Wrestling with the Fundamental Theorem of Calculus

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The Fundamental Theorem of Calculus:

1. If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Are these statements ***always*** true? What assumptions do we have to make about *f* and *F* in order for these statements to be true?

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What if a function is not the derivative of some identifiable function?

$$e^{\left(-x^2\right)}$$

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How do you define area?

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A.-L. Cauchy: First to define the integral as the limit of the summation $\sum f(x_{i-1})(x_i - x_{i-1})$

Bernhard Riemann (1852, 1867) *On the representation of a function as a trigonometric series*

Defined $\int_a^b f(x)\,dx$ as limit of $\sum f\left(x_i^*\right)\left(x_i - x_{i-1}\right)$

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Key to convergence: on each interval, look at the **variation** of the function

$$V_i = \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)$$

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Integral exists if and only if $\sum V_i \left(x_i - x_{i-1}\right)$ can be made as small as we wish by taking sufficiently small intervals.

Any continuous function is integrable:

Can make V_i as small as we want by taking sufficiently small intervals:

$$\sum V_i\left(x_i - x_{i-1}\right) < \frac{\varepsilon}{b-a}\sum\left(x_i - x_{i-1}\right) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Any piecewise continuous function is integrable:

If $\lim\limits_{x \to c^-} f(x) - \lim\limits_{x \to c^+} f(x) = J$,

then take an interval of length $< \dfrac{\varepsilon}{2\left(|J| + 1\right)}$

where the variation is less than $|J| + 1$:

$$\sum V_i\left(x_i - x_{i-1}\right) < \left(|J| + 1\right)\frac{\varepsilon}{2\left(|J| + 1\right)} + \sum_{\text{other intervals}} V_i\left(x_i - x_{i-1}\right)$$

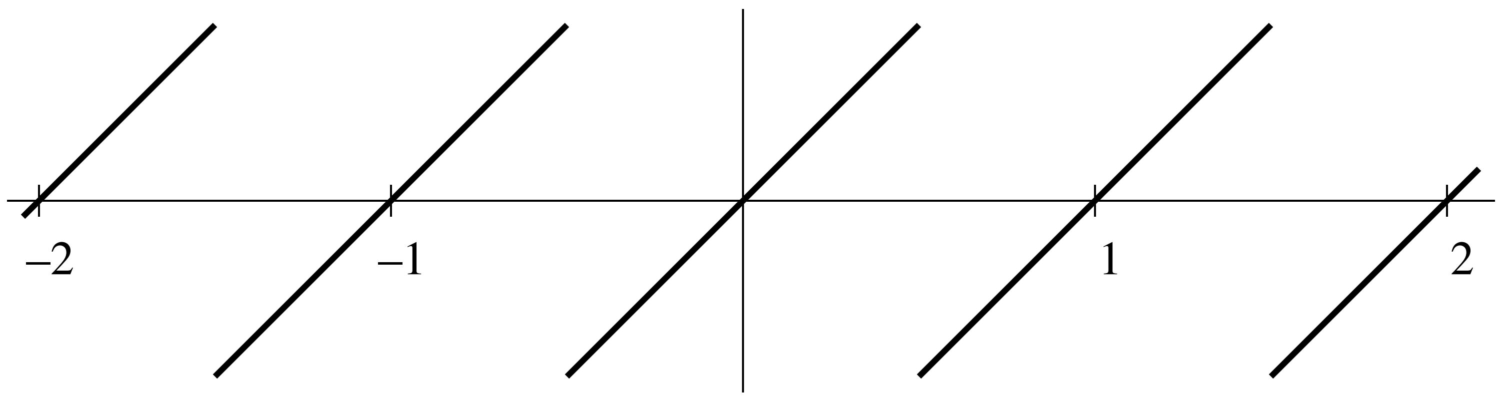
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Bernhard Riemann (1852, 1867) *On the representation of a function as a trigonometric series*

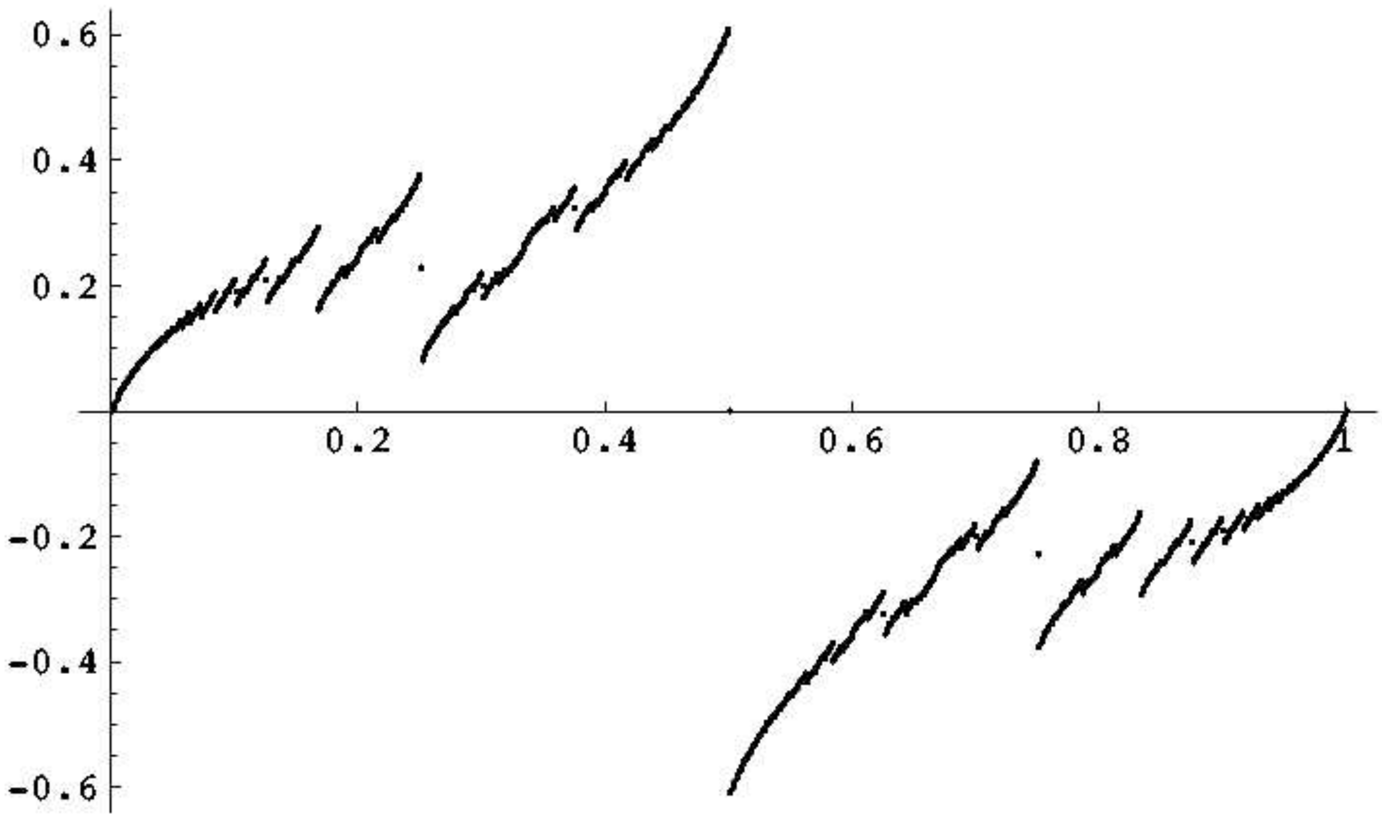
Riemann gave an example of a function that has a jump discontinuity in *every* subinterval of [0,1], but which can be integrated over the interval [0,1].

Riemann's function: $f(x)=\sum_{n=1}^{\infty}\frac{\{nx\}}{n^2}$

$$\{x\}=\begin{cases} x-(\text{nearest integer}), \text{ when this is } < \tfrac{1}{2}, \\ 0, \text{ when distance to nearest integer is } \tfrac{1}{2} \end{cases}$$



$\frac{\{nx\}}{n^2}$ has n jumps of size $\frac{2}{n^2}$ between 0 and 1



At $x = \frac{a}{2b}$, $\gcd(a,2b) = 1$, the function jumps by $\frac{\pi^2}{8b^2}$

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At $x = a/2b$, $\gcd(a,2b) = 1$, the function jumps by $\pi^2/8b^2$

The key to the integrability is that given any positive number, no matter how small, there are only a finite number of places where is jump is larger than that number.

