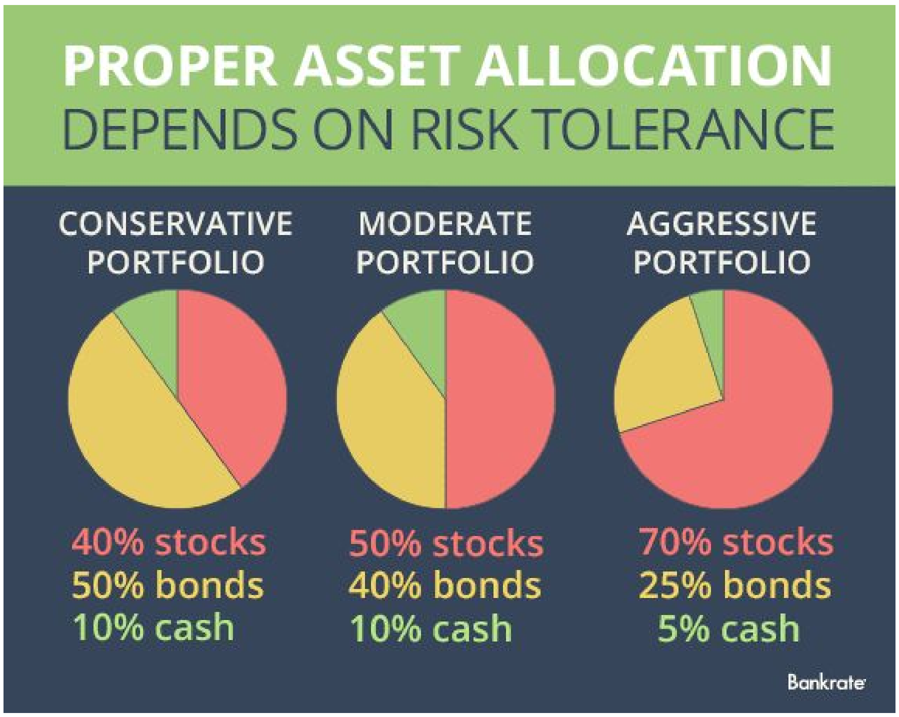
Portfolio Optimization

Strategies to allocate the weight of assets



PY538 Econophysics

Name: Sicheng Yi (Tiger Yi)

BU ID: U43188754

Email: tigeryi@bu.edu

INTRODUCTION

“Risk vs. Reward” has always been at the center theme for capital asset management. The various types of assets and their respective weights in your portfolio could play an indispensable role in profit maximization and risk minimization. In general, higher risks directly correlate with higher rewards. But the actual risk reward dynamic is much more subtle and in depth. How to balance out the risk vs. reward, not just qualitatively, but rather quantitatively, could be really insightful for us to score that extra profit margin without being exposed to the undesirable level of risks.

Before going to some specific strategies to construct a portfolio, it is imperative to point out why to construct a portfolio in the first place. Why don’t we simply invest 100% of our capital into only one type of assets, like stocks? The purpose of asset diversification is very evident: to dilute the overall risks into many more assets. The construction of a portfolio is an art of hedging. A well balanced portfolio tends to contain both the high risk high reward assets and the low risk low reward assets, across many different economic sectors. In general, there are always fewer risks in a well diversified portfolio than any single asset. For example, the stock indices such as the S&P 500 always tend to outperform any random individual stock over the long term on average.

But by no means it is wise to invest as many assets as possible either. It is empirically impractical to include every single asset on the market. Perhaps more importantly, there always exist some statistical outliers which are extremely volatile yet unable to generate enough profits to justify the risks associated with. Sometimes, the presumed less risky assets can sometimes unexpectedly outperform the general markets.

A systematic and numerical approach to quantify risk and returns will be very critical to the process of our portfolio decision making. 2 of our important metrics, namely volatility and returns, can be mathematically quantified as the standard deviation and percentage change of the adjusted daily prices. The data is publicly available and can be easily extrapolated from Yahoo Finance. We will closely examine several portfolio weight allocation models with different optimization objectives, and discuss their merits and drawbacks respectively.

THEORY

There are mainly 2 major models that will be examined closely here. The 1st model is the Markowitz model within the context of the Modern Portfolio Theory. The 2nd model is the risk parity model. Both models try to adjust the weights of individual assets in a given portfolio, yet they differ drastically from each other in terms of objective functions and methodology.

The goal of the Markowitz Model is to strike the desirable balance between the volatility (standard deviation) and the expected return (adjusted daily percentage change of asset prices). Modern Portfolio Theory not only enables those extremely risk averse investors to choose weights of portfolio at the minimum volatility, but also more importantly, allows those rational investors to optimize the portfolio at the maximum Sharpe Ratio: to attain the marginal profit with the least amount of marginal risks.

Risk parity strategy on the hand has some similarities and differences compared to the volatility minimization part of the Markowitz model. In short, the distinctive goal of the risk parity strategy is to have each individual asset contributing the same way to the overall portfolio volatility. Therefore, for the risk parity strategy, investors must be more thoughtful of what assets to be included in the portfolio because the returns aren't really the primary focus of risk parity strategy. Nevertheless, risk parity is unique as investors will instead allocate the total risks across the assets. By leveraging and deleveraging the risks across the entire portfolio, Risk parity is much more immune and resilient to unexpected downturns in the market.

Now we are going into the details of the 1st model: Modern Portfolio Theory and Markowitz Portfolio Model. The key metrics are the risks and returns.

In general, the total portfolio return should be the linear sums of all individuals.

$E[R_p] = \sum_i w_i E[R_i]$; R_p: return of portfolio; w_i: weights; R_i: return of asset

Portfolio Variance: $\sigma_p^2 = \sum_i w_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_{ij}$; $\sigma_{i,j} = \sigma_i \sigma_j \rho_{ij}$ sample covariance.

Or $\sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_{ij}$ and portfolio volatility is square root of variance $\sigma_p = \sqrt{\sigma_p^2}$

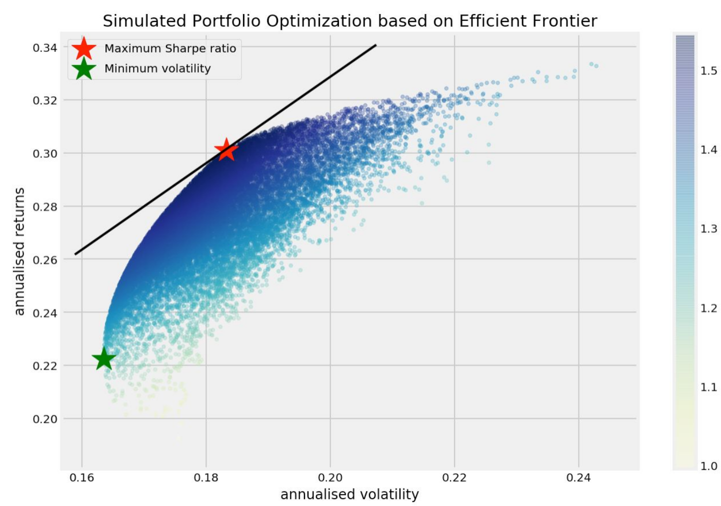
We can express the variance of portfolio return in the matrix form: $w^T \Sigma w$, w is the column vector of weights of assets, w^T is transpose or the row vector of weights, and Σ is the covariance matrix for the returns of our entire portfolio.

