Homework 3 (Due 10/5)

(A)

Three events, A, B, and C, are said too be mutually independent if

$P(A \cap B) = P(A) \cdot P(B)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap C) = P(A) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

A : Head appears on the first toss.

B : Head appears on the second toss.

C : Both tosses yield the same outcome

Are A, B, and C mutually independent?

Ans :

$U = \{(+,+),\ (+,-),\ (-,-),\ (-,+)\}$

$A = \{(+,+),\ (+,-)\},\ \ P(A) = \frac{1}{2}$

$B = \{(+,+),\ (-,+)\},\ \ P(B) = \frac{1}{2}$

$C = \{(+,+),\ (-,-)\},\ \ P(C) = \frac{1}{2}$

$A \cap B = \{(+,+)\},\ \ P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B)$

$B \cap C = \{(+,+)\},\ \ P(B \cap C) = \frac{1}{4} = P(B) \cdot P(C)$

$A \cap C = \{(+,+)\},\ \ P(A \cap C) = \frac{1}{4} = P(A) \cdot P(C)$

$A \cap B \cap C = \{(+,+)\}\ \ P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C)$

$\implies A, B$, and C are not mutually indenpendent

(B)

Referring to the following theorem, given $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma}$, show that $Z \sim N(0,1)$, i.e., unit normal.

Ans :

$$E[Z] = E[\frac{X-\mu}{\sigma}] = \frac{1}{\sigma}(E[X] - \mu) = \frac{1}{\sigma}(\mu - \mu) = 0$$

$$
\begin{aligned}
Var[Z^2] &= E[Z^2] - E[Z]^2 \\
&= E[(\frac{X-\mu}{\sigma})^2] - 0^2 \\
&= \frac{1}{\sigma^2}(E[X^2] - 2\mu E[X] + \mu^2) \\
&= \frac{1}{\sigma^2}((E[X^2] - E[X]^2) + E[X]^2 - 2\mu E[X] + \mu^2) \\
&= \frac{1}{\sigma^2}(Var[X] + \mu^2 - 2\mu\mu + \mu^2) \\
&= \frac{1}{\sigma^2}(\sigma^2) \\
&= 1
\end{aligned}
$$