If $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ exist,

then they must be equal and they

must equal $f'(c)$.

Mean Value Theorem:

$$f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^-} f'(k), \quad x < k < c$$

$$f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} f'(k), \quad c < k < x$$

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The derivative of a function cannot have any jump discontinuities!

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The key to the integrability is that given any positive number, no matter how small, there are only a finite number of places where is jump is larger than that number.

Conclusion: $F(x) = \int_0^x f(t)\,dt$ exists and is well-defined for all x, but F is *not* differentiable at any rational number with an even denominator.

The Fundamental Theorem of Calculus:

1. If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2.If f does not have a jump discontinuity at x, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

If F is differentiable at $x = a$, can $F\,'(x)$ be discontinuous at $x = a$?

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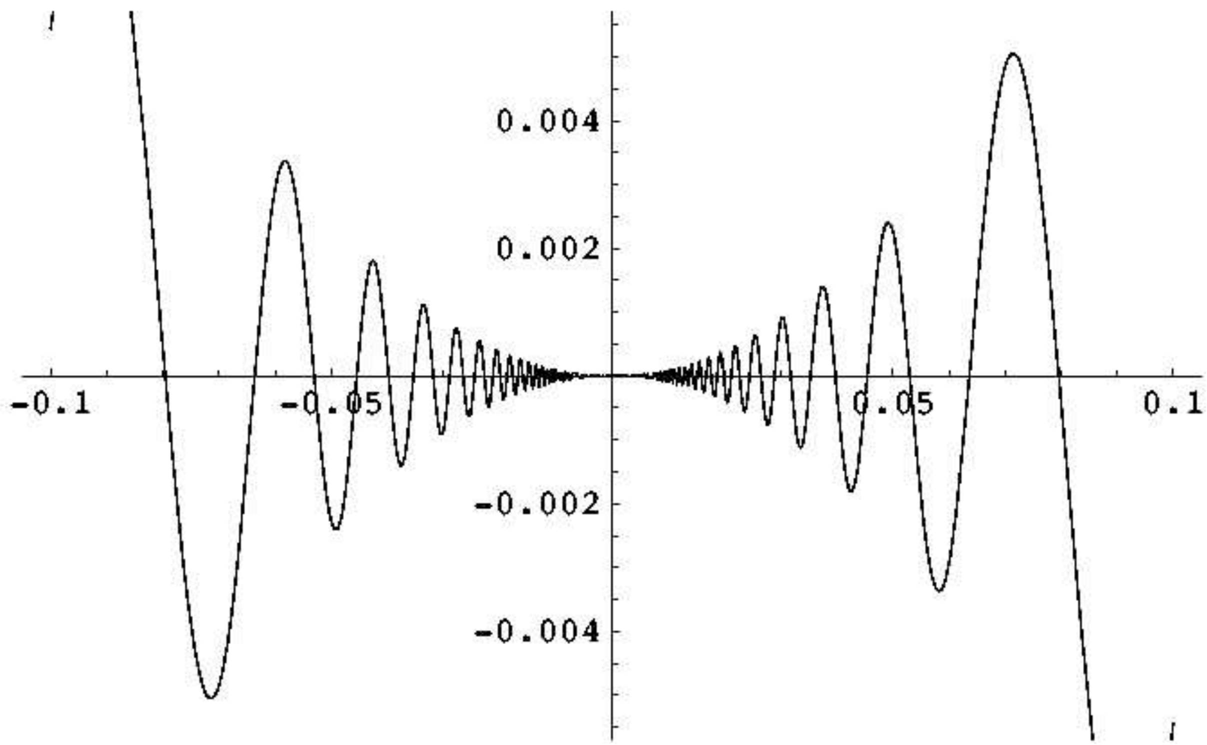
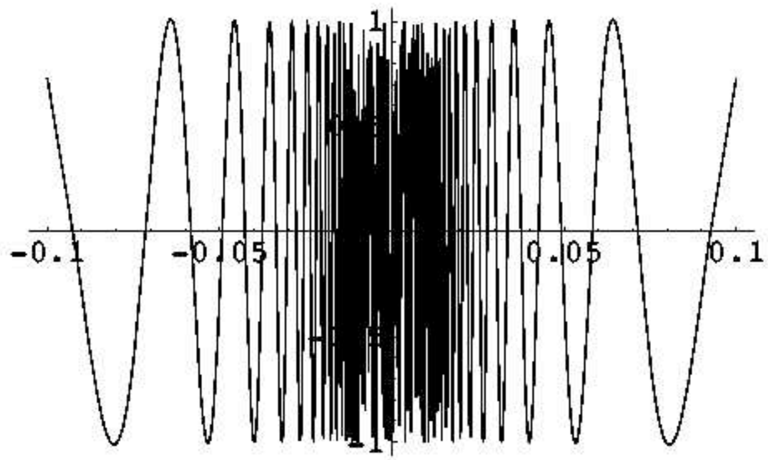
Yes!

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$



$$F(x) = \begin{cases} x^2 \sin\left(1/x\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$F'(x) = 2x \sin\left(1/x\right) - \cos\left(1/x\right), \quad x \neq 0.$$



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$$F'(0) = \lim_{h\to 0} \frac{F(h) - F(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0\,.$$

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$$= \lim_{h\to 0} h\sin\left(1/h\right) = 0 \ .$$

$\lim\limits_{x\to 0} F'(x)$ does not exist, but

$F'(0)$ does exist (and equals 0).

$$f(x) = \begin{cases} 2x\sin\left(1/x\right) - \cos\left(1/x\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is f integrable?

$$\int_0^x f(t)\,dt \stackrel{?}{=} \begin{cases} x^2 \sin\left(1/x\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

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$$V_i = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Integral exists if and only if $\sum V_i \left(x_i - x_{i-1}\right)$ can be made as small as we wish by taking sufficiently small intervals.

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Is f integrable? $\boxed{\text{Yes!}}$

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Integral exists if and only if $\sum V_i (x_i - x_{i-1})$ can be made as small as we wish by taking sufficiently small intervals.

Consider:

$$F(x)=\begin{cases} x^2\sin\left(1/x^2\right), & x\neq 0, \\ 0, & x=0. \end{cases}$$

$$F'(x)=\begin{cases} 2x\sin\left(1/x^2\right)-\dfrac{2}{x}\cos\left(1/x^2\right), & x\neq 0, \\ 0, & x=0. \end{cases}$$

Is F' integrable?

$$\int_0^x F'(t)\,dt \stackrel{?}{=} \begin{cases} x^2\sin\left(1/x^2\right), & x\neq 0, \\ 0, & x=0. \end{cases}$$

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No!

Consider:

$$F(x) = \begin{cases} x^2 \sin\left(1/x^2 \right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$F'(x) = \begin{cases} 2x \sin\left(1/x^2 \right) - \frac{2}{x} \cos\left(1/x^2 \right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is F' integrable?

$$\int_0^x F'(t)\,dt \stackrel{?}{=} \begin{cases} x^2 \sin\left(1/x^2 \right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

But the right side *does* equal $\lim_{\varepsilon \to 0} \int_\varepsilon^x F'(t)\,dt$

What if the derivative, $F\,'$, is bounded? Will the derivative always be integrable?

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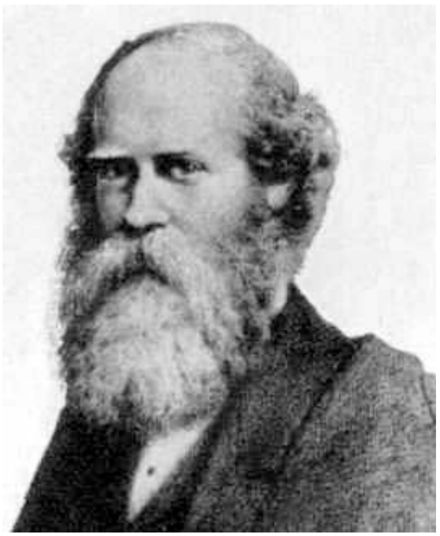
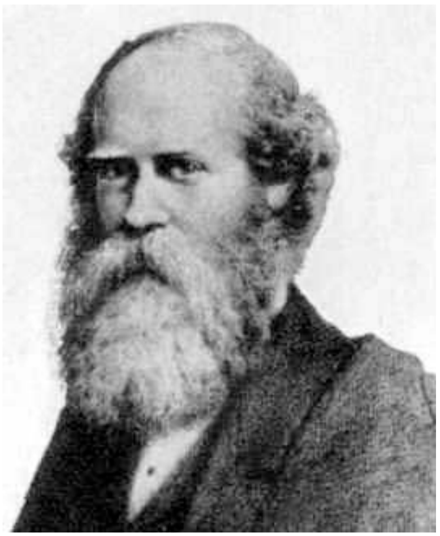
No!