Here is a quick illustration. Assume we have a portfolio for 2 types of assets: 1 and 2. Then the variance and expected returns of this portfolio can be computed as:

$$\sigma_p^2 = w^T \Sigma w = [w_1 \ w_2] \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}$$

The essential aspect of this model involves the concept of efficient frontier and the tangential line of the capital allocation line (CAL), or sometimes, the capital market line (CML). The efficient frontier is mostly a parabolic curve in the “risk - expected return space”, and the CAL is the upward sloping straight line that is tangent to the upper part of the efficient frontier. The upper part of an efficient frontier or sometimes the “Markowitz Bullet” really represents the portfolios for which there is a minimum level of risks given any level of the expected returns. In other words, those portfolios which lie directly on the efficient frontier curve offer the best expected returns for any given risk level. For CML: $R_p = R_f + (R_m - R_f)\frac{\sigma_p}{\sigma_m}$; R_p: expected return of portfolio; R_m: return on market portfolio; R_f: risk free interest rate; σ_m: standard deviation of market portfolio; σ_p: standard deviation of portfolio. And all the points on the line of CML have the same Sharpe Ratio as that of the market portfolio i.e.:

$$\frac{E[R] - R_f}{\sigma_p} = \frac{E[R_m] - R_f}{\sigma_m}$$



Therefore, the slope of CML is both Sharpe Ratio of the market portfolio and portfolio. The goal of Markowitz Portfolio Model is to maximize the Sharpe Ratio of the portfolio so the highest the highest level CML that is tangent to the efficient frontier will strike the perfect balance point of risk vs. return, which corresponds to the red star in the image. The green star corresponds to the lowest volatility point for the portfolio.

To determine the efficient frontier curve, given risk tolerance $q \epsilon [0, \infty)$, efficient frontier can be found by the minimization of this objective function: $w^T \Sigma w - qR^T w$ object to $\sum_i w_i = 1$ (for simplicity this paper will disallow shorting assets). Σ is the covariance matrix. $w^T \Sigma w$ is the variance of portfolio return. R is the expected returns. w is the weight vector, so $R^T w$ is the expected return of the portfolio after weighted by each asset. $q \geq 0$ is the risk tolerance factor. $q = 0$ means the absolute minimum risks of the portfolio; $q \mapsto \infty$ as risk blows up unbounded. Alternative approach could be run minimization on portfolio variance $w^T \Sigma w$ subject to expected return $R^T w = \mu$ for the target return budget μ. Then solve the Lagrange Multiplier with linear system of equations $\begin{bmatrix} 2\Sigma & -R \\ R^T & 0 \end{bmatrix} * \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \mu \end{bmatrix}$

Now move onto the Risk Parity Model, specifically for the equally-weighted risk contribution portfolio. Assume N is the total number of assets in a given portfolio. w_i is the weight of each individual asset, which forms w the weight vector. Σ is the covariance matrix. The volatility of the portfolio is defined as $\sigma(w) = \sqrt{w^T \Sigma w}$. The risk contribution from each individual asset is $\sigma_i(w) = w_i \times \partial_{w_i} \sigma(w) = \frac{w_i (\Sigma w)_i}{\sqrt{w^T \Sigma w}}$, for which together form the overall portfolio risk $\sigma(w) = \sum_{i=1}^{N} \sigma_i(w)$. Risk Parity strategy is aiming for equal risk contribution $\sigma_i(w) = \sigma_j(w)$ for all i, j. $\sigma_i(w) = \frac{\sigma(w)}{N}$. For the Risk Parity strategy, we can either compute $w_i = \frac{\sigma(w)^2}{(\Sigma w)_i N}$ directly or run the minimization for $arg_w \ min \sum_{i=1}^{N} [w_i - \frac{\sigma(w)^2}{(\Sigma w)_i N}]^2$ subject to $\sum_i w_i = 1$ and $w_i \geq 0$to make the overall weights to sum up to 100% (again disallowing to short assets for simplicity).

Later, all Python computational implementations in Jupyter Notebook will primarily be based upon the foundation that's been laid out in this theoretical discussion.

DATA

Data gathering of asset returns and volatility will be vital for this paper. We will mostly analyze the publicly accessible data from Yahoo Finance. Yahoo Finance enables us to pull public trading data from various assets and financial instruments such as the company stock price, major global stock indices, and as well as the U.S. government bond and commodity over an extended period of time.

The data from the U.S. Treasury Department will also be quite useful to determine the Risk Free Interest Rate R_f over any period of time. The U.S. treasury yield is the effective risk free interest rate because the U.S. government bond will always be honored by the FED. The risk free interest rate itself increases as the time period increases for 1 year, 5 year, 10 year, and 30 year risk free interest rate.

The method to extrapolate data from Yahoo Finance will require specific “Yahoo ticketers” for the specific assets and companies. Here are a few potential candidates for the portfolio that we intend to optimize.

Here are all the assets that will be part of the Python analysis. For the best performance of codes, I won’t put every asset in a solo portfolio. Instead I will build multiple portfolios with different assets to compare and contrast for the different portfolio optimization strategies.

From the data from the U.S. Treasury Department, I will use 0.17% Treasury Yield as the 1 year risk free interest rate on May 1st, 2020. We will mostly examine the data from May 1st, 2019 up to the most recent trade date May 1st, 2020.

Yahoo Tickers	Asset Name	Type of Asset
^IXIC	NASDAQ Composite (U.S.)	Stock Market Index
^DJI	Dow Jones Industrial Average (U.S.)	Stock Market Index
^GSPC	S&P 500 (U.S.)	Stock Market Index

^N225	Nikkei 225 (Japan)	Stock Market Index
^HSI	HANG SENG INDEX (Hong Kong)	Stock Market Index
^GDAXI	DAX PERFORMANCE INDEX (Germany)	Stock Market Index
TY=F	Ten-Year US Treasury Note Future	Future
GC=F	Gold Future (June 2020)	Future
SI=F	Silver Future (July 2020)	Future
CL=F	Crude Oil Future (June 2020)	Future
BTC=F	Bitcoin Futures (May 2020)	Future
GOOGL	Alphabet Inc. (Google)	Equity/Stock
AAPL	Apple Inc.	Equity/Stock
AMZN	Amazon.com Inc.	Equity/Stock
FB	Facebook Inc.	Equity/Stock
MSFT	Microsoft Corporation	Equity/Stock
JNJ	Johnson & Johnson	Equity/Stock
BRK-A	Berkshire Hathaway Inc.	Equity/Stock
JPM	JPMorgan Chase & Co.	Equity/Stock
ZM	Zoom Video Communication	Equity/Stock

Code Implementation

I will NOT include all the Python code from Jupyter Notebook in this paper. For the details of the code implementation itself, please refer to the .ipynb source code with detailed instructions of how the theory is actually implemented computationally.

Nevertheless, it is still worthwhile to point out a few quick highlights of this complicated Python data visualization.

Again we will use the 0.17% one year treasury yield from the U.S. Treasury Department on May 1st 2020 as the risk free interest rate from May 2019 to May 2020. For the simplicity, this paper will NOT discuss shorting and longing assets via borrowing and lending, so all the weights of individual asset must sum up to 100%

```
$ constraints = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1})
```

The code begins with the data importation from Yahoo Finance, along with the graphs of actual adjusted closing price of all assets and daily percentage returns.

The first bulk of the code randomly generates 50,000 potential portfolios with different weights of the assets we have imported earlier. It will calculate all the annualized adjusted returns, volatility of each portfolio, and the Sharpe Ratio. All the portfolio will be represented by the “dot” on the “return-volatility” graph. Out of all 50,000 dots, the “red star” represents the portfolio in which maximize the Sharpe Ratio, and the “green star” represents the portfolio with minimum volatility. Those optimal solutions will be reported in detail as what exact weight of each asset in the portfolio is.

The previous bulk of the code is more realistic in practice. However, in addition to the 1st part, the 2nd part of the code will instead numerically compute the theoretical portfolios that will maximize Sharpe Ratio or minimize the volatility. Therefore, the “red star” and the “green star” here could be very different portfolios from the 50,000 randomly generated portfolios. The graph will also show the theoretical curve of the efficient frontier.

