Time Series Model

$y_0, y_1, \dots, y_t, \dots$ — is a time series, $y_i \in \mathbb{R}$
The model of time series:

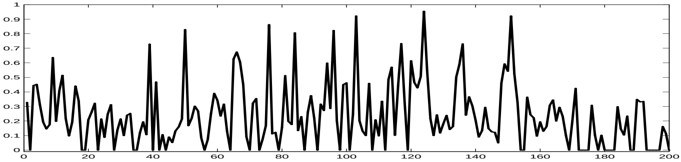
$$y_t = l_t + \varepsilon_t$$

where l_t — level of time series (changing slowly),
ε_t — (unobserved) error component (noise),

Forecasting model:

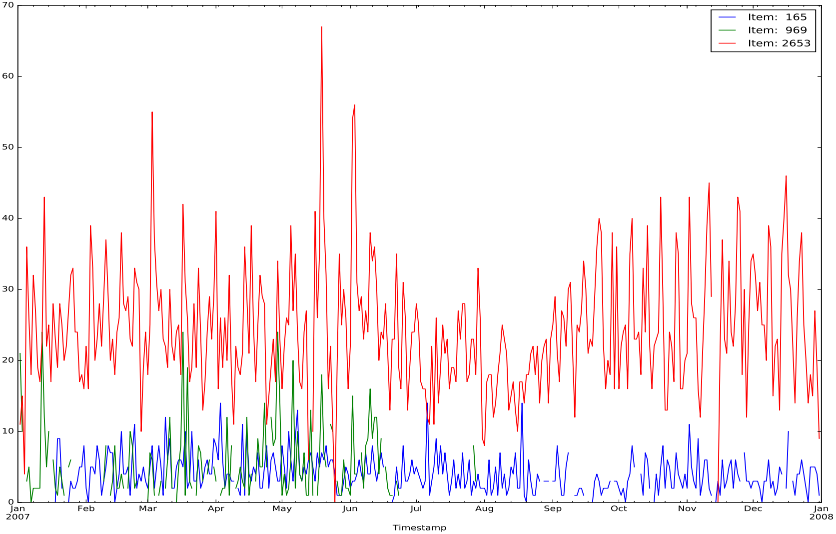
$$\hat{y}_{t+d} = \hat{l}_t$$

where $\hat{l}_t$ — an estimation of level,



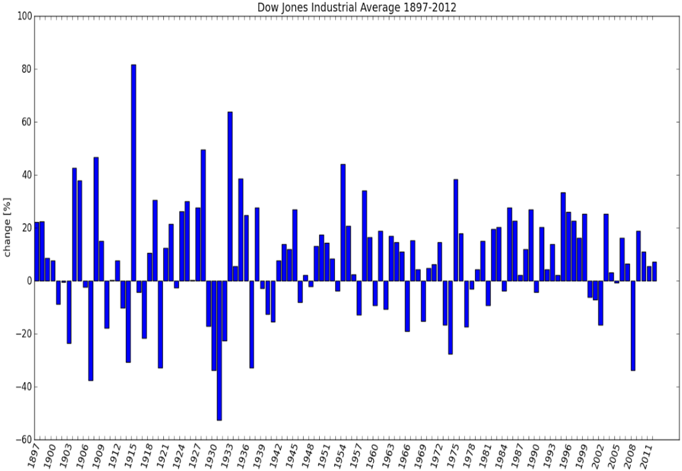
Time Series Model

Real time series of in Retail Chain:



Time Series Model

Jndex Dow-Jones:



Simple Exponential Smoothing

Weighted average with exponentially diminishing weights forecast:

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \dots$$

$\alpha \uparrow 1 \;\Rightarrow$ greater weight to last points,
$\alpha \downarrow 0 \;\Rightarrow$ greater smoothing.

Time point	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2	0.4	0.6	0.8
y_{T-1}	0.16	0.24	0.24	0.16
y_{T-2}	0.128	0.144	0.096	0.032
y_{T-3}	0.1024	0.0864	0.0384	0.0064
y_{T-4}	0.08192	0.05184	0.01536	0.00128
y_{T-5}	0.065536	0.031104	0.006144	0.000256

We find the optimal α^* using moving control:

$$Q(\alpha) = \sum_{t=T_0}^{T_1} \left(\hat{y}_t(\alpha) - y_t\right)^2 \to \min_{\alpha}$$

Empirical rules:
if $\alpha^* \in (0, 0.3)$ the series is stationary, ES works;
if $\alpha^* \in (0.3, 1)$ the series is non-stationary, we need a trend model.