“Cantor’s Set”



First described by H.J.S. Smith, 1875

"Cantor's Set"



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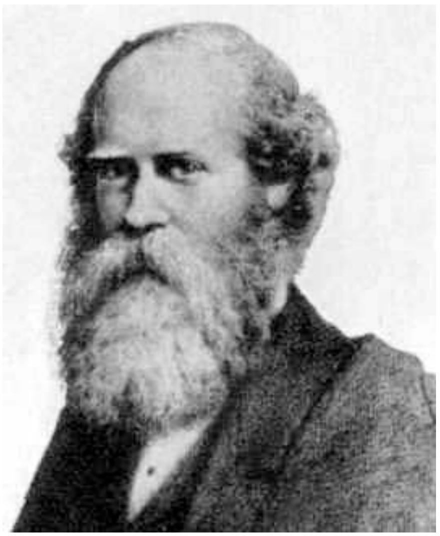
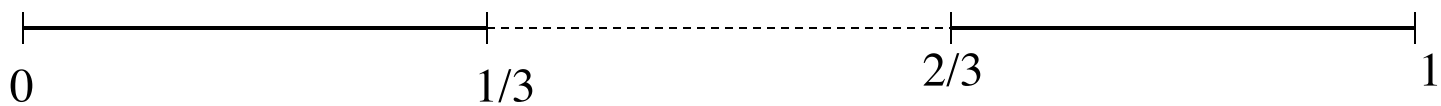


First described by H.J.S. Smith, 1875

Then by Vito Volterra, 1881

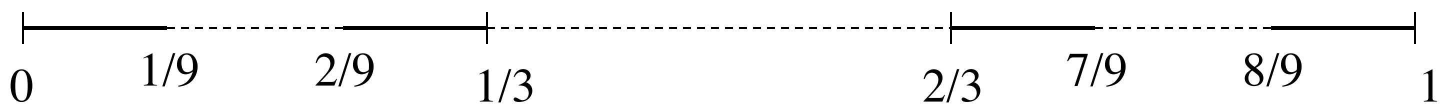
And finally by Georg Cantor, 1883

Cantor's Set



Remove [1/3,2/3]

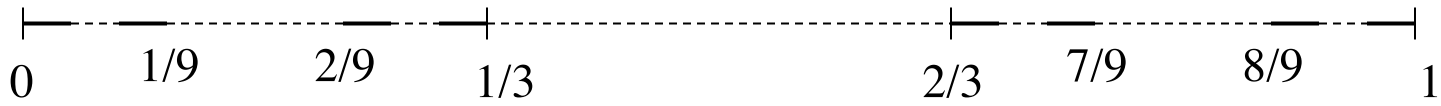
Cantor's Set



Remove [1/3,2/3]

Remove [1/9,2/9] and [7/9,8/9]

Cantor's Set



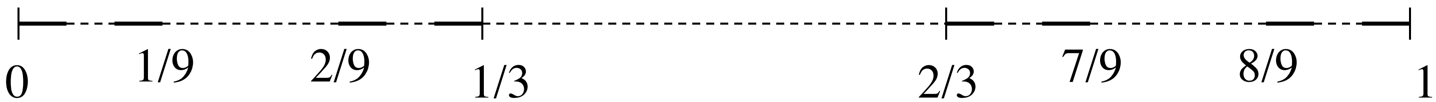
Remove [1/3,2/3]

Remove [1/9,2/9] and [7/9,8/9]

Remove [1/27,2/27], [7/27,8/27], [19/27,20/27], and [25/27,26/27], and so on …

What's left?

Cantor's Set



$$1-\left(\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots\right)$$

$$=1-\frac{1}{3}\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^{2}+\left(\frac{2}{3}\right)^{3}+\cdots\right)$$

$$=1-\frac{1}{3}\cdot\frac{1}{1-2/3}=1-1=0\ .$$

What's left has ***measure*** 0.

A set has ***measure*** 0 if for we can put the set inside a union of intervals whose total lengths are as close to 0 as we wish.

Examples:

Any finite set has measure zero.

The Cantor set has measure 0.

$$\left\{ \frac{1}{2^n} \,\middle|\, n = 1,2,\ldots \right\} \subset \left(\frac{1}{2} - \frac{1}{10}, \frac{1}{2} + \frac{1}{10}\right) \cup \left(\frac{1}{4} - \frac{1}{100}, \frac{1}{4} + \frac{1}{100}\right) \cup \left(\frac{1}{8} - \frac{1}{1000}, \frac{1}{8} + \frac{1}{1000}\right) \cup \cdots$$

$$\text{total length} < \frac{2}{10}\,\frac{1}{1 - 1/10} = \frac{2}{9}$$

$$[0,0.1),(.99,1],\left(\frac{1}{2}-\frac{1}{10^3},\frac{1}{2}+\frac{1}{10^3}\right),\left(\frac{1}{3}-\frac{1}{10^4},\frac{1}{3}+\frac{1}{10^4}\right),$$

$$\left(\frac{2}{3}-\frac{1}{10^5},\frac{2}{3}+\frac{1}{10^5}\right),\left(\frac{1}{4}-\frac{1}{10^6},\frac{1}{4}+\frac{1}{10^6}\right),\left(\frac{3}{4}-\frac{1}{10^7},\frac{3}{4}+\frac{1}{10^7}\right),$$

$$\left(\frac{1}{5}-\frac{1}{10^8},\frac{1}{5}+\frac{1}{10^8}\right),\left(\frac{2}{5}-\frac{1}{10^9},\frac{2}{5}+\frac{1}{10^9}\right),\left(\frac{3}{5}-\frac{1}{10^{10}},\frac{3}{5}+\frac{1}{10^{10}}\right),\ldots$$

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Every rational number in [0,1] is in at least one of these intervals. The sum of the lengths of these intervals is less than

$$\frac{1}{10}+\frac{1}{100}+\frac{2}{10^3}\left(1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)$$

$$=\frac{11}{100}+\frac{2}{1000}\cdot\frac{10}{9}=\frac{101}{900}$$

Base three numbers: 10211

1's

3's

9's

27's

81's

$$1 \cdot 1 + 1 \cdot 3 + 2 \cdot 9 + 0 \cdot 27 + 1 \cdot 81 = 103$$

Base three numbers: 10211.2012

1/3’s

1/9’s

1/27’s

1/81’s

$$1\cdot 1 + 1\cdot 3 + 2\cdot 9 + 0\cdot 27 + 1\cdot 81 = 103$$

$$2\cdot\frac{1}{3} + 0\cdot\frac{1}{9} + 1\cdot\frac{1}{27} + 2\cdot\frac{1}{81} = \frac{59}{81}$$

Base three numbers: $10211.2012 = 103\frac{59}{81}$

1/3's

1/9's

1/27's

1/81's

$\boxed{\text{"tercimal number"}}$

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Cantor’s set eliminates all tercimals from 0.1 to 0.2.

From 0.01 to 0.02 and 0.21 to 0.22

From 0.001 to 0.002, 0.021 to 0.022,
0.201 to 0.202, and 0.221 to 0.222

Cantor’s set eliminates all tercimals that terminate or contain a 1.

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Cantor's set eliminates all tercimals that terminate or contain a 1.

What's left?

$$0.020202\overline{02} = \frac{2}{9} + \frac{2}{81} + \frac{2}{9^3} + \frac{2}{9^4} + \cdots = \frac{2}{9} \cdot \frac{1}{1 - \frac{1}{9}} = \frac{1}{4}$$

The size of what's left is still uncountable.