Then, instead of simulating all 50,000 randomly generated portfolios, the graph will just simply show the average return and volatility of each individual asset as the “blue dot” in the 3rd graph. The “green” and “red” stars will concur with the 2nd graph.

Last part of the Python code is the Risk Parity strategy to choose the weights of each asset, so that each asset will contribute the same toward the overall portfolio volatility. Again for the simplicity of the model, we set a subjective function to restrict the weights of the overall portfolio to 100% without allowing shorting the assets.

```
$ constraints = ({'type': 'eq', 'fun': lambda x: np.sum(x) - 1.0},
                 {'type': 'ineq', 'fun': lambda x: x})
```

We will try to minimize the objective function of our risk budget in order to find the weight of each asset in the portfolio $arg_w \ min \sum_{i=1}^{N} [w_i - \frac{\sigma(w)^2}{(\Sigma w)_i N}]^2$ subject to $\sum_i w_i = 1$

RESULTS

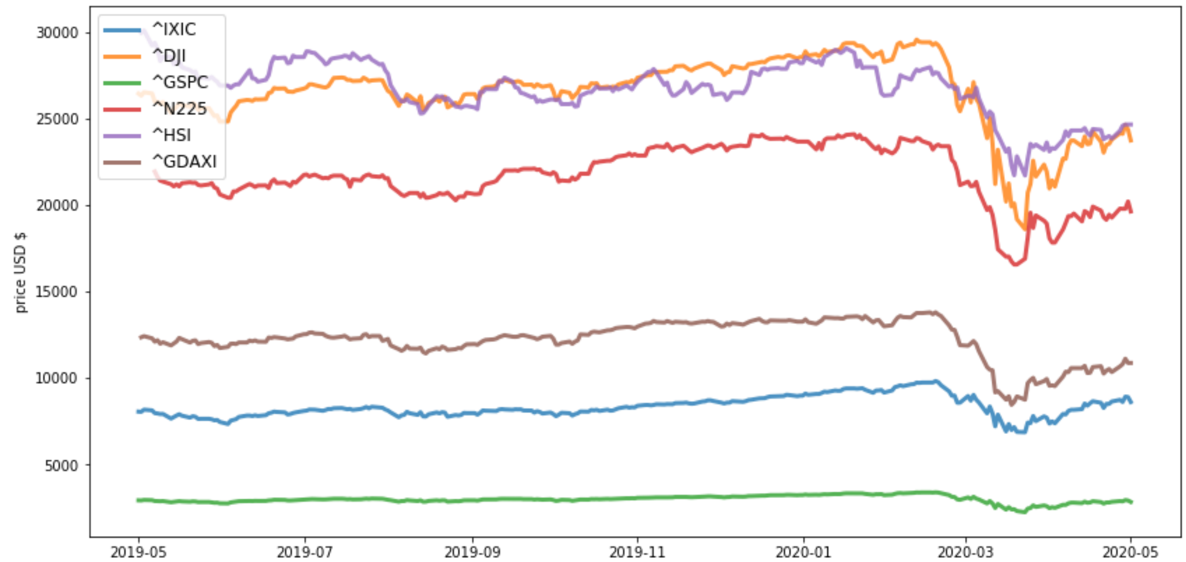
Instead of putting every possible asset in one single portfolio, multiple portfolios with different assets will be constructed for comparison and contrast between the models. Only the final results of the generated weights and graphs will be analyzed here; for specific Python code, please refer to the source code .ipynb directly.

The data for all the assets will be one year from May 1st, 2019 to May 1st, 2020. Keep in mind that due to the coronavirus global pandemic in 2020, the price of the majority of the assets plummeted since roughly Feb 20th, 2020. The price of assets has recently recovered some grounds due to the vast economic stimulus packages and aggressive cut in FED interest rate, yet it is still far from the pre-crisis level.

Portfolio 1

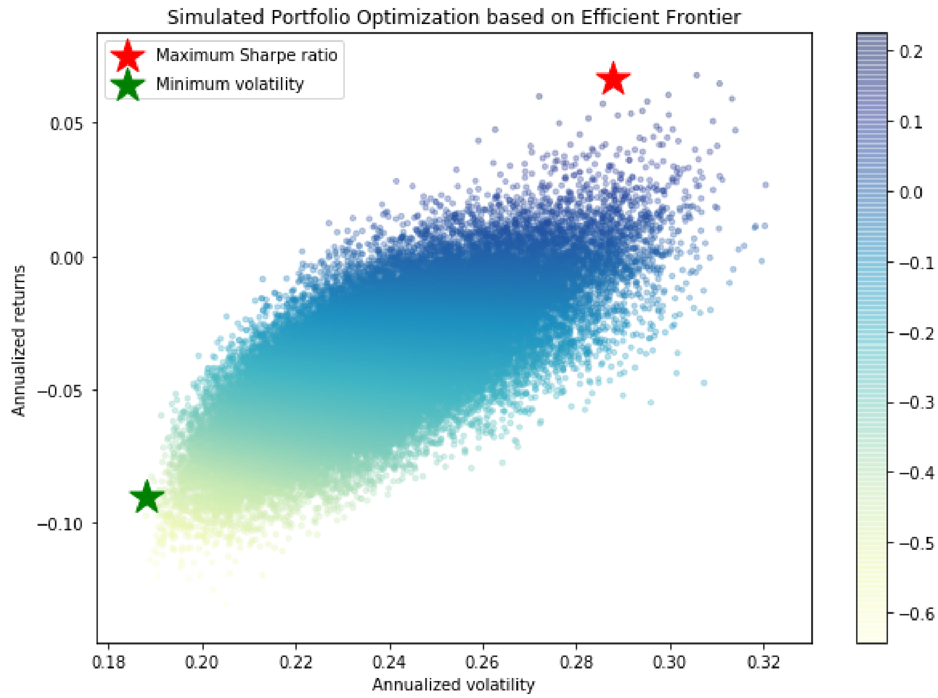
All major stock market indices

Assets: U.S. Dow Jones (^DJI), U.S. Nasdaq (^IXIC), U.S. S&P 500 (^GSPC), Hong Kong Hang Seng (^HSI), Japan Nikkei 225 (^N225), Germany Dax (^GDAXI)



Each major stock index is made of hundreds and thousands of publicly traded companies on the stock market, but each index is mainly composed of its national companies. Here we are trying to develop a strategy to put weights on each index in our investment portfolio. The stock markets around the world start to crash and become much more volatile since March 2020 due to the Covid-19 pandemic.

