

AFOSR -1525

SOME USEFUL INFORMATION
FOR THE DESIGN
OF AIR-CORE SOLENOIDS

by

D. Bruce Montgomery
J. Terrell

November, 1961
(2nd Printing, December 1961)

Sponsored by the Solid State Sciences Division,
Air Force Office of Scientific Research (OAR).

Air Force Contract AF 19(604)-7344

Principal Investigators: Benjamin Lax and Francis Bitter

National Magnet Laboratory
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

AFOSR - 1525

**SOME USEFUL INFORMATION
FOR THE DESIGN
OF AIR-CORE SOLENOIDS**

**PART I: Relationships between Magnetic Field, Power,
Ampere-Turns and Current Density**

PART II: Homogenous Magnetic Fields

by

D. Bruce Montgomery

J. Terrell

November, 1961

(2nd Printing, December 1961)

**Sponsored by the Solid State Sciences Division,
Air Force Office of Scientific Research (OAR).**

Air Force Contract AF 19(604)-7344

Principal Investigators: Benjamin Lax and Francis Bitter

**National Magnet Laboratory
Massachusetts Institute of Technology
Cambridge 39, Massachusetts**

ACKNOWLEDGEMENT

THE AUTHORS ARE INDEBTED
TO F. BITTER FOR HIS MANY
CONTRIBUTIONS TO THIS PAPER.

ABSTRACT

Relationships relating power, magnetic field, current density, and ampere-turns in terms of certain dimensionless factors are summarized for many types of coil geometries and current distributions. A number of plots of these factors are presented.

The field homogeneity in magnet structures is presented in terms of a series expansion about the origin utilizing Legendre Polynomials. A number of tables to facilitate design of homogeneous fields are presented. A method of achieving homogeneity in long solenoid structures by the use of determinants is discussed. Expressions for the axial field from uniform and radially varying current density coils are given.

PART I
RELATIONSHIPS BETWEEN MAGNETIC FIELD, POWER, AMPERE-TURNS
AND CURRENT DENSITY

It has been customary in the literature on magnet design to write the relationships between fields, power, ampere-turns and current density in terms of a number of dimensionless factors. In this section we will treat in summary form twelve cases of coil construction (Cases I - XII). This work is in part a summary, and enlargement of parts of the following three papers: F. Bitter, R.S.I. Vol. 7, 1936 Part II; W. F. Gauster, AIEE, Fall General Meeting, Chicago, October 1959; F. Gaume, Journal de Recherche du Centre Nationale de Recherche Scientifique, No. 43, June 1958. (English translation, Lincoln Laboratory, M81-12, part I and II, Dr. H. H. Kolm.)

A. The most commonly used dimensionless factor is the G factor first suggested by Fabry, which connects a given magnetic field with the power required to produce this field and is a function of the type of current distribution and the geometry of the coil.

$$H = G \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} \quad (1)$$

with

W = watts

ρ = resistivity in ohm - cm

a_1 = inner radius in cm

λ = fractional volume of conductor

The space factor λ is assumed constant throughout the volume. If λ is a function of x and y, more complicated formulas are needed.

B. A second dimensionless factor J has been suggested by W. F. Gauster and connects the current density at the innermost winding of a magnet to the power of the magnet. J is also a function of the current distribution and the geometry of the coil.

$$j_1 = J \left(\frac{W}{\rho \lambda a_1^3} \right)^{1/2} \quad (2)$$

C. For any particular type of coil a maximum G factor can be found which will give the most magnetic field for the least power used. However, in many types of coil construction this maximizing of the G factor may lead to excessively high current densities at the innermost winding which may make cooling the coil difficult. To illustrate this point a ratio, γ , can be formed which compares the maximum current density in any coil to the current density in a uniform current density coil, where both have the same inner radius and both develop the same field. Quantities for the uniform case are designated as primed:

$$H = G \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} \quad H' = G' \left(\frac{W'\lambda}{\rho a_1} \right)^{1/2}$$

$$\text{if } H = H'$$

$$\left(\frac{W}{W'} \right)^{1/2} = \frac{G'}{G} \quad (3)$$

from (2)

$$\frac{j_1}{j'_1} = \frac{J}{J'} \left(\frac{W}{W'} \right)^{1/2}$$

and using (3)

$$\frac{j_1}{j'_1} = \frac{J}{J'} \frac{G'}{G} = \gamma \quad (4)$$

This ratio indicates the difficulty of cooling and supporting any coil compared with a coil of uniform current density at the same field and bore.

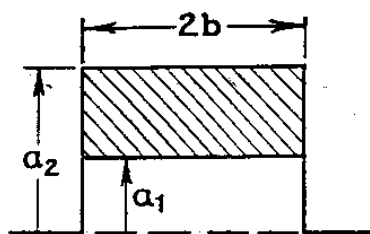
D. The ratio in equation (3) is also useful and represents the power saving in using any current distribution compared with uniform current density when generating the same field in the same bore. We give it in the more useful form.

$$\frac{W}{W'} = \left(\frac{G'}{G} \right)^2 = P \quad (5)$$

These dimensionless factors and ratios are presented in Table I for nine types of magnet construction and following Table I for three additional more complicated magnet constructions.

One of the most interesting new coil constructions presented is that of case V. This coil was first suggested by F. Gaume and achieves a relatively high G factor without excessive current densities and can be quite simply constructed of plates in the usual Bitter magnet style, with the introduction of axial current variations by making the plates thicker towards the ends of the magnet.

A number of plots of the geometry factor, G, against α and β are given in Figures 1 through 6. Cockcroft's curves for case VIII are reprinted in Figure 1, Bitter's curves for case VI in Figure 2, new curves for cases VIII and VI in the small α and β region are given in Figures 3 and 4. New curves for case V (Gaume's axial current variation) and for case X (the ellipsoidal shell) are given in Figures 5 and 6. The variables α and β are defined:



$$\alpha = a_2/a_1$$

$$\beta = b/a_1$$

E. It is sometimes instructive to write the magnetic field in terms of ampere-turns rather than power. This arises from the practical standpoint that it is difficult to always predict the resistance of a magnet exactly, especially as a function of power, and one may wish to know how much field will be produced for each ampere available in the power supply. This relationship between ampere-turns and field can be related to the J factor and the G factor and miscellaneous α and β terms and numerical constants. The relationships are given here for three cases: Case VI, Case VII and Case VIII. All the others are easily derivable by integrating the current density over the coil to obtain the total current in terms of J and W. Total current I and the factor J can now be substituted for W in equation (1), and the ampere-turns related to the field.

Case VI:

$$\begin{aligned} H_6 &= \frac{I\lambda}{t} \left[\frac{1}{J_6 \ln \alpha} \right] G_6 \\ &= \frac{NI}{2b} \left[\frac{1}{J_6 \ln \alpha} \right] G_6 \end{aligned} \quad (6)$$

Case VII:

$$\begin{aligned} H_7 &= \frac{I\lambda}{t} \left[\frac{1}{J_7} \right] G_6 \\ &= \frac{NI}{k a_1} \left[\frac{1}{(\alpha-1) J_7} \right] G_7 \end{aligned} \quad (7)$$

Case VIII:

$$\begin{aligned} H_8 &= \frac{I\lambda}{t} \left[\frac{1}{J_8 (\alpha-1)} \right] G_8 \\ &= \frac{NI}{2b} \left[\frac{1}{J_8 (\alpha-1)} \right] G_8 \end{aligned} \quad (8)$$

F. Useful geometry factors can be written for very long solenoids (long enough so that end effects can be neglected).

For a constant current density coil, the ampere-turns per unit length can be written:

$$\frac{NI}{2b} = \left(\frac{(W/2b) \lambda}{2\pi\rho} \right)^{1/2} \left(\frac{2(\alpha-1)}{\alpha+1} \right)^{1/2} \quad (9)$$

where $W/2b$ is the power per unit length (watts/centimeter)

For the disk construction, with a radial current distribution we write

$$\frac{NI}{2b} = \left(\frac{(W/2b) \lambda}{2\pi\rho} \right)^{1/2} \left(\ln \alpha \right)^{1/2} \quad (10)$$

Rewriting (9) and (10) in terms of the field in gauss and power per length as watts per meter, we obtain

$$H(\text{gauss}) = \frac{(2\pi)^{1/2}}{50} G_5 \left(\frac{W(\text{watts/meter}) \lambda}{\rho(\text{ohm cm.})} \right)^{1/2}$$

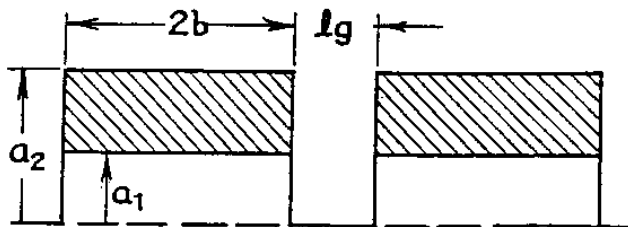
where

$$G_5(\text{uniform}) = \left(\frac{2(\alpha-1)}{\alpha+1} \right)^{1/2}$$

$$G_5(\text{radial}) = (\ln \alpha)^{1/2}$$

These factors are plotted in Figure 7 and it is interesting to note that below an α of 2.5 there is negligible power saving with the disks, but at greater α , the saving is increasingly pronounced; at an α of 10, 45% more power would be required by the uniform wound solenoid for the same magnetic field.

G. It is useful to use the G factor notation when dealing with split coils (coils separated by a gap). If we take a pair of coils with parameters as defined in Figure 8, we can write the field in the gap by simple addition and subtraction of coils.



$$\begin{aligned} \alpha &= a_2/a_1 \\ \beta_1 &= \frac{\text{total length}}{2a_1} = \frac{4b + l_g}{2a_1} \\ \beta_2 &= \frac{\text{gap}}{2a_1} = \frac{l_g}{2a_1} \end{aligned}$$










Figure 8

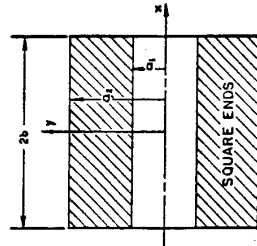
$$H(0,0) = \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} \left(\frac{G_1 \beta_1^{1/2} - G_2 \beta_2^{1/2}}{(\beta_1 - \beta_2)^{1/2}} \right)$$

$$G_1 = G_1(\alpha, \beta_1), \quad G_2 = G_2(\alpha, \beta_2)$$

This relationship (11) holds for any current distribution having no axial variation.

TABLE 1

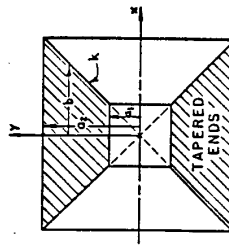
Shape of Coil	Current Distribution	G	G _{max.}	J	J _{max.}	Y	P
I Spherical Shell 	Optimum Current Distribution $i = f(x, y) \sim \frac{y}{x^2 + y^2}$	$\frac{1}{5} \left(\frac{2x}{y} \right)^{1/2} \left(\frac{\alpha-1}{\alpha} \right)^{1/2}$	$\alpha = \infty$ 0.289	$\left(\frac{\alpha}{2\pi(\alpha-1)} \right)^{1/2}$	$\alpha = 10$ G = 0.275 0.421	2.74	0.421
II Cylindrical Bore Square Ends 	Optimum Current Distribution $i = f(x, y) \sim \frac{y}{(x^2 + y^2)^{1/2}}$	$\frac{(2x)^{1/2}}{10} \left(\frac{1}{4} \left(\frac{1}{y^2} - \frac{\alpha}{\alpha-1} \right) \right)^{1/2}$ $-\frac{3}{4\alpha} \ln \left(\frac{\alpha-1}{\alpha} \right) + \frac{3}{4} \ln \left(\frac{\alpha}{\alpha-1} \right)$	$\alpha, \beta = \infty$ 0.272	$\frac{1}{10G}$	$\alpha = \beta = 10$ G = 0.255 0.392	2.76	0.495
III Cylindrical Bore Tapered Ends 	Optimum Current Distribution $i \sim \frac{1}{y^2}$ For Tapered Ends	$\frac{1}{5} \left(\frac{2x}{y} \right)^{1/2} \left(\frac{\alpha-1}{\alpha} \right)^{1/2}$ (Max. when $k=1$)	$k=1, \alpha = \infty$ 0.250	$\left(\frac{\alpha}{(G-1)4\pi k} \right)^{1/2}$	$k=1, \alpha = 10$ G = 0.238 0.298	2.24	0.565
IV Cylindrical Bore Square Ends 	Optimum Radial Current Distribution $i = f(y) \sim \frac{1}{y(1+y^2)^{1/2}}$	$\frac{1}{10} \left(\frac{2x}{y} \right)^{1/2} \left(\frac{\alpha-1}{\alpha} \right)^{1/2}$	$\alpha = \infty, \beta = 2$ 0.225	$\frac{1}{10G}$	$\alpha = 10, \beta = 3$ G = 0.216 0.463	3.85	0.681
V FIGURE 5 Cylindrical Bore Square Ends 	$i \sim \frac{1}{y} \left[\frac{1}{(x^2 + y^2)^{1/2}} - \frac{1}{(x^2 + y^2)^{3/2}} \right] \frac{\alpha-1}{\alpha}$	$\frac{(2x)^{1/2}}{5} \left(\frac{1}{4} \left(\frac{1}{y^2} - \frac{\alpha}{\alpha-1} \right) \right)^{1/2}$ $-\frac{3}{4\alpha} \ln \left(\frac{\alpha-1}{\alpha} \right) + \frac{3}{4} \ln \left(\frac{\alpha}{\alpha-1} \right)$ $F(\phi, k)$ - Elliptic of 1st kind $\phi = \tan^{-1} \frac{y}{x}, k^2 = \frac{\alpha-1}{\alpha}$	$\alpha = \beta = 8$ 0.232	$\frac{\alpha-1}{10G\alpha\beta\alpha}$	$\alpha = \beta = 8$ G = 0.232 0.181	1.40	0.592
VI FIGURES 2, 4 Cylindrical Bore Square Ends 	$i \sim \frac{1}{y}$	$\frac{(2x)^{1/2}}{5} \left(\frac{1}{4} \left(\frac{1}{y^2} - \frac{\alpha}{\alpha-1} \right) \right)^{1/2}$ $\frac{1}{\alpha} \ln \left(\frac{\alpha}{\alpha-1} \right)$	$\alpha = 6, \beta = 2$ 0.209	$\left(\frac{1}{4\pi\beta\alpha\alpha} \right)^{1/2}$	$\alpha = 6, \beta = 2$ G = 0.209 0.149	1.30	0.726
VII Cylindrical Bore Tapered Ends 	$i \sim \frac{1}{y}$	$\frac{1}{5} \left(\frac{2x}{y} \right)^{1/2} \left(\frac{\alpha}{\alpha-1} \right)^{1/2}$ (Max. when $k=1$)	$k=1, \alpha = 4.5$ 0.201	$\left(\frac{1}{4\pi k(\alpha-1)} \right)^{1/2}$	$k=1, \alpha = 4.5$ G = 0.201 0.151	1.36	0.796
VIII FIGURES 1, 3 Cylindrical Bore Square Ends 	$i = \text{Constant}$	$\frac{(2x)^{1/2}}{5} \left(\frac{1}{4} \left(\frac{1}{y^2} - \frac{\alpha}{\alpha-1} \right) \right)^{1/2}$ $\frac{1}{\alpha} \ln \left(\frac{\alpha}{\alpha-1} \right)$	$\alpha = 3, \beta = 2$ 0.179	$\left(\frac{1}{2\pi\beta(\alpha-1)} \right)^{1/2}$	$\alpha = 3, \beta = 2$ G = 0.179 0.099	1.00	1.000
IX Cylindrical Bore Tapered Ends 	$i = \text{Constant}$	$\frac{1}{5} \left(\frac{2x}{y} \right)^{1/2} \left(\frac{\alpha-1}{\alpha} \right)^{1/2}$ (Max. when $k=1$)	$k=1, \alpha = 2.7$ 0.172	$\left(\frac{3}{4\pi k(\alpha-1)} \right)^{1/2}$	$k=1, \alpha = 2.7$ G = 0.172 0.135	1.17	1.085



$$\alpha = \frac{a^2}{b^2}$$

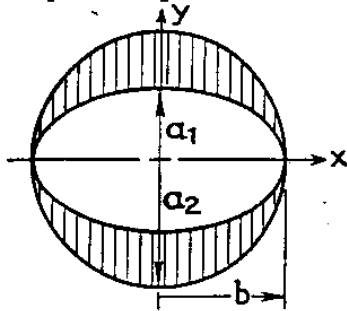
$$\beta = \frac{b}{a}$$

$$k = \frac{a}{b}$$



CASE X (FIGURE 6)

Coil Shape: Ellipsoidal shell, $\alpha = \text{constant}$



$$\alpha = a_2/a_1$$

$$\beta = \frac{b_{\max}}{a_1}$$

Current Distribution: $i = i_0/y$

Geometry Factor:

$$\text{i) } \begin{matrix} \beta > 1 \\ \beta > \alpha \end{matrix} \quad G = \frac{\pi^{1/2}}{5 \ln \alpha} \beta^{1/2} \left(\frac{1}{(\beta^2 - 1)^{1/2}} \ln(\beta + (\beta^2 - 1)^{1/2}) - \frac{1}{(\beta^2 - \alpha^2)^{1/2}} \ln \left(\frac{\beta + (\beta^2 - \alpha^2)^{1/2}}{\alpha} \right) \right)$$

$$\text{ii) } \begin{matrix} \beta < 1 \\ \alpha > \beta \end{matrix} \quad G = \frac{\pi^{1/2}}{5} \beta^{1/2} \left(\frac{1}{(\beta^2 - 1)^{1/2}} \sec^{-1} \frac{1}{\beta} - \frac{1}{(\alpha^2 - \beta^2)^{1/2}} \sec^{-1} \frac{\alpha}{\beta} \right)$$

$$\text{iii) } \begin{matrix} \beta > 1 \\ \alpha > \beta \end{matrix} \quad G = \frac{\pi^{1/2}}{5} \beta^{1/2} \left(\frac{1}{(\beta^2 - 1)^{1/2}} \ln(\beta + (\beta^2 - 1)^{1/2}) - \frac{1}{(\alpha^2 - \beta^2)^{1/2}} \sec^{-1} \frac{\alpha}{\beta} \right)$$

$$G_{\max} = .204 \text{ at } \alpha = 6, \beta = 2$$

Current Density Factor J:

$$J = \left(\frac{1}{4 \pi \beta \ln \alpha} \right)^{1/2}$$

$$\text{at } \alpha = 6, \beta = 2, G_{\max} = 0.204$$

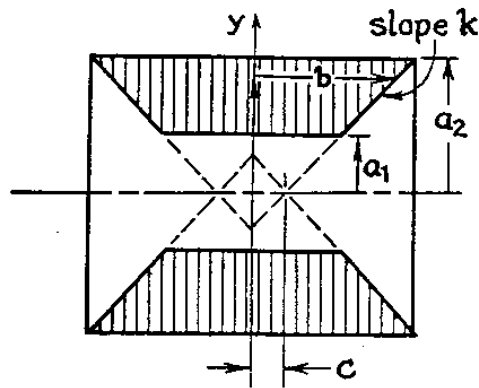
$$\text{and } J_{\max} = 0.149$$

Current Density Ratio γ for same Field referred to case VIII: = 1.31

Power Ratio P for same Field referred to case VIII = 0.769

CASE XI

Coil Shape: Cylindrical Bore, tapered ends, non-coinciding apexes.