Each element of the Cantor set corresponds to a unique number in base 2:

$$\frac{1}{4} = 0.020202\overline{02}_3 \Rightarrow 0.010101\overline{01}_2 = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$$

We get all of the base 2 numbers *except* those that terminate (rational numbers with denominators a power of 2).

The Fundamental Theorem of Calculus:

1. If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. If f does not have a jump discontinuity at x, then
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Vito Volterra, 1881: There exists a function, $F(x)$, whose derivative, $F'(x)$, exists and is bounded for all x, but the derivative, $F'(x)$, cannot be integrated.

Create a new set like the Cantor set except

the first middle piece only has length 1/4

each of the next two middle pieces only have length 1/16

the next four pieces each have length 1/64, etc.

The amount left has size $1-\left(\frac{1}{4}+\frac{2}{16}+\frac{4}{64}+\frac{2^3}{4^4}+\cdots\right)$

$$=1-\frac{1}{4}\left(1+\left(\frac{2}{4}\right)+\left(\frac{2}{4}\right)^2+\left(\frac{2}{4}\right)^3+\cdots\right)$$

$$=1-\frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}}=\frac{1}{2}$$

We'll call this set SVC (for Smith-Volterra-Cantor).

It has some surprising characteristics:

1. SVC contains no intervals - no matter how small a subinterval of [0,1] we take, there will be points in that subinterval that are not in SVC. SVC is **nowhere dense.**

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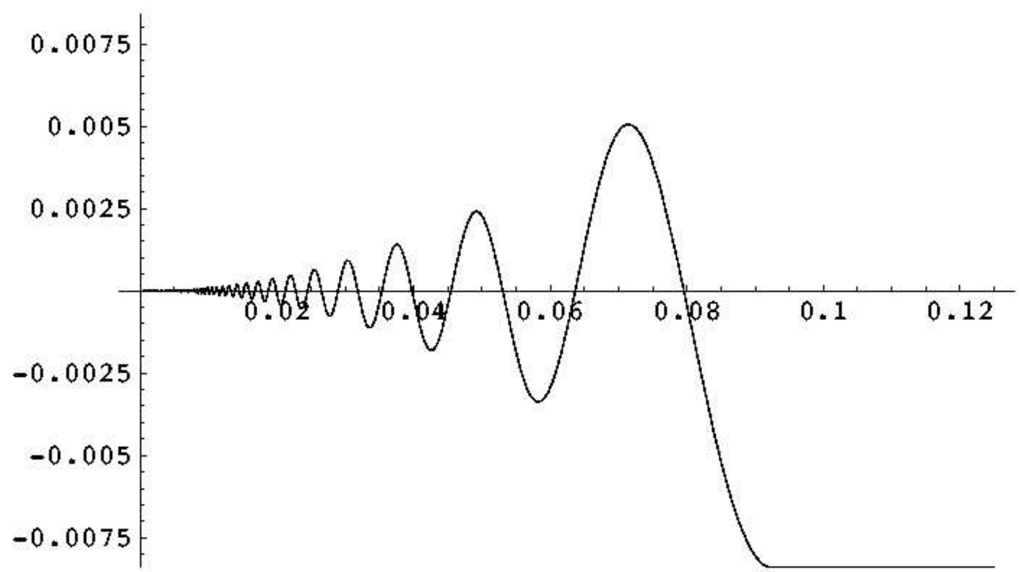
2. Given *any* collection of disjoint subintervals of [0,1], if the sum of the lengths is greater than 1/2, then they must contain at least one point in SVC.

3. Given *any* partition of [0,1] into subintervals, the sum of the lengths of the intervals that contain points in SVC will always be at least 1/2. SVC has **measure 1/2**.

Volterra's construction:

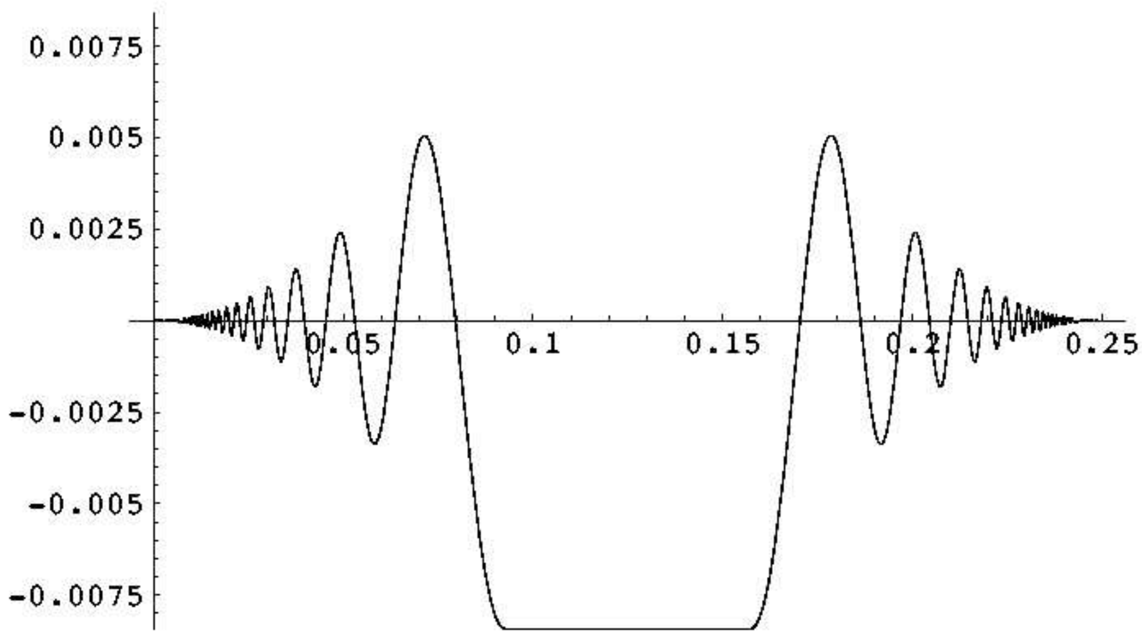
Start with the function $F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$

Restrict to the interval [0,1/8], except find the largest value of x on this interval at which $F'(x) = 0$, and keep F constant from this value all the way to $x = 1/8$.



Volterra's construction:

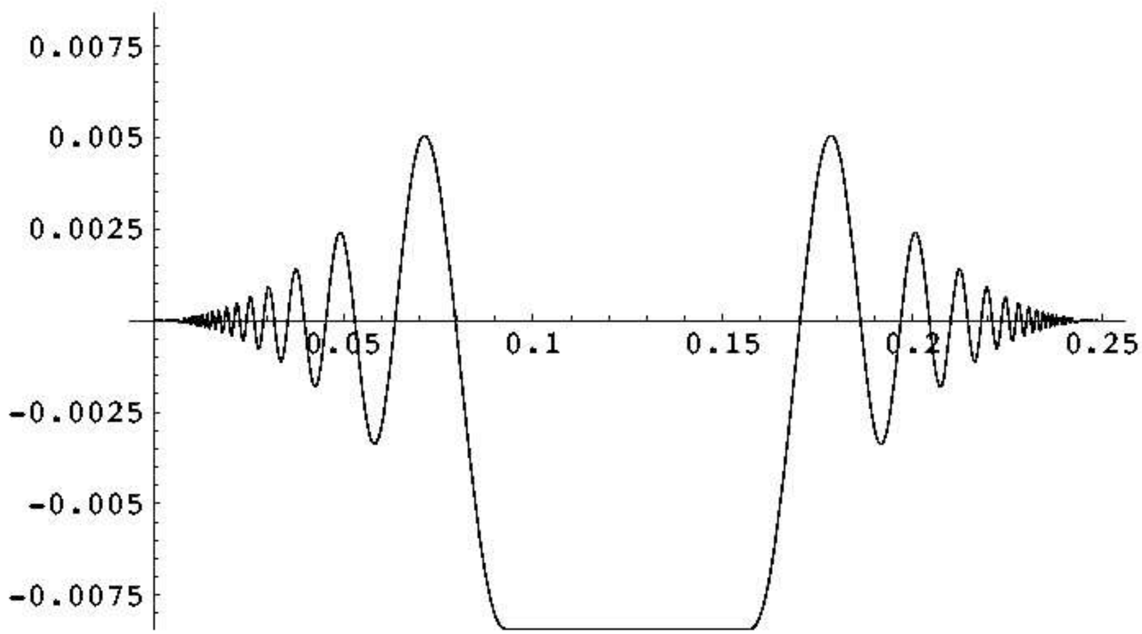
To the right of $x = 1/8$, take the mirror image of this function: for $1/8 < x < 1/4$, and outside of [0,1/4], define this function to be 0. Call this function $f_1(x)$.



