Introduction to Digital Systems

Part I (4 lectures)

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Introduction

Number Systems and Codes

Combinational Logic Design Principles

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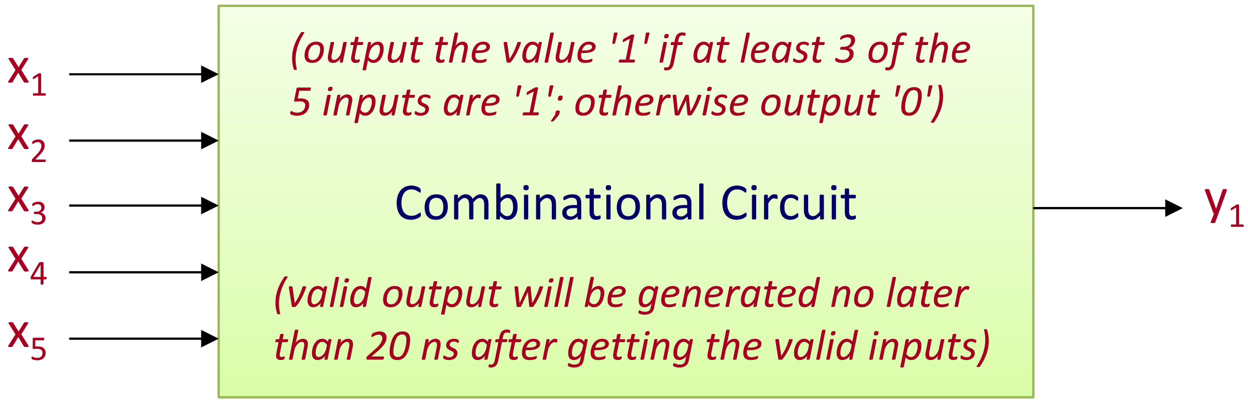
Lecture 3 contents

- Combinational circuits
- Boolean algebra
  - axioms
  - theorems
  - duality
  - algebraic simplification of logic functions
  - canonical forms
- Standard representations of logic functions

Combinational Circuits

- A logic circuit whose outputs depend only on its current inputs is called a **combinational circuit.**
- A combinational circuit is characterized by
  - one or more inputs
  - one or more outputs
  - a functional specification describing each output as a function of the inputs
  - a time specification that includes at least the maximum time it will take the circuit to produce valid output values for an arbitrary set of input values -> **propagation delay**.



Boolean Algebra

- Formal analysis techniques for digital circuits have their roots in the work of an English mathematician, George Boole.
- In 1854, he invented a two-valued algebraic system, now called **Boolean algebra**.
- In 1938, Bell Labs researcher Claude E. Shannon showed how to adapt Boolean algebra to analyze and describe the behavior of circuits.
- In switching algebra we use a symbolic variable, such as *x*, to represent the condition of a logic signal.
- A logic signal is in one of two possible conditions: low or high, off or on, and so on, depending on the technology.
- We say that *x* has the value “0" for one of these conditions and "1" for the other.

Axioms

- The **axioms** (or **postulates**) of a mathematical system are a minimal set of basic definitions that we assume to be true, from which all other information about the system can be derived.
- A variable *x* can take on only one of two values:
  - $x = 0 \; if \; x \neq 1$
  - $x = 1 \; if \; x \neq 0$
- Inversion (definition of NOT):
  - $if \; x = 0 \; then \; \bar{x} = 1$
  - $if \; x = 1 \; then \; \bar{x} = 0$
- Definition of the AND and OR operations:
  - $0 \cdot 0 = 0, \qquad 1 \cdot 1 = 1, \qquad 0 \cdot 1 = 1 \cdot 0 = 0$
  - $1 + 1 = 1, \qquad 0 + 0 = 0 \qquad 1 + 0 = 0 + 1 = 1$
- These pairs of axioms completely define switching algebra.
- All other facts about the system can be proved using these axioms as a starting point.