$$b = ky + c$$

$$k = \text{slope}$$

$$\gamma = \frac{c}{a_1} \quad \alpha = \frac{a_2}{a_1}$$

Current Distribution: $i = i_0/y$

Geometry Factor:

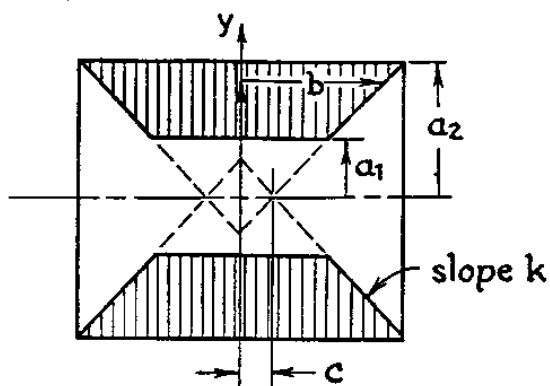
$$G = \frac{\pi^{1/2}}{5} \frac{1}{k(\alpha-1) + \gamma \ln \alpha} \left[\frac{k}{k^2+1} \cdot \ln \frac{((k^2+1)\alpha^2 + 2k\gamma\alpha + \gamma^2)^{1/2} + \alpha(k^2+1) + \gamma(k^2/k^2+1)^{1/2}}{((k^2+1) + 2k\gamma + \gamma^2)^{1/2} + (k^2+1)^{1/2} + \gamma(k^2/k^2+1)^{1/2}} + \ln \left(\alpha \frac{((k^2+1) + 2k\gamma + \gamma^2)^{1/2} + \gamma + k}{((k^2+1)\alpha^2 + 2k\gamma\alpha + \gamma^2)^{1/2} + \gamma + k\alpha} \right) \right]$$

G is a maximum when $k=1$ and $\alpha = 4.5$ which reduces case XI to case VII Current Density Factor J:

$$J = \left(\frac{1}{4\pi(k(\alpha-1) + \gamma \ln \alpha)} \right)^{1/2}$$

CASE XII

Coil Shape: Cylindrical Bore, tapered ends, non-coinciding apexes.



$$b = ky + c$$

$$k = \text{slope}$$

$$\gamma = \frac{c}{a_1} \quad \alpha = a_2/a_1$$

Current Distribution: $i = i_0/b = \frac{i_0}{ky+c}$

Geometry Factor:

$$G = \frac{\pi^{1/2}}{5} \frac{k}{(k^2+1)^{1/2}} \left[\left(\frac{1}{k(\alpha-1) + \gamma \ln \left(\frac{\gamma+k}{\gamma+\alpha k} \right)} \right) \cdot \ln \frac{((k^2+1)\alpha^2 + 2k\gamma\alpha + \gamma^2) + \alpha(k^2+1)^{1/2} + \gamma \left(\frac{k^2}{k^2+1} \right)^{1/2}}{((k^2+1) + 2k\gamma + \gamma^2)^{1/2} + (k^2+1)^{1/2} + \gamma \left(\frac{k^2}{k^2+1} \right)^{1/2}} \right]$$

G is a maximum when $k = 1.0$ and $\alpha = 4.5$ which reduces case XII to case VII Current Density Factor J:

$$J = \left(\frac{1}{4\pi (k(\alpha-1) + \gamma \ln \left(\frac{k+\gamma}{k\alpha+\gamma} \right))} \right)^{1/2}$$

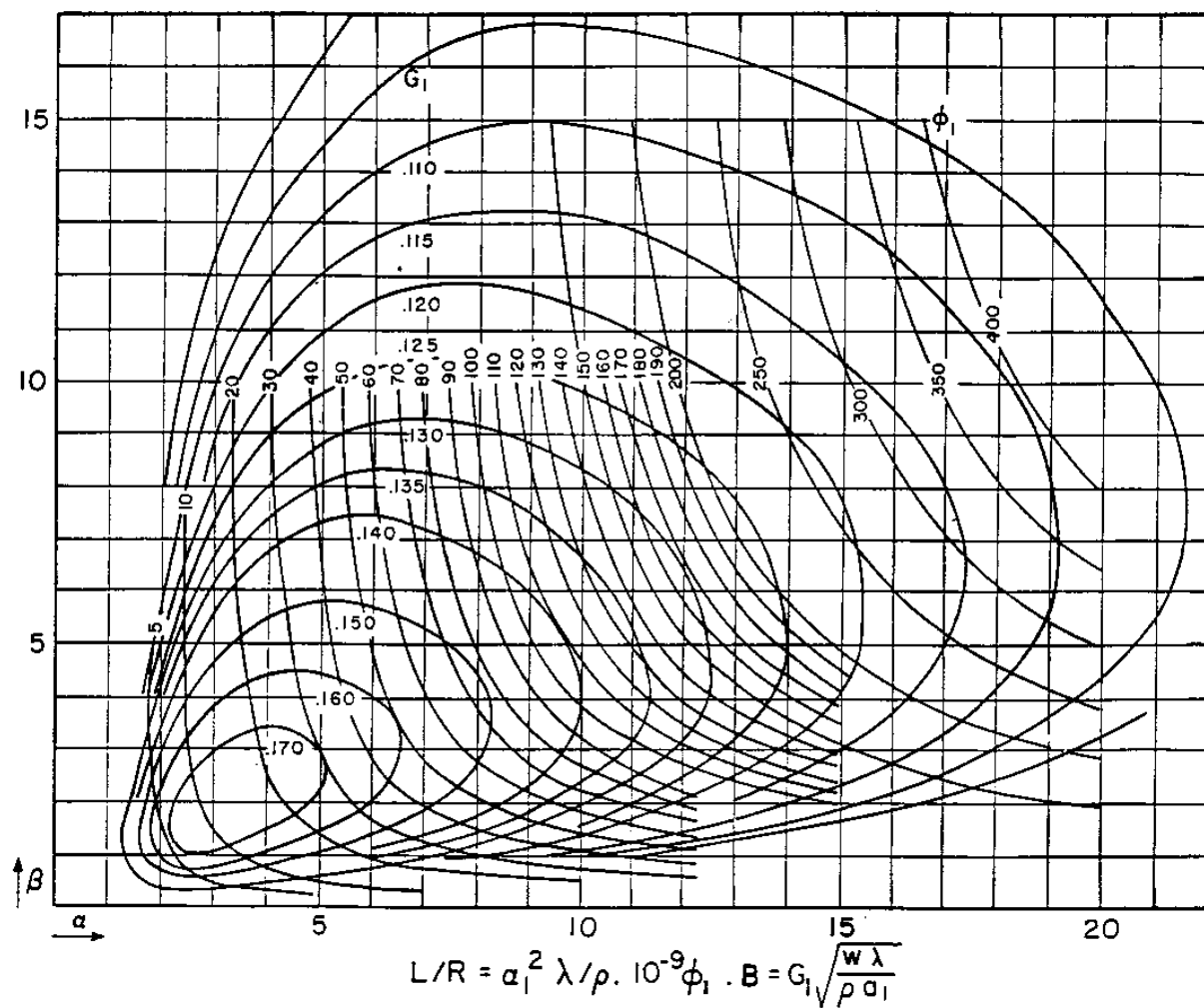


Fig. 1 Reproduction of Cockcroft's G-factor curves for uniformly wound coils with a square cross-section (Case VIII)

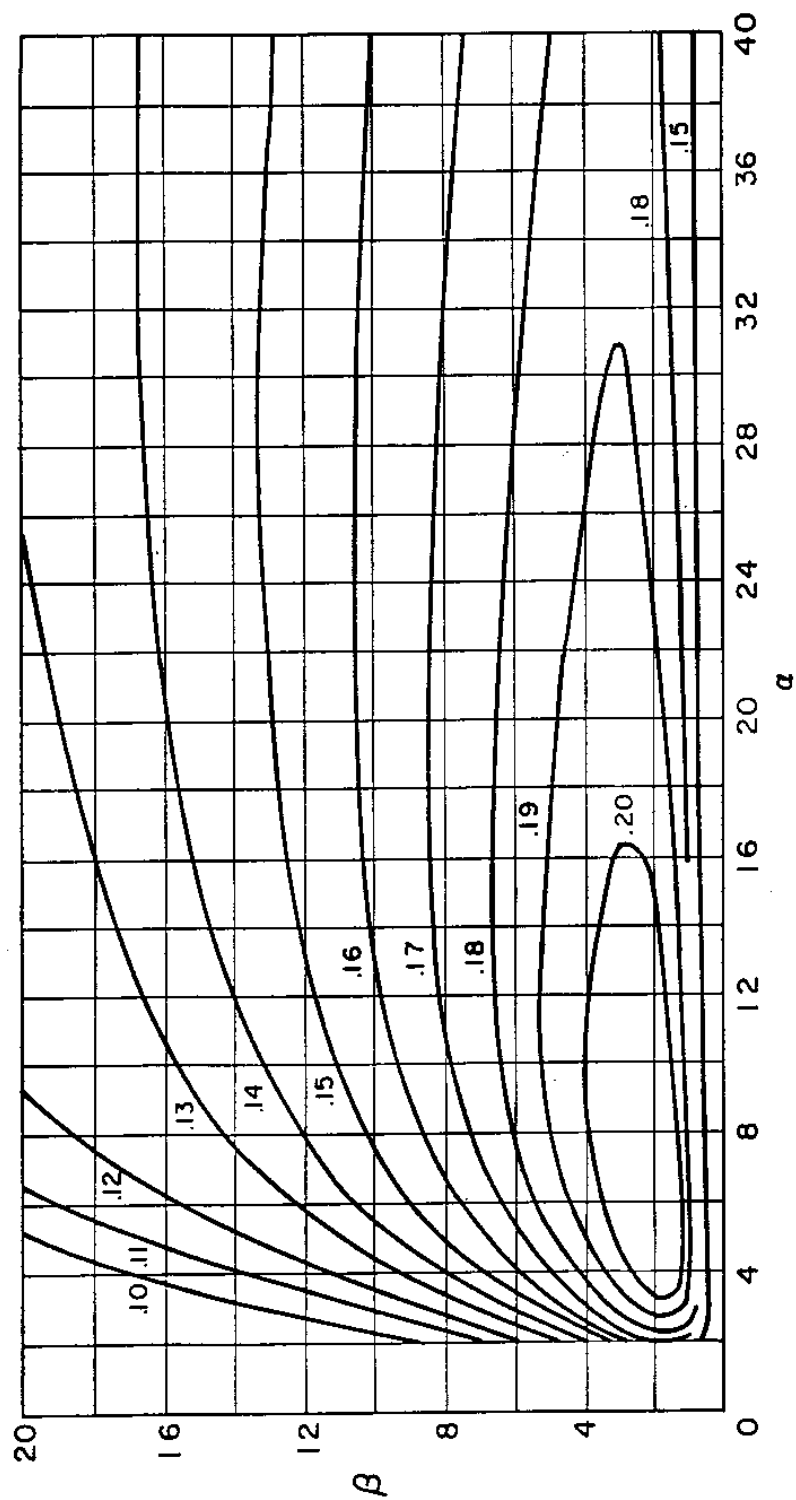


Fig. 2 Reproduction of Bitter's G-factor curves for plate magnets, $l = i_0/r$
(Case VI)

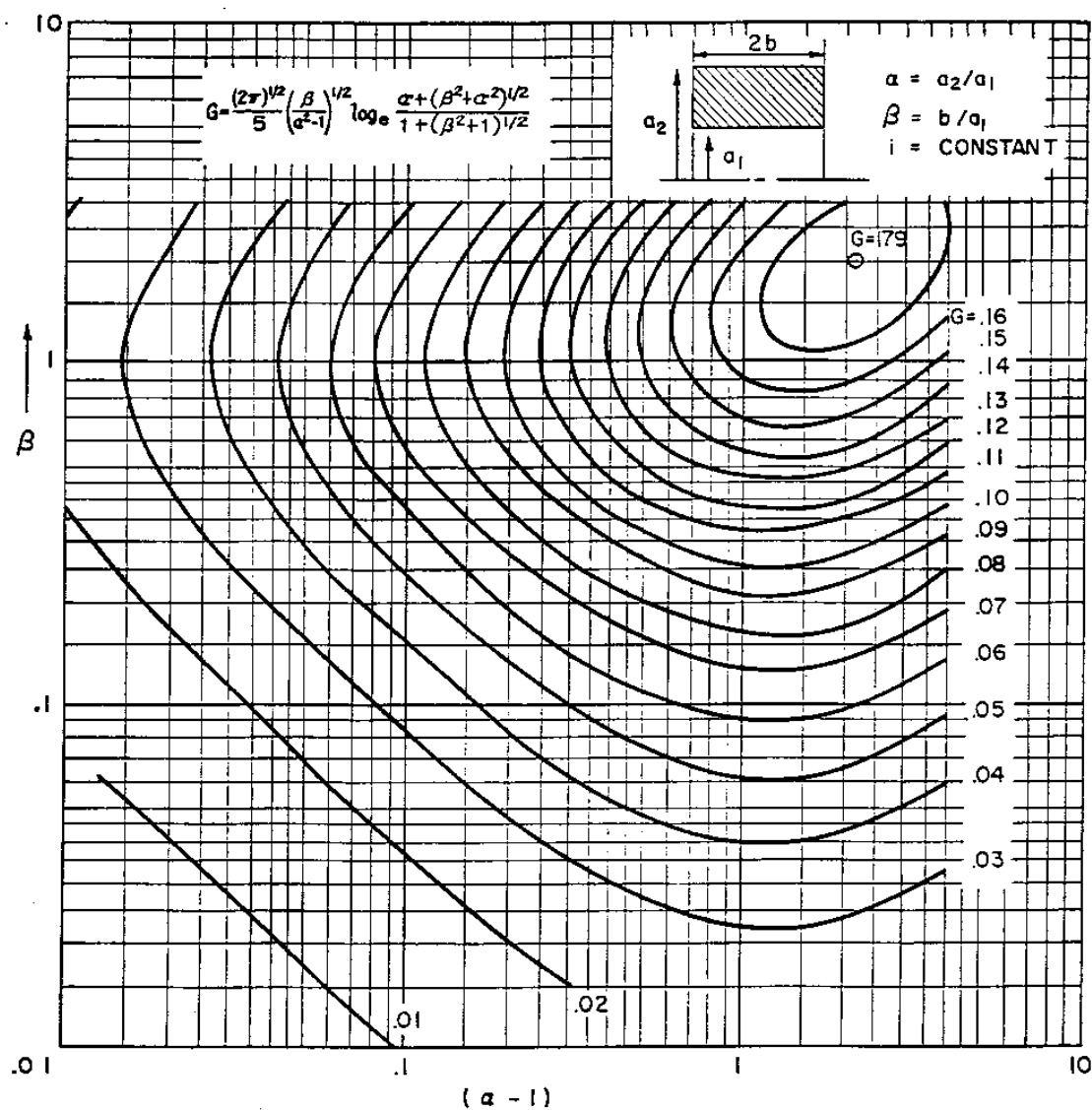


Fig. 3 G-factors for uniform current density magnets with small α and β 's (Case VIII)

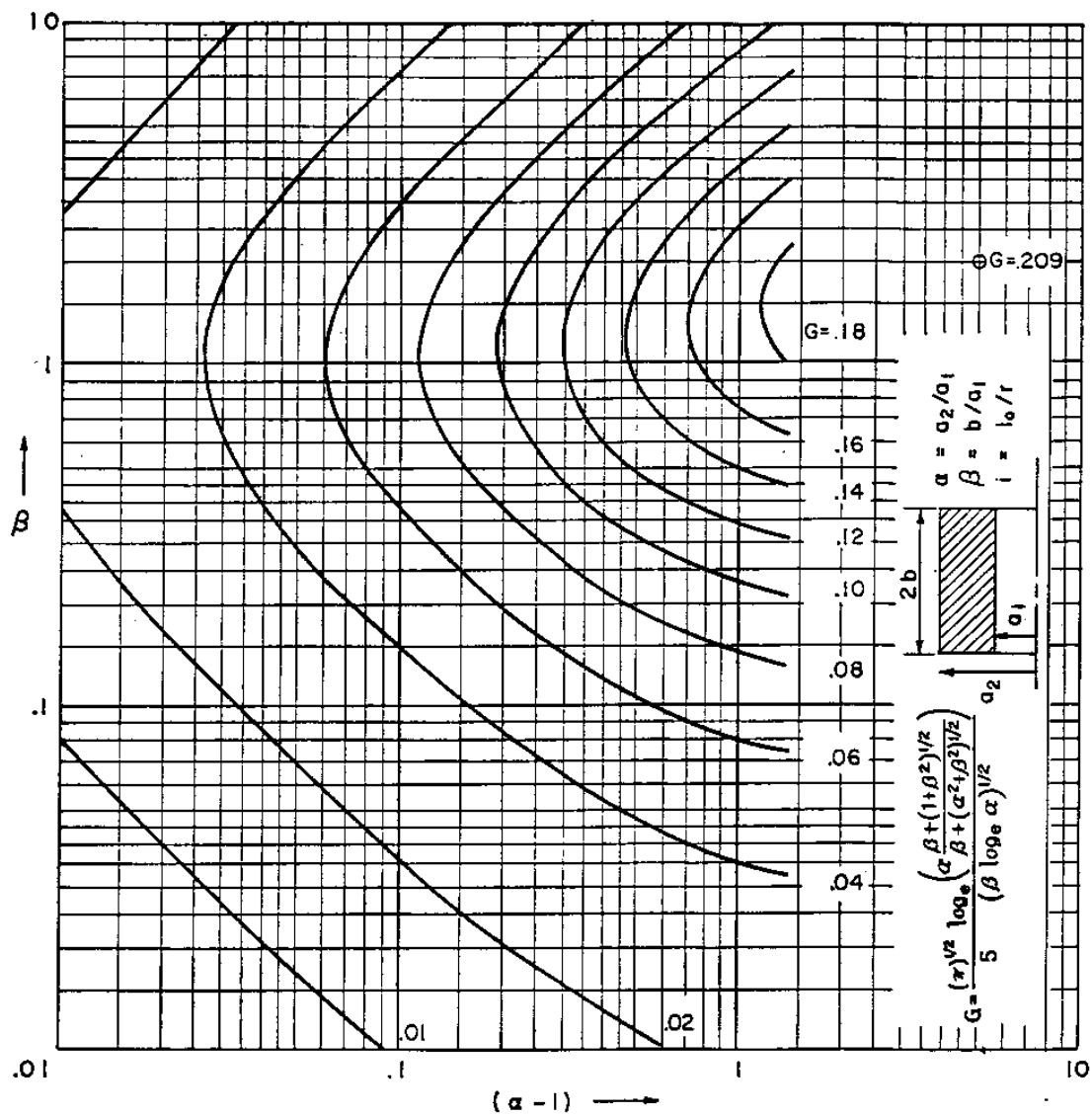


Fig. 4 G-factors for radial current density magnets with small α and β 's (Case VI)