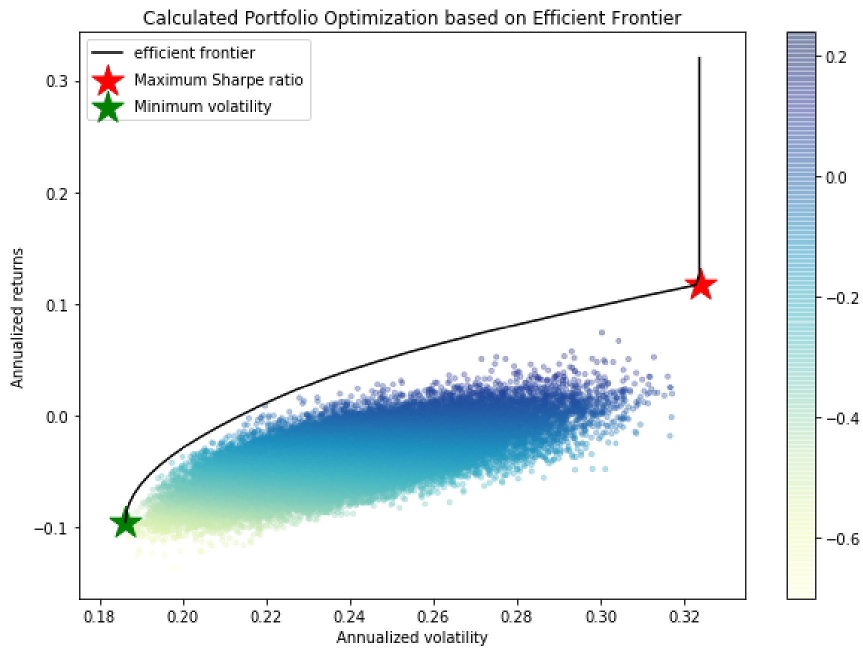
The graph below shows the 50,000 randomized portfolios for stock indices. And the exact weight of the “red” max Sharpe ratio and “green” min volatility are shown:




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.07
Annualized Volatility: 0.29


            ^IXIC  ^DJI  ^GSPC  ^N225  ^HSI  ^GDAXI
allocation  68.21  3.89  11.53  11.98  0.14    4.24
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: -0.09
Annualized Volatility: 0.19


            ^IXIC  ^DJI  ^GSPC  ^N225   ^HSI  ^GDAXI
allocation  11.36  1.07   4.14  39.01  41.75    2.67
```

So in order to minimize the portfolio volatility made of index, investors should put emphasis on the Hong Kong and Japanese stock market. But the U.S. stock market index like Nasdaq and S&P will give the investor a much higher Sharpe ratio due to significantly higher returns compared to the stock market index of all other countries.

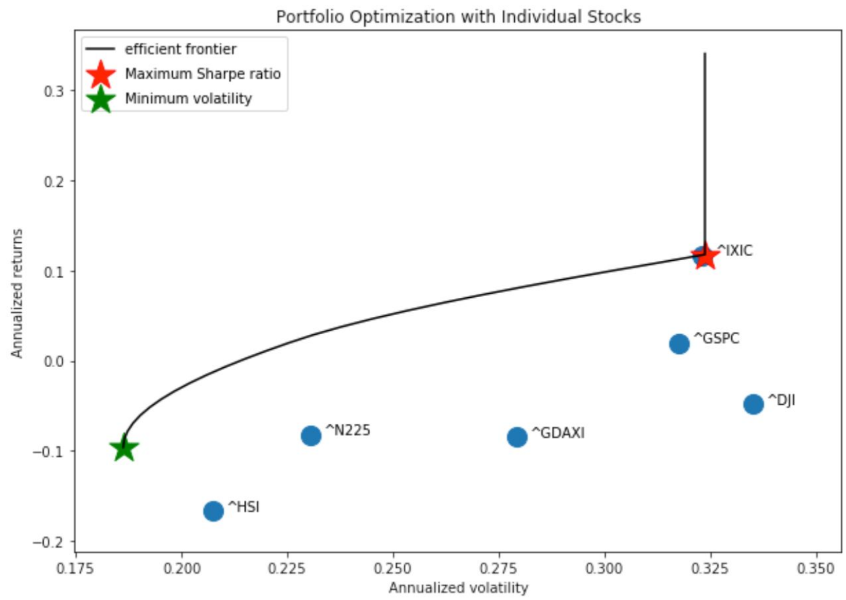
The theoretical optimization and efficient frontier will be shown below:




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualised Return: 0.12
Annualised Volatility: 0.32


            ^IXIC  ^DJI  ^GSPC  ^N225  ^HSI  ^GDAXI
allocation  100.0   0.0    0.0    0.0   0.0     0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualised Return: -0.1
Annualised Volatility: 0.19


            ^IXIC  ^DJI  ^GSPC  ^N225   ^HSI  ^GDAXI
allocation  15.03   0.0    0.0  32.93  52.04     0.0
```

Compared to the randomly generated portfolio, the theoretical model is more extreme. For Sharpe ratio maximization, it asks the investors to invest all 100% of money into Nasdaq and nothing else! For volatility minimization, the theoretical model asks the investors to invest even more heavily in the Hong Kong and Japanese market, despite the more negative annual return (-0.1) than previously (-0.99).

The intuition behind those 2 different strategies become very clear if we instead run the average annual return of those stock market indices:




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.12
Annualized Volatility: 0.32


            ^IXIC  ^DJI  ^GSPC  ^N225  ^HSI  ^GDAXI
allocation  100.0   0.0    0.0    0.0   0.0     0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: -0.1
Annualized Volatility: 0.19


            ^IXIC  ^DJI  ^GSPC  ^N225   ^HSI  ^GDAXI
allocation  15.03   0.0    0.0  32.93  52.04     0.0
--------------------------------------------------------------------------------
Individual Stock Returns and Volatility

^IXIC : Annualized return 0.12 , Annualized volatility: 0.32
^DJI : Annualized return -0.05 , Annualized volatility: 0.34
^GSPC : Annualized return 0.02 , Annualized volatility: 0.32
^N225 : Annualized return -0.08 , Annualized volatility: 0.23
^HSI : Annualized return -0.17 , Annualized volatility: 0.21
^GDAXI : Annualized return -0.08 , Annualized volatility: 0.28
--------------------------------------------------------------------------------
```

As shown in the results here, the Hong Kong and Japan stock index are low risk and low return, so they are preferable to minimize the portfolio volatility. However, the U.S. stock indices such as the Nasdaq and S&P 500 have much higher returns to justify the higher risk according to the Sharpe ratio maximization. Depending on how risk averse the investors are, they now have 2 different strategies with distinctive objectives: either maximize profit given risk, or choose safety at minimum volatility.

The Risk Parity strategy on the other hand cares much less of the returns itself. The main focus of Risk Parity is to let every single asset of the portfolio contribute the same way to the overall portfolio volatility. Therefore, the Risk Parity strategy will advise the investors to spread their capital more evenly across all the stock market indices.

```
# get weights to minimize the risk

get_weights(yahoo_tickers=yahoo_tickers,
            start_date=start_date,
            end_date=end_date)
```