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Operator Precedence

- **Operator precedence** is an ordering of logical operators designed to allow the dropping of parentheses in logical expressions.
- The following list gives a hierarchy of precedences for the Boolean operators (from highest to lowest):
  - NOT
  - AND
  - OR

Example:

$$x \cdot \bar{y} + z = (x \cdot (\bar{y})) + \mathrm{z}$$

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Single-Variable Theorems

- Switching algebra **theorems** are statements, known to be always true, that permit us to manipulate algebraic expressions to allow simpler analysis or more efficient synthesis of the corresponding circuits.

- Identities:
  - $x + 0 = x$
  - $x \cdot 1 = x$
- Null elements:
  - $x + 1 = 1$
  - $x \cdot 0 = 0$
- Idempotency:
  - $x + x = x$
  - $x \cdot x = x$
- Involution:
  - $\bar{\bar{x}} = x$
- Complements:
  - $x + \bar{x} = 1$
  - $x \cdot \bar{x} = 0$

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Two- and Three-Variable Theorems

- Commutativity:
  - $x + y = y + x$
  - $x \cdot y = y \cdot x$
- Associativity:
  - $(x + y) + z = x + (y + z)$
  - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- Distributivity:
  - $x \cdot y + x \cdot z = x \cdot (y + z)$
  - $(x + y) \cdot (x + z) = x + y \cdot z$
- Covering:
  - $x + x \cdot y = x$
  - $x \cdot (x + y) = x$
- Combining:
  - $x \cdot y + x \cdot \bar{y} = x$
  - $(x + y) \cdot (x + \bar{y}) = x$
- Simplification:
  - $x + \bar{x} \cdot y = x + y$
  - $x \cdot (\bar{x} + y) = x \cdot y$
- Consensus:
  - $x \cdot y + \bar{x} \cdot z + y \cdot z = x \cdot y + \bar{x} \cdot z$
  - $(x + y) \cdot (\bar{x} + z) \cdot (y + z) = (x + y) \cdot (\bar{x} + z)$

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Resume of Theorems

- Most theorems in switching algebra are exceedingly simple to prove using a technique called **perfect induction**:
  - prove a theorem by proving that it is true for all possible values (“0” and “1”) of all the variables
- In all of the theorems, **it is possible to replace each variable with an arbitrary logic expression**:
  - $x + x \cdot (a + b \cdot c \cdot \bar{d} \cdot e) = x$
- When realizing the AND(OR) operation, we can connect gate inputs in any order: either one n-input gate or (*n* - 1) 2-input gates interchangeably, though propagation delay and cost are likely to be higher with multiple 2-input gates:
  - $w \cdot x \cdot y \cdot z = (w \cdot x) \cdot (y \cdot z) = \big(w \cdot (x \cdot (y \cdot z))\big) \ldots$
- In Boolean algebra, logical addition distributes over logical multiplication:
  - $(x + y) \cdot (x + z) = x + y \cdot z$

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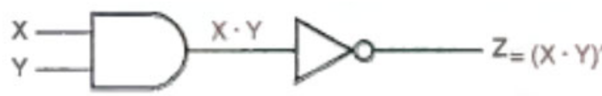
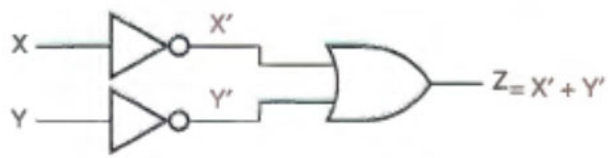
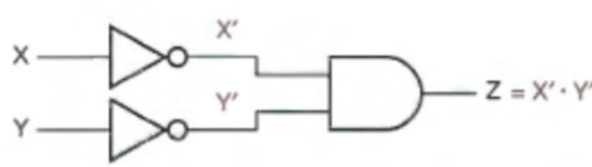
DeMorgan's Theorems