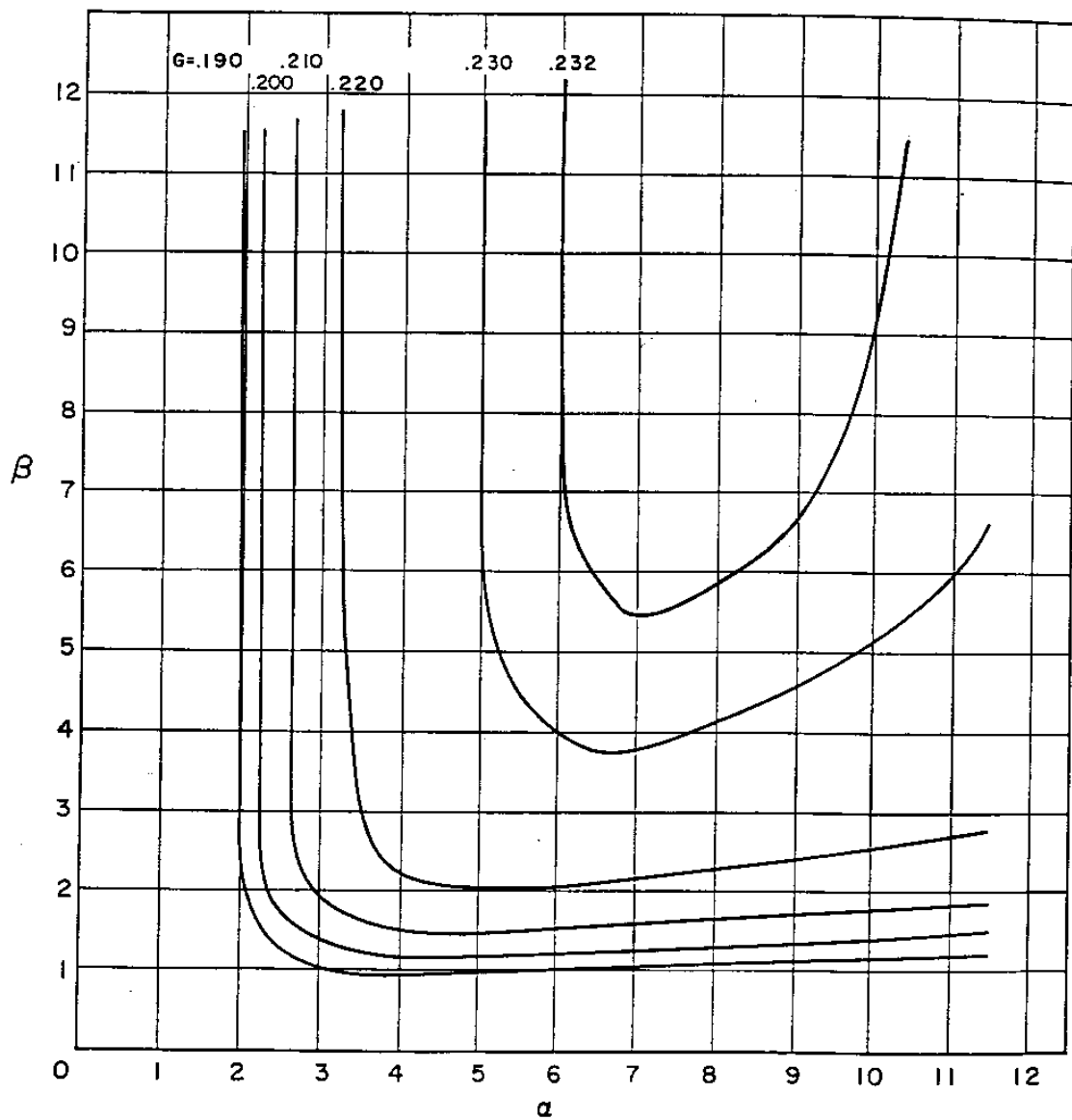


Fig. 5 G-factors for a Gaume current distribution (Case V)

ELLIPSOID
G VALUES
 $i = \frac{I_0}{r} \left(\frac{a_1}{a} \right)$

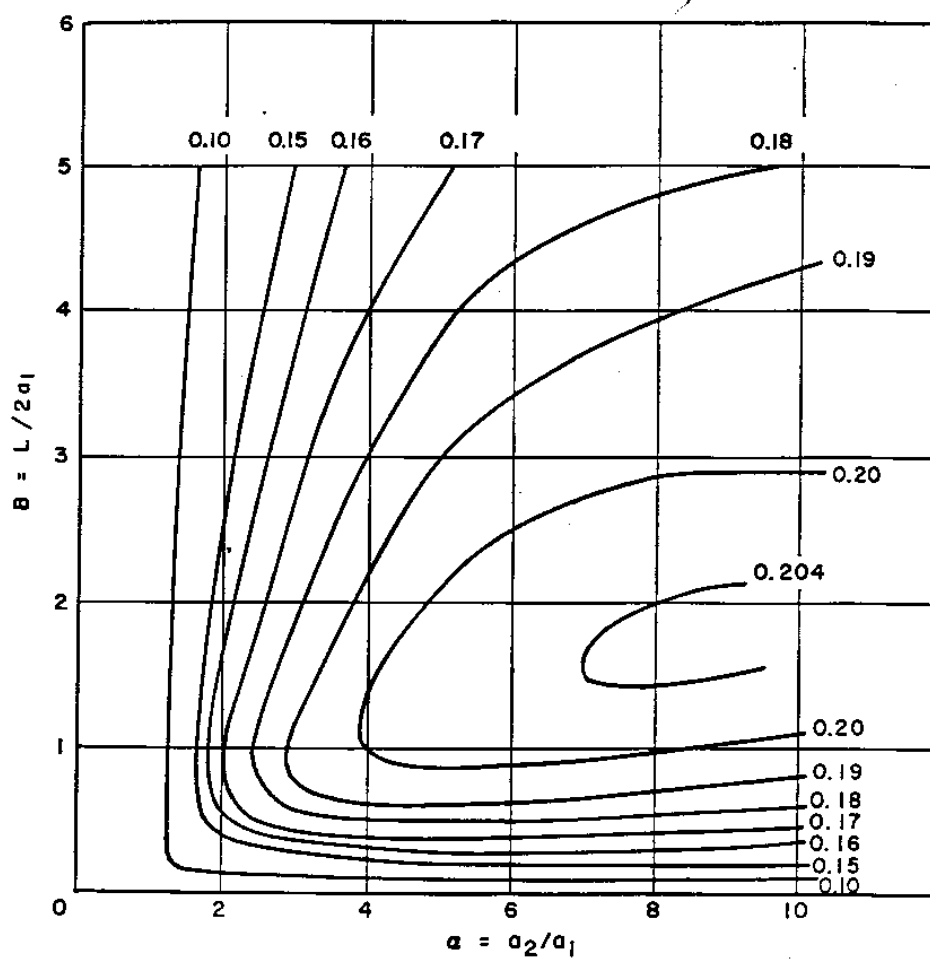
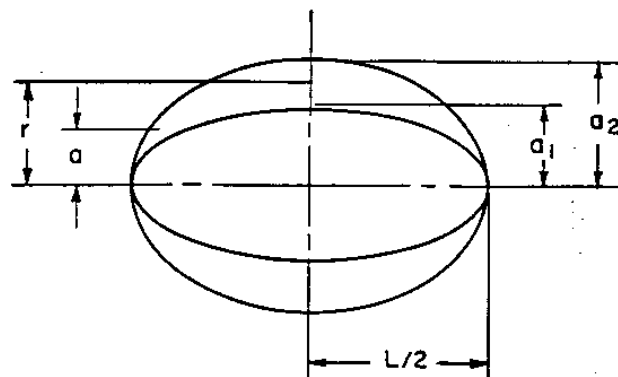


Fig. 6 G-factors for an ellipsoid of constant α
(Case X)

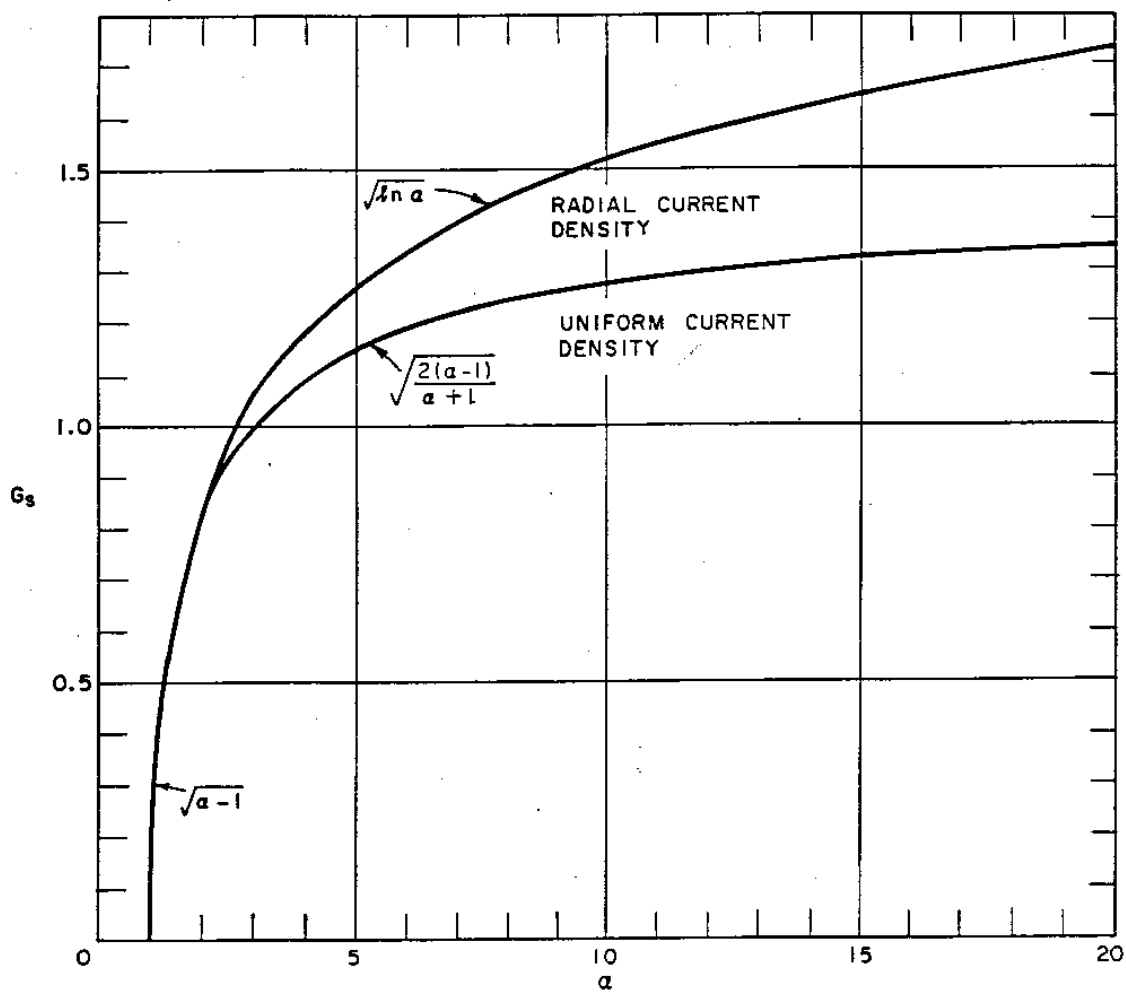


Fig. 7 G-factors for long solenoids

PART II

HOMOGENEOUS MAGNETIC FIELDS

A.

A magnetic field can be written as a power series involving Legendre polynomials expanded about the origin of the field. If this origin is on the axis and on the plane of symmetry the expansion will have no odd terms and the axial component H_z and the radial component H_ρ can be written:

$$H_z(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!} \left[H_z^{(2n-2)}(z, 0) \right] r^{2n-2} P_{2n-2}(u) \quad (1)$$

$$H_\rho(r, \theta) = - \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} \left[H_z^{(2n-2)}(z, 0) \right] r^{2n-2} P'_{2n-2}(u) \quad (2)$$

$$u = \cos \theta$$

In these equations the coefficients are defined as follows:

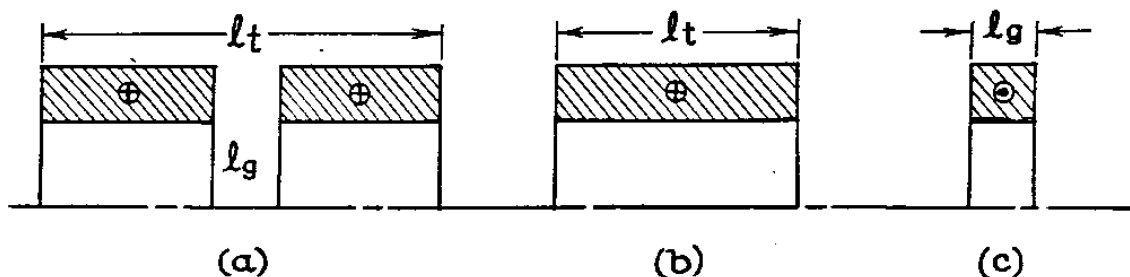
$$H_z^{(2n-2)}(z, 0) \equiv \left. \frac{d^{2n-2} H_z(z, 0)}{dz^{2n-2}} \right]_{z=0}$$

and $P_{2n-2}(u)$ are the Legendre polynomials tabulated in Table 2.

The importance of this approach, due to W. Garrett¹, is that the field at any point on the axis of a system of coaxial coils can be given explicitly in simple algebraic form. From this, the derivatives of equations (1) and (2) can be obtained, and so a complete expression for the field in the vicinity of the origin can be obtained. If sufficient derivative terms are used the field can be found anywhere within a sphere whose center lies at the origin and which does not extend far enough to include any corners of the coil. However, out to within a few percent of the inner radius of the coil, the field can be found quite accurately with only a few derivative terms. Compensation can be designed for each component of the field separately. In general, if we have a coil system of the assumed symmetry having N variable parameters such as size, or ampere-turns, it is possible to specify N coefficients in the expansions 1 and 2.

For example, to examine the standard Helmholtz pair of coils, where the coils are so spaced that there is no second derivative of the field at the origin, we can think of the coil system in the following way:

¹. M. W. Garrett, J. Appl. Phys. 22, 9, Sept. 1951



$$a = b - c$$

Figure 1

As shown in Fig. 1, we consider a single coil having length equal to the coils and the gap separation, from which we subtract a second coil of the length of the gap. The gap coil is thought of as having a current in the reverse direction to the original long coil. We now merely need select the length of the gap coil so that it will have a second derivative equal and opposite to that of the original long coil and the two will thus cancel at the origin.

There are many other combinations which would also cancel the second derivative at the origin, for instance, the hypothetical gap coil with its reverse current could have been somewhat longer and had less reverse current in it, or shorter and had more reverse current in it. Both of these solutions, of course, lead to a continuous solenoid with no gap but with an altered current density in the center section. These various solutions will have both an effect on the amount of field in the center of the magnet and on the size of the higher derivatives. The greater the distance over which one compensates, the more parallel will be the flux lines from the compensating coil, hence the flatter will be the flux distribution at the origin, and hence the smaller will be the sum of the higher derivatives. This compensation may not be compatible with obtaining the highest central field for the least power,²

B.

Equations (1) and (2) can be rewritten in the following form:

$$H_z(r_1, 0) = H_z(0, 0) \left[1 + \epsilon_2 \left(\frac{r}{a_1} \right)^2 P_2(u) + \epsilon_4 \left(\frac{r}{a_1} \right)^4 P_4(u) + \dots \right] \quad (3)$$

$$H_\rho(r_1, 0) = H_z(0, 0) \left[0 + \epsilon_2 \left(\frac{r}{a_1} \right)^2 P_2'(u) + \epsilon_4 \left(\frac{r}{a_1} \right)^4 P_4'(u) + \dots \right] \quad (4)$$

$$H_z(0, 0) = G \left(\frac{W \lambda}{\rho a_1} \right)^{1/2}$$

2. R. S. Ingarden and J. Michalczyk, Bulletin De L'Academie Polonaise Des Sciences (Serie des sci. math, astr., and phys) VIII, 5, 1960.

The expansion has been normalized to the inside radius of the coils a_1 . A tabulation of the \mathcal{E}_n coefficients for three useful geometries is given in Table 1. The geometries treated are (i) a cylindrical thin current sheet, (ii) a finite solenoid with a uniform current density, and (iii) a finite solenoid with a radial current distribution ($i = i_0/r$). The value of the first eight Legendre polynomials and their derivatives are given for convenience in Table 2. From equations (3) and (4) the magnetic field can be found at any point near the origin of a magnet, at a distance r from the origin and at an angle θ . The derivative terms in equation (3) have their largest value for $\theta = 0$ ($P_n(1) = 1$ for all n when $\theta = 0$ and $\cos \theta = 1$). The maximum value of the terms therefore can be obtained from the simpler form of (3) where $\theta = 0$ and $z = r$:

$$H_z(z, 0) = H_z(0, 0) \left[1 + \mathcal{E}_2 \left(\frac{z}{a_1} \right)^2 + \mathcal{E}_4 \left(\frac{z}{a_1} \right)^4 + \dots \right] \quad (5)$$

C.

The most commonly used method of achieving some degree of homogeneity in coils is to use the spaced helmholtz pair. Garrett has compiled a number of tables dealing with this method of compensation and since these tables have not been published we are making them part of this report. These tables are for constant current density only. The first table, Table 3 is used to determine the proper spacing between finite coils to achieve cancellation of the second derivative at the origin. The second table, Table 4 is used to determine how far from the origin one can move before the field deviates by more than 0.1 percent of the field at the origin. The third table, Table 5 is particularly useful and can show how accurately the coils must be spaced in order to cancel the second derivative to the desired degree. The two remaining tables, Table 6 and Table 7 give the resultant fourth and the sixth \mathcal{E} coefficients remaining after the proper spacing of the coils. The tables are used with the notation of equations (3) and (4) except that Garrett has normalized the expansion to the mean radius of the coil rather than to the inner radius as in Section B above ($(\frac{z}{a_0})$ rather than $(\frac{z}{a_1})$ where $a_0 = (\frac{a_1 + a_2}{2})$).

To illustrate the use of Garrett's tables, we work out the following examples:

Assume we have two coils, each with an $\alpha = 3$ and a $\beta = 1$ and we wish to separate them to cancel the second order derivative at the origin.

$$X = \frac{4\beta}{\alpha+1} = 1.0$$

$$A = \frac{2(\alpha-1)}{\alpha+1} = 1.0$$

$$\text{From Table 3, } K = .58260 = \frac{\Delta X}{a_1 + a_2} = \frac{\Delta X}{a_1(1+\alpha)}$$

therefore
$$\frac{\Delta X}{a_1} = 2.33040 = 2\beta + \frac{lg}{a_1}$$

and the spacing should be

$$\frac{lg}{a_1} = .33040$$

Using Table 4 we see that $V = \frac{z}{a_0} = .181$ or $\frac{z}{a_1(1+\alpha)} = .181$ and $\frac{z}{a_1} = .724$. This means that out to a distance along the axis of 72.4% of the inner radius, the field is within 0.1% of the field at the origin. Using Table 5, we can examine the effect of improper spacing.

$$H = H_0 \left(1 + Q \left(\frac{d}{a_0} \right) \cdot \left(\frac{z}{a_0} \right)^2 + \epsilon_4 \left(\frac{z}{a_0} \right)^4 + \dots \right)$$

where Q is the value in table 5 and d/a_0 is the displacement error in locating one coil. For the above example, we can find the error that will make the uncanceled 2nd derivative error equal the fourth at $\left(\frac{z}{a_0} \right) = 0.1$

$$Q = 4.21 \text{ (at } X=1, A=1)$$

$$\epsilon_4 = .922 \text{ from table 6}$$

$$4.21 \left(\frac{d}{a_0} \right) \times 10^{-2} = .922 \times 10^{-4}$$

$$\frac{d}{a_0} = 2.18 \times 10^{-3}$$

D.