```
^IXIC     16.43
^DJI      13.86
^GSPC     15.40
^N225     14.68
^HSI      24.73
^GDAXI    14.90
Name: weight, dtype: float64
```

As you can see, the final result of Risk Parity strategy is very different from the volatility minimization or the Sharpe Ratio maximization strategy. All 3 models have very different objective functions to optimize. The risk parity strategy in particular is much more resistant to the unexpected downturn of the market, such as the most recent coronavirus pandemic. Because the Risk Parity is essentially leveraging out the risk and spreading it equally across every single asset in the portfolio.

All those 3 strategies are valid with respective objectives, which one to choose is largely up to the risk vs. reward preference of the investors themselves. For the optimal level of Sharpe Ratio, the model suggests to invest all 100% of money into Nasdaq alone, but the 0.32 volatility does seem to be the most risky strategy out of all 3.

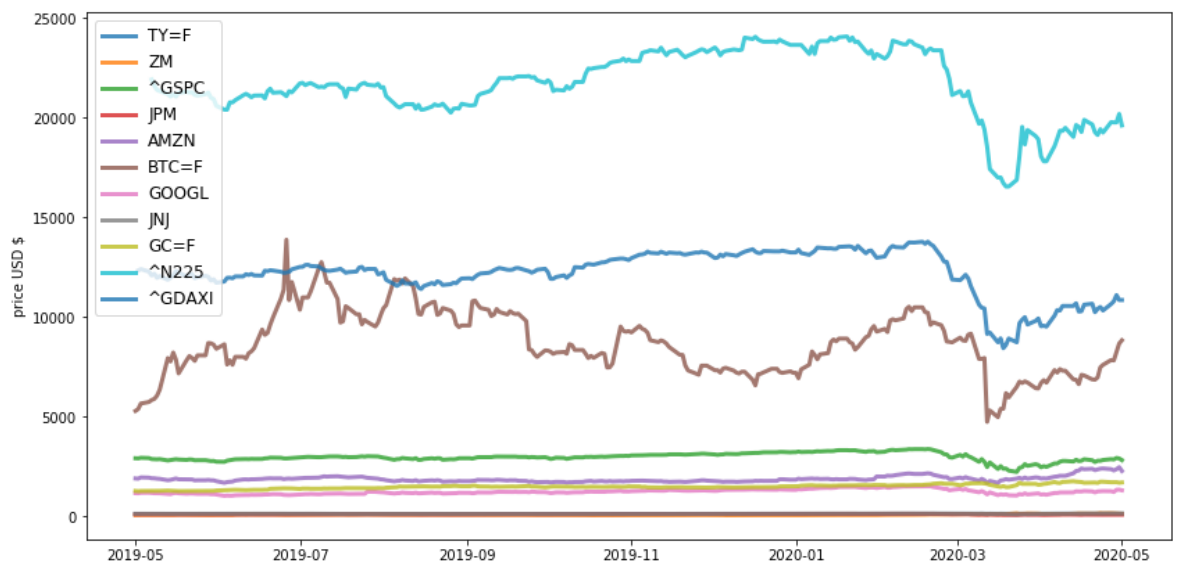
And of course, the Sample Portfolio 1 is built solely off the global stock market indices, and the assets themselves are almost homogeneous and highly correlated. In our next portfolio, we are going to diversify into many assets such as the equity, index, futures to compare and contrast the 3 models we have established.

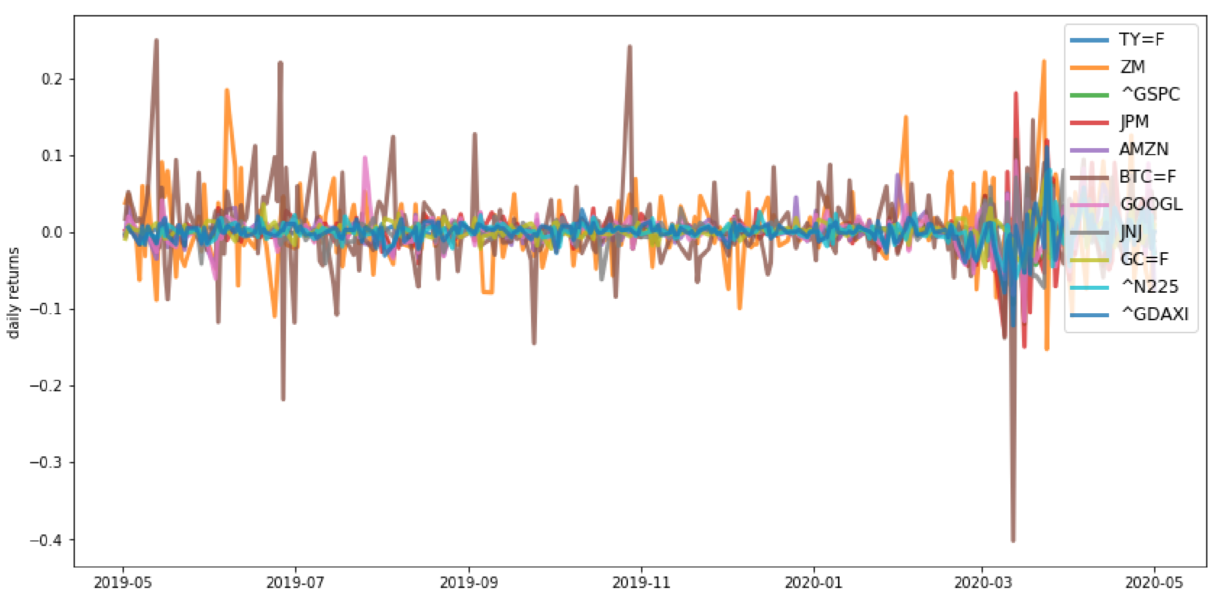
Portfolio 2

Various forms of the assets

Ten-Year US Treasury Note Future (TY=F), Zoom Video Communication (ZM), S&P 500 (^GSPC), JPMorgan Chase & Co. (JPM), Amazon.com Inc. (AMZN), Bitcoin Futures (BTC=F), Alphabet Inc. (GOOGL), Johnson & Johnson (JNJ), Gold Future (GC=F), Nikkei 225 (^N225), Dax Index (^GDAXI)

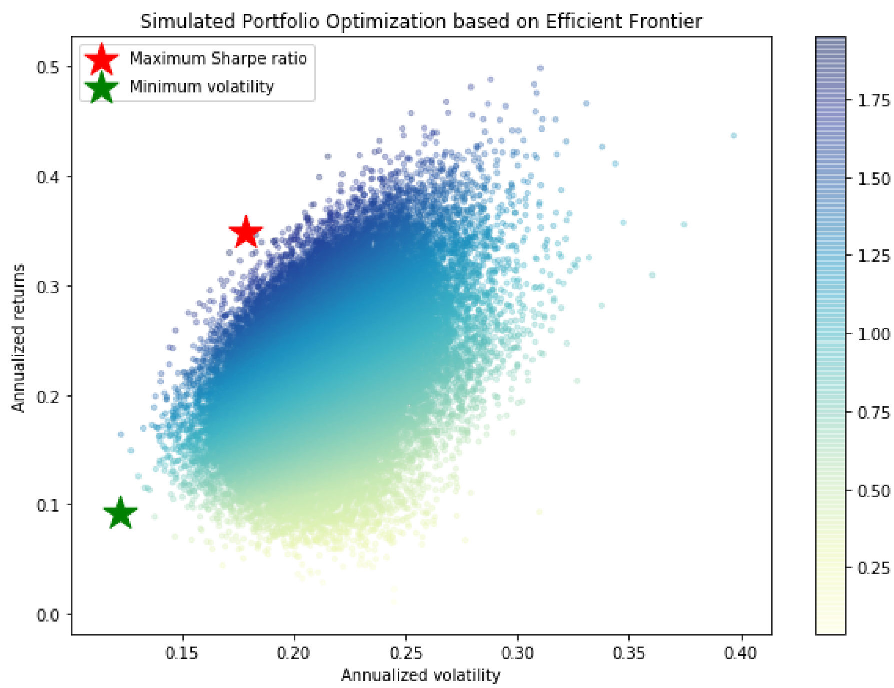
We again look at the daily adj close price and percentage returns of our assets. It's clear some assets have huge volatility like the Bitcoin, not so much with treasury.





Again randomly simulate 50,000 portfolios with various weights of those assets.

The green star represents minimum volatility strategy, Red is maximum Sharpe ratio.

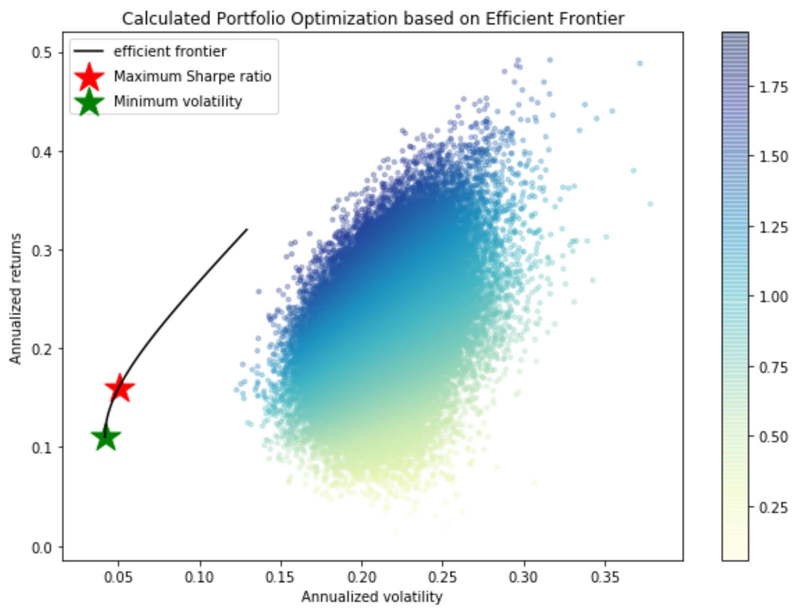