- An *n*-input AND gate whose output is complemented is equivalent to an *n*-input OR gate whose inputs are complemented:
  - $\overline{x \cdot y} = \bar{x} + \bar{y}$
  - $\overline{\prod_{i=0}^{n-1} x_i} = \sum_{i=0}^{n-1} \overline{x_i}$





- An *n*-input OR gate whose output is complemented is equivalent to an *n*-input AND gate whose inputs are complemented:
  - $\overline{x + y} = \bar{x} \cdot \bar{y}$
  - $\overline{\sum_{i=0}^{n-1} x_i} = \prod_{i=0}^{n-1} \overline{x_i}$



Generalized DeMorgan's Theorem

- Given any *n*-variable <u>fully parenthesized</u> logic expression, its complement can be obtained by swapping + and · and complementing all variables:
  - $\overline{F(x_0, x_1, \ldots, x_{n-1}, +, \ \cdot)} = F(\overline{x_0}, \overline{x_1}, \ldots, \overline{x_{n-1}}, \ \ \cdot, +)$

Example:

$$F(w, x, y, z) = \bar{w} \cdot x + x \cdot y + w \cdot (\bar{x} + \bar{z})$$

$$\overline{F(w, x, y, z)} = (w + \bar{x}) \cdot (\bar{x} + \bar{y}) \ \cdot (\bar{w} + x \cdot z)$$

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Principle of Duality

- The **principle of duality** states that any theorem or identity in switching algebra remains true if 0 and 1 are swapped and · and + are swapped throughout.
  - Duals of all the axioms are true.
  - Duals of all switching-algebra theorems are true.

- If $F(x_0, x_1, \dots, x_{n-1}, +, \ \cdot)$ is a fully parenthesized logic expression involving the variables x_0, x_1, ..., x_{n-1}, and the operators + and ·, then the dual of F, written F^D is the same expression with + and · swapped:
  - $F^D(x_0, x_1, \dots, x_{n-1}, +, \ \cdot) = F(x_0, x_1, \dots, x_{n-1}, \ \cdot, +)$

- The generalized DeMorgan's theorem may now be restated as follows:
  - $\overline{F(x_0, x_1, \dots, x_{n-1})} = F^D(\overline{x_0}, \overline{x_1}, \dots, \overline{x_{n-1}})$

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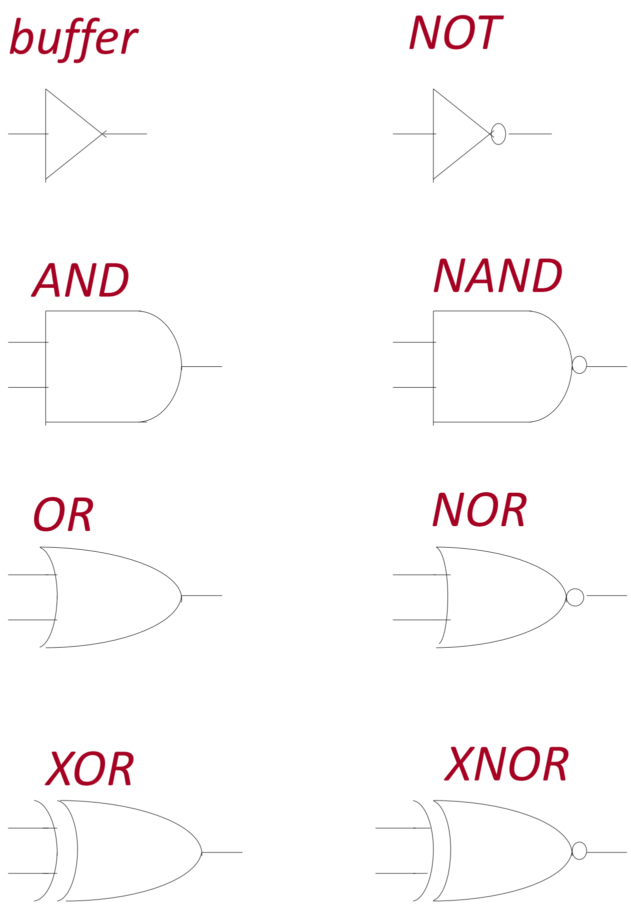
NAND and NOR Gates