Simple Exponential Smoothing

- The method suits forecasting of time series without trend and seasonality:

$$\hat{y}_{t+1|t} = l_t,$$
$$l_t = \alpha y_t + (1-\alpha)\, l_{t-1} = \hat{y}_{t|t-1} + \alpha \cdot e_t.$$

  where $e_t = y_t - \hat{y}_{t|t-1}$ — forecast error at time point t
  Proof:

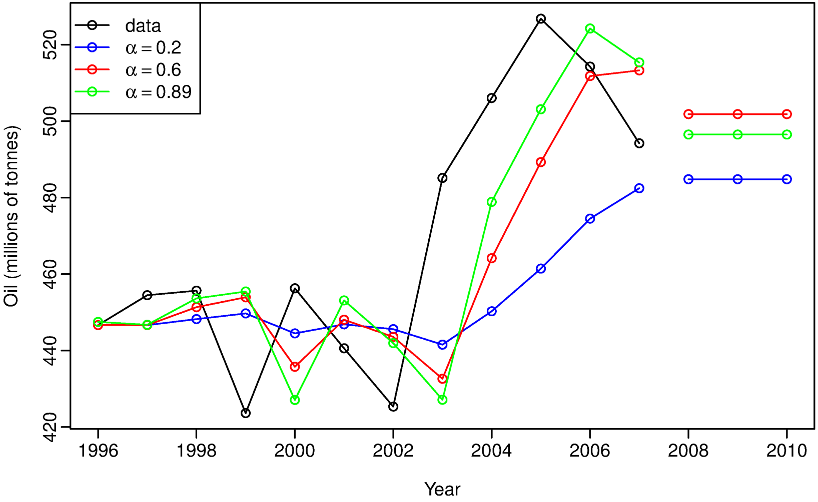
$$\hat{y}_{t+1} := \alpha y_t + (1-\alpha)\hat{y}_t = \hat{y}_t + \alpha \cdot e_t$$

- The forecast depends on l_0:

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha\,(1-\alpha)^j\, y_{T-j} + (1-\alpha)^T\, l_0.$$

  We can take $l_0 = y_1$ or optimize it.
- Forecast turns out flat, i.e. $\hat{y}_{t+d|t} = \hat{y}_{t+1|t}$.

Simple Exponential Smoothing



Simple ES applied to data on oil production in Saudi Arabia (1996–2007).

Double (Brown) Exponential Smoothing

Model of time series:

$$y_t = l_t + b_t t + \varepsilon_t,$$

where l_t, b_t — adaptive coefficients, ε_t — noise

Two successive ES:

$$S_t^{[1]} := \alpha y_t + (1-\alpha)S_{t-1}^{[1]};$$
$$S_t^{[2]} := \alpha S_t^{[1]} + (1-\alpha)S_{t-1}^{[2]}.$$

Smoothed formula ($l_t = 2 \cdot S_t^{[1]} - S_{t-1}^{[2]}, \quad b_t = \frac{\alpha}{1-\alpha}(S_t^{[1]} - S_{t-1}^{[2]})$):

$$l_t := \alpha y_t + (1-\alpha)(l_{t-1} + b_{t-1});$$
$$b_t := \alpha(l_t - l_{t-1}) + (1-\alpha)b_{t-1}.$$

Forecast:

$$\hat{y}_{t+d} = l_t + (d - 1 + 1/\alpha)\, b_t,$$

Double (Brown) Exponential Smoothing

Model of time series:

$$y_t = l_t + b_t t + \varepsilon_t,$$

where l_t, b_t — adaptive coefficients, ε_t — noise

Error-correction form:

$$l_t := \alpha y_t + (1-\alpha)(l_{t-1} + b_{t-1}) = \hat{y}_t + \alpha e_t;$$
$$b_t := \alpha(l_t - l_{t-1}) + (1-\alpha) b_{t-1} = b_{t-1} + \alpha^2 e_t.$$

Proof of the last equation:

$$\alpha(\hat{y}_t + \alpha e_t - l_{t-1}) - \alpha b_{t-1} + b_{t-1} =$$
$$b_{t-1} + \alpha\left(\hat{y}_t + \alpha e_t - l_{t-1} - b_{t-1}\right) = b_{t-1} + \alpha^2 e_t.$$