When maximum homogeneity and not maximum efficiency is of the greatest importance, the higher derivatives must be handled. Since the number of derivatives that can be cancelled depends upon the number of variables, we must now pick solutions which have more variables at our disposal. For instance, if we wish to cancel both the second and the fourth derivative of the field we can superimpose two coils on each other each with two geometry variables or each with one geometry variable and one current variable. To proceed, then, we would pick the length of one coil and the current of one coil and the variables then would be the length of the second coil and the ratio of the two current densities. Garrett has calculated a set of so called sixth-order solenoids (no second or fourth derivatives) by making the following choice of variables: He constructs an overwound end solenoid by winding N layers of turns over a certain central region of the solenoid and $2N$ layers over a certain section of both ends (i.e. a notch on the center section of the o.d. of the coil). He now solves uniquely for the length of the overall coil and the length of the notch. The results of his calculation are shown in Table 8 for several thicknesses of coil.

There are other choices of two variables that one can make to construct sixth-order solenoids and Table 9 illustrates several methods. Table 9 was constructed for the specific example of a rather long solenoid. Several general remarks can be drawn from the table. Probably the most important of which is that since the solenoid is very long

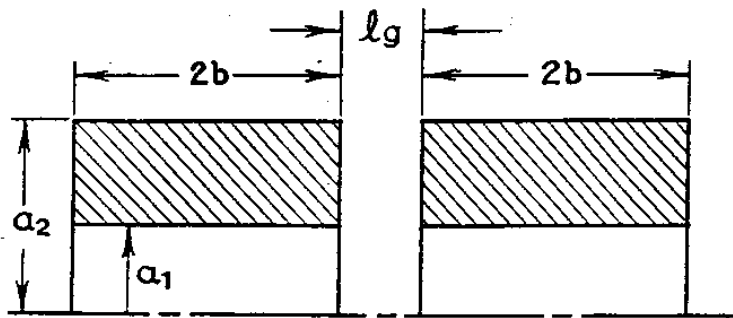
the job of compensating it for high homogeneities is much simplified, the higher derivative having been quite small to begin with. We notice, however, that in all but one case in the table, cancelling the second and fourth derivatives of the field, have increased the sixth derivative. The single exception is the current sheet of the o.d. The current sheet on the o.d., however, requires more power than any of the other methods of solution. Perhaps the most attractive method of solution is that represented by the last column, that of current sheets on either end of the i.d. of the coil. This method takes little power and leaves the center section of the magnet open for further compensation if desired.

Cancellation of higher derivatives than the fourth is of course possible, but is often of decreasing practical importance because of the unrealistic accuracy necessary in the cancellation of lower derivatives, if the leftover uncompensated part of say the second, is to be smaller than the sixth derivative or eighth derivative, etc. It probably makes sense to design coils to cancel the second and the fourth derivative and use a small auxiliary set of coils driven from the separate power source to tune out any random errors which creep in. Some of these random errors are of course going to be nonsymmetrical and will give rise to odd derivative terms. It is quite possible to construct a set of auxiliary compensating coils which will operate independently of each other. One coil can be designed to have a second derivative and no fourth and a second coil to have a fourth derivative and no second, etc. Each one can then be tuned independently of the other.

E.

We have constructed a table of \mathcal{E} coefficients up to the eighth order for coils of varying α and β with both constant current density and with current density equal $i = i_0/r$. The tables also include the G factor and the multiplication of the G factor and the \mathcal{E} coefficients. The tables are to be used with the expansion of equation (3) and (4) where z is referred to the inner radius a_1 ; \mathcal{E} is tabulated as D for uniform currents and as A for the radial current distribution. Table 10A gives the coefficients for the uniform current density case for several α 's and for β up to 10. Table 10B gives the coefficients over the same range of α and β for the radial current distribution case. Tables 11A, B, C, D show more detail for the constant current and the radial current cases in the small β range.

The use of the tables is probably best illustrated by the following example: Let us assume we have a coil we wish to separate for purposes of access and we want to know the resultant field in the gap and the resultant homogeneity (see Fig. 2)



W = power
 λ = space factor
 ρ = resistivity

$$\alpha = a_2/a_1$$

$$\beta_t = \frac{4b + l_g}{2a_1} \quad G_t(\alpha, \beta_t)$$

$$\beta_g = \frac{l_g}{2a_1} \quad G_g(\alpha, \beta_g)$$

Figure 2

If we now subtract from the field expansions for a coil of length $4b + l_g$, a coil of length l_g we can write:

$$H_{gap} = (H_{0t} - H_{0g}) + (H_{2t} - H_{2g})\left(\frac{z}{a_1}\right)^2 + (H_{4t} - H_{4g})\left(\frac{z}{a_1}\right)^4 + \dots$$

$$H_{gap} = \left(\frac{W\lambda}{\rho a_1}\right)^{1/2} \left(\frac{\beta_t}{\beta_t - \beta_g}\right)^{1/2} \left[\left(G_t - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}}\right) + \left(G_t \mathcal{E}_2 - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}} \mathcal{E}_2'\right) \left(\frac{z}{a_1}\right)^2 + \left(G_t \mathcal{E}_4 - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}} \mathcal{E}_4'\right) \left(\frac{z}{a_1}\right)^4 + \dots \right]$$

where the primed quantities refer to the error coefficients for the gap. Taking a specific geometry, let the current be constant ($\mathcal{E} = D$).

let

$$\beta_t = 4$$

$$\alpha = 3$$

$$\beta_g = 0.2$$

What is the field in the gap and the second error coefficient, and what would be the second error coefficient if there were no access gap?

The D_2 listed for $\alpha = 3$, $\beta = 4$ in Table 10A is $-.1415 \times 10^{-1}$, therefore the second error coefficient of the whole coil would be $\mathcal{E}_2 = -1.415 \times 10^{-2}$ where

$$H = H_0 \left(1 + \mathcal{E}_2 \left(\frac{z}{a_1} \right)^2 + \dots \right)$$

To solve for the split coil, we use (6) and the G_1 , D_{11} columns in Table 10A

$$\begin{aligned} H_{gap} &= \left(\frac{W\lambda}{\rho_1 a_1} \right)^{1/2} \left(\frac{4}{3.8} \right)^{1/2} \left[\left(0.1580 - 0.08639 \left(\frac{0.2}{4} \right)^{1/2} \right) + \right. \\ &\quad \left. + \left(-0.2235 \times 10^{-2} + 0.5004 \times 10^{-1} \left(\frac{0.2}{4} \right)^{1/2} \right) \left(\frac{z}{a_1} \right)^2 + \dots \right] \\ H_{gap} &= \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} (1.03) \left[.1387 + 0.8965 \times 10^{-2} \left(\frac{z}{a_1} \right)^2 \right] \end{aligned}$$

this gives a central field of

$$H_{gap}(0,0) = \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} (0.143)$$

and if the second order coefficient is written in the following form:

$$H = H_0 \left(1 + \mathcal{E}_2 \left(\frac{z}{a_1} \right)^2 + \dots \right)$$

the coefficient for the above example would be

$$\mathcal{E}_2 = \frac{0.8965 \times 10^{-2}}{.1387} = 6.46 \times 10^{-2}$$

The loss in field for this case has been 9.5% and the second order error coefficient has been increased by a factor of 4.55. If we had been trying to homogenize the field by separating the coils, we note that we would have separated them by too much. To use these tables for synthesis (i.e. cancellation of derivatives) it would be necessary to use trial and error, in the manner of the above example.

It is interesting to note the oscillatory behavior of the \mathcal{E} coefficients, particularly in the small β region. The second derivative is always negative, the fourth derivative goes through zero once, the sixth derivative goes through zero twice and the eighth derivative three times. This means that there are coils which can be built with no fourth, no sixth and no eighth derivatives. This may be useful in the design of independent compensating coils.

F.

If one wishes to compensate a very long solenoid over a considerable length of the axis such as a multicoil structure used for plasma experiments, expansion of the derivatives around the origin has limited usefulness. One might wish to use the following procedure instead; namely specifying that the field at any number of intervals along the axis be constant and then solving for the current distribution in the coils that would make the field be equal at the specified points. In the case of a multicoil structure one could ask that the total field in the center of each of the coils be equal to a constant and then solve for the current density required in each of the coils to make this true. The same procedure could be used with a single long solenoid where the solenoid could be broken up into an arbitrary number of pieces.

The most useful method of solution of this type of problem is basically the following: taking any single coil we first find the field at the center of that coil and at distances away from the coil represented by the center lines of each of the other coils, for a given current. Then by properly adding up the coils in matrix form and demanding that the field matrix times the current matrix be a constant we can invert the current matrix and solve for the individual currents required in each of the coils to achieve the desired results. The method of solution and a useful notation are developed in the following example. Mr. R. Bradshaw of Conesco, Inc., Arlington, brought this method of solution to our attention. Consider the following system of coils arranged so that they have a common axis. They may in general be of various sizes but for illustration consider them to be all identical with the same separation. For ease in calculation choose an even number of coils so that we have a plane of symmetry passing between coils.

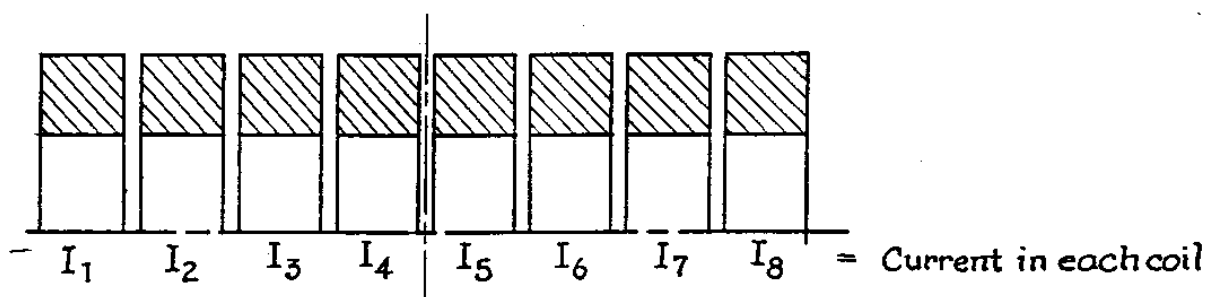


Figure 3

Our object is to find what ratio of currents is needed in the various coils so that a uniform magnetic field is obtained throughout the interior. If the coils are thin and close together, to a good approximation uniformity is obtained by requiring that the field at the center of each coil be the same. The field falls off rapidly on each side of the center of a single coil and we may construct a matrix the elements of which represent the contributions of the coils at their respective centers. Let F_{ij} be the field at the center of coil i due to the field of coil j . For example, for 4 coils, 2 coils on each

side of the plane of symmetry, we have

$$\begin{array}{|c|c|c|c|} \hline F_{11} & F_{12} & F_{13} & F_{14} \\ \hline F_{21} & F_{22} & F_{23} & F_{24} \\ \hline F_{31} & F_{32} & F_{33} & F_{34} \\ \hline F_{41} & F_{42} & F_{43} & F_{44} \\ \hline \end{array}$$

These matrix elements may be expressed in % of an infinite solenoidal sheet of the same mean radius, or whatever one desires, since we need only ratios.

$F_{ij} I_j = \mathcal{F}_i$ where \mathcal{F}_i is the field at the center of the i^{th} coil or

$$\begin{array}{|c|c|c|c|} \hline F_{11} & F_{12} & F_{13} & F_{14} \\ \hline F_{21} & F_{22} & F_{23} & F_{24} \\ \hline F_{31} & F_{32} & F_{33} & F_{34} \\ \hline F_{41} & F_{42} & F_{43} & F_{44} \\ \hline \end{array} \begin{array}{|c|} \hline I_1 \\ \hline I_2 \\ \hline I_3 \\ \hline I_4 \\ \hline \end{array} = \begin{array}{|c|} \hline \mathcal{F}_1 \\ \hline \mathcal{F}_2 \\ \hline \mathcal{F}_3 \\ \hline \mathcal{F}_4 \\ \hline \end{array}$$

If one now specifies the \mathcal{F}_i 's the F_{ij} matrix can be inverted and the I_j matrix found.

If the current in the coils are arranged so that $I_1 = I_4$ and $I_2 = I_3$ then it is convenient to partition the matrix as follows:

$$f_{11} = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, f_{12} = \begin{vmatrix} F_{13} & F_{14} \\ F_{23} & F_{24} \end{vmatrix}, f_{21} = \begin{vmatrix} F_{31} & F_{32} \\ F_{41} & F_{42} \end{vmatrix}, f_{22} = \begin{vmatrix} F_{33} & F_{34} \\ F_{43} & F_{44} \end{vmatrix}$$

$$i_1 = \begin{vmatrix} I_1 \\ I_2 \end{vmatrix}, i_2 = \begin{vmatrix} I_3 \\ I_4 \end{vmatrix}; \mathcal{F}_1' = \begin{vmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{vmatrix}, \mathcal{F}_2' = \begin{vmatrix} \mathcal{F}_3 \\ \mathcal{F}_4 \end{vmatrix}$$

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \end{vmatrix} = \begin{vmatrix} \mathcal{F}_1' \\ \mathcal{F}_2' \end{vmatrix} \quad \text{and then} \quad i_2 = k i_1 \quad \text{and} \quad \mathcal{F}_2' = k \mathcal{F}_1' \quad \text{where} \quad k = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = i_2 \quad \text{and} \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \begin{vmatrix} i_1 \\ k i_1 \end{vmatrix} = \begin{vmatrix} \mathcal{F}_1' \\ k \mathcal{F}_1' \end{vmatrix}$$

it follows that

$$(f_{11} + f_{12}k) i_1 = \mathcal{F}_1'$$

Since we desire the fields at the centers of each coil to be a constant

$$\mathcal{F}_1 = \mathcal{F}_2 = \text{constant} = C \quad \text{where } C = c \left| \begin{matrix} 1 \\ 1 \end{matrix} \right|$$

$$\text{i.e., } (f_{11} + f_{12}k)i_1 = C$$

It is clear that $i_1 = (f_{11} + f_{12}k)^{-1}C$ and this gives us the desired ratio of currents.

As an example consider the field on the axis due to a system of four current loops of radius $y = 1$ cm

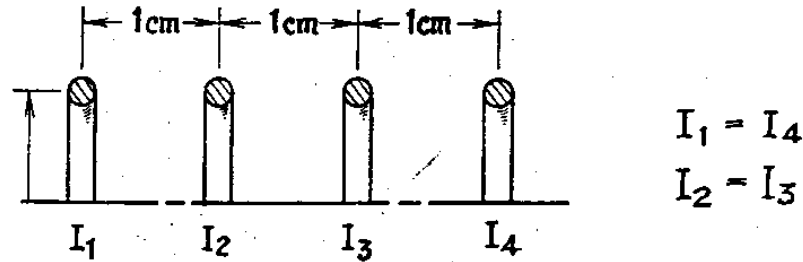


Figure 4

Separation between centers of the coils is 1 cm. We need calculate the field of one loop at the center and 1, 2, and 3 cm. on the axis away from the center. For a single loop

$$H_x = \frac{2\pi i}{10} \frac{y^2}{(x^2 + y^2)^{3/2}}$$

x	H_x/i
0	.628
1	.222
2	.0562
3	.0199

$$\text{then } f_{11} = \begin{vmatrix} 0.628 & 0.222 \\ 0.222 & 0.628 \end{vmatrix} \quad f_{12} = \begin{vmatrix} 0.056 & 0.019 \\ 0.222 & 0.056 \end{vmatrix}$$

$$f_{12}k = \begin{vmatrix} 0.056 & 0.019 \\ 0.222 & 0.056 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 0.019 & 0.056 \\ 0.056 & 0.222 \end{vmatrix}$$

$$f_{11} + f_{12}k = \begin{vmatrix} 0.628 & 0.222 \\ 0.222 & 0.628 \end{vmatrix} + \begin{vmatrix} 0.019 & 0.056 \\ 0.056 & 0.222 \end{vmatrix} = \begin{vmatrix} 0.647 & 0.278 \\ 0.278 & 0.850 \end{vmatrix}$$

We now need the inverse of $(f_{11} + f_{12}k)$

By definition, the inverse of a matrix A with matrix elements a_{ij} is obtained if for each element $(a_{ij})^{-1} = \frac{(-1)^{i+j} A_{ji}}{\det A}$; A_{ji} is the minor of the determinant A.

If $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ then $A^{-1} = \begin{vmatrix} (a_{11})^{-1} (a_{12})^{-1} \\ (a_{21})^{-1} (a_{22})^{-1} \end{vmatrix}$

For 2 x 2 matrices this is very simple:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\det (f_{11} + f_{12} k) = (.850)(.647) - (.278)^2 = .473$$

$$(f_{11} + f_{12} k)^{-1} = \frac{1}{.473} \begin{vmatrix} .850 & -.278 \\ -.278 & .647 \end{vmatrix} = \begin{vmatrix} 1.797 & -.587 \\ -.587 & 1.367 \end{vmatrix}$$

$$\therefore \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} 1.797 & -.587 \\ -.587 & 1.367 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} c = c \begin{vmatrix} 1.210 \\ .780 \end{vmatrix}$$

$$\therefore I_1 = c(1.210) \quad I_2 = c(0.780)$$

The current in coil 1 must then be 1.55 times greater than the current in coil 2.

Ingarden and Michalczyk⁽²⁾ have also considered the problem of homogeneities over a certain length of the axis rather than strictly at the origin. They have used the requirement that the mean square deviation over a certain interval be minimized. Their notation is useful, but solution of the equations, which are transcendental in character, presents a number of problems.

G.

It is often of interest to know the field along the axis of a solenoid both inside the solenoid and outside the solenoid. The following two equations can be used to find the field along the axis of a uniform current solenoid and a radially distributed current solenoid (Bitter discs) respectively.