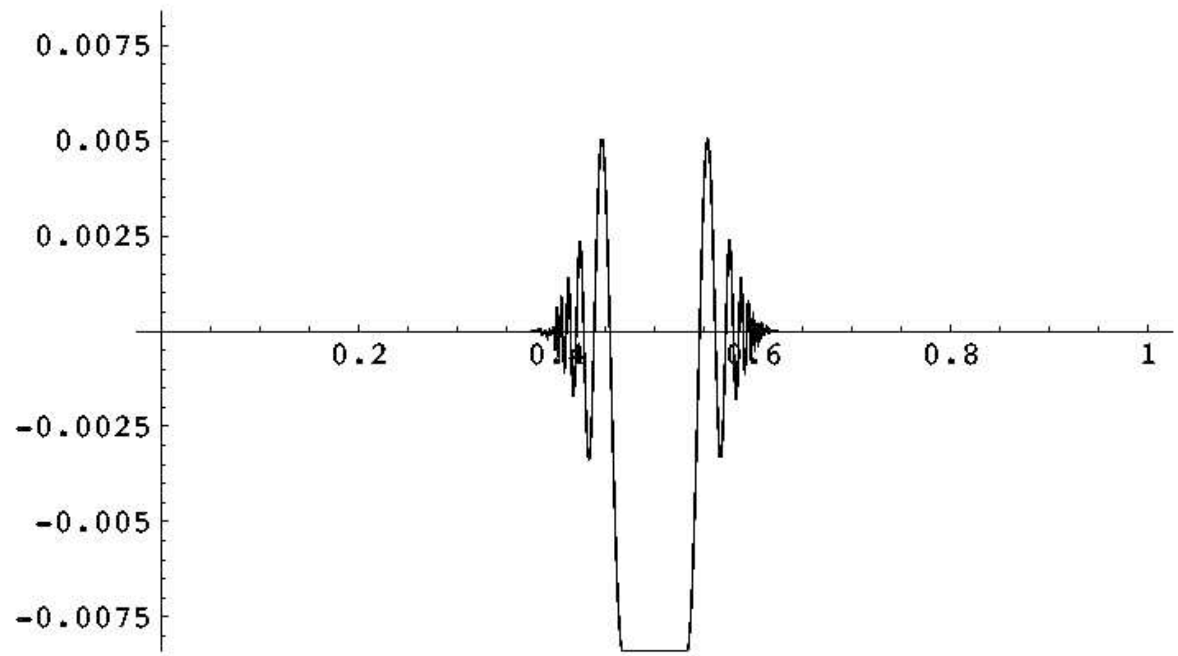
Volterra's construction:

$f_1(x)$ is a differentiable function for all values of x, but

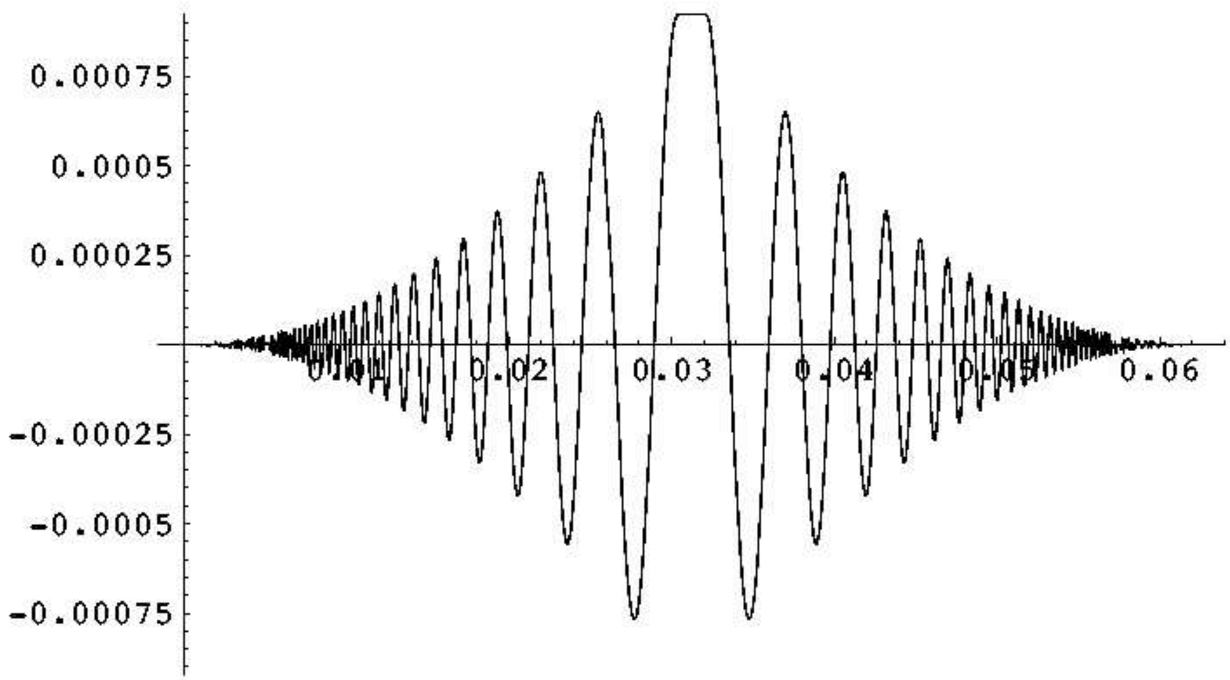
$\lim_{x \to 0^+} f_1'(x)$ and $\lim_{x \to \frac{1}{4}^-} f_1'(x)$ do not exist



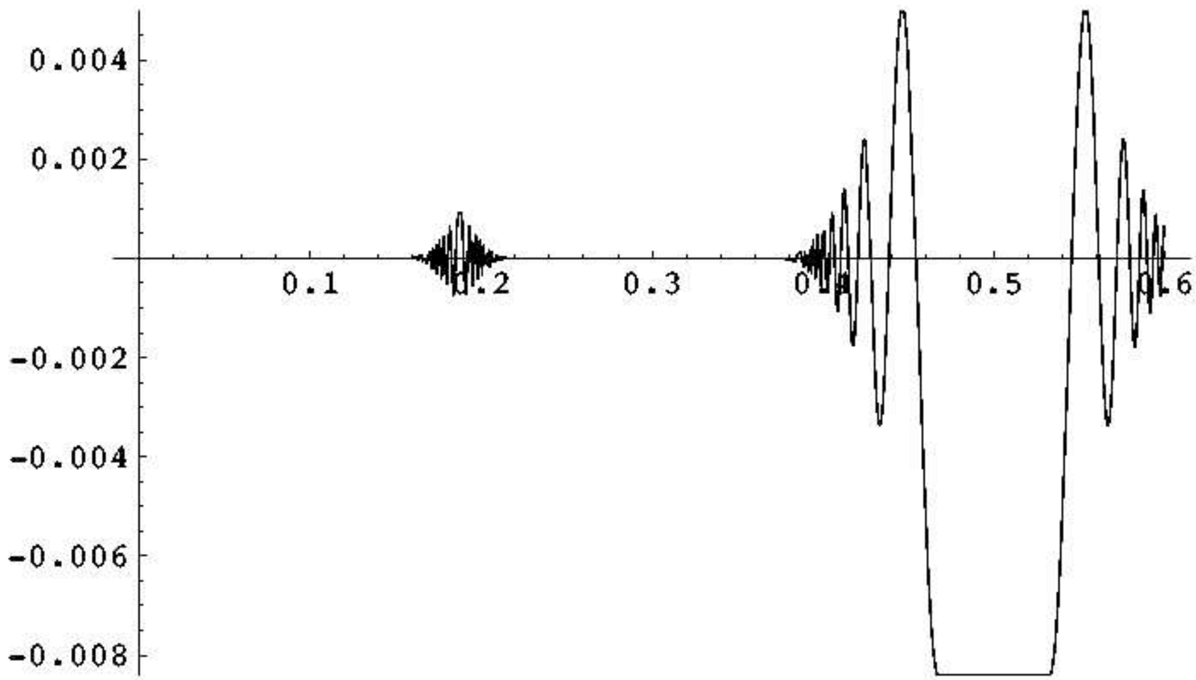
Now we slide this function over so that the portion that is not identically 0 is in the interval [3/8,5/8], that middle piece of length 1/4 taken out of the SVC set.