Tracking Signal

Tracking signal [Trigg, 1964]

$$K_t = \frac{\hat{e}_t}{\tilde{e}_t} \qquad \begin{aligned} \hat{e}_{t+1} &:= \gamma e_t + (1-\gamma)\hat{e}_t; \\ \tilde{e}_{t+1} &:= \gamma |e_t| + (1-\gamma)\tilde{e}_t. \end{aligned}$$

Recommendation: $\gamma = 0.05 \ldots 0.1$

Statistics adequacy test (at $\gamma \geq 0.1,\ t \to \infty$):

hypotheses H_0: $\mathsf{E}\varepsilon_t = 0,\ \mathsf{E}\varepsilon_t\varepsilon_{t+d} = 0$
is accepted at significance level δ if

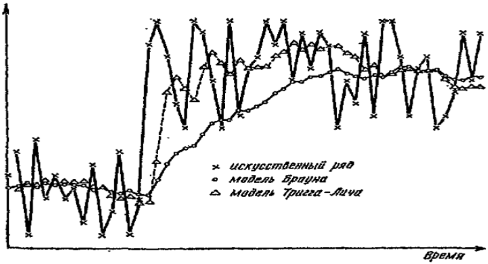
$$|K_t| \leq 1.2\Phi_{1-\delta/2}\sqrt{1-\gamma/(1+\gamma)},$$

$\Phi_{1-\delta/2}$ — normal distribution quantile,
$\Phi_{1-\delta/2} = \Phi_{0.975} = 1.96$ at $\delta = 0.05$

Trigg-Leach Model [Trigg, Leach, 1967]

Problem: adaptive models adjust poorly to sharp changes of structure

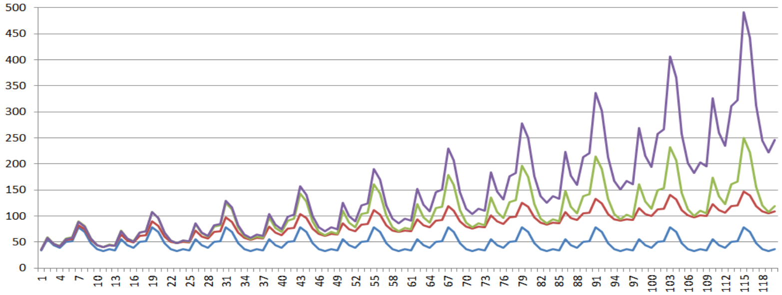
Solution: $\alpha = |K_t|$



Drawbacks:
1) reacts poorly to single outliers; $(\alpha_t = |K_{t-1}|)$
2) requires fitting γ given recommended $\gamma = 0.05\dots0.1$.

Examples of Trend and Seasonality

Example: Combination of trend and seasonality (model data)



Ряд 1 — seasonality and no trend
Ряд 2 — linear trend, additive seasonality
Ряд 3 — linear trend, multiplicative seasonality
Ряд 4 — exponential trend, multiplicative seasonality

Holt Model = Linear Trend

Linear trend with no seasonality effect:

$$\hat{y}_{t+d} = l_t + b_t d,$$

where l_t, b_t — adaptive coefficients of linear trend

Recursive formula:

$$l_t := \alpha y_t + (1-\alpha)(l_{t-1} + b_{t-1}) = \hat{y}_t + \alpha e_t;$$
$$b_t := \beta(l_t - l_{t-1}) + (1-\beta) b_{t-1} = b_{t-1} + \alpha\beta e_t.$$

Particular case — Brown linear growth model:

$$\alpha = \alpha, \quad \beta = \alpha$$

Other Methods that Account Trend

Multiplicative linear (exponential) trend:

$$\hat{y}_{t+d|t} = l_t b_t^d,$$
$$l_t = \alpha y_t + (1-\alpha)\left(l_{t-1} b_{t-1}\right),$$
$$b_t = \beta \frac{l_t}{l_{t-1}} + (1-\beta)\, b_{t-1}.$$

$\alpha, \beta \in [0, 1]$.