(i) ^{Radial} Uniform current $k = z/a_1$ (7)

$$H = \frac{\pi i \lambda a_1}{5} \left[\ln \frac{(k-\beta) + (\alpha^2 + (k-\beta)^2)^{1/2}}{(k+\beta) + (\alpha^2 + (k+\beta)^2)^{1/2}} - \ln \frac{(k-\beta) + (1 + (k-\beta)^2)^{1/2}}{(k+\beta) + (1 + (k+\beta)^2)^{1/2}} \right]$$

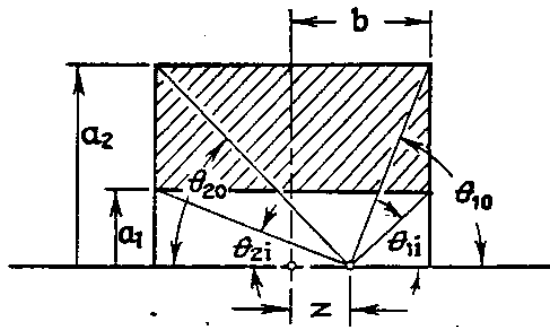
(ii) ^{Uniform} Radial current $i = \frac{i_1 a_1}{r}$; $k = \frac{z}{a_1}$

$$H = \frac{\pi i_1 \lambda a_1}{5} \left[(k+\beta) \ln \left(\frac{\alpha + (\alpha^2 + (k+\beta)^2)^{1/2}}{1 + (1 + (k+\beta)^2)^{1/2}} \right) - (k-\beta) \ln \frac{\alpha + (\alpha^2 + (k-\beta)^2)^{1/2}}{1 + (1 + (k-\beta)^2)^{1/2}} \right] \quad (8)$$

* ETD

Tabulated values of equation (7) for a wide variety of coil shapes have been published by D. E. Mapother and James Snyder in a University of Illinois circular #66 entitled, "The Axial Variation of the Magnetic Field in Solenoids of Finite Thickness". There is no available tabulation of equation (8)

Equations (7) and (8) can be written in a much simpler form utilizing the following angle notation:



$$\frac{a_2}{a_1} = \alpha$$

$$\frac{b}{a_1} = \beta$$

$$\frac{z}{a_1} = \gamma$$

Figure 5

i) uniform current density

$$H = \frac{\pi i_1 \lambda a_1}{5} \left[(\beta - \gamma) \ln \frac{\tan\left(\frac{\pi}{4} + \frac{\theta_{10}}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\theta_{1i}}{2}\right)} + (\beta + \gamma) \ln \frac{\tan\left(\frac{\pi}{4} + \frac{\theta_{20}}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\theta_{2i}}{2}\right)} \right] \quad (9)$$

ii) radial current density

$$H = \frac{\pi i_1 \lambda a_1}{5} \ln \left[\frac{\tan \frac{\theta_{10}}{2} \tan \frac{\theta_{20}}{2}}{\tan \frac{\theta_{1i}}{2} \tan \frac{\theta_{2i}}{2}} \right] \quad (10)$$

General expressions for the off axis fields arising from a loop of current can be written in terms of elliptic integrals of the first and second kind.⁽³⁾ They can be written in the following way, where the parameters are defined in Figure 6.

$$H_z = \frac{2I}{Q^{1/2}} \left[F(k) + \frac{(a^2 - \rho^2 - z^2)}{Q} s(k) \right] \quad (11)$$

$$H_\rho = \frac{2I}{Q^{1/2}} \left[-F(k) + \frac{(a^2 + \rho^2 + z^2)}{Q} s(k) \right] \times \frac{z}{\rho} \quad (12)$$

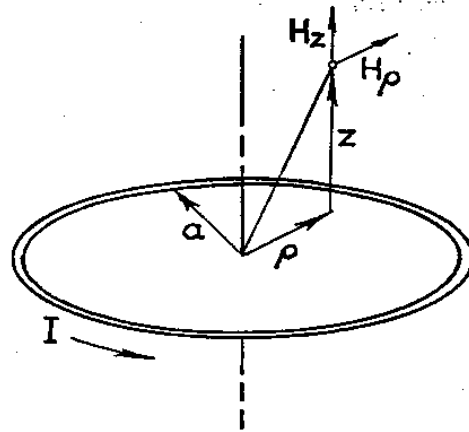
3. W. R. Smythe, STATIC AND DYNAMIC ELECTRICITY, p. 266, McGraw-Hill, New York, 1950

$$Q = (a + \rho)^2 + z^2$$

$$k = \left(\frac{4ap}{(a+\rho)^2 + z^2} \right)^{1/2}$$

$$F(k) = \text{Elliptic integral of the first kind}$$

$$S(k) = \frac{E(k)}{1-k^2} \quad \text{where } E(k) = \text{elliptic integral of the second kind}$$



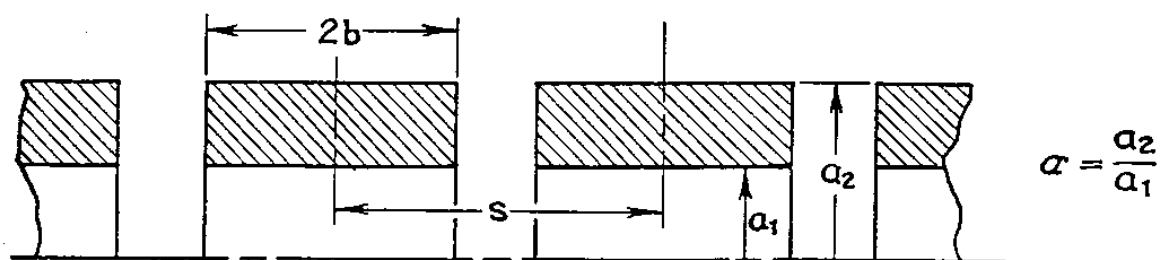
Using equations (11) and (12) the field at any point produced by a finite coil can be found by dividing up the coil into loops and summing all the contributions. A number of laboratories, including our own have written simple computer programs utilizing equations (11) and (12) to find the fields in and outside the conductor volume.

On-axis and off-axis fields from any solenoid with any radial current distribution can be calculated by means of a lengthy set of tables published by the Oak Ridge National Laboratory Report #ORNL-2828, entitled, "Tables for a Semi-Infinite Circular Current Sheet," by N. B. Alexander and A. C. Downing. These tables are for the field around the end of a semi-infinite current sheet and the coil in question is built up out of these current sheets in the method shown in the introduction of the table.

Attention is also called to a tabulation by E. E. Callaghan and S. H. Maslen, NASA D465 "The Magnetic Field of a Finite Solenoid". This paper plots graphs of the radial and axial components in and around current sheets of various lengths. While not as accurate as the Oak Ridge tables, it is very useful for approximate results.

A final paper of particular relevance is that of G. R. North, "Some Parameters of Lumped Solenoids", (Oak Ridge ORNL-2975). He derives an expression for the field ripple resulting from the separation of coils in a long multicoil structure (end effects neglected). The equation is as follows (see Figure 7)

$$H(z) = H_{INF} \left(\frac{2b}{s} \right) \left[1 + \left(\frac{s}{a_1} \right)^{1/2} \frac{1}{\alpha - 1} \left(e^{-\frac{2\pi a_1}{s}} - \alpha^{1/2} e^{-\frac{2\pi \alpha a_1}{s}} \right) \cdot \frac{\sin \frac{2b\pi}{s}}{\frac{2b\pi}{s}} \cdot \cos \frac{2\pi z}{s} \right] \quad (13)$$



$$H_{INF} = \frac{4\pi NI}{10} = \text{Infinite Solenoid}$$

Figure 7

Figure 8 plots the peak value of the ripple component of eq. (13) for several values of a_1/s . The peak value occurs when $z = 0$, $\cos \frac{2\pi z}{s} = 1$.

$$\delta = \left(\frac{s}{a_1}\right)^{1/2} \frac{1}{\alpha - 1} \left(e^{-\frac{2\pi a_1}{s}} - \alpha^{1/2} e^{-\frac{2\pi \alpha a_1}{s}} \right) \quad (14)$$

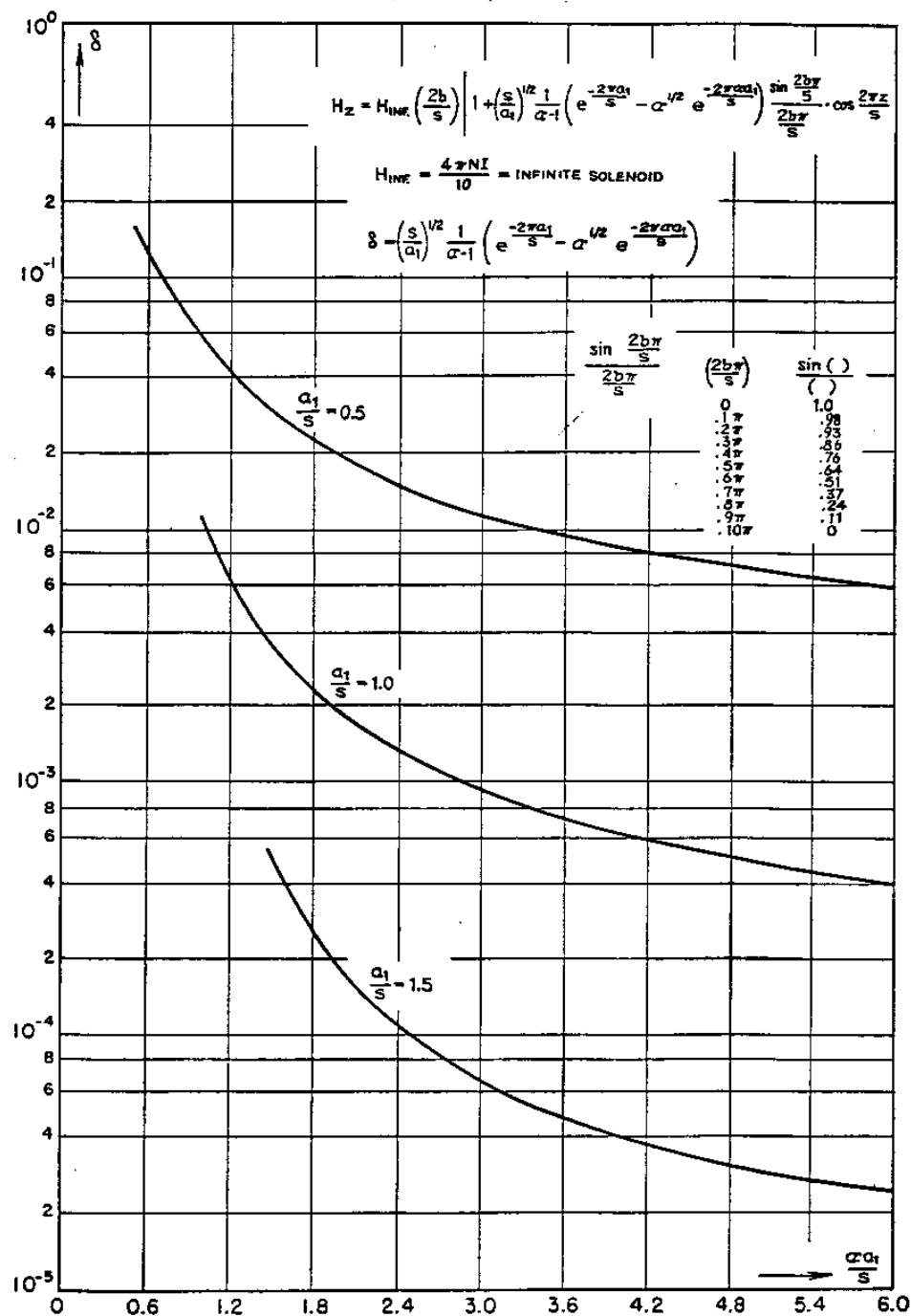
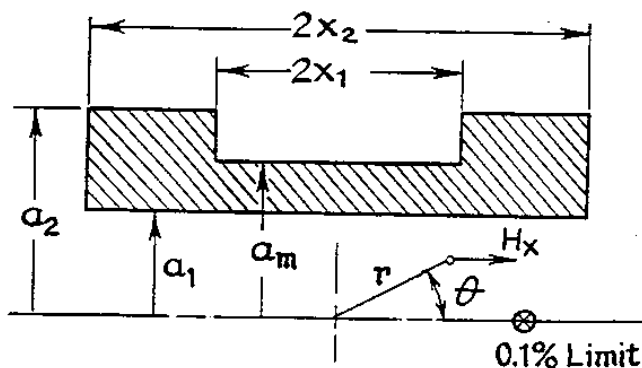


FIG. 8 PEAK VALUE OF FIELD RIPPLE IN MULTICOIL SOLENOIDS
 (MULTIPLY BY 10^2 FOR RIPPLE IN PERCENT OF CENTRAL FIELD)

Typical 6th order solenoids (No 2nd or 4th derivative on axis) where length is fixed.
(W. Garrett)

a_1	x_1/a_2	x_2/a_2	$\mathcal{E}_{6 \times 10^{-3}}$	$\mathcal{E}_{8 \times 10^{-3}}$	0.1 % Limits*
1.0	1.39327	1.82234	-5.504	-2.634	75
0.9	1.41768	1.82451	-4.918	-2.566	77
0.8	1.44747	1.82933	-4.255	-2.403	79
0.7	1.48310	1.83709	-3.551	-2.151	81
0.6	1.52472	1.84784	-2.849	-1.831	84
0.5	1.57193	1.86124	-2.194	-1.478	88

5102



$$a_2 \equiv 1.0$$

$$a_m \equiv \frac{a_2 + a_1}{2}$$

$$u = \cos \theta$$

$$H_X = H_0 \left[1 + \varepsilon_6 \left(\frac{r}{a_2} \right)^6 \rho_6(u) + \varepsilon_8 \left(\frac{r}{a_2} \right)^8 \rho_8(u) + \dots \right]$$

TABLE 1
TABLE OF ERROR COEFFICIENTS

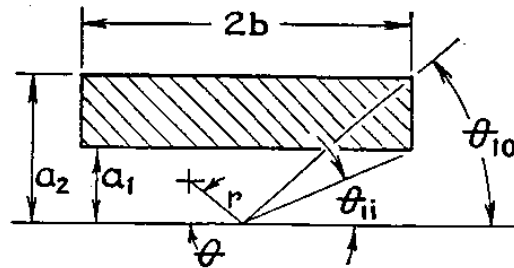
THE \mathcal{E} COEFFICIENTS ARE CONSTRUCTED FROM THE
FOLLOWING VARIABLES:

$$c_1 = \frac{1}{1+\beta^2} = \sin^2 \theta_{ii}$$

$$c_2 = \frac{\beta^2}{1+\beta^2} = \cos^2 \theta_{ii}$$

$$c_3 = \frac{\alpha^2}{\alpha^2 + \beta^2} = \sin^2 \theta_{10}$$

$$c_4 = \frac{\beta^2}{\alpha^2 + \beta^2} = \cos^2 \theta_{10}$$



$$\frac{a_2}{a_1} = \alpha$$

$$\frac{b}{a_1} = \beta$$

$$c_5 = \ln \left(\frac{\alpha + (\beta^2 + \alpha^2)^{1/2}}{1 + (\beta^2 + 1)^{1/2}} \right) = \ln \left(\frac{\tan(\pi/4 + \theta_{10}/2)}{\tan(\pi/4 + \theta_{ii}/2)} \right)$$

$$c_6 = \ln \left(\alpha \frac{\beta + (\beta^2 + 1)^{1/2}}{\beta + (\beta^2 + \alpha^2)^{1/2}} \right) = \ln \left(\frac{\tan(\theta_{10}/2)}{\tan(\theta_{ii}/2)} \right)$$

THE COEFFICIENTS ARE USED IN THE FOLLOWING SENSE:

$$H_z = H_0 \left(1 + \mathcal{E}_2 \left(\frac{r}{a_1} \right)^2 \rho_2(u) + \mathcal{E}_4 \left(\frac{r}{a_1} \right)^4 \rho_4(u) + \dots \right)$$

$$u = \cos \theta$$

TABLE 1 (continued)

CASE 1: CURRENT SHEET (I = AMP TURNS/cm.)

$$H(r, \theta) = \frac{4\pi I}{10} \cos \theta \left[1 + \epsilon_2 \left(\frac{r}{a} \right)^2 \rho_2(u) + \epsilon_4 \left(\frac{r}{a} \right)^4 \rho_4(u) + \dots \right]$$

$$\epsilon_2 = -\frac{3}{2} c_1^2$$

$$\epsilon_4 = -\frac{5}{8} (4c_1^2 - 7c_1^4)$$

$$\epsilon_6 = -\frac{1}{24} \left(\frac{693}{2} c_2^2 - 315c_2 + \frac{105}{2} \right)$$

CASE 2: UNIFORM CURRENT DENSITY SOLENOID

$$H(0,0) = G \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} = H_0$$

$$H(r, \theta) = H_0 \left(1 + \epsilon_2 \left(\frac{r}{a_1} \right)^2 \rho_2(u) + \epsilon_4 \left(\frac{r}{a_1} \right)^4 \rho_4(u) + \dots \right)$$

$$\epsilon_2 = \frac{1}{2\beta^2 c_5} \left(c_1^{3/2} - c_3^{3/2} \right)$$

$$\epsilon_4 = \frac{1}{12\beta^4 c_5} \left[c_1^{3/2} \left(1 + \frac{3}{2} c_2 + \frac{15}{2} c_2^2 \right) - c_3^{3/2} \left(1 + \frac{3}{2} c_4 + \frac{15}{2} c_4^2 \right) \right]$$

TABLE I

CASE 2 (continued)

$$\mathcal{E}_6 = \frac{1}{30\beta^6 c_5} \left[c_1^{3/2} \left(1 + \frac{3}{2} c_2 + \frac{15}{8} c_2^2 - \frac{35}{4} c_2^3 + \frac{315}{8} c_2^4 \right) \right. \\ \left. - c_3^{3/2} \left(1 + \frac{3}{2} c_4 + \frac{15}{8} c_4^2 - \frac{35}{4} c_4^3 + \frac{315}{8} c_4^4 \right) \right]$$

$$\mathcal{E}_8 = \frac{1}{56\beta^8 c_5} \left[c_1^{3/2} \left(1 + \frac{3}{2} c_2 + \frac{15}{8} c_2^2 + \frac{35}{16} c_2^3 + \frac{315}{16} c_2^4 \right. \right. \\ \left. \left. - \frac{2079}{16} c_2^5 + \frac{3003}{16} c_2^6 \right) \right. \\ \left. - c_3^{3/2} \left(1 + \frac{3}{2} c_4 + \frac{15}{8} c_4^2 + \frac{35}{16} c_4^3 + \frac{315}{16} c_4^4 \right. \right. \\ \left. \left. - \frac{2079}{16} c_4^5 + \frac{3003}{16} c_4^6 \right) \right]$$

TABLE 1 (continued)

CASE 3: RADIAL CURRENT DISTRIBUTION ($i = i_0/r$)

$$H(0,0) = G \left(\frac{W\lambda}{\rho a_1} \right)^{1/2} = H_0$$

$$H(r,\theta) = H_0 \left(1 + \varepsilon_2 \left(\frac{r}{a_1} \right)^2 \rho_2(u) + \varepsilon_4 \left(\frac{r}{a_1} \right)^4 \rho_4(u) + \dots \right)$$

$$\varepsilon_2 = \frac{1}{2\beta^2 C_6} \left(C_4^{3/2} - C_2^{3/2} \right)$$

$$\varepsilon_4 = \frac{1}{4\beta^4 C_6} \left[\frac{35}{14} \left(C_4^{7/2} - C_2^{7/2} \right) - \frac{21}{14} \left(C_4^{5/2} - C_2^{5/2} \right) \right]$$

$$\varepsilon_6 = \frac{1}{6\beta^6 C_6} \left[\frac{126}{16} \left(C_4^{11/2} - C_2^{11/2} \right) - \frac{140}{16} \left(C_4^{9/2} - C_2^{9/2} \right) + \frac{30}{16} \left(C_4^{7/2} - C_2^{7/2} \right) \right]$$

$$\begin{aligned} \varepsilon_8 = \frac{1}{8\beta^8 C_6} & \left[\frac{6235}{240} \left(C_4^{15/2} - C_2^{15/2} \right) - \frac{7161}{240} \left(C_4^{13/2} - C_2^{13/2} \right) \right. \\ & + \frac{315}{240} \left(C_4^{11/2} - C_2^{11/2} \right) - \frac{525}{240} \left(C_4^{9/2} - C_2^{9/2} \right) \\ & \left. - \frac{70}{240} \left(C_4^{7/2} - C_2^{7/2} \right) \right] \end{aligned}$$