$$\overline{x \cdot y}$$

x	y	x NAND y
0	0	1
0	1	1
1	0	1
1	1	0

$$\overline{x + y}$$

x	y	x NOR y
0	0	1
0	1	0
1	0	0
1	1	0

Functional Completeness

- A **functionally complete set** of Boolean operators is one which can be used to describe the behavior of any digital circuit.
- Examples:
  - {AND, OR, NOT}
  - {AND, NOT}
  - {OR, NOT}
  - {NAND}
  - {NOR}

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NAND and NOR Gates

- To write a Boolean expression only with the operators **NAND**, first put the expression in the **sum-of-products** form and then apply <u>the **involution** theorem</u> ($\bar{\bar{x}} = x$) followed by the **DeMorgan's** theorem ($\overline{\sum_{i=0}^{n-1} x_i} = \prod_{i=0}^{n-1} \overline{x_i}$).

- To write a Boolean expression only with the operators **NOR**, first put the expression in the **product-of-sums** form and then apply <u>the **involution** theorem</u> ($\bar{\bar{x}} = x$) followed by the **DeMorgan's** theorem ($\overline{\prod_{i=0}^{n-1} x_i} = \sum_{i=0}^{n-1} \overline{x_i}$).

- Always assume that complemented versions of input variables are available.

Examples:

$$x + (y \cdot \bar{z}) = \overline{\overline{x + (y \cdot \bar{z})}} = \overline{\bar{x} \cdot \overline{y \cdot \bar{z}}}$$

$$x + (y \cdot \bar{z}) = (x + y) \cdot (x + \bar{z}) = \overline{\overline{(x + y) \cdot (x + \bar{z})}} = \overline{\overline{x + y} + \overline{x + \bar{z}}}$$

Boolean Functions

- A Boolean function $f(x_0, x_1, \dots, x_{n-1})$ is a match that associates an element of the set {0,1} with each of the 2^n possible combinations that variables can assume.
- There are $2^{m \times 2^n}$ different Boolean functions that can be implemented in a digital system with *n* inputs and *m* outputs.

Examples:

For n=1, m=1: $2^{1 \times 2^1} = 4$

x	constant ‘0’	x	$\overline{x}$	constant ‘1’
0	0	0	1	1
1	0	1	0	1

For n=4, m=3: $2^{3 \times 2^4} = 2^{48}$= 281 474 976 710 656

Truth Table

- The most basic representation of a logic function is the **truth table**.
- A truth table simply lists the output of the circuit for every possible input combination.
- Traditionally, the input combinations are arranged in rows in ascending binary counting order, and the corresponding output values are written in a column next to the rows.
- The truth table for an n-variable logic function has 2^n rows.

Example (n=3):

	x	y	z	f(x,y,z)
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

Minterms and Maxterms

- A **literal** is a variable or the complement of a variable. Examples: $\mathrm{x}, \mathrm{y}, \bar{x}$.
- A **product term** is a single literal or a logical product of two or more literals. Examples: $\bar{z},\ x \cdot y,\ x \cdot \bar{y} \cdot z$.
- A **sum term** is a single literal or a logical sum of two or more literals. Examples: $\bar{z},\ \ x + y,\ x + \bar{y} + z$.
- A **normal term** is a product or sum term in which no variable appears more than once.
- An *n*-variable **minterm** is a normal product term with *n* literals. There are 2^n such product terms.
  - A minterm m_i corresponds to row *i* of the truth table.
  - In minterm m_i, a particular variable appears complemented if the corresponding bit in the binary representation of *i* is 0; otherwise, it is uncomplemented.
- An *n*-variable **maxterm** is a normal sum term with *n* literals. There are 2^n such sum terms.
  - A maxterm M_i corresponds to row *i* of the truth table.
  - In maxterm M_i, a particular variable appears complemented if the corresponding bit in the binary representation of *i* is 1; otherwise, it is uncomplemented.