Other Methods that Account Trend

Additive damped trend:

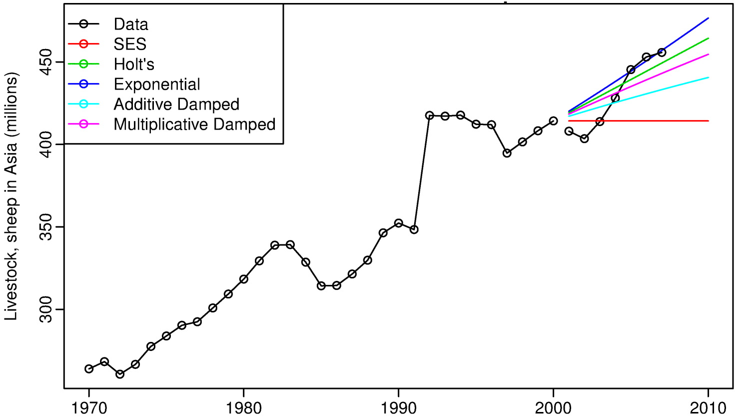
$$\begin{aligned}\hat{y}_{t+d|t} &= l_t + \left(\phi + \phi^2 + \dots + \phi^d\right) b_t,\\ l_t &= \alpha y_t + (1-\alpha)\left(l_{t-1} + \phi b_{t-1}\right),\\ b_t &= \beta\left(l_t - l_{t-1}\right) + (1-\beta)\,\phi b_{t-1}.\end{aligned}$$

Multiplicative damped trend:

$$\begin{aligned}\hat{y}_{t+d|t} &= l_t b_t^{\left(\phi + \phi^2 + \dots + \phi^d\right)},\\ l_t &= \alpha y_t + (1-\alpha)\, l_{t-1} b_{t-1}^{\phi},\\ b_t &= \beta \frac{l_t}{l_{t-1}} + (1-\beta)\, b_{t-1}^{\phi}.\end{aligned}$$

$\alpha, \beta \in [0,1]\,,\ \phi \in (0,1)\,.$

Other Methods that Account Trend



Forecast of sheep population in Asia with regard for trend.

	SES	Holt's	Exponential	Additive damped	Multiplicative damped
α	1	0.98	0.98	0.99	0.98
β		0	0	0	0.00
ϕ				0.98	0.98

Winters Model = Multiplicative Seasonality

Multiplicative Seasonality of Period p:

$$\hat{y}_{t+d} = l_t \cdot s_{t-p+(d \bmod p)},$$

$s_0, \ldots, s_{p-1}$ — seasonality profile of period p.
Recursive formula:

$$l_t := \alpha(y_t/s_{t-p}) + (1-\alpha)l_{t-1} = l_{t-1} + \alpha e_t/s_{t-p};$$
$$s_t := \beta(y_t/l_t) + (1-\beta)s_{t-p} = s_{t-p} + \beta(1-\alpha)e_t/l_t.$$

Proof of the last equation:

$$s_t := s_{t-p} + \beta\left(y_t/l_t - s_{t-p}\right) = s_{t-p} + \beta\left(y_t - s_{t-p}l_t\right)/l_t =$$
$$s_{t-p} + \beta\big(y_t - s_{t-p}(l_{t-1} + \alpha e_t/s_{t-p})\big)/l_t = s_{t-p} +$$
$$+\beta\left(\underbrace{y_t - s_{t-p}l_{t-1}}_{e_t} - \alpha e_t\right)/l_t$$

Additive Seasonality ES Model

Additive seasonality with period of length p:

$$\begin{aligned}\hat{y}_{t+d|t} &= l_t + s_{t-p+(d \mod p)},\\ l_t &= \alpha\left(y_t - s_{t-p}\right) + (1-\alpha)\left(l_{t-1}\right) = l_{t-1} + \alpha e_t;\\ s_t &= \gamma\left(y_t - l_{t-1}\right) + (1-\gamma)\, s_{t-p} = s_{t-p} + \gamma(1-\alpha)e_t.\end{aligned}$$

Theil-Wage Model

Linear trend with additive seasonality of period s:

$$\hat{y}_{t+d} = (l_t + b_t d) + s_{t+(d \bmod s)-p}.$$

$l_t + b_t d$ — trend cleaned of seasonality,
$s_0, \ldots, s_{p-1}$ — seasonality profile of period p.