TABLE 2

TABLE OF LEGENDRE POLYNOMIALS
AND THEIR DERIVATIVES (EVEN NO'S)

$$\rho_0(u) = 1 \quad u = \cos \theta \quad u' = -\sin \theta$$

$$\rho_0'(u) = 0$$

$$\rho_2(u) = \frac{1}{2}(3u^2 - 1)$$

$$\rho_2'(u) = \frac{1}{2}(6u) u'$$

$$\rho_4(u) = \frac{1}{8}(35u^4 - 30u^2 + 3)$$

$$\rho_4'(u) = \frac{1}{8}(140u^3 - 60u) u'$$

$$\rho_6(u) = \frac{1}{16}(231u^6 - 315u^4 + 105u^2 - 5)$$

$$\rho_6'(u) = \frac{1}{16}(1386u^5 - 1260u^3 + 210u) u'$$

$$\rho_8(u) = \frac{1}{128}(6435u^8 - 12,012u^6 + 6,930u^4 - 1,260u^2 + 35)$$

$$\rho_8'(u) = \frac{1}{128}(51,480u^7 - 72,072u^5 + 27,720u^3 - 2,520u) u'$$

SPACING OF HELMHOLTZ PAIRS

TABLE 3

TABULATED VALUES OF K VS X AND A

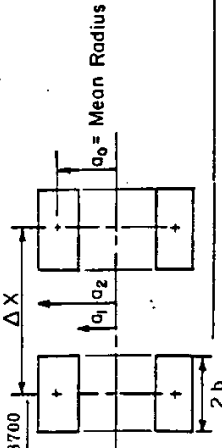
USED TO DETERMINE PROPER SPACING BETWEEN
HELMHOLTZ COILS FOR NO 2ND DERIVATIVE AT THE
ORIGIN. (UNIFORM CURRENT)

$$X = \frac{2b}{a_2 + a_1} = \frac{4\beta}{a+1}$$

$$K = \frac{\Delta X}{2a_0} = \frac{\Delta X}{a_1(1+\alpha)}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1} = \frac{2(a-1)}{a+1}$$

A \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
0.0	.50000	.50150	.50600	.51350	.52399	.53746	.55390	.57325	.59547	.62047	.64814	.67835	.71092	.74569	.78246	.82102	.86118	.90275	.94553
0.1	.49871	.50022	.50474	.51227	.52281	.53635	.55285	.57228	.59458	.61966	.64741	.67770	.71035	.74519	.78202	.82084	.86085	.90246	.94529
0.2	.49483	.49636	.50095	.50860	.51929	.53300	.54972	.56938	.59192	.61724	.64524	.67576	.70864	.74369	.78071	.81951	.85987	.90161	.94450
0.3	.48835	.48992	.49463	.50246	.51341	.52744	.54450	.56455	.58750	.61324	.64164	.67256	.70582	.74121	.77855	.81763	.85825	.90021	.94330
0.4	.47926	.48089	.48576	.49387	.50518	.51965	.53723	.55783	.58135	.60768	.63688	.66814	.70192	.73780	.77588	.81505	.85601	.89827	.94160
0.5	.46753	.46924	.47434	.48281	.49461	.50968	.52793	.54925	.57353	.60062	.63036	.66255	.69700	.73350	.77184	.81181	.85321	.89586	.94000
0.6	.45315	.45495	.46035	.46929	.48172	.49754	.51664	.53888	.56411	.59215	.62280	.65587	.69113	.72838	.76739	.80796	.84990	.89300	
0.7	.43608	.43801	.44378	.45332	.46653	.48329	.50344	.52680	.55318	.58235	.61410	.64820	.68442	.72254	.76233	.80359	.84610	.88970	
0.8	.41632	.41840	.42463	.43491	.44909	.46701	.48843	.51313	.54086	.57136	.60438	.63967	.67698	.71608	.75674	.79877	.84200	.88600	
0.9	.39385	.39613	.40293	.41412	.42949	.44880	.47174	.49802	.52733	.55936	.59382	.63044	.66895	.70913	.75074	.79360			
1.0	.36868	.37120	.37872	.39014	.40786	.42884	.45358	.48169	.51280	.54655	.58260	.62068	.66050	.70183	.74445	.78800			
1.1	.34085	.34368	.35210	.36580	.38439	.40736	.43420	.46442	.49755	.53319	.57098	.61061	.65181	.69435	.73800				
1.2	.31043	.31366	.32321	.33864	.35937	.38470	.41397	.44656	.48191	.51959	.55922	.60047	.64309	.68700					
1.3	.27755	.28129	.29229	.30989	.33321	.36133	.39336	.42856	.46631	.50613	.54760	.59050	.63500						
1.4	.24239	.24681	.25969	.28003	.30652	.33786	.37296	.41096	.45120	.49320									
1.5	.20518	.21054	.22597	.24981	.28009	.31510	.35351	.39440	.43710										
1.6	.16622	.17297	.19195	.22029	.25505	.29410	.33590	.38000											
1.7	.12585	.13480	.15890	.19300	.23300	.27600	.32100												
1.8	.08444	.08690	.12900	.17000	.21600														



AXIAL ERROR LIMITS

TABLE 4

TABULATION OF V VS X AND A

WHERE $V = z/a_0$ AND WHERE Z REPRESENTS
THE POINT AT WHICH THE FIELD DIFFERS FROM
THE CENTRAL FIELD BY 0.1 % (UNIFORM CURRENT)

$$X = \frac{2b}{a_2 + a_1} = \frac{4\beta}{a + 1}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1} = \frac{2(a - 1)}{a + 1}$$

A \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
0.0	.172	.172	.173	.176	.179	.183	.188	.193	.200	.207	.216	.224	.234	.244	.255	.266	.278	.290	.302
0.1	.171	.172	.173	.175	.178	.182	.187	.193	.200	.207	.215	.224	.234	.244	.254	.266	.277	.289	
0.2	.170	.171	.172	.174	.177	.181	.186	.192	.199	.206	.214	.223	.233	.243	.253	.265	.276	.288	
0.3	.168	.169	.170	.172	.176	.180	.185	.190	.197	.204	.212	.221	.231	.241	.252	.263	.274	.286	
0.4	.166	.166	.167	.170	.173	.177	.182	.188	.195	.202	.210	.219	.228	.238	.249	.260	.271		
0.5	.162	.163	.164	.166	.170	.174	.179	.185	.192	.199	.207	.216	.225	.235	.246	.257	.268		
0.6	.158	.158	.160	.162	.166	.170	.175	.181	.188	.195	.203	.212	.222	.231	.242	.252			
0.7	.153	.153	.155	.157	.161	.165	.170	.177	.183	.191	.199	.208	.217	.227	.237	.247			
0.8	.147	.148	.149	.152	.155	.160	.165	.171	.178	.186	.194	.203	.212	.221	.231				
0.9	.140	.141	.143	.145	.149	.154	.159	.165	.172	.180	.188	.197	.206	.215	.225				
1.0	.133	.133	.135	.138	.142	.147	.152	.159	.166	.173	.181	.190	.199	.208					
1.1	.124	.125	.127	.130	.134	.139	.145	.152	.159	.166	.174	.182	.191						
1.2	.115	.116	.118	.121	.126	.131	.137	.144	.151	.158	.166	.174							
1.3	.104	.105	.108	.111	.116	.122	.128	.135	.142	.149									
1.4	.093	.094	.097	.101	.106	.112	.119	.125	.132										
1.5	.081	.082	.085	.090	.096														
1.6	.067																		

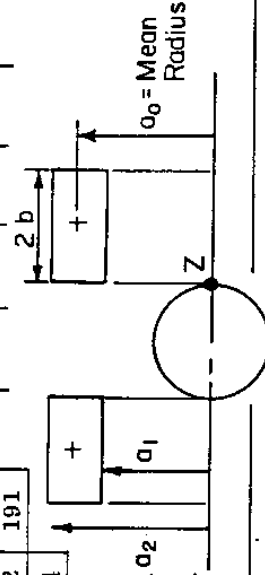


TABLE 5

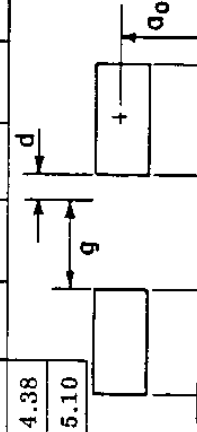
AXIAL DISPLACEMENT ERROR
COEFFICIENT Q FOR IMPROPERLY SPACED
HELMHOLTZ COILS (UNIFORM CURRENT)

$$H = H_0 \left(1 + Q \left(\frac{d}{a_0} \right) \left(\frac{z}{a_0} \right)^2 + \mathcal{E}_4 \left(\frac{z}{a_0} \right)^4 + \dots \right)$$

$$X = \frac{4b}{(a_2 + a_1)}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1}$$

$\frac{X}{A}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	3.84	3.83	3.77	3.68	3.56	3.42	3.27	3.11	2.95	2.79	2.64	2.51	2.38	2.27	2.17	2.09	2.02	1.95
0.1	3.86	3.84	3.78	3.69	3.57	3.43	3.28	3.12	2.96	2.80	2.65	2.52	2.39	2.28	2.18	2.10	2.03	1.96
0.2	3.90	3.88	3.82	3.73	3.61	3.47	3.31	3.15	2.99	2.83	2.68	2.55	2.42	2.31	2.21	2.13	2.05	1.99
0.3	3.98	3.96	3.90	3.80	3.68	3.54	3.38	3.21	3.05	2.89	2.74	2.60	2.47	2.36	2.26	2.17	2.10	2.03
0.4	4.09	4.07	4.01	3.91	3.78	3.63	3.47	3.30	3.13	2.97	2.81	2.67	2.54	2.43	2.33	2.24	2.16	2.09
0.5	4.24	4.22	4.16	4.06	3.92	3.77	3.60	3.42	3.24	3.07	2.92	2.77	2.64	2.52	2.42	2.33	2.26	
0.6	4.46	4.44	4.37	4.26	4.11	3.95	3.77	3.58	3.40	3.22	3.06	2.91	2.77	2.65	2.55	2.46	2.37	
0.7	4.75	4.72	4.64	4.52	4.36	4.18	3.99	3.79	3.59	3.41	3.24	3.08	2.94	2.82	2.71	2.62		
0.8	5.13	5.10	5.01	4.87	4.70	4.50	4.28	4.07	3.86	3.66	3.48	3.31	3.17	3.04	2.93			
0.9	5.64	5.61	5.51	5.35	5.14	4.91	4.67	4.43	4.20	3.99	3.79	3.62	3.46	3.33	3.21			
1.0	6.37	6.32	6.19	5.99	5.75	5.47	5.19	4.92	4.66	4.42	4.21	4.02	3.85	3.71	3.61			
1.1	7.38	7.32	7.15	6.89	6.58	6.24	5.91	5.58	5.29	5.02	4.78	4.57	4.38					
1.2	8.87	8.79	8.55	8.19	7.77	7.34	6.91	6.51	6.16	5.84	5.57	5.33	5.10					
1.3	11.2	11.1	10.7	10.1	9.54	8.93	8.37	7.87	7.43	7.05								
1.4	15.0	14.8	14.1	13.2	12.3	11.4	10.6	9.93	9.36									
1.5	21.9	21.4	20.1	18.4	16.8													



g = Proper Spacing
 d = Displacement Error
 a_0 = Mean Radius

TABLE 6

FOURTH ORDER ERROR

COEFFICIENTS \mathcal{E}_4 FOR PROPERLY SPACED
HELMHOLTZ COILS. (UNIFORM CURRENT)

$$H = H_0 \left(1 + \mathcal{E}_4 \left(\frac{z}{a_0} \right)^4 + \dots \right)$$

$$X = \frac{4b}{a_2 + a_1}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1}$$

$X \backslash A$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	1.15	1.14	1.10	1.05	.978	.895	.805	.714	.625	.541	.464	.395	.334	.282	.237	.200	.168	.142
0.1	1.16	1.15	1.11	1.06	.986	.902	.811	.719	.629	.544	.466	.397	.336	.284	.239	.201	.169	.142
0.2	1.19	1.18	1.14	1.09	1.01	.923	.830	.735	.642	.555	.475	.404	.342	.288	.243	.204	.172	.145
0.3	1.25	1.23	1.20	1.13	1.05	.961	.862	.762	.664	.573	.490	.417	.352	.297	.250	.210	.177	.149
0.4	1.33	1.32	1.27	1.20	1.12	1.02	.909	.802	.698	.601	.513	.435	.367	.309	.260	.219	.184	.153
0.5	1.45	1.43	1.38	1.30	1.21	1.09	.976	.857	.743	.638	.544	.460	.388	.326	.274	.231	.194	
0.6	1.61	1.59	1.53	1.44	1.33	1.20	1.06	.931	.805	.688	.584	.493	.415	.349	.295	.246	.207	
0.7	1.83	1.81	1.74	1.63	1.49	1.34	1.18	1.03	.885	.754	.636	.537	.451	.379	.318	.267		
0.8	2.14	2.11	2.02	1.89	1.72	1.53	1.34	1.16	.991	.840	.708	.594	.498	.417	.350			
0.9	2.58	2.54	2.42	2.25	2.03	1.79	1.56	1.33	1.13	.954	.800	.669	.560	.468	.392			
1.0	3.23	3.17	3.00	2.76	2.46	2.15	1.85	1.57	1.32	1.11	.922	.769	.641	.536	.452			
1.1	4.20	4.11	3.87	3.51	3.09	2.66	2.26	1.89	1.58	1.31	1.09	.904	.753					
1.2	5.76	5.62	5.22	4.66	4.03	3.41	2.84	2.35	1.94	1.60	1.32	1.09	.907					
1.3	8.42	8.16	7.45	6.49	5.48	4.53	3.71	3.02	2.47	2.02								
1.4	13.4	12.8	11.4	9.62	7.85	6.31	5.05	4.05	3.27									
1.5	23.7	22.4	19.1	15.3	12.0													

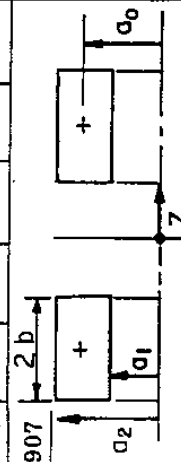


TABLE 7
SIXTH ORDER ERROR
COEFFICIENTS ε_6 FOR PROPERLY SPACED
HELMHOLTZ COILS (UNIFORM CURRENT)

$$H = H_0 \left(1 + \dots + \varepsilon_6 \left(\frac{z}{a_0} \right)^6 + \dots \right)$$
[illegible]

TABLE 9

Sixth Order Long Solenoids: $a_1 = 6 \text{ cm}$, $a_2 = 12 \text{ cm}$, $b = 36 \text{ cm}$

Cancelled Second & Fourth Derivatives ($H = H_0(1 + \epsilon_2 Z^2 + \epsilon_4 Z^4)$
 $\epsilon_2 = .607 \times 10^{-2} \%$, $\epsilon_4 = .640 \times 10^{-5} \%$

$i \propto \frac{1}{\gamma}$: $a_1 = 6 \text{ cm}$ $a_2 = 12 \text{ cm}$ $b = 36 \text{ cm}$

	Current sheet on I.D.	Current sheet on O.D.	One notch on O.D.	Uniformly reduced current in central region ends on I.D.	Two separate sheets placed at ends on I.D.
—	$\frac{I}{I_1} = -207$ (-0.45%)	$\frac{I}{I_1} = -41$ (-12.76%)	$\frac{I}{I_1} = -1$ (-12.94%)	$\frac{I}{I_1} = -71.5$ (-11.01%)	$\frac{I}{I_1} = +207$ (+0.27%)
$(.536 \times 10^{-9}) Z^6 (\%)$	$(.758 \times 10^{-6}) Z^6 (\%)$	$(.303 \times 10^{-8}) Z^6 (\%)$	$(.484 \times 10^{-6}) Z^6 (\%)$	$(1.84 \times 10^{-4}) Z^6 (\%)$	$(.758 \times 10^{-6}) Z^6 (\%)$

Change in field
at center of
solenoid on axis
in % of main field
(-): decrease in field
(+): increase in field

6th order error
due to correction
coil in % of main
field at center

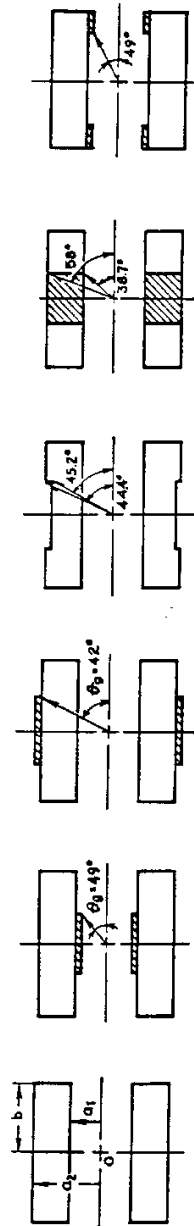
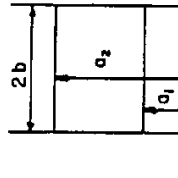


TABLE 10A
ERROR COEFFICIENTS AND GEOMETRY FACTORS
FOR UNIFORM CURRENT DENSITY VS α AND β .

$$H = H_0 \left(1 + D_2 \left(\frac{z}{a_1} \right)^2 R_2(u) + D_4 \left(\frac{z}{a_1} \right)^4 P_4(u) + \dots \right)$$

