Exercise 7

Deadline: 12.06.2013, 16:00

1 Natural Image Statistics (7pt.)

The space of natural images is a tiny subspace of the space of all possible images. To draw samples from this space, download some photos from the flickr photostream.

For each of you images, compute the gradients in x and y direction, and plot a histogram (log-space) of the gradient values for the two directions seperately. How does the distribution look like? Compare to a Gaussian distribution. Is there a difference between $\partial/\partial x$ and $\partial/\partial y$? Explain.

In comparison, how would the histogram look for random images, where each pixel is drawn independently from the same normal distribution? Explain by a theoretical argument.

Hint: In your convolutions, use a gradient filter $[-1, 1]$.

2 Factor Graphs (10 pt.)

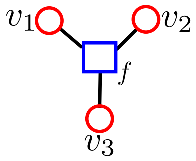
A factor graph $G(\mathcal{V}, \mathcal{F}, \mathcal{E})$ represents a probability distribution p over $|\mathcal{V}|$ random variables which factorizes in a certain way. We say that G is higher-order if the scope of any factor $f \in \mathcal{F}$ is greater than two.

1. Show why a Gaussian Markov Random Field cannot have order greater than two.

In the following, we will deal with discrete graphical models.

2. Let G be a factor graph consisting of N binary variables and a single factor $f \in \mathcal{F}$, which is connected to all N variables.
   (a) Draw the factor graph for $N = 5$.
   (b) How many entries are in the value table of f (as a function of N)?
   (c) Explain why the degrees of freedom in specifying the entries of the value table are one less than the toal number of entries.
3. Let G be a factor graph consisting of N binary variables and, for each pair of variables, a factor connecting them.
   (a) Draw the factor graph for $N = 5$.
   (b) How many factors does the graph have (as a function of N)?
   (c) How many entries do the value tables of all factors have in total (as a function of N)?
4. Let G be a factor graph representing a Markov random field consisting of N binary variables corresponding to pixels of a $\sqrt{N} \times \sqrt{N}$ image. For each pixel, there is a single-side potential and there are pair-wise potentials according to the 4-neighbourhood.
   (a) Draw the factor graph for $N = 3$.
   (b) How many factors (of each type) does the graph have (as a function of N)?
   (c) How many entries do the value tables of all factors have in total (as a function of N)?
5. Argue why, in general, there need not exist a factor graph of order 2 with N variables that represents the same probability distribution as a factor graph of order N with N variables.

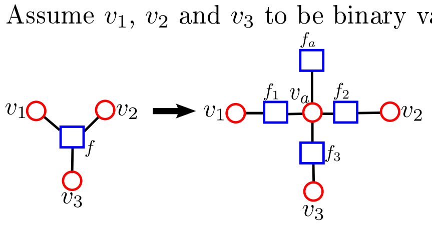
6. Let us examine the factor graph below:



The label space is $v_1 \in \mathcal{L}_1 = \{1, 2, 3\}$, $v_2 \in \mathcal{L}_2 = \{\text{Apple}, \text{Banana}\}$ and $v_3 \in \mathcal{L}_3 =$ {✓, ☺, 🍃, ✂, ⊠}. What is the domain of f? How many entries does the corresponding value table have?

In the lecture, it was briefly mentioned that it is always possible to convert higher-order factor graphs into pair-wise factor graphs by introducing additional nodes. We will look into this in more detail here.

In the paper "Belief propagation and its generalizations" (JS Yedidia, WT Freeman, Y Weiss, 2002), the authors describe how to do the transformation (see Fig. 8).
Assume v_1, v_2 and v_3 to be binary variables.



7. Give the number of entries in the value tables of f_1, f_2, f_3 and f_a.
8. Let the function represented by the factor graph be $2 \cdot (v_1 + 2 \cdot v_2 + 4 \cdot v_3) + 1$. Construct the pairwise factor graph. Explicitly write down all value tables.

3 Integer Linear Programming (3 pt.)

Let x and y be integer variables. Find

$$\begin{aligned} \max_{x,y} \{4x + 3y\} & \\ \text{subject to } y &\leq +0.05 \cdot x + 3.28 \\ y &\leq -1.71 \cdot x + 7.51 \\ y &\leq -3.83 \cdot x + 14.08 \end{aligned}$$

by constructing the constraints geometrically (use your favorite plotting program) and reading off the optimum.

We now relax the constraint $x, y \in \mathbb{N}$ to be $x, y \in \mathbb{R}^+$, which gives us a linear program. Indicate the solution of the LP in your drawing, and read off the value. What do you obtain if you round the result to the nearest integer solution?