```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.35
Annualized Volatility: 0.18


            TY=F     ZM  ^GSPC   JPM  AMZN  BTC=F  GOOGL   JNJ   GC=F  ^N225  \
allocation  25.0  21.56   0.73  3.19  1.49   6.73   1.58  4.01  25.36   1.58

            ^GDAXI
allocation    8.77
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: 0.09
Annualized Volatility: 0.12


             TY=F    ZM  ^GSPC  JPM  AMZN  BTC=F  GOOGL   JNJ   GC=F  ^N225  \
allocation  26.85  2.19   9.62  2.3   2.8   1.92   3.79  0.07  14.02  35.59

            ^GDAXI
allocation    0.85
```

It seems that the 10 year U.S. treasury note future, Gold future are the favors for both min volatility and max Sharpe ratio strategy, from May 2019 to May 2020. For max sharpe ratio, Zoom seems to be really profitable due to the coronavirus pandemic. And for min volatility, Japan stock index seems to be relatively safe for investment.

Nevertheless, those 50,000 randomly generated portfolios with various weights can't count for all the possible samples. Now with especially 11 assets, those data with extreme weights on a particular asset will be tossed out. In another word, those 50,000 portfolios don't include the extreme outliers where all money is invested in 1 asset.

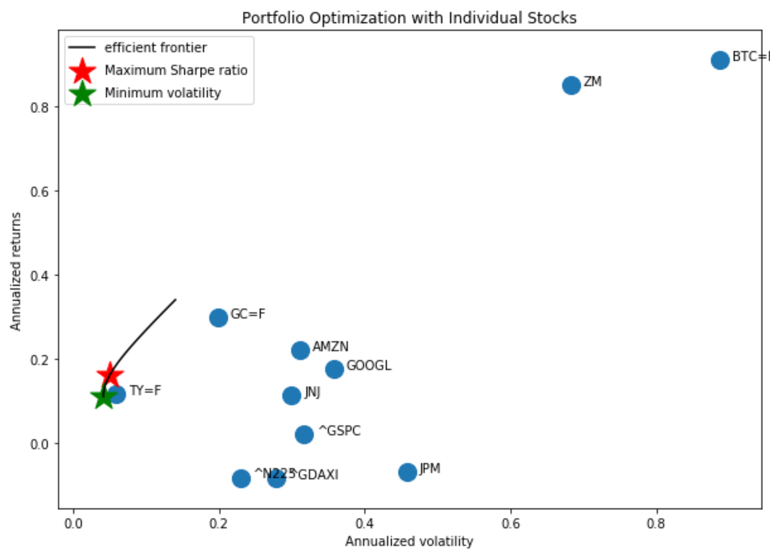
Now we are going to see the theoretical optimization for those 2 models:




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualised Return: 0.16
Annualised Volatility: 0.05


             TY=F    ZM  ^GSPC   JPM  AMZN  BTC=F  GOOGL   JNJ  GC=F  ^N225  \
allocation  79.12  2.45    0.0  1.15  4.35   1.78   1.59  4.37   5.2    0.0

            ^GDAXI
allocation     0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualised Return: 0.11
Annualised Volatility: 0.04


             TY=F    ZM  ^GSPC   JPM  AMZN  BTC=F  GOOGL   JNJ  GC=F  ^N225  \
allocation  85.75  0.04    0.0  3.71  4.27   0.11   0.55  3.37   0.0    2.2

            ^GDAXI
allocation     0.0
```

The theoretical result is quite remarkable as you can tell the efficient frontier seems to be off from the main cluster of the 50,000 random portfolios. Clearly there is one particular outlier that is extremely safe but very lucrative at the same.

If we look at the actual weights of both strategies, we can see the model put a huge emphasis on the U.S. 10 year treasury notes. And in fact, both maximum Sharpe ratio and minimum volatility strategy ask the investors to put most money in the government bond. It is reasonable to speculate that the U.S. 10 year treasury in particular is not volatile over the last year, but very profitable at the same time. Let's check the average performance of each asset to confirm our speculation:




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.16
Annualized Volatility: 0.05


             TY=F    ZM  ^GSPC   JPM  AMZN  BTC=F  GOOGL   JNJ  GC=F  ^N225  \
allocation  79.12  2.45    0.0  1.15  4.35   1.78   1.59  4.37   5.2    0.0

            ^GDAXI
allocation     0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: 0.11
Annualized Volatility: 0.04


             TY=F    ZM  ^GSPC   JPM  AMZN  BTC=F  GOOGL   JNJ  GC=F  ^N225  \
allocation  85.75  0.04    0.0  3.71  4.27   0.11   0.55  3.37   0.0    2.2

            ^GDAXI
allocation     0.0
--------------------------------------------------------------------------------
Individual Stock Returns and Volatility

TY=F : Annualized return 0.12 , Annualized volatility: 0.06
ZM : Annualized return 0.85 , Annualized volatility: 0.68
^GSPC : Annualized return 0.02 , Annualized volatility: 0.32
JPM : Annualized return -0.07 , Annualized volatility: 0.46
AMZN : Annualized return 0.22 , Annualized volatility: 0.31
BTC=F : Annualized return 0.91 , Annualized volatility: 0.89
GOOGL : Annualized return 0.18 , Annualized volatility: 0.36
JNJ : Annualized return 0.11 , Annualized volatility: 0.3
GC=F : Annualized return 0.3 , Annualized volatility: 0.2
^N225 : Annualized return -0.08 , Annualized volatility: 0.23
^GDAXI : Annualized return -0.08 , Annualized volatility: 0.28
--------------------------------------------------------------------------------
```

Indeed, it is very clear that on average, the 10 year U.S. treasury is not only the safest asset over the past year (volatility = 0.12), but also, it is also extremely profitable (return = 0.12). It is just a no brainer to invest heavily into assets that are both safe and profitable. The U.S. treasury just dwarfs everything else theoretically.

It is also important to point out that even though Zoom communication benefited heavily from the coronavirus pandemic since March 2020, yet according to the model, Zoom is so volatile that the return doesn't justify the risk which investors are exposed to. In fact, Zoom is quite similar to Bitcoin in the high reward, higher risks category. None of the models recommend investing in Zoom and Bitcoin.

Lastly, we look at our 3rd strategy of Risk Parity, you will notice all 3 models recommend to invest heavily in the U.S. 10 year treasury. This result is remarkable because in our 1st portfolio, the risk parity model asks us to invest evenly across all assets. But since the treasury is so safe, all models point to it.

Obviously, Risk Parity isn't as aggressive as other 2 models in terms of the investment in the 10 year U.S. treasury bond. It still includes all assets with minor weights.