Example:

	x	y	z	f(x,y,z)
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

$$m_0 = \bar{x} \cdot \bar{y} \cdot \bar{z} \qquad M_0 = x + y + z$$

$$m_5 = x \cdot \bar{y} \cdot z \qquad M_5 = \bar{x} + y + \bar{z}$$

$$m_i = \overline{M_i} \qquad i = 0,1,\ldots,2^n - 1$$

Algebraic Representations

- Any Boolean function can be presented as:
  - a sum of the minterms corresponding to truth-table rows (input combinations) for which the function produces a 1 -> **canonical sum**
  - a product of the maxterms corresponding to truth-table rows (input combinations) for which the function produces a 0 -> **canonical product**

Example:

	x	y	z	f(x,y,z)
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	1

$$f(x,y,z)$$
$$= \bar{x}\cdot\bar{y}\cdot z + x\cdot\bar{y}\cdot\bar{z} + x\cdot\bar{y}\cdot z + x\cdot y\cdot\bar{z} + x\cdot y\cdot z$$

$$= \sum_{x,y,z} m(1,4,5,6,7)$$

$$f(x,y,z) = (x+y+z)\cdot(x+\bar{y}+z)\cdot(x+\bar{y}+\bar{z})$$

$$= \prod_{x,y,z} M(0,2,3)$$

Shannon's Expansion Theorems

- Any Boolean function $f(x_0, x_1, \dots, x_{n-1})$ can be presented in the following forms:
  - $\overline{x_0} \cdot f(0, x_1, \dots, x_{n-1}) + x_0 \cdot\ f(1, x_1, \dots, x_{n-1})$
  - $(\overline{x_0} + f(1, x_1, \dots, x_{n-1})) \cdot (x_0 +\ f(0, x_1, \dots, x_{n-1}))$

Perfect induction:

If x_0 = 0 then: $f(0, x_1, \dots, x_{n-1}) = 1 \cdot f(0, x_1, \dots, x_{n-1}) + 0 \cdot f(1, x_1, \dots, x_{n-1})$

If x_0 = 1 then: $f(1, x_1, \dots, x_{n-1}) = 0 \cdot f(0, x_1, \dots, x_{n-1}) + 1 \cdot f(1, x_1, \dots, x_{n-1})$

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Canonical Sum

- Extending to 2 variables:

$$f(x_0, x_1, \ldots, x_{n-1}) = \overline{x}_0 \cdot f(0, x_1, \ldots, x_{n-1}) + x_0 \cdot f(1, x_1, \ldots, x_{n-1}) =$$

$$= \overline{x}_0 \cdot \overline{x}_1 \cdot f(0,0, x_2, \ldots, x_{n-1}) + \overline{x}_0 \cdot x_1 \cdot f(0,1, x_2, \ldots, x_{n-1}) +$$

$$+ x_0 \cdot \overline{x}_1 \cdot f(1,0, x_2, \ldots, x_{n-1}) + x_0 \cdot x_1 \cdot f(1,1, x_2, \ldots, x_{n-1})$$

- Extending to *n* variables:

$$f(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{2^n - 1} m_i \cdot f_i \qquad f_i = f((x_0, x_1, \ldots, x_{n-1}) = i)$$

Canonical Product

- Extend the Shannon expansion theorem $f(x_0, x_1, \ldots, x_{n-1}) = (\overline{x_0} + f(1, x_1, \ldots, x_{n-1})) \cdot \big(x_0 + f(0, x_1, \ldots, x_{n-1})\big)$ to *n* variables:
  - $f(x_0, x_1, \ldots, x_{n-1}) = \prod_{i=0}^{2^n-1}(f_i + M_i)$