NOTE: E-03 EXPONENT = 10^{-3}

ALPHA	BETA	D2	D4	D6	D8	G1	G102	G104	G106	G108
2	5	-0.6068E-00	0.2612E-00	-0.6392E-01	-0.1640E-01	0.1335E-00	-0.8086E-01	0.3480E-01	-0.8518E-02	-0.2185E-02
3	5	-0.4460E-00	0.1793E-00	-0.4264E-01	-0.3738E-02	0.1314E-00	-0.6121E-01	0.2356E-01	-0.5602E-02	-0.1279E-02
4	5	-0.3925E-00	0.1436E-00	-0.3378E-01	-0.7634E-02	0.1220E-00	-0.4786E-01	0.1752E-01	-0.4120E-02	-0.9316E-03
5	5	-0.3469E-00	0.1238E-00	-0.2899E-01	-0.6545E-02	0.1125E-00	-0.3902E-01	0.1392E-01	-0.3262E-02	-0.7363E-03
2	1	-0.3219E-00	0.2027E-01	0.1713E-01	-0.2553E-02	0.1627E-00	-0.5239E-01	0.3299E-01	0.2788E-02	-0.4156E-03
3	1	-0.2669E-00	0.2096E-01	0.9748E-02	-0.1589E-02	0.1661E-00	-0.4433E-01	0.3482E-02	0.1619E-02	-0.2631E-03
4	1	-0.2306E-00	0.1815E-01	0.7402E-02	-0.1210E-02	0.1571E-00	-0.3621E-01	0.2851E-02	0.1163E-02	-0.1917E-03
5	1	-0.2059E-00	0.1595E-01	0.6250E-02	-0.1031E-02	0.1444E-00	-0.3015E-01	0.2336E-02	0.9153E-03	-0.1510E-03
2	2	-0.8250E-01	-0.8532E-02	0.1017E-03	0.1396E-03	0.1638E-00	-0.1351E-01	0.1398E-02	0.1665E-04	0.2284E-04
3	2	-0.8524E-01	-0.4561E-02	0.2520E-03	0.7333E-04	0.1789E-00	-0.1525E-01	-0.8157E-03	0.4508E-04	0.1312E-04
4	2	-0.8542E-01	-0.2070E-02	0.1986E-03	0.1986E-03	0.1762E-00	-0.1433E-01	-0.4946E-03	0.3500E-04	0.8194E-05
5	2	-0.8132E-01	-0.2028E-02	0.1601E-03	0.4291E-04	0.1687E-00	-0.1286E-01	-0.3423E-03	0.2701E-04	0.1240E-05
2	3	-0.7259E-01	-0.2402E-02	-0.4377E-04	0.1925E-05	0.1492E-00	-0.3873E-02	-0.3585E-03	0.1861E-04	-0.1200E-05
3	3	-0.3229E-01	-0.1886E-02	-0.1175E-04	0.1298E-05	0.1791E-00	-0.5491E-02	-0.3206E-03	0.7443E-05	-0.3273E-06
4	3	-0.3479E-01	-0.1017E-02	-0.1175E-04	0.9240E-06	0.1695E-00	-0.5893E-02	-0.2352E-03	0.3126E-05	0.2242E-06
5	3	-0.1005E-01	-0.6920E-03	-0.3311E-04	-0.1074E-05	0.1353E-00	-0.1360E-02	-0.9346E-04	-0.4481E-05	-0.1835E-06
2	4	-0.1415E-01	-0.6995E-03	-0.2056E-04	-0.2459E-06	0.1580E-00	-0.2745E-02	-0.1105E-03	-0.3248E-05	-0.3889E-07
3	4	-0.1673E-01	-0.4758E-03	-0.1121E-04	-0.5149E-07	0.1438E-00	-0.2745E-02	-0.7792E-04	-0.1639E-05	-0.8515E-08
4	4	-0.4566E-02	-0.2315E-03	-0.8796E-05	-0.4398E-07	0.1238E-00	-0.3634E-03	-0.2867E-04	-0.1138E-05	-0.7202E-08
5	4	-0.6931E-02	-0.2766E-03	-0.7444E-05	-0.2648E-06	0.1487E-00	-0.1020E-02	-0.4050E-04	-0.1089E-05	-0.3279E-07
2	5	-0.8848E-02	-0.2686E-03	-0.5007E-05	-0.1327E-06	0.1346E-00	-0.1368E-02	-0.4152E-04	-0.1092E-05	-0.1950E-07
3	5	-0.2331E-01	-0.2365E-03	-0.3233E-05	-0.5276E-07	0.1562E-00	-0.1584E-02	-0.3694E-04	-0.7739E-06	-0.8155E-08
4	5	-0.3744E-02	-0.8914E-04	-0.2644E-05	-0.6574E-07	0.1145E-00	-0.2876E-03	-0.1021E-04	-0.3028E-06	-0.4508E-08
5	5	-0.5021E-02	-0.1190E-03	-0.2733E-05	-0.2516E-07	0.1369E-00	-0.5125E-03	-0.1629E-04	-0.3741E-06	-0.7529E-08
2	6	-0.6045E-02	-0.1234E-03	-0.2193E-05	-0.1328E-07	0.1436E-00	-0.7320E-03	-0.1874E-04	-0.3194E-06	-0.6498E-08
3	6	-0.1309E-02	-0.3855E-04	-0.9018E-06	-0.1819E-07	0.1485E-00	-0.8980E-03	-0.1833E-04	-0.2342E-06	-0.1973E-08
4	6	-0.2171E-02	-0.5553E-04	-0.1067E-05	-0.1650E-07	0.1285E-00	-0.1399E-03	-0.4121E-05	-0.9440E-07	-0.1945E-08
5	6	-0.3030E-02	-0.6508E-04	-0.9830E-06	-0.1093E-07	0.1413E-00	-0.4169E-03	-0.8955E-05	-0.1372E-06	-0.2121E-08
2	7	-0.7846E-03	-0.6715E-04	-0.7865E-06	-0.6362E-08	0.1005E-00	-0.7904E-04	-0.9489E-05	-0.1111E-06	-0.8991E-09
3	7	-0.1336E-02	-0.1832E-04	-0.3437E-06	-0.5654E-08	0.1213E-00	-0.1621E-03	-0.4842E-05	-0.3655E-07	-0.6484E-09
4	7	-0.1918E-02	-0.2784E-04	-0.4473E-06	-0.5987E-08	0.1305E-00	-0.2504E-03	-0.3378E-05	-0.5428E-07	-0.7292E-09
5	7	-0.2467E-02	-0.3467E-04	-0.4576E-06	-0.4698E-08	0.1347E-00	-0.3323E-03	-0.5113E-05	-0.5473E-07	-0.8132E-09
2	8	-0.4993E-03	-0.9406E-05	-0.1440E-06	-0.1952E-08	0.9512E-01	-0.4750E-04	-0.6948E-06	-0.1169E-07	-0.1835E-09
3	8	-0.6630E-03	-0.1495E-04	-0.2006E-06	-0.2308E-08	0.1152E-00	-0.9318E-04	-0.1710E-05	-0.2310E-07	-0.2638E-09
4	8	-0.1266E-02	-0.1937E-04	-0.2224E-06	-0.2606E-08	0.1288E-00	-0.1357E-03	-0.2407E-05	-0.2164E-07	-0.1974E-09
5	8	-0.1666E-02	-0.2223E-04	-0.2119E-06	-0.1533E-08	0.1288E-00	-0.1357E-03	-0.2407E-05	-0.2164E-07	-0.1974E-09
2	9	-0.3317E-03	-0.5145E-05	-0.6526E-07	-0.1389E-09	0.8048E-01	-0.3001E-04	-0.4655E-06	-0.5905E-08	-0.6685E-10
3	9	-0.5806E-03	-0.8360E-05	-0.9568E-07	-0.9477E-09	0.1088E-01	-0.6373E-04	-0.9176E-06	-0.1050E-07	-0.1040E-09
4	9	-0.8656E-03	-0.1129E-04	-0.1129E-06	-0.9477E-09	0.1088E-01	-0.6373E-04	-0.9176E-06	-0.1050E-07	-0.1040E-09
5	9	-0.1140E-02	-0.1347E-04	-0.1147E-06	-0.7682E-09	0.1234E-00	-0.1431E-03	-0.1642E-05	-0.1416E-07	-0.9480E-10



$$\alpha = a_2/a_1$$

$$\beta = b/a_1$$

$$I = \text{Constant}$$

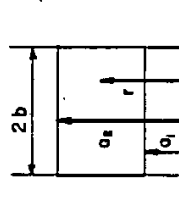
TABLE 10B

ERROR COEFFICIENTS AND GEOMETRY FACTORS
FOR RADIAL CURRENT DENSITY $j = \left(\frac{1}{r} \right) P_4(u) + \dots$
VS. α AND β

$$H = H_0 \left(1 + A_2 \left(\frac{z}{a_1} \right)^2 P_2(u) + A \left(\frac{z}{a_1} \right)^4 P_4(u) + \dots \right)$$

ALPHA BETA

A2	A4	A6	A8	G2	G2A2	G2A4	G2A6	G2A8
-0.6432E-00	0.2867E-00	-0.6871E-01	0.4245E-00	0.1408E-00	-0.9053E-01	0.4035E-01	-0.9671E-02	0.5975E-01
-0.5392E-00	0.2248E-00	-0.5307E-01	0.3167E-00	0.1508E-00	-0.8131E-01	0.3390E-01	-0.8004E-02	0.4776E-01
-0.4910E-00	0.2002E-00	-0.4707E-01	0.2801E-00	0.1518E-00	-0.7454E-01	0.3039E-01	-0.7146E-02	0.4253E-01
-0.4639E-00	0.1875E-00	-0.4402E-01	0.2619E-00	0.1507E-00	-0.6991E-01	0.2825E-01	-0.6635E-02	0.3947E-01
-0.4300E-00	0.1643E-01	0.2021E-01	0.8493E-01	0.1704E-00	-0.5623E-01	0.2600E-02	0.3644E-02	0.1447E-01
-0.2906E-00	0.1643E-01	0.1423E-01	0.8202E-01	0.1873E-00	-0.5444E-01	0.3602E-02	0.2668E-02	0.1162E-01
-0.2676E-00	0.1698E-01	0.1234E-01	0.5422E-01	0.1909E-00	-0.5108E-01	0.3241E-02	0.2360E-02	0.1035E-01
-0.2534E-00	0.1603E-01	0.1147E-01	0.5035E-01	0.1908E-00	-0.4834E-01	0.3059E-02	0.2188E-02	0.9609E-02
-0.8048E-01	-0.9182E-02	-0.3120E-03	0.1947E-02	0.1693E-00	-0.1362E-01	-0.1554E-02	-0.6297E-05	0.3296E-03
-0.8312E-01	-0.4937E-02	0.1202E-03	0.1487E-02	0.2049E-00	-0.1629E-01	-0.1213E-02	0.2352E-04	0.2911E-03
-0.7881E-01	-0.4366E-02	0.9818E-04	0.1166E-02	0.2082E-00	-0.1666E-01	-0.1012E-02	0.2245E-04	0.2613E-03
-0.2474E-01	-0.2397E-02	0.1373E-03	0.7170E-04	0.1533E-00	-0.1641E-01	-0.9088E-03	0.2044E-04	0.2428E-03
-0.2986E-01	-0.2044E-02	-0.7536E-04	0.6974E-04	0.1830E-00	-0.3794E-02	-0.3674E-03	-0.2106E-04	0.1099E-04
-0.3169E-01	-0.1709E-02	-0.5466E-04	0.6185E-04	0.1956E-00	-0.5427E-02	-0.3740E-03	-0.1379E-04	0.1276E-04
-0.3190E-01	-0.1498E-02	-0.4773E-04	0.5631E-04	0.2016E-00	-0.6432E-02	-0.3020E-03	-0.1609E-04	0.1210E-04
-0.9481E-02	-0.6701E-03	-0.3354E-04	0.4533E-05	0.1386E-00	-0.1314E-02	-0.9289E-04	-0.4650E-05	0.1135E-04
-0.1258E-01	-0.6865E-03	-0.2461E-04	0.5984E-05	0.1884E-00	-0.2119E-02	-0.1156E-03	-0.4146E-05	0.1008E-05
-0.1441E-01	-0.6255E-03	-0.1827E-04	0.5956E-05	0.1827E-00	-0.2632E-02	-0.1143E-03	-0.3337E-05	0.1088E-05
-0.1536E-01	-0.5613E-03	-0.1525E-04	0.5582E-05	0.1903E-00	-0.2922E-02	-0.1068E-03	-0.2902E-05	0.1063E-05
-0.4283E-02	-0.2209E-03	-0.8620E-05	0.4312E-06	0.1267E-00	-0.5425E-03	-0.2798E-04	-0.1092E-05	0.5461E-07
-0.6073E-02	-0.2577E-03	-0.7176E-05	0.7453E-06	0.1556E-00	-0.9449E-03	-0.4009E-04	-0.1210E-05	0.1140E-06
-0.7380E-02	-0.2569E-03	-0.6253E-05	0.8680E-06	0.1703E-00	-0.1257E-02	-0.4375E-04	-0.1265E-05	0.1679E-06
-0.8236E-02	-0.2414E-03	-0.5174E-05	0.8717E-06	0.1788E-00	-0.1473E-02	-0.4317E-04	-0.9253E-06	0.1559E-06
-0.2195E-02	-0.8434E-04	-0.2846E-05	0.5558E-07	0.1170E-00	-0.2557E-03	-0.9871E-05	-0.2880E-06	0.4505E-08
-0.3236E-02	-0.1076E-03	-0.2354E-05	0.1635E-06	0.1447E-00	-0.4683E-03	-0.1551E-04	-0.3851E-06	0.1738E-07
-0.4109E-02	-0.1158E-03	-0.2354E-05	0.1803E-06	0.1594E-00	-0.6550E-03	-0.1846E-04	-0.3752E-06	0.2601E-07
-0.4765E-02	-0.1147E-03	-0.2005E-05	0.1803E-06	0.1684E-00	-0.8023E-03	-0.1930E-04	-0.3373E-06	0.3035E-07
-0.1221E-02	-0.3629E-04	-0.8598E-06	0.8960E-08	0.1092E-00	-0.1933E-03	-0.5962E-05	-0.9387E-07	0.9783E-09
-0.1863E-02	-0.4925E-04	-0.9776E-06	0.2330E-07	0.1356E-00	-0.2526E-03	-0.6677E-05	-0.1353E-06	0.3187E-08
-0.2443E-02	-0.5635E-04	-0.9826E-06	0.3684E-07	0.1500E-00	-0.3665E-03	-0.8453E-05	-0.1444E-06	0.5526E-08
-0.2922E-02	-0.5861E-04	-0.8600E-06	0.4483E-07	0.1591E-00	-0.4648E-03	-0.9323E-05	-0.1368E-06	0.7131E-08
-0.3727E-02	-0.1719E-04	-0.3256E-06	0.1694E-08	0.1026E-00	-0.7519E-04	-0.1764E-05	-0.3342E-07	0.1738E-09
-0.1141E-02	-0.2439E-04	-0.4074E-06	0.5365E-08	0.1278E-00	-0.1458E-03	-0.3117E-05	-0.5207E-07	0.6859E-09
-0.1533E-02	-0.2925E-04	-0.4226E-06	0.9515E-08	0.1419E-00	-0.2174E-03	-0.4150E-05	-0.5995E-07	0.1350E-08
-0.1878E-02	-0.3174E-04	-0.3978E-06	0.1272E-07	0.1509E-00	-0.2834E-03	-0.4790E-05	-0.6003E-07	0.1919E-08
-0.4648E-03	-0.8803E-05	-0.1357E-06	0.3539E-09	0.9709E-01	-0.4513E-04	-0.8547E-06	-0.1318E-07	0.3436E-10
-0.7347E-03	-0.1290E-04	-0.1795E-06	0.1385E-08	0.1212E-00	-0.8904E-04	-0.1563E-05	-0.2176E-07	0.1678E-09
-0.1005E-02	-0.1605E-04	-0.1974E-06	0.2745E-08	0.1348E-00	-0.1355E-03	-0.2164E-05	-0.2661E-07	0.3701E-09
-0.1255E-02	-0.1805E-04	-0.1952E-06	0.3997E-08	0.1437E-00	-0.1804E-03	-0.2593E-05	-0.2805E-07	0.5744E-09
-0.3085E-03	-0.4807E-05	-0.6134E-07	0.7605E-10	0.9234E-01	-0.2849E-04	-0.4439E-06	-0.5664E-08	0.7023E-11
-0.4931E-03	-0.7216E-05	-0.8458E-07	0.3940E-09	0.1154E-00	-0.5692E-04	-0.8339E-06	-0.9764E-08	0.4548E-10
-0.6838E-03	-0.9238E-05	-0.9745E-07	0.8679E-09	0.1286E-00	-0.8796E-04	-0.1188E-05	-0.1254E-07	0.1116E-09
-0.8673E-03	-0.1070E-04	-0.1006E-06	0.1366E-08	0.1374E-00	-0.1191E-03	-0.1469E-05	-0.1382E-07	0.1876E-09



SIXTH ERROR COEFFICIENTS

SMALL β 's. ($0.1 < \beta < 3$)UNIFORM CURRENT: $H = H_0 \left(1 + \dots + D_6 \left(\frac{Z}{a_1} \right)^6 P_6(u) + \dots \right)$ RADIAL CURRENT: $H = H_0 \left(1 + \dots + A_6 \left(\frac{Z}{a_1} \right)^6 P_6(u) + \dots \right)$

ALPHA	BETA	D ₆	A ₆	G ₁	G ₂	G ₁₀₆	G _{2A6}
0.11E 01	0.20E-00	-0.114744E 01	-0.115192E 01	0.458034E-01	0.458542E-01	-0.528566E-01	-0.528202E-01
0.11E 01	0.40E-00	-0.327958E-00	-0.328502E-00	0.616108E-01	0.616748E-01	-0.202057E-01	-0.202603E-01
0.11E 01	0.60E 00	0.429624E-01	0.435002E-01	0.700961E-01	0.701627E-01	0.301150E-02	0.305209E-02
0.11E 01	0.80E 00	0.825581E-01	0.828536E-01	0.741416E-01	0.742052E-01	0.612099E-02	0.614816E-02
0.11E 01	1.00E 00	0.444103E-01	0.444703E-01	0.754556E-01	0.755141E-01	0.335101E-02	0.335814E-02
0.11E 01	0.12E 01	0.160882E-01	0.160742E-01	0.751609E-01	0.752140E-01	0.120920E-02	0.120901E-02
0.11E 01	0.14E 01	0.368124E-02	0.366027E-02	0.739675E-01	0.740156E-01	0.272292E-03	0.2720917E-03
0.11E 01	0.16E 01	-0.433870E-03	-0.447750E-03	0.723502E-01	0.723502E-01	-0.313716E-04	-0.323948E-04
0.11E 01	0.18E 01	-0.134460E-02	-0.135209E-02	0.704315E-01	0.704716E-01	-0.947023E-04	-0.952843E-04
0.11E 01	0.20E 01	-0.125399E-02	-0.125769E-02	0.684891E-01	0.685262E-01	-0.858849E-04	-0.861845E-04
0.11E 01	0.22E 01	-0.949194E-03	-0.950915E-03	0.665616E-01	0.665961E-01	-0.631798E-04	-0.633273E-04
0.11E 01	0.24E 01	-0.666185E-03	-0.666947E-03	0.646936E-01	0.647259E-01	-0.430979E-04	-0.431688E-04
0.11E 01	0.26E 01	-0.453687E-03	-0.453998E-03	0.629076E-01	0.629381E-01	-0.285403E-04	-0.285738E-04
0.11E 01	0.28E 01	-0.305942E-03	-0.306049E-03	0.612135E-01	0.612424E-01	-0.187478E-04	-0.187432E-04
0.11E 01	0.30E 01	-0.206385E-03	-0.206405E-03	0.596135E-01	0.596410E-01	-0.123102E-04	-0.123102E-04
0.30E 01	0.20E-00	-0.243637E-00	-0.340836E-00	0.863901E-01	0.998805E-01	-0.210478E-01	-0.340429E-01
0.30E 01	0.40E-00	-0.543680E-01	-0.125969E-00	0.119413E-00	0.137482E-00	-0.112687E-01	-0.173185E-01
0.30E 01	0.60E 00	-0.112737E-01	-0.100209E-01	0.141226E-00	0.161610E-00	-0.159215E-02	-0.161947E-02
0.30E 01	0.80E 00	0.103659E-01	0.172626E-01	0.156097E-00	0.177366E-00	0.161808E-02	0.306180E-02
0.30E 01	1.00E 00	0.974784E-02	0.124310E-01	0.166092E-00	0.187341E-02	0.161904E-02	0.266605E-02
0.30E 01	0.12E 01	0.577425E-02	0.776106E-02	0.172561E-00	0.193252E-00	0.996410E-03	0.149984E-02
0.30E 01	0.14E 01	0.294275E-02	0.359257E-02	0.176446E-00	0.196308E-00	0.519289E-03	0.705248E-03
0.30E 01	0.16E 01	0.139454E-02	0.148599E-02	0.178497E-00	0.197366E-00	0.248921E-03	0.293284E-03
0.30E 01	0.18E 01	0.622447E-03	0.527236E-03	0.179171E-00	0.197033E-00	0.11524E-03	0.103883E-03
0.30E 01	0.20E 01	0.252046E-03	0.120160E-03	0.178861E-00	0.195741E-00	0.450815E-04	0.235201E-04
0.30E 01	0.22E 01	0.787438E-04	-0.386301E-04	0.177843E-00	0.193794E-00	0.140040E-04	-0.748627E-05
0.30E 01	0.24E 01	0.527543E-06	-0.902879E-04	0.174320E-00	0.191409E-00	0.930165E-07	-0.172819E-04
0.30E 01	0.26E 01	-0.320600E-04	-0.979137E-04	0.174445E-00	0.188743E-00	-0.559270E-05	-0.184805E-04
0.30E 01	0.28E 01	-0.429428E-04	-0.890411E-04	0.172328E-00	0.185906E-00	-0.740024E-05	-0.165532E-04
0.30E 01	0.30E 01	-0.437708E-04	-0.753634E-04	0.170053E-00	0.182977E-00	-0.744337E-05	-0.137897E-04
0.50E 01	0.20E-00	-0.166245E-00	-0.283817E-00	0.732226E-01	0.991589E-01	-0.121729E-01	-0.281430E-01
0.50E 01	0.40E-00	-0.641562E-01	-0.104585E-00	0.101821E-00	0.137015E-00	-0.653269E-02	-0.143297E-01
0.50E 01	0.60E 00	-0.778613E-02	-0.839155E-02	0.121531E-00	0.162012E-00	-0.946502E-03	-0.135953E-02
0.50E 01	0.80E 00	0.669620E-02	0.140163E-01	0.135873E-00	0.179117E-00	0.909831E-03	0.251057E-02
0.50E 01	1.00E 00	0.625015E-02	0.114679E-01	0.146445E-00	0.190760E-00	0.915302E-03	0.218761E-02
0.50E 01	0.12E 01	0.363998E-02	0.618905E-02	0.154235E-00	0.198509E-00	0.561414E-03	0.122858E-02
0.50E 01	0.14E 01	0.182079E-02	0.263399E-02	0.159928E-00	0.203455E-00	0.291196E-03	0.576263E-03
0.50E 01	0.16E 01	0.850042E-03	0.115950E-02	0.164017E-00	0.206372E-00	0.139421E-03	0.239289E-03
0.50E 01	0.18E 01	0.379212E-03	0.410272E-03	0.166865E-00	0.207813E-00	0.632772E-04	0.852599E-04
0.50E 01	0.20E 01	0.160062E-03	0.981818E-04	0.168747E-00	0.208175E-00	0.270099E-04	0.204390E-04
0.50E 01	0.22E 01	0.604136E-04	-0.207735E-04	0.169870E-00	0.207746E-00	0.102625E-04	-0.431561E-05
0.50E 01	0.24E 01	0.162126E-04	-0.583425E-04	0.170396E-00	0.206737E-00	0.276256E-05	-0.120615E-04
0.50E 01	0.26E 01	-0.253418E-05	-0.635182E-04	0.170449E-00	0.205303E-00	-0.431948E-06	-0.130405E-04
0.50E 01	0.28E 01	-0.971601E-05	-0.571566E-04	0.170127E-00	0.203561E-00	-0.165295E-05	-0.116306E-04
0.50E 01	0.30E 01	-0.117469E-04	-0.477299E-04	0.169508E-00	0.201598E-00	-0.199119E-05	-0.962224E-05

