

# 1 Orders of Growth

When we talk about the efficiency of a function, we are often interested in the following: as the size of the input grows, how does the runtime of the function change? And what do we mean by “runtime”?

- `square(1)` requires one primitive operation: `*` (multiplication). `square(100)` also requires one. No matter what input `n` we pass into `square`, it always takes one operation.

input	function call	return value	number of operations
1	<code>square(1)</code>	$1 \cdot 1$	1
2	<code>square(2)</code>	$2 \cdot 2$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	<code>square(100)</code>	$100 \cdot 100$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	<code>square(<math>n</math>)</code>	$n \cdot n$	1

- `factorial(1)` requires one multiplication, but `factorial(100)` requires 100 multiplications. As we increase the input size of `n`, the runtime (number of operations) increases linearly proportional to the input.

input	function call	return value	number of operations
1	<code>factorial(1)</code>	$1 \cdot 1$	1
2	<code>factorial(2)</code>	$2 \cdot 1 \cdot 1$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	<code>factorial(100)</code>	$100 \cdot 99 \cdots 1 \cdot 1$	100
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	<code>factorial(<math>n</math>)</code>	$n \cdot (n-1) \cdots 1 \cdot 1$	$n$

For expressing complexity, we use what is called big  $\Theta$  (Theta) notation. For example, if we say the running time of a function `foo` is in  $\Theta(n^2)$ , we mean that the running time of the process will grow proportionally with the square of the size of the input as it becomes very large.

- **Ignore lower order terms:** If a function requires  $n^3 + 3n^2 + 5n + 10$  operations with a given input  $n$ , then the runtime of this function is in  $\Theta(n^3)$ . As  $n$  gets larger, the lower order terms (10,  $5n$ , and  $3n^2$ ) all become insignificant compared to  $n^3$ .
- **Ignore constants:** If a function requires  $5n$  operations with a given input  $n$ , then the runtime of this function is in  $\Theta(n)$ . We are only concerned with how the runtime grows asymptotically with the input, and since  $5n$  is still asymptotically linear; the constant factor does not make a difference in runtime analysis.

## Kinds of Growth

Here are some common orders of growth, ranked from no growth to fastest growth:

- $\Theta(1)$  — constant time takes the same amount of time regardless of input size
- $\Theta(\log n)$  — logarithmic time
- $\Theta(n)$  — linear time
- $\Theta(n \log n)$  — linearithmic time
- $\Theta(n^2)$ ,  $\Theta(n^3)$ , etc. — polynomial time
- $\Theta(2^n)$ ,  $\Theta(3^n)$ , etc. — exponential time (considered “intractable”; these are really, really horrible)

In addition, some programs will never terminate if they get stuck in an infinite loop.

## Questions

What is the order of growth for the following functions?

```
1.1 def sum_of_factorial(n):
    if n == 0:
        return 1
    else:
        return factorial(n) + sum_of_factorial(n - 1)
```

n

```
1.2 def bonk(n):
    total = 0
    while n >= 2:
        total += n
        n = n / 2
    return total
```

n

```
1.3 def mod_7(n):
    if n % 7 == 0:
        return 0
    else:
        return 1 + mod_7(n - 1)
```

n

## 2 Linked Lists

There are many different implementations of sequences in Python. Today, we'll explore the linked list implementation.

A linked list is either an empty linked list, or a `Link` object containing a `first` value and the `rest` of the linked list.

To check if a linked list is an empty linked list, compare it against the class attribute `Link.empty`:

```
if link is Link.empty:
    print('This linked list is empty!')
else:
    print('This linked list is not empty!')
```

### Implementation

```
class Link:
    empty = ()
    def __init__(self, first, rest=empty):
        assert rest is Link.empty or isinstance(rest, Link)
        self.first = first
        self.rest = rest

    def __repr__(self):
        if self.rest:
            rest_str = ', ' + repr(self.rest)
        else:
            rest_str = ''
        return 'Link({0}{1})'.format(repr(self.first), rest_str)

    @property
    def second(self):
        return self.rest.first

    @second.setter
    def second(self, value):
        self.rest.first = value

    def __str__(self):
        string = '<'
        while self.rest is not Link.empty:
            string += str(self.first) + ' '
            self = self.rest
        return string + str(self.first) + '>'
```

## Questions

- 2.1 Write a function that takes in a Python list of linked lists and multiplies them element-wise. It should return a new linked list.