3rd and 4th Canonical Forms

- 3rd canonical form:

$$f(x_0,x_1,\dots,x_{n-1}) = \overline{\overline{f(x_0,x_1,\dots,x_{n-1})}} = \overline{\overline{\sum_{i=0}^{2^n-1} f_i \cdot m_i}} = \overline{\prod_{i=0}^{2^n-1} \overline{f_i \cdot m_i}}$$

- 4th canonical form:

$$f(x_0,x_1,\dots,x_{n-1}) = \overline{\overline{f(x_0,x_1,\dots,x_{n-1})}} = \overline{\overline{\prod_{i=0}^{2^n-1} f_i + M_i}} = \overline{\sum_{i=0}^{2^n-1} \overline{f_i + M_i}}$$

Canonical Forms

canonical sum of products: $$f(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{2^n-1} m_i \cdot f_i$$ AND-OR

canonical product of sums: $$f(x_0, x_1, \ldots, x_{n-1}) = \prod_{i=0}^{2^n-1} (f_i + M_i)$$ OR-AND

3rd canonical form: $$f(x_0, x_1, \ldots, x_{n-1}) = \overline{\prod_{i=0}^{2^n-1} \overline{f_i \cdot m_i}}$$ NAND-NAND

4th canonical form : $$f(x_0, x_1, \ldots, x_{n-1}) = \overline{\sum_{i=0}^{2^n-1} \overline{f_i + M_i}}$$ NOR-NOR

Canonical Forms (cont.)

Example: Derive the canonical forms of function f(x,y,z):

$$f(x,y,z) = x \cdot y + \overline{z}$$

	x	y	z	f(x,y,z)
0	0	0	0	1
1	0	0	1	0
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	1
7	1	1	1	1

1st: $f(x,y,z) = \sum m(0,2,4,6,7)$

2nd: $f(x,y,z) = \prod M(1,3,5)$

3rd: $f(x,y,z) = \overline{\prod \overline{m(0,2,4,6,7)}}$

4th: $f(x,y,z) = \overline{\sum \overline{M(1,3,5)}}$

Standard Representations of Logic Functions

- A truth table
- Algebraic
- Logic circuit

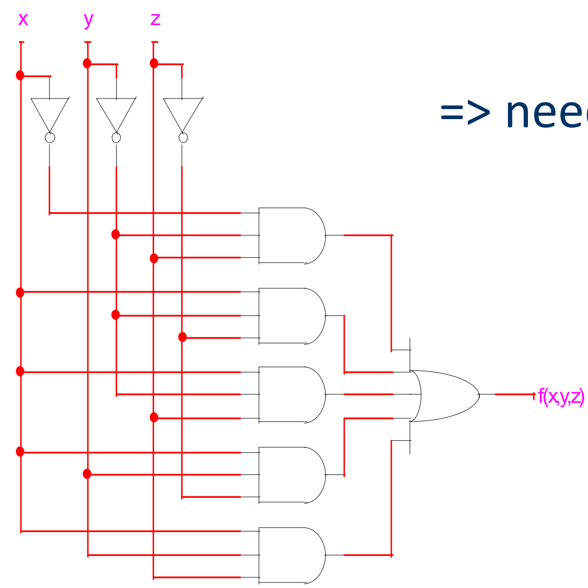
An algebraic representation frequently includes redundant terms:

$$f(x,y,z) = \bar{x}\cdot\bar{y}\cdot z + x\cdot\bar{y}\cdot\bar{z} + \\ + x\cdot\bar{y}\cdot z + x\cdot y\cdot\bar{z} + x\cdot y\cdot z$$

A truth table representation is unique:

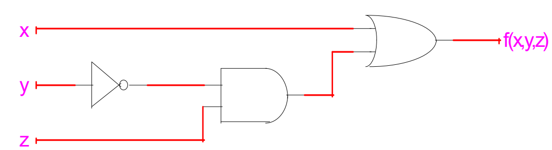
x	y	z	f(x,y,z)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Logic circuit:



=> need for simplification

$$f(x,y,z) = x + \bar{y}\cdot z$$



Exercises

- Are the following expressions correct?

$$[x + y \cdot z]^D = x \cdot y + z$$

$$[x + y \cdot z]^D = \overline{x} \cdot (\overline{y} + \overline{z})$$

- A self-dual logic function is a function *f* such that $f = f^D$. Which of the following functions are self-dual?

$$f_1(x, y, z) = \overline{x} \cdot y + \overline{x} \cdot z + y \cdot z$$

$$f_2(x, y) = \overline{x} \cdot y + x \cdot \overline{y}$$

Exercises (cont.)

- Express the function *y* in the simplest form using only the operator NAND.

$$y = x_1 \cdot (x_2 + \bar{x}_3 \cdot x_4) + x_2$$

$$y = x_1 \cdot x_2 + x_1 \cdot \bar{x}_3 \cdot x_4 + x_2 = x_2 + x_1 \cdot \bar{x}_3 \cdot x_4$$

$$y = \overline{\overline{x_2 + x_1 \cdot \bar{x}_3 \cdot x_4}} = \overline{\bar{x}_2 \cdot \overline{x_1 \cdot \bar{x}_3 \cdot x_4}}$$

- Express the function *y* in the simplest form using only the operator NOR.

$$y = x_1 \cdot (x_2 + \bar{x}_3 \cdot x_4) + x_2$$

$$y = (x_1 + x_2) \cdot (x_2 + \bar{x}_3 \cdot x_4 + x_2) = (x_1 + x_2) \cdot (x_2 + \bar{x}_3 \cdot x_4)$$

$$y = (x_1 + x_2) \cdot (x_2 + \bar{x}_3) \cdot (x_2 + x_4)$$

$$y = \overline{\overline{(x_1 + x_2) \cdot (x_2 + \bar{x}_3) \cdot (x_2 + x_4)}} = \overline{\overline{(x_1 + x_2)} + \overline{(x_2 + \bar{x}_3)} + \overline{(x_2 + x_4)}}$$

Exercises (cont.)

- Determine all the canonical forms of the function *f*:

$$f(x,y,z) = x \cdot y + \overline{x} \cdot \overline{z} + y \cdot z$$

$$f(x,y,z) = \sum m(0,2,3,6,7) = \overline{x} \cdot \overline{y} \cdot \overline{z} + \overline{x} \cdot y \cdot \overline{z} + \overline{x} \cdot y \cdot z + x \cdot y \cdot \overline{z} + x \cdot y \cdot z$$

$$f(x,y,z) = \prod M(1,4,5) = (x + y + \overline{z}) \cdot (\overline{x} + y + z) \cdot (\overline{x} + y + \overline{z})$$

$$f(x,y,z) = \overline{\prod \overline{m(0,2,3,6,7)}} = \overline{\overline{(\overline{x} \cdot \overline{y} \cdot \overline{z})} \cdot \overline{(\overline{x} \cdot y \cdot \overline{z})} \cdot \overline{(\overline{x} \cdot y \cdot z)} \cdot \overline{(x \cdot y \cdot \overline{z})} \cdot \overline{(x \cdot y \cdot z)}}$$

$$f(x,y,z) = \overline{\sum \overline{M(1,4,5)}} = \overline{\overline{(x + y + \overline{z})} + \overline{(\overline{x} + y + z)} + \overline{(\overline{x} + y + \overline{z})}}$$

- Minimize this function.
- Minimize the following functions:

$$f(a,b,c) = \overline{a} \cdot b + \overline{a} \cdot \overline{c} + a \cdot c + a \cdot \overline{b} + b + c$$

$$f(a,b,c) = \overline{a} \cdot \overline{b} \cdot \overline{c} + \overline{a} \cdot b \cdot c + a \cdot b \cdot \overline{c} + a \cdot \overline{b} \cdot c$$