We follow the same procedure to create a new function, $f_2(x)$, that occupies the interval [0,1/16] and is 0 outside this interval.



We slide one copy of $f_2(x)$ into each interval of length 1/16 that was removed from the SVC set.



Volterra's function, $V(x)$, is what we obtain in the limit as we do this for *every* interval removed from the SVC set. It has the following properties:

1. V is differentiable at every value of x, and its derivative is bounded (below by –1.013 and above by 1.023).

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1. V is differentiable at every value of x, and its derivative is bounded (below by –1.013 and above by 1.023).

2. If a is a left or right endpoint of one of the removed intervals, then the derivative of V at a exists (and equals 0), but we can find points arbitrarily close to a where the derivative is +1, and points arbitrarily close to a where the derivative is –1.

We'll call this set SVC (for Smith-Volterra-Cantor).

It has some surprising characteristics:

1. SVC contains no intervals - no matter how small a subinterval of [0,1] we take, there will be points in that subinterval that are not in SVC. SVC is **nowhere dense.**

2. Given *any* collection of disjoint subintervals of [0,1], if the sum of the lengths is greater than 1/2, then they must contain at least one point in SVC.

3. Given *any* partition of [0,1] into subintervals, the sum of the lengths of the intervals that contain points in SVC will always be at least 1/2. SVC has **measure 1/2**.

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No matter how we partition [0,1], the pieces that contain endpoints of removed intervals must have lengths that add up to at least 1/2.

The pieces on which the variation of V' is at least 2 must have lengths that add up to at least 1/2.

Bernhard Riemann (1852, 1867) *On the representation of a function as a trigonometric series*

Defined definite integral as limit of $\sum f\left(x_i^{\ *}\right)\left(x_i - x_{i-1}\right)$

Key to convergence: on each interval, look at the **variation** of the function

$$V_i = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Integral exists if and only if $\sum V_i\left(x_i - x_{i-1}\right)$ can be made as small as we wish by taking sufficiently small intervals.

Conclusion: Volterra's function V can be differentiated and has a bounded derivative, but its derivative, V', cannot be integrated:

$$\frac{d}{dx}V(x) = v(x), \text{ but } \int_0^x v(t)\,dt \neq V(x) - V(0).$$

The Fundamental Theorem of Calculus:

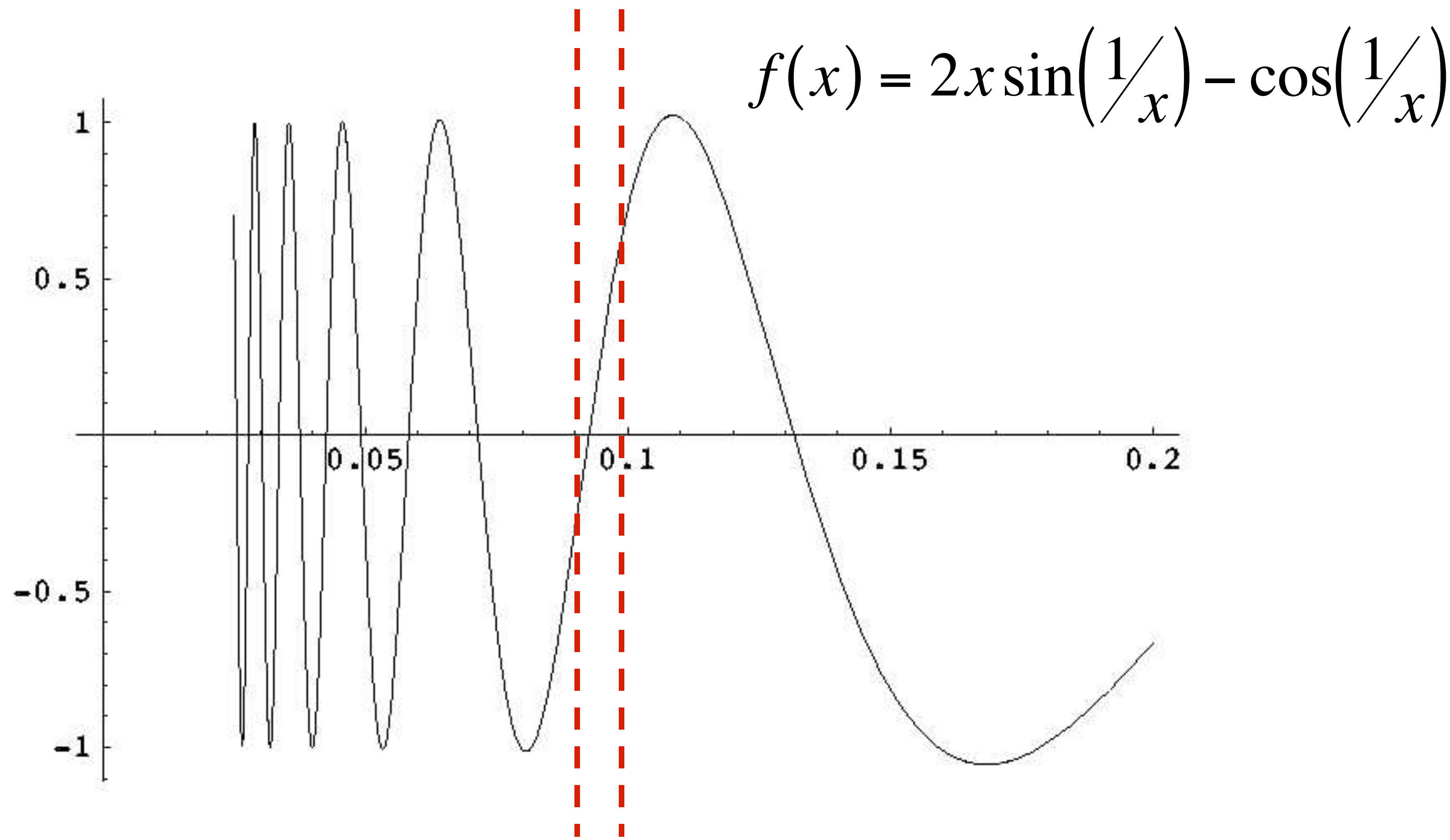
1. If F has bounded variation and $F'(x) = f(x)$,
then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

2.If f does not have a jump discontinuity at x, then
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