If not all of the `Link` objects are of equal length, return a linked list whose length is that of the shortest linked list given. You may assume the `Link` objects are shallow linked lists, and that `lst_of_lns` contains at least one linked list.

```
def multiply_lns(lst_of_lns):
    """
    >>> a = Link(2, Link(3, Link(5)))
    >>> b = Link(6, Link(4, Link(2)))
    >>> c = Link(4, Link(1, Link(0, Link(2))))
    >>> p = multiply_lns([a, b, c])
    >>> p.first
    48
    >>> p.rest.first
    12
    >>> p.rest.rest.rest
    ()
    """
```

- 2.2 Write a function that takes a sorted linked list of integers and mutates it so that all duplicates are removed.

```
def remove_duplicates(lnk):
    """
    >>> lnk = Link(1, Link(1, Link(1, Link(1, Link(5)))))
    >>> remove_duplicates(lnk)
    >>> lnk
    Link(1, Link(5))
    """
```

### 3 Midterm Review

- 3.1 Write a function that takes a list and returns a new list that keeps only the even-indexed elements of `lst` and multiplies them by their corresponding index.

```
def even_weighted(lst):
    """
    >>> x = [1, 2, 3, 4, 5, 6]
    >>> even_weighted(x)
    [0, 6, 20]
    """

    return [_____]
```

- 3.2 The **quicksort** sorting algorithm is an efficient and commonly used algorithm to order the elements of a list. We choose one element of the list to be the **pivot** element and partition the remaining elements into two lists: one of elements less than the pivot and one of elements greater than the pivot. We recursively sort the two lists, which gives us a sorted list of all the elements less than the pivot and all the elements greater than the pivot, which we can then combine with the pivot for a completely sorted list.

First, implement the `quicksort_list` function. Choose the first element of the list as the pivot. You may assume that all elements are distinct.

*Note: in computer science, “sorting” refers to placing elements in order from least to greatest, not putting things in categories*

```
def quicksort_list(lst):
    """
    >>> quicksort_list([3, 1, 4])
    [1, 3, 4]
    """

    if _____:

        _____

    pivot = lst[0]

    less = _____

    greater = _____

    return _____
```

- 3.3 Write a function that takes in a list and returns the maximum product that can be formed using nonconsecutive elements of the list. The input list will contain only numbers greater than or equal to 1.

```
def max_product(lst):
    """Return the maximum product that can be formed using lst
    without using any consecutive numbers
    >>> max_product([10,3,1,9,2]) # 10 * 9
    90
    >>> max_product([5,10,5,10,5]) # 5 * 5 * 5
    125
    >>> max_product([])
    1
    """
```

- 3.4 Complete `redundant_map`, which takes a tree `t` and a function `f`, and applies `f` to each node ( $2^d$ ) times, where `d` is the depth of the node. The root has a depth of 0. It should mutate the existing tree rather than creating a new tree.

```
def redundant_map(t, f):
    """
    >>> double = lambda x: x*2
    >>> tree = Tree(1, [Tree(1), Tree(2, [Tree(1, [Tree(1)])])])
    >>> redundant_map(tree, double)
    >>> print_levels(tree)
    [2] # 1 * 2 ^ (1) ; Apply double one time
    [4, 8] # 1 * 2 ^ (2), 2 * 2 ^ (2) ; Apply double two times
    [16] # 1 * 2 ^ (2 ^ 2) ; Apply double four times
    [256] # 1 * 2 ^ (2 ^ 3) ; Apply double eight times
    """

    t.label = _____

    new_f = _____

    _____

    _____
```