Recursive formula:

$$l_t := \alpha(y_t - s_{t-p}) + (1-\alpha)(l_{t-1} + b_{t-1}) = l_{t-1} + b_{t-1} + \alpha e_t;$$
$$b_t := \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1} = b_{t-1} + \alpha\beta e_t;$$
$$s_t := \gamma(y_t - l_t) + (1-\gamma)s_{t-p} = s_{t-p} + \gamma(1-\alpha)e_t.$$

Winters Model with Linear Trend

Multiplicative seasonality of period s with a linear trend:

$$\hat{y}_{t+d} = (l_t + b_t d) \cdot s_{t+(d \bmod p)-p},$$

$l_t + b_t d$ — trend cleaned of seasonality,
$s_0, \dots, s_{p-1}$ — seasonality profile of period s.

Recursive formula:

$$l_t := \alpha(y_t/s_{t-p}) + (1-\alpha)(l_{t-1} + b_{t-1}) = l_{t-1} + b_{t-1} + \alpha e_t/s_{t-p};$$
$$b_t := \beta(l_t - l_{t-1}) + (1-\beta)b_{t-1} = b_{t-1} + \alpha\beta e_t/s_{t-p};$$
$$s_t := \gamma(y_t/l_t) + (1-\gamma)s_{t-p} = s_{t-p} + \gamma(1-\alpha)e_t/l_t.$$

Winters Model with exponential trend

Multiplicative trend model exponential trend:

$$\hat{y}_{t+d} = l_t (r_t)^d \cdot s_{t+(d \bmod p)-p},$$

$l_t (r_t)^d$ — exponential trend without seasonality,
$s_0, \ldots, s_{p-1}$ — seasonal trend p.

Recurrent version:

$$l_t := \alpha(y_t/s_{t-p}) + (1-\alpha) l_{t-1} r_{t-1} = l_{t-1} r_{t-1} + \alpha e_t / s_{t-1};$$
$$r_t := \beta(l_t/l_{t-1}) + (1-\beta) r_{t-1} = r_{t-1} + \alpha\beta e_t / st - 1;$$
$$s_t := \gamma(y_t/l_t) + (1-\gamma) s_{t-p} = s_{t-p} + \gamma(1-\alpha) e_t / l_t.$$

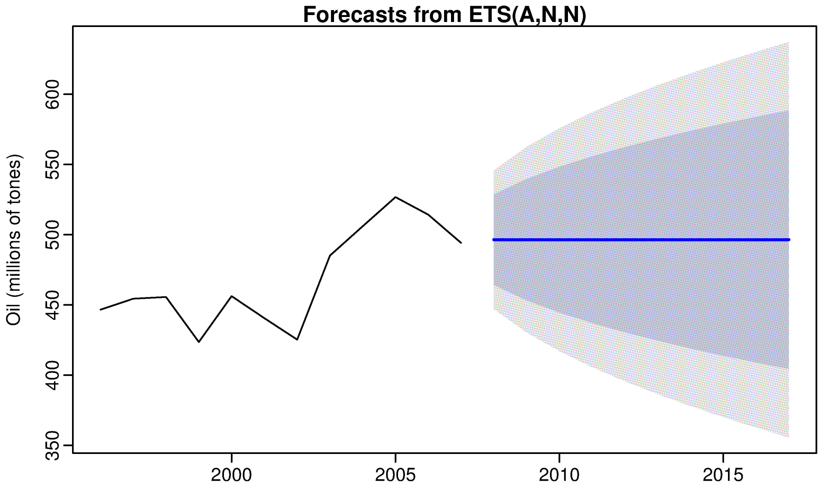
ES Models

	Seasonality		
Trend	N (None)	A (Additive)	M (Multiplicative)
N (None)	(N,N)	(N,A)	(N,M)
A (Additive)	(A,N)	(A,A)	(A,M)
Ad (Additive damped)	(Ad,N)	(Ad,A)	(Ad,M)
M (Multiplicative)	(M,N)	(M,A)	(M,M)
Md (Multiplicative damped)	(Md,N)	(Md,A)	(Md,M)

We may additionally suggest an additive (A) or a multiplicative (M) error (the type of error does not influence single-value prediction). Multiplicative error is suitable only for strictly positive time series.

The final model may be written as $ESM\left(\cdot,\cdot,\cdot\right)$.

Examples of Forecast



For the data on oil production in Saudi Arabia function ESM selects simple ES.