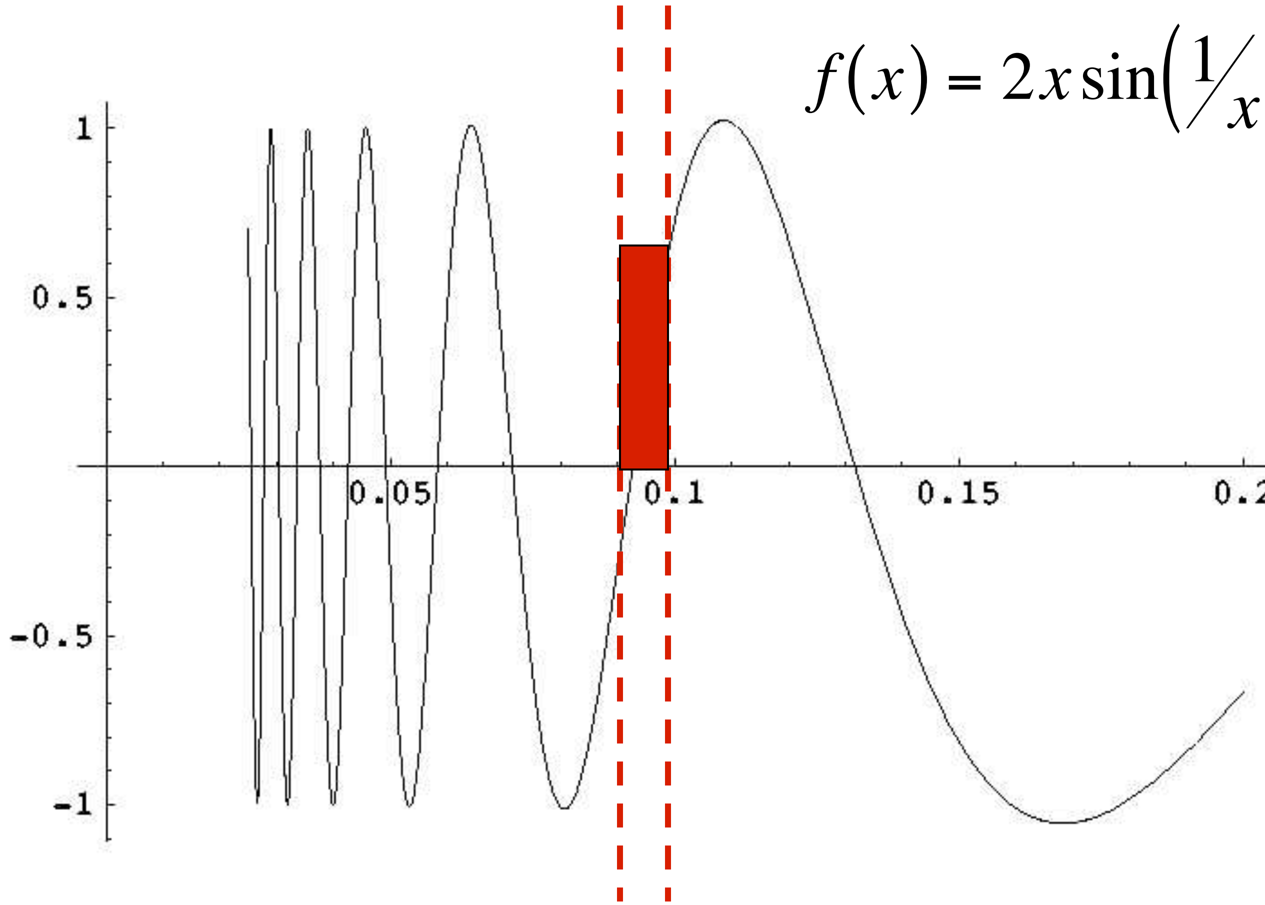
The Lebesgue Integral

Henri Lebesgue

Ph.D. thesis: *On a generalization of the definite integral*, 1901


$f(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$
1
0.5
-0.5
-1
0.05
0.1
0.15
0.2

$$f(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$



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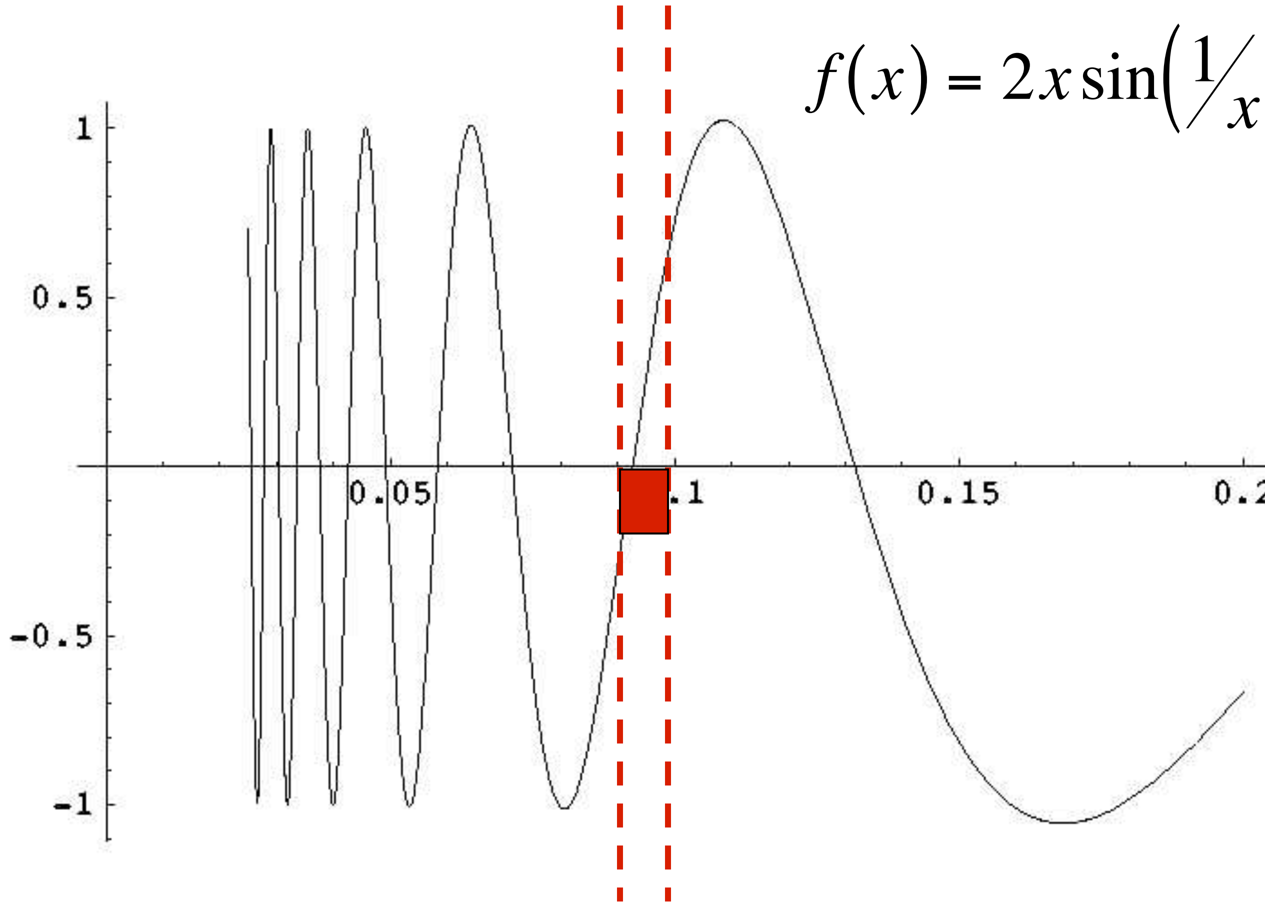

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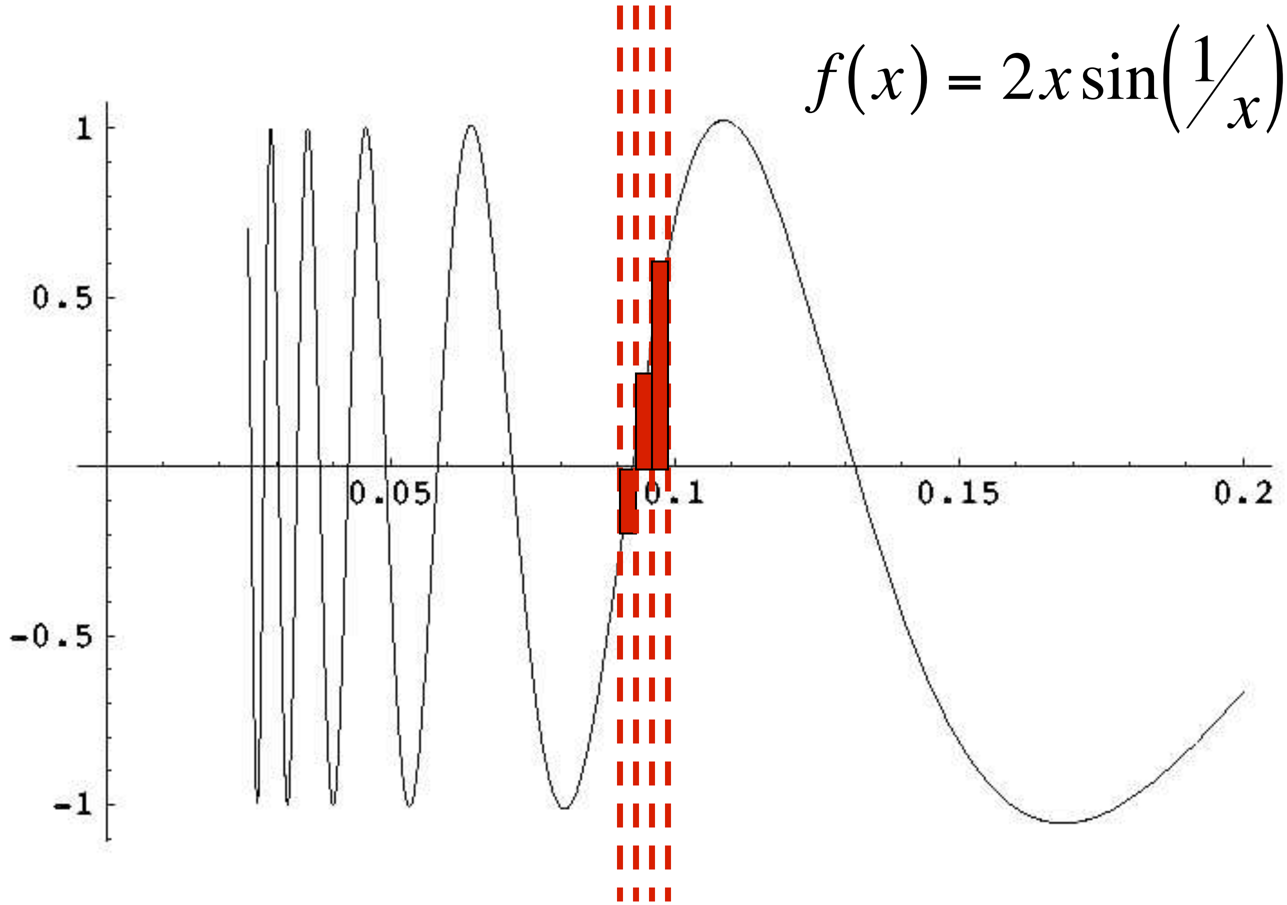

