx (49.79 \times 10^6)^5 \times 84$
$\approx 306 \times 10^6 \times 10^{30} \times 84$
≈ 2.57 x10^{40}

27^2
$3^2 \times 9^2$
9×81
729

72^2
$9^2 \times 8^2$
81×64
5,184

25^{25}
$(25^5)^5$
$(5^5 \times 5^5)^5$
$(5^5 \times 5^5)^5$
$(3125 \times 3125)^5$
$\approx (9.77 \times 10^6)^5$
$\approx (89{,}017 \times 10^{30})$
$\approx (8.90 \times 10^4 \times 10^{30})$
≈ (8.90x10^{34})

56^2
$7^2 \times 8^2$
49×64
3,136

116^3
$2^3 \times 58^3$
$8 \times 195{,}112$
≈ 1.56 x 10^{6}

12^{12}
$12^6 \times 12^6$
$2^6 \times 6^6 \times 2^6 \times 6^6$
$(2^6 \times 6^6)^2$
$(64 \times 46{,}656)^2$
$(2.99 \times 10^6)^2$
≈ 8.94 x 10^{12}

PRIME BASE

17^{19}

19^2

43^7

2.39×10^{23}	19×19 **361**	**2.72×10^{11}**
11^0 **1**	31^5 **$\approx 2.86 \times 10^7$**	37^8 **3.51×10^{12}**

MULTIPLE OF BASE TEN

200^2 **40,000**	7000^2 49 **49×10^6**	$47{,}000^3$ 103,823 **$103{,}823 \times 10^9$**	$4{,}000^4$ 256 **256×10^{12}**	27000^6 $(3^3 \times 9^3)^2$ $(27 \times 729)^2$ $19{,}683^2$ $\approx 20{,}000^2$ $\approx 4 \times 10^8$ $\approx 4 \times 10^8 \times 10^{18}$ **$\approx 4 \times 10^{26}$**
350^2 $5^2 \times 7^2$ 25×49 1225 **122,500**	$14{,}000^2$ $2^2 \times 7^2$ 4×49 196 **196×10^6**	9000^3 729 **729×10^9**	$66{,}000^4$ $11^4 \times 2^4 \times 3^4$ $14641 \times 16 \times 81$ $234{,}256 \times 81$ $\approx 1.88 \times 10^7$ $\approx 1.88 \times 10^7 \times 10^{12}$ **$\approx 1.88 \times 10^{19}$**	11000^7 $\approx 1.95 \times 10^7$ $\approx 1.95 \times 10^7 \times 10^{21}$ **$\approx 1.95 \times 10^{16}$**
470^2 47^2 2209 **220,900**	$38{,}000^2$ $19^2 \times 2^2$ 361×4 **1,444**	$4{,}000^3$ 64 **64×10^9**	72000^5 $9^5 \times 8^5$ $59{,}049 \times 32{,}768$ $\approx 1.93 \times 10^9$ $\approx 1.93 \times 10^9 \times 10^{15}$ **$\approx 1.93 \times 10^{24}$** Or if you don't require as much precision: 72000^5 $9^5 \times 8^5$ $59{,}049 \times 32{,}768$ round as 59×33 1947	$21{,}000^8$ $(3^4 \times 7^4)^2$ $(81 \times 2401)^2$ $(194{,}481)^2$ $\approx (200{,}000)^2$ $\approx 4 \times 10^{10}$ $\approx 4 \times 10^{10} \times 10^{24}$ **$\approx 4 \times 10^{34}$**

$\approx 1.95 \times 10^{9}$
$\approx 1.95 \times 10^{9} \times 10^{15}$
$\approx \mathbf{1.95 \times 10^{24}}$