```
# get weights to minimize the risk

get_weights(yahoo_tickers=yahoo_tickers,
            start_date=start_date,
            end_date=end_date)
```

```
TY=F      60.62
ZM         2.88
^GSPC      3.37
JPM        3.89
AMZN       4.64
BTC=F      1.85
GOOGL      4.80
JNJ        4.96
GC=F       6.40
^N225      3.17
^GDAXI     3.43
Name: weight, dtype: float64
```

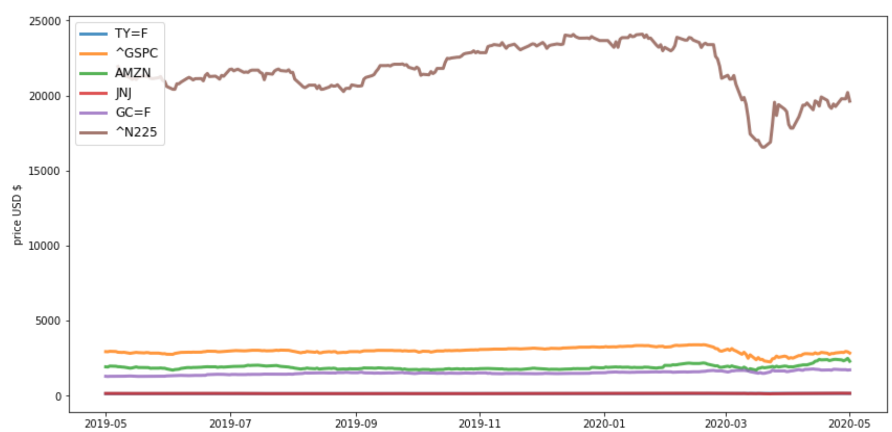
And just qualitatively, Risk parity seems the most reasonable model, if the treasury becomes unprofitable, investors still can leverage against the risks with the other assets. Once the situation on Covid-19 improves, it is expected that the 10 year treasury future will start to underperform as the FED Fund interest rate goes back up to normal.

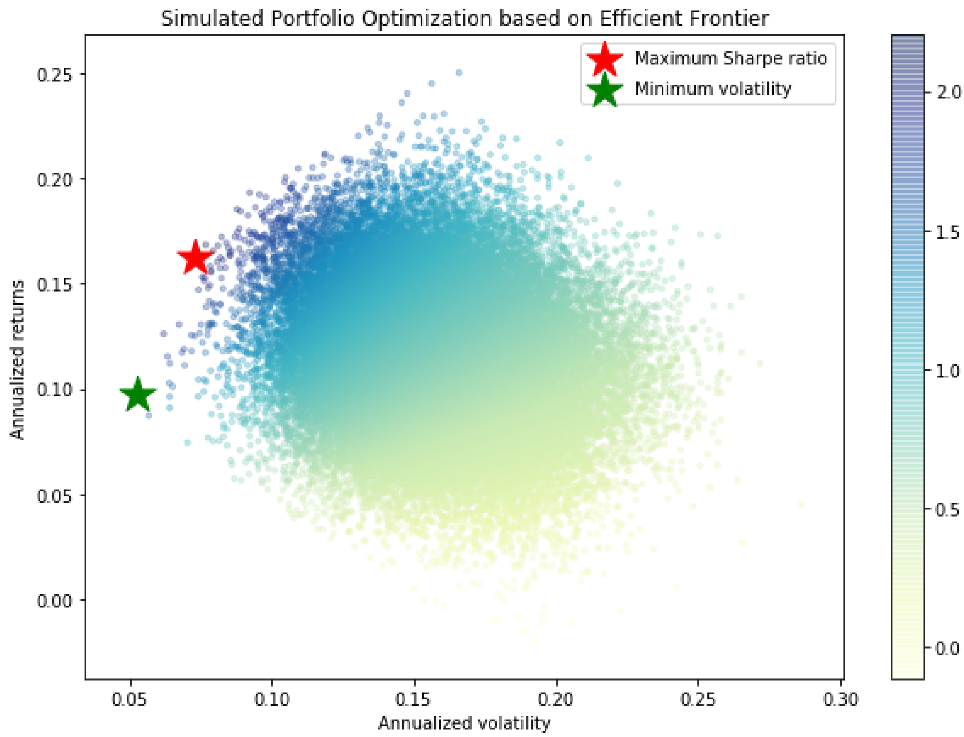
It is pretty obvious though in portfolio 2, there are some of the assets that none of the 3 models has put significant weights in those outliers. Therefore, it is reasonable to get rid of some of the assets such as the bitcoin or Zoom in the next portfolio.

Portfolio 3

Redacted from portfolio 2 to get rid of some underperforming outliersAssets: 10 year U.S. Treasury note futures, S&P 500 Index, Amazon, Johnson & Johnson, Gold Future, Nikkei 225 Index

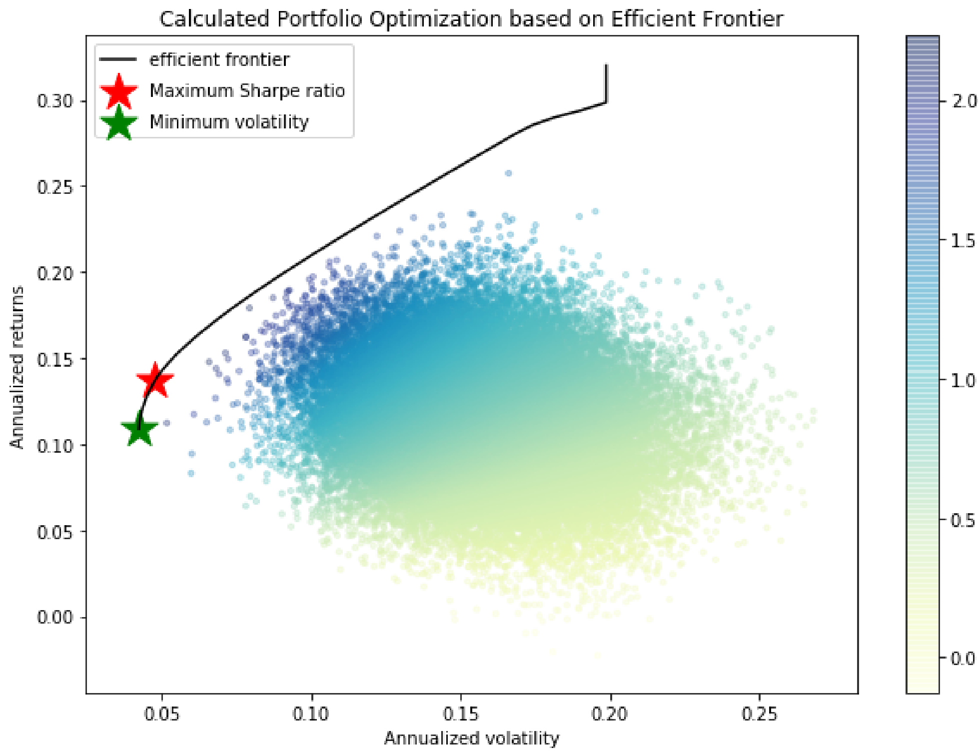
The result of this portfolio will not be discussed in detail. The main takeaway is that what asset to be included in the portfolio will have a huge impact on how accurate the prediction will be for all those 3 models. Clearly, those high risk low reward assets should be eliminated from our portfolio. Therefore, the portfolio optimization should also be run iteratively for many trials in order to achieve a desirable portfolio.





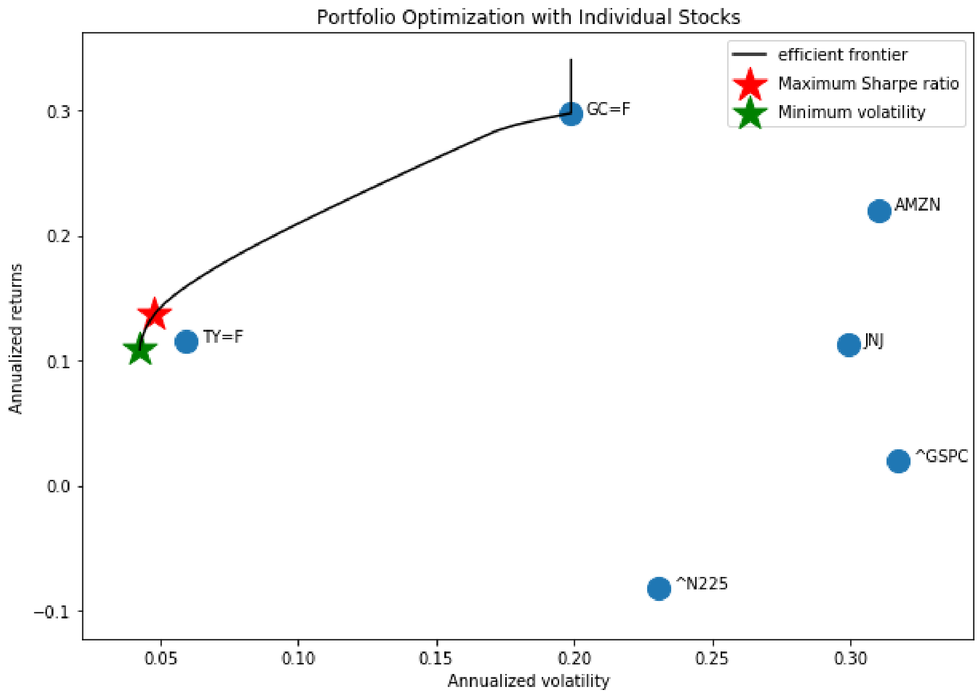