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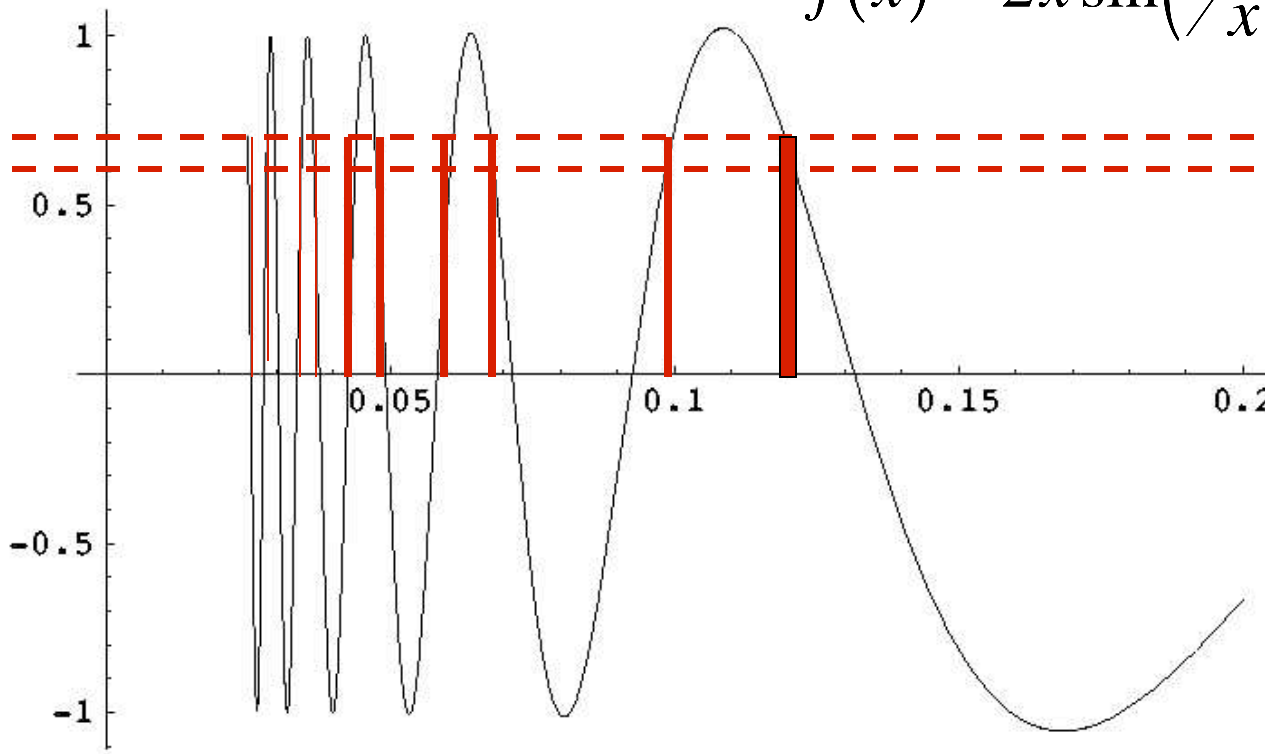

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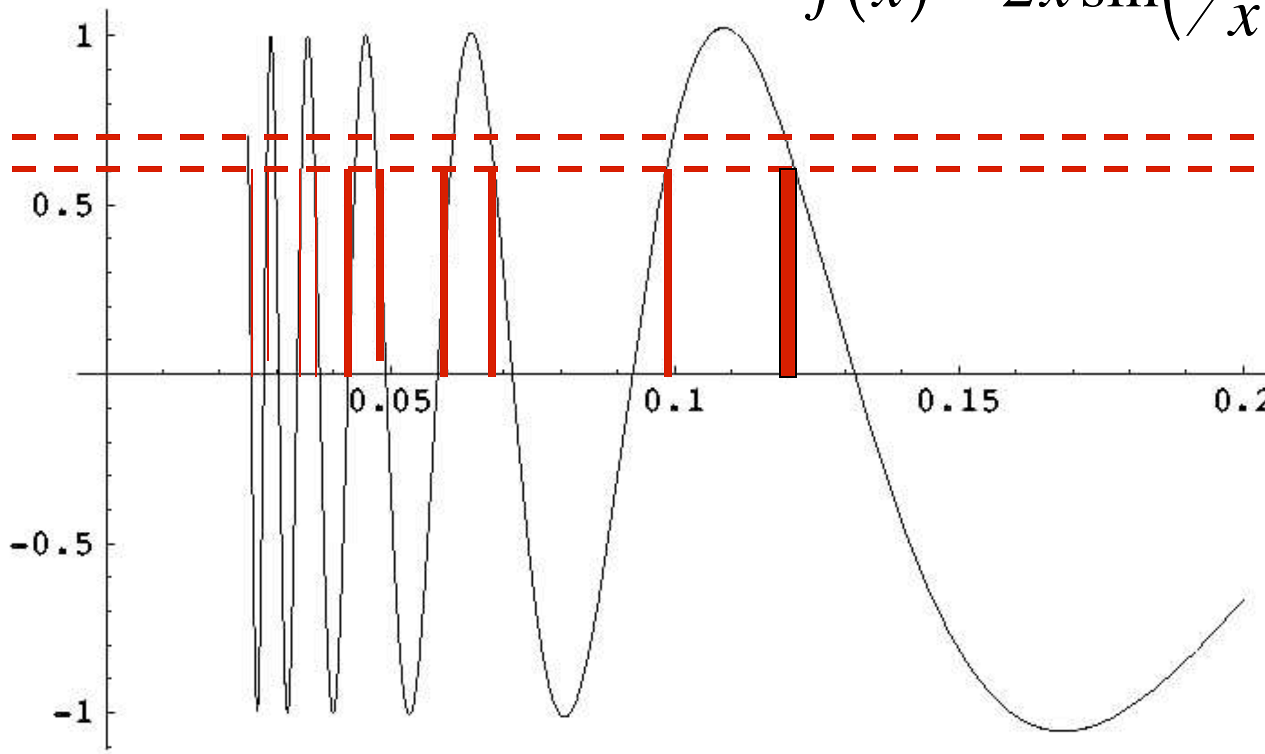

With Lebesgue's definition, the derivative of Volterra's function *is* integrable.

Theorem If f' exists and is bounded on $[a,b]$, then f' is integrable (in the Lebesgue sense) and

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Theorem If f is integrable on $[a,b]$, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, except, possibly, on a set of measure 0.