APPENDIX: WORKED OUT ANSWERS TO PROBLEMS CHAPTERS 2 AND 4

SQUARE AND CUBE ROOTS TO MEMORIZE

$\sqrt[2]{4}$
2

$\sqrt[2]{49}$
7

$\sqrt[2]{196}$
14

$\sqrt[2]{289}$
17

$\sqrt[2]{361}$
19

$\sqrt[3]{1}$
1

$\sqrt[3]{512}$
8

$\sqrt[3]{4{,}913}$
17

SQUARE ROOTS OF THREE TO FOUR DIGIT BASES

$^2\sqrt{484}$
$^2\sqrt{484}$
4 -> **2**
$\mathbf{2}^2 \le 4 < 3^2$
22

$^2\sqrt{3,025}$
$^2\sqrt{3,025}$
5 -> **5**
$5^2 \le 30 < 6^2$
55

$^2\sqrt{1369}$
$^2\sqrt{1,369}$
9 -> **7**
$\mathbf{3}^2 \le 13 < 4^2$
37

$^2\sqrt{6,561}$
$^2\sqrt{6,561}$
1 -> **1**
$\mathbf{8}^2 \le 65 < 9^2$
81

$^2\sqrt{4624}$
$^2\sqrt{4,624}$
4 -> **8**
$\mathbf{6}^2 \le 46 < 7^2$
68

$^2\sqrt{8,649}$
$^2\sqrt{8,649}$
9 -> **3**
$\mathbf{9}^2 \le 86 < 10^2$
93

$^2\sqrt{9,801}$
$^2\sqrt{9,801}$
1 -> **9**
$\mathbf{9}^2 \le 98 < 10^2$
99

CUBE ROOTS OF THREE TO SIX DIGIT BASES

$\sqrt[3]{9,261}$
$\sqrt[3]{9,261}$
1 -> **1**
$2^3 \leq 9 < 3^3$
21

$\sqrt[3]{117,649}$
$\sqrt[3]{117,649}$
9 -> **9**
$40^3 \leq 117k < 50^3$
49

$\sqrt[3]{21,952}$
$\sqrt[3]{21,952}$
2 -> **8**
$20^3 \leq 21k < 30^3$
28

$\sqrt[3]{373,248}$
$\sqrt[3]{373,248}$
8 -> **2**
$70^3 \leq 373k < 80^3$
72

$\sqrt[3]{46,656}$
$\sqrt[3]{46,656}$
6 -> **6**
$30^3 \leq 46k < 40^3$
36

$\sqrt[3]{857,375}$
$\sqrt[3]{857,375}$
5 -> **5**
$90^3 \leq 857k < 100^3$
95

$\sqrt[3]{405,224}$
$\sqrt[3]{405,224}$
4 -> **4**
$7^3 \leq 405 < 8^3$
74

$\sqrt[3]{571,787}$
$\sqrt[3]{571,787}$
7 -> **3**
$70^3 \leq 571k < 80^3$
83

SQUARE ROOTS OF MORE THAN FOUR DIGIT BASES

$\sqrt[2]{110,889}$
$\sqrt[2]{110,889}$
9 -> **3** or 7
30^3 $\leq$ 110k < 40^2
DSSR = 1+1+8+8+9 = 9
9 -> 3, 6 or **9 (DSR)**
333, 363 or 393
333

$\sqrt[2]{855,625}$
$\sqrt[2]{855,625}$
5 -> **5**
90^3 $\leq$ 855k < 100^3
DSSR = 8+5+5+6+2+5 = 4
4 -> 2, **7** (DSR)
965, 925
925

$\sqrt[2]{200,704}$
$\sqrt[2]{200,704}$
4 -> **8**
40^3 $\leq$ 200k < 50^2
DSSR = 2+7+4 = 4
4 -> 2, **7 (DSR)**
488, 448
448

$\sqrt[2]{474,721}$
$\sqrt[2]{474,721}$
1 -> **9**
60^3 $\leq$ 474k < 70^3
DSSR = 4+7+4+7+2+1 = 7
7 -> **5 (DSR)**
689

$\sqrt[2]{405,769}$
$\sqrt[2]{405,769}$
9 -> 3 or **7**
60^3 $\leq$ 405k < 70^2
4 -> 2, **7 (DSR)**
677, 637
637

$\sqrt[2]{528,529}$
$\sqrt[2]{528,529}$
9 -> 3 or **7**
70^3 < 528k < 80^2
DSSR = 5+2+8+5+2+9 = 4
4 -> 2, **7** (DSR)
727, 767
727

$\sqrt[2]{664,225}$
$\sqrt[2]{664,225}$
5 -> **5**
80^3 $\leq$ 664k < 90^2
DSSR = 6+6+4+2+2+5 = 7
7 -> **5 (DSR)**
815

CUBE ROOTS OF MORE THAN SIX DIGIT BASES

$^3\sqrt{}$6,539,203
$^3\sqrt{}$6,539,203
3 -> **7**
100^3 ≤ 6m < 200^3
DSSR = 6+5+3+9+2+3 = 1
1 -> 1, 4, or **7 (DSR)**
127, **157**, **187**
187

$^3\sqrt{}$125,000,000
$^3\sqrt{}$125,000,000
500^3 ≤ 125m < 600^3
0 -> **0** or 5
550, 500
500

$^3\sqrt{}$63,044,792
$^3\sqrt{}$63,044,792
2 -> **8**
300^3 ≤ 63m < 400^3
DSSR = 6+3+4+4+7+9+2 = 8
8 -> **2,** 5, or 8 **(DSR)**
398, **338**, **368**
398

$^3\sqrt{}$334,255,384
$^3\sqrt{}$334,255,384
4 -> **4**
600^3 ≤ 334m < 700^3
DSSR = 3+3+4+2+5+5+3+8+4 = 1
1 -> **1**, 4, or 7
694, **634**, **664**
694

$^3\sqrt{}$296,740,963
$^3\sqrt{}$296,740,963
3 -> **7**
600^3 ≤ 296m < 700^3
DSSR = 2+9+6+7+4+9+6+3 = 1
1 -> **1**, 4, or 7 **(DSR)**
667, **697**, **637**
667

$^3\sqrt{}$359,425,431
$^3\sqrt{}$359,425,431
1 -> **1**
700^3 ≤ 359m < 800^3
DSSR = 3+5+9+4+2+5+4+3+1 = 9
9 -> 3, 6, **or 9**
741, **771**, **711**
711

$^3\sqrt{}$149,721,291
$^3\sqrt{}$149,721,291
1 -> **1**
500^3 ≤ 149m < 600^3
DSSR = 1+4+9+7+2+1+2+9+1 = 9
9 -> 3, 6, or **9 (DSR)**
561, 501, **531**
531

SIMPLIFY AND ESTIMATE ROOTS

$\sqrt[5]{1.0x10^{5}}$ **10**	$\sqrt[5]{1.0x10^{20}}$ **10,000**	$\sqrt[9]{1.0x10^{81}}$ **1,000,000,000**
$\sqrt[19]{1.0x10^{19}}$ **10**	$\sqrt[5]{1.0x10^{40}}$ **100,000,000**	$\sqrt[11]{1.0x10^{121}}$ **100,000,000,000**
$\sqrt[9]{1.0x10^{9}}$ **10**	$\sqrt[7]{1.0x10^{21}}$ **1,000**	$\sqrt[8]{1.0x10^{48}}$ **1,000,000**
$\sqrt[5]{1.0x10^{10}}$ **100**	$\sqrt[7]{1.0x10^{49}}$ **10,000,000**	