TABLE 11D
EIGHTH ERROR COEFFICIENT ...
SMALL β 's. (0.1 β 4.0)

$$\begin{aligned} \text{UNIFORM CURRENT, } H &= H_0 \left(1 + \dots + D_8 \left(\frac{z}{a_1} \right)^8 P_8(u) + \dots \right) \\ \text{RADIAL CURRENT, } H &= H_0 \left(1 + \dots + A_8 \left(\frac{z}{a_1} \right)^8 P_8(u) + \dots \right) \end{aligned}$$

ALPHA	BETA	00	A0	G1	G2	G100
-0.11E 01	-0.20E-00	-0.55943E 00	-0.45337E 01	-0.635034E-01	-0.635034E-01	-0.424115E-01
-0.11E 01	-0.40E-00	-0.51749E 00	-0.42007E 01	-0.616749E-01	-0.616749E-01	-0.434923E-01
-0.11E 01	-0.60E-00	-0.48202E 00	-0.38801E 01	-0.600611E-01	-0.600611E-01	-0.450296E-01
-0.11E 01	-0.80E-00	-0.44649E 00	-0.35444E 00	-0.585452E-01	-0.585452E-01	-0.466969E-01
-0.11E 01	-1.00E-00	-0.422601E-01	-0.33244E-01	-0.572037E-01	-0.572037E-01	-0.483702E-01
-0.11E 01	-1.20E-00	-0.405959E-02	-0.316324E-01	-0.561141E-01	-0.561141E-01	-0.501399E-01
-0.11E 01	-1.40E-00	-0.395344E-02	-0.303246E-01	-0.551609E-01	-0.551609E-01	-0.520166E-02
-0.11E 01	-1.60E-00	-0.387674E-02	-0.292345E-01	-0.543173E-01	-0.543173E-01	-0.539815E-02
-0.11E 01	-1.80E-00	-0.381239E-02	-0.283505E-01	-0.535813E-01	-0.535813E-01	-0.560485E-02
-0.11E 01	-2.00E-00	-0.376760E-01	-0.276491E-01	-0.529435E-01	-0.529435E-01	-0.577103E-01
-0.11E 01	-2.20E-00	-0.373701E-04	-0.271379E-04	-0.524515E-01	-0.524515E-01	-0.577103E-03
-0.11E 01	-2.40E-00	-0.371745E-04	-0.268072E-02	-0.520677E-02	-0.520677E-02	-0.577103E-03
-0.11E 01	-2.60E-00	-0.370319E-04	-0.265949E-04	-0.518546E-01	-0.518546E-01	-0.577103E-03
-0.11E 01	-2.80E-00	-0.369376E-04	-0.264694E-04	-0.516801E-01	-0.516801E-01	-0.577103E-03
-0.11E 01	-3.00E-00	-0.368874E-04	-0.264030E-04	-0.515351E-01	-0.515351E-01	-0.577103E-03
-0.11E 01	-3.20E-00	-0.368590E-04	-0.263540E-04	-0.514135E-01	-0.514135E-01	-0.577103E-03
-0.11E 01	-3.40E-00	-0.368280E-04	-0.262967E-04	-0.512291E-01	-0.512291E-01	-0.577103E-03
-0.11E 01	-3.60E-00	-0.368220E-04	-0.262822E-04	-0.511616E-01	-0.511616E-01	-0.577103E-03
-0.11E 01	-3.80E-00	-0.368210E-04	-0.262760E-04	-0.511044E-01	-0.511044E-01	-0.577103E-03
-0.11E 01	-4.00E-00	-0.368190E-04	-0.262700E-04	-0.510566E-01	-0.510566E-01	-0.577103E-03
-0.11E 01	-4.20E-00	-0.368170E-04	-0.262640E-04	-0.510173E-01	-0.510173E-01	-0.577103E-03
-0.11E 01	-4.40E-00	-0.368150E-04	-0.262580E-04	-0.509860E-01	-0.509860E-01	-0.577103E-03
-0.11E 01	-4.60E-00	-0.368130E-04	-0.262520E-04	-0.509623E-01	-0.509623E-01	-0.577103E-03
-0.11E 01	-4.80E-00	-0.368110E-04	-0.262460E-04	-0.509458E-01	-0.509458E-01	-0.577103E-03
-0.11E 01	-5.00E-00	-0.368090E-04	-0.262400E-04	-0.509360E-01	-0.509360E-01	-0.577103E-03
-0.11E 01	-5.20E-00	-0.368070E-04	-0.262340E-04	-0.509326E-01	-0.509326E-01	-0.577103E-03
-0.11E 01	-5.40E-00	-0.368050E-04	-0.262280E-04	-0.509439E-01	-0.509439E-01	-0.577103E-03
-0.11E 01	-5.60E-00	-0.368030E-04	-0.262220E-04	-0.509578E-01	-0.509578E-01	-0.577103E-03
-0.11E 01	-5.80E-00	-0.368010E-04	-0.262160E-04	-0.509770E-01	-0.509770E-01	-0.577103E-03
-0.11E 01	-6.00E-00	-0.367990E-04	-0.262100E-04	-0.509999E-01	-0.509999E-01	-0.577103E-03
-0.11E 01	-6.20E-00	-0.367980E-04	-0.262040E-04	-0.510355E-01	-0.510355E-01	-0.577103E-03
-0.11E 01	-6.40E-00	-0.367970E-04	-0.261980E-04	-0.510841E-01	-0.510841E-01	-0.577103E-03
-0.11E 01	-6.60E-00	-0.367960E-04	-0.261920E-04	-0.511452E-01	-0.511452E-01	-0.577103E-03
-0.11E 01	-6.80E-00	-0.367950E-04	-0.261860E-04	-0.512111E-01	-0.512111E-01	-0.577103E-03
-0.11E 01	-7.00E-00	-0.367940E-04	-0.261800E-04	-0.512811E-01	-0.512811E-01	-0.577103E-03
-0.11E 01	-7.20E-00	-0.367930E-04	-0.261740E-04	-0.513551E-01	-0.513551E-01	-0.577103E-03
-0.11E 01	-7.40E-00	-0.367920E-04	-0.261680E-04	-0.514331E-01	-0.514331E-01	-0.577103E-03
-0.11E 01	-7.60E-00	-0.367910E-04	-0.261620E-04	-0.515151E-01	-0.515151E-01	-0.577103E-03
-0.11E 01	-7.80E-00	-0.367900E-04	-0.261560E-04	-0.516011E-01	-0.516011E-01	-0.577103E-03
-0.11E 01	-8.00E-00	-0.367890E-04	-0.261500E-04	-0.516911E-01	-0.516911E-01	-0.577103E-03
-0.11E 01	-8.20E-00	-0.367880E-04	-0.261440E-04	-0.517851E-01	-0.517851E-01	-0.577103E-03
-0.11E 01	-8.40E-00	-0.367870E-04	-0.261380E-04	-0.518831E-01	-0.518831E-01	-0.577103E-03
-0.11E 01	-8.60E-00	-0.367860E-04	-0.261320E-04	-0.519851E-01	-0.519851E-01	-0.577103E-03
-0.11E 01	-8.80E-00	-0.367850E-04	-0.261260E-04	-0.520911E-01	-0.520911E-01	-0.577103E-03
-0.11E 01	-9.00E-00	-0.367840E-04	-0.261200E-04	-0.522011E-01	-0.522011E-01	-0.577103E-03
-0.11E 01	-9.20E-00	-0.367830E-04	-0.261140E-04	-0.523151E-01	-0.523151E-01	-0.577103E-03
-0.11E 01	-9.40E-00	-0.367820E-04	-0.261080E-04	-0.524331E-01	-0.524331E-01	-0.577103E-03
-0.11E 01	-9.60E-00	-0.367810E-04	-0.261020E-04	-0.525551E-01	-0.525551E-01	-0.577103E-03
-0.11E 01	-9.80E-00	-0.367800E-04	-0.260960E-04	-0.526811E-01	-0.526811E-01	-0.577103E-03
-0.11E 01	-1.00E-00	-0.367790E-04	-0.260900E-04	-0.528111E-01	-0.528111E-01	-0.577103E-03
-0.11E 01	-1.20E-00	-0.367780E-04	-0.260840E-04	-0.529451E-01	-0.529451E-01	-0.577103E-03
-0.11E 01	-1.40E-00	-0.367770E-04	-0.260780E-04	-0.530831E-01	-0.530831E-01	-0.577103E-03
-0.11E 01	-1.60E-00	-0.367760E-04	-0.260720E-04	-0.532251E-01	-0.532251E-01	-0.577103E-03
-0.11E 01	-1.80E-00	-0.367750E-04	-0.260660E-04	-0.533711E-01	-0.533711E-01	-0.577103E-03
-0.11E 01	-2.00E-00	-0.367740E-04	-0.260600E-04	-0.535211E-01	-0.535211E-01	-0.577103E-03
-0.11E 01	-2.20E-00	-0.367730E-04	-0.260540E-04	-0.536751E-01	-0.536751E-01	-0.577103E-03
-0.11E 01	-2.40E-00	-0.367720E-04	-0.260480E-04	-0.538331E-01	-0.538331E-01	-0.577103E-03
-0.11E 01	-2.60E-00	-0.367710E-04	-0.260420E-04	-0.539951E-01	-0.539951E-01	-0.577103E-03
-0.11E 01	-2.80E-00	-0.367700E-04	-0.260360E-04	-0.541611E-01	-0.541611E-01	-0.577103E-03
-0.11E 01	-3.00E-00	-0.367690E-04	-0.260300E-04	-0.543311E-01	-0.543311E-01	-0.577103E-03
-0.11E 01	-3.20E-00	-0.367680E-04	-0.260240E-04	-0.545051E-01	-0.545051E-01	-0.577103E-03
-0.11E 01	-3.40E-00	-0.367670E-04	-0.260180E-04	-0.546831E-01	-0.546831E-01	-0.577103E-03
-0.11E 01	-3.60E-00	-0.367660E-04	-0.260120E-04	-0.548651E-01	-0.548651E-01	-0.577103E-03
-0.11E 01	-3.80E-00	-0.367650E-04	-0.260060E-04	-0.550511E-01	-0.550511E-01	-0.577103E-03
-0.11E 01	-4.00E-00	-0.367640E-04	-0.260000E-04	-0.552411E-01	-0.552411E-01	-0.577103E-03
-0.11E 01	-4.20E-00	-0.367630E-04	-0.259940E-04	-0.554351E-01	-0.554351E-01	-0.577103E-03
-0.11E 01	-4.40E-00	-0.367620E-04	-0.259880E-04	-0.556331E-01	-0.556331E-01	-0.577103E-03
-0.11E 01	-4.60E-00	-0.367610E-04	-0.259820E-04	-0.558351E-01	-0.558351E-01	-0.577103E-03
-0.11E 01	-4.80E-00	-0.367600E-04	-0.259760E-04	-0.560411E-01	-0.560411E-01	-0.577103E-03
-0.11E 01	-5.00E-00	-0.367590E-04	-0.259700E-04	-0.562511E-01	-0.562511E-01	-0.577103E-03
-0.11E 01	-5.20E-00	-0.367580E-04	-0.259640E-04	-0.564651E-01	-0.564651E-01	-0.577103E-03
-0.11E 01	-5.40E-00	-0.367570E-04	-0.259580E-04	-0.566831E-01	-0.566831E-01	-0.577103E-03
-0.11E 01	-5.60E-00	-0.367560E-04	-0.259520E-04	-0.569051E-01	-0.569051E-01	-0.577103E-03
-0.11E 01	-5.80E-00	-0.367550E-04	-0.259460E-04	-0.571311E-01	-0.571311E-01	-0.577103E-03
-0.11E 01	-6.00E-00	-0.367540E-04	-0.259400E-04	-0.573611E-01	-0.573611E-01	-0.577103E-03
-0.11E 01	-6.20E-00	-0.367530E-04	-0.259340E-04	-0.575951E-01	-0.575951E-01	-0.577103E-03
-0.11E 01	-6.40E-00	-0.367520E-04	-0.259280E-04	-0.578331E-01	-0.578331E-01	-0.577103E-03
-0.11E 01	-6.60E-00	-0.367510E-04	-0.259220E-04	-0.580751E-01	-0.580751E-01	-0.577103E-03
-0.11E 01	-6.80E-00	-0.367500E-04	-0.259160E-04	-0.583211E-01	-0.583211E-01	-0.577103E-03
-0.11E 01	-7.00E-00	-0.367490E-04	-0.259100E-04	-0.585711E-01	-0.585711E-01	-0.577103E-03
-0.11E 01	-7.20E-00	-0.367480E-04	-0.259040E-04	-0.588251E-01	-0.588251E-01	-0.577103E-03
-0.11E 01	-7.40E-00	-0.367470E-04	-0.258980E-04	-0.590831E-01	-0.590831E-01	-0.577103E-03
-0.11E 01	-7.60E-00	-0.367460E-04	-0.258920E-04	-0.593451E-01	-0.593451E-01	-0.577103E-03
-0.11E 01	-7.80E-00	-0.367450E-04	-0.258860E-04	-0.596111E-01	-0.596111E-01	-0.577103E-03
-0.11E 01	-8.00E-00	-0.367440E-04	-0.258800E-04	-0.598811E-01	-0.598811E-01	-0.577103E-03
-0.11E 01	-8.20E-00	-0.367430E-04	-0.258740E-04	-0.601551E-01	-0.601551E-01	-0.577103E-03
-0.11E 01	-8.40E-00	-0.367420E-04	-0.258680E-04	-0.604331E-01	-0.604331E-01	-0.577103E-03
-0.11E 01	-8.60E-00	-0.367410E-04	-0.258620E-04	-0.607151E-01	-0.607151E-01	-0.577103E-03
-0.11E 01	-8.80E-00	-0.367400E-04	-0.258560E-04	-0.610011E-01	-0.610011E-01	-0.577103E-03
-0.11E 01	-9.00E-00	-0.367390E-04	-0.258500E-04	-0.612911E-01	-0.612911E-01	-0.577103E-03
-0.11E 01	-9.20E-00	-0.367380E-04	-0.258440E-04	-0.615851E-01	-0.615851E-01	-0.577103E-03
-0.11E 01	-9.40E-00	-0.367370E-04	-0.258380E-04	-0.618831E-01	-0.618831E-01	-0.577103E-03
-0.11E 01	-9.60E-00	-0.367360E-04	-0.258320E-04	-0.621851E-01	-0.621851E-01	-0.577103E-03
-0.11E 01	-9.80E-00	-0.367350E-04	-0.258260E-04	-0.624911E-01	-0.624911E-01	-0.577103E-03
-0.11E 01	-1.00E-00	-0.367340E-04	-0.258200E-04	-0.628011E-01	-0.628011E-01	-0.577103E-03
-0.11E 01	-1.20E-00	-0.367330E-04	-0.258140E-04	-0.631151E-01	-0.631151E-01	-0.577103E-03
-0.11E 01	-1.40E-00	-0.367320E-04	-0.258080E-04	-0.634331E-01	-0.634331E-01	-0.577103E-03
-0.11E 01	-1.60E-00	-0.367310E-04	-0.258020E-04	-0.637551E-01	-0.637551E-01	-0.577103E-03
-0.11E 01	-1.80E-00	-0.367300E-04	-0.257960E-04	-0.640811E-01	-0.640811E-01	-0.577103E-03
-0.11E 01	-2.00E-00	-0.367290E-04	-0.257900E-04	-0.644111E-01	-0.644111E-01	-0.577103E-03
-0.11E 01	-2.20E-00	-0.367280E-04	-0.257840E-04	-0.647451E-01	-0.647451E-01	-0.577103E-03
-0.11E 01	-2.40E-00	-0.367270E-04	-0.257780E-04	-0.650831E-01	-0.650831E-01	-0.577103E-03
-0.11E 01	-2.60E-00	-0.367260E-04	-0.257720E-04	-0.654251E-01	-0.654251E-01	-0.577103E-03
-0.11E 01	-2.80E-00	-0.367250E-04	-0.257660E-04	-0.657711E-01	-0.657711E-01	-0.577103E-03
-0.11E 01	-3.00E-00	-0.367240E-04	-0.257600E-04	-0.661211E-01	-0.661211E-01	-0.577103E-03
-0.11E 01	-3.20E-00	-0.367230E-04	-0.257540E-04	-0.664751E-01	-0.664751E-01	-0.577103E-03
-0.11E 01	-3.40E-00	-0.367220E-04	-0.257480E-04	-0.668331E-01	-0.668331E-01	-0.577103E-03
-0.11E 01	-3.60E-00	-0.3672				