```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.16
Annualized Volatility: 0.07


             TY=F  ^GSPC   AMZN   JNJ  GC=F  ^N225
allocation  53.77   1.53  12.64  8.85  21.5   1.71
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: 0.1
Annualized Volatility: 0.05


             TY=F  ^GSPC  AMZN   JNJ  GC=F  ^N225
allocation  71.07  11.82  0.23  0.31  6.71   9.86
```

Compared to Portfolio 2, which contains a lot of data outliers. In portfolio 3, we see that the max Sharpe ratio model also puts a lot more weights on gold future at 21.5% and Amazon at 12.64%. So clearly portfolio 3 is more rational than portfolio 2.




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualised Return: 0.14
Annualised Volatility: 0.05


             TY=F  ^GSPC  AMZN   JNJ  GC=F  ^N225
allocation  79.91    0.0  8.62  4.63  6.84    0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualised Return: 0.11
Annualised Volatility: 0.04


             TY=F  ^GSPC  AMZN   JNJ  GC=F  ^N225
allocation  85.42   6.12  3.48  2.74   0.0   2.25
```

It is very clear that, compared to portfolio 2, the efficient frontier becomes a lot more accurate and reliable after we get rid of these assets that are considered to be outliers in our overall portfolio. Portfolio 3 has the advantage of diversification over portfolio 1, and 3 also has the edge over portfolio 2 in terms of accuracy and asset selection.




```
--------------------------------------------------------------------------------
Maximum Sharpe Ratio Portfolio Allocation

Annualized Return: 0.14
Annualized Volatility: 0.05


             TY=F  ^GSPC  AMZN   JNJ  GC=F  ^N225
allocation  79.91    0.0  8.62  4.63  6.84    0.0
--------------------------------------------------------------------------------
Minimum Volatility Portfolio Allocation

Annualized Return: 0.11
Annualized Volatility: 0.04


             TY=F  ^GSPC  AMZN   JNJ  GC=F  ^N225
allocation  85.42   6.12  3.48  2.74   0.0   2.25
--------------------------------------------------------------------------------
Individual Stock Returns and Volatility

TY=F : Annualized return 0.12 , Annualized volatility: 0.06
^GSPC : Annualized return 0.02 , Annualized volatility: 0.32
AMZN : Annualized return 0.22 , Annualized volatility: 0.31
JNJ : Annualized return 0.11 , Annualized volatility: 0.3
GC=F : Annualized return 0.3 , Annualized volatility: 0.2
^N225 : Annualized return -0.08 , Annualized volatility: 0.23
--------------------------------------------------------------------------------
```

Here the average expected return of each asset clearly shows the tradeoff between the low risk low reward of U.S. treasury future versus the high risk high reward of the Gold prices. The maximum Sharpe Ratio and the minimum volatility will differ in the amount of the weight that is distributed to each asset in the portfolio.

```
# get weights to minimize the risk

get_weights(yahoo_tickers=yahoo_tickers,
            start_date=start_date,
            end_date=end_date)
```

```
TY=F      63.22
^GSPC      5.95
AMZN       7.80
JNJ        7.34
GC=F       9.66
^N225      6.03
Name: weight, dtype: float64
```

For Portfolio 3, the Risk Parity strategy still put most weight on the 10 year U.S. Treasury Bond. However, it has been constantly better that all other assets now have a statistically more significant amount of shares in the overall portfolio. This could be immensely beneficial in case that the equity market starts to rally and the interest rate will increase in the post Covid-19 crisis. Risk parity is really great at leveraging the risks across to all assets in the portfolio.

The biggest takeaway from portfolio 3 is that all three portfolio optimization models enable the investors to parse out the assets which are statistical outliers. Not only the model is useful to determine what are the weights for each asset, but also more importantly, what specific assets to be included in the portfolio. It is an iterative algorithmic process to build up the final version of our market portfolio.

CONCLUSION

To sum up, this paper explored mainly the 3 types of portfolio optimization strategies to allocate specific weight on each individual asset in the portfolio basket: the minimal volatility, the maximum Sharpe ratio, and the risk parity strategy. Each model has its unique objective function for optimization. The minimal volatility strategy is most suitable for the risk averse investors. The maximum Sharpe ratio strategy emphasizes the marginal returns based on the given risk level, but sometimes might be too aggressive to over invest in a small number of assets. Risk Parity strategy mainly focuses on leveraging and hedging the risk equally across every single asset in the unexpected events of economic collapse.

There isn't really a correct or best model compared to the other ones. Depending on the investor's target of returns and risk preference, any one of the 3 models would be reasonable. Perhaps it is wise to analyze various markets with all of the 3 models to determine the trade off between risk and rewards.

Since May 2019, all 3 portfolio optimization models overwhelmingly recommend to invest heavily in the U.S. treasury bond and Gold future due to the slash in risk free interest and the massive economic stimulus packages.

Nevertheless, on the microscopic scale, each model still differs from one another in the actual number of weights of various assets.

It is important that all the 3 models should be applied iteratively until we add and subtract assets from our portfolio. These 3 models are not only useful in determining what is the exact weight for each asset in the portfolio, but perhaps more importantly, the models give investors the insight of what kinds of the assets to be included in the portfolio in the first place.

REFERENCES

1. https://en.wikipedia.org/wiki/Modern_portfolio_theory
2. https://en.wikipedia.org/wiki/Portfolio_optimization
3. https://en.wikipedia.org/wiki/Efficient_frontier
4. https://en.wikipedia.org/wiki/Capital_asset_pricing_model
5. https://en.wikipedia.org/wiki/Risk_parity
6. Chaves, Denis, et al. "Efficient Algorithms for Computing Risk Parity Portfolio Weights." *Journal of Investing* 21.3 (2012): 150,163,11. *ProQuest.* Web.
7. Spinu, Florin. (2013). An Algorithm for Computing Risk Parity Weights. SSRN Electronic Journal. 10.2139/ssrn.2297383.
8. Bruder, Benjamin and Roncalli, Thierry, Managing Risk Exposures Using the Risk Budgeting Approach (January 20, 2012). https://ssrn.com/abstract=2009778