Excercise Sheet 1

Exercise 1 (Computation of Fourier transforms)

Compute $\hat{f}_i$ for every of the following functions, which are defined on $\mathbb{R}^2$:

a) $f_1(x) = \delta\left(3x - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$,

b) $f_2(x) = \mathcal{X}_{[-1,1]\times[-1,1]}(x)$,

c) $f_3(x) = \cos(x_1 - 2) + \cos(x_2 + 2)$,

Exercise 2 (Properties of the Fourier transform)

Show:

a) For $f, g \in L^1(\mathbb{R}^d)$ it holds:

1. $\mathcal{F}E^{\omega_0} f(\omega) = T^{\omega_0}\mathcal{F}f(\omega)$,
2. $\mathcal{F}(f * g)(\omega) = (\mathcal{F}f)(\omega) \cdot (\mathcal{F}g)(\omega)$, where $f * g(x) = \int f(y)g(x-y)\,dy$.

b) It holds for $f \in L^1(\mathbb{R})$:

1. f is *even* (which means $f(x) = f(-x)$) $\Rightarrow$ $\hat{f} = \mathcal{F}f$ is real (which means $\operatorname{Im}\hat{f} = 0$),
2. f is *odd* (which means $f(x) = -f(-x)$) $\Rightarrow$ $\hat{f} = \mathcal{F}f$ is imaginary (which means $\operatorname{Re}\hat{f} = 0$).

Compute $\hat{f}_i$ for $f_1(x) = \cos(\alpha x)$ and for $f_2(x) = \sin(\alpha x)$. Use these results to compute $\hat{f}$ for real $f(x) = \sum_{i=1}^{k} \cos(\alpha_i x) + \sum_{j=1}^{l} \sin(\beta_j x)$.

c) For an invertible matrix $A \in \mathbb{R}^{d\times d}$ define $(D^A f)(x) := \frac{1}{|\det A|^{1/2}} f(A^{-1}x)$.
Show that $\mathcal{F}D^A f = D^{A^{-\top}}\mathcal{F}f$ holds.

d) It holds that $\mathcal{F}D^a f = D^{1/a}\mathcal{F}f$ where the dilation operator D^a is defined as $D^a(f)(x) = f(a\cdot x)$.

e) For a rotation matrix $R \in \mathrm{SO}(d)$ (which means $R \in \mathbb{R}^{d\times d}$ with pairwise orthogonal columns and $\det R = 1$) and $g(x) := f(Rx)$ it holds that $(\mathcal{F}g)(\omega) = (\mathcal{F}f)(R\omega)$. How then does a rotation change the Fourier transform of an image? What does this mean for rotationally invariant pictures?

f) Show that the Fourier transform of a function $f \in L^1$ is bounded and uniformly continuous.

Exercise 3 (Relationship between $\mathcal{F}$ and $\mathcal{F}^{-1}$)

a) Show that

$$(2\pi)^d\,\mathcal{F}^{-1}f \quad = \quad \overline{\mathcal{F}\overline{f}}$$

holds. $\bar{f}$ is the complex confugate of f, which means $f(x) = a(x) + ib(x) \Leftrightarrow \bar{f}(x) = a(x) - ib(x)$ for real valued a and b.

b) Assuming all following expressions to be valid, compute with help of the exercise above:

1. $\mathcal{F}^{-1}(\partial^a f)$,
2. $\mathcal{F}^{-1} T^y f$ für $y \in \mathbb{R}^d$,
3. $\mathcal{F}^{-1} D^a f$ für $a > 0$,
4. $\mathcal{F}^{-1}(f * g)$.

Exercise 4 ($\mathcal{F}$ and the central limit theorem)

a) Show that $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ is an eigenvector of the Fourier transform, i.e. $\mathcal{F}(f) = \alpha \cdot f$ for some $\alpha \in \mathbb{R}$

Hint: Write $y = \frac{1}{2}x^2 - i\omega x = \left(\frac{1}{\sqrt{2}}t - i\sqrt{2}\frac{\omega}{2}\right) + \frac{\omega^2}{2}$ in the definition of $\mathcal{F}(f)$ and then substitute for y.

b) You have learned in the lecture that for a real function g with $\int_{\mathbb{R}} g(x)\, dx = 1$ and $\int_{\mathbb{R}} x g(x)\, dx = 0$ the properly normalized sequence

$$\frac{g\left(\frac{\cdot}{\sqrt{n}}\right) * \cdots * g\left(\frac{\cdot}{\sqrt{n}}\right)}{s_n}$$

converges to f as defined above. Compute s_n as a function of $\int_{\mathbb{R}} g(x)^2\, dx$.

Hint: Use the identity $\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1) \cdot \mathcal{F}(f_1)$ and use the fact that f is an eigenvector of $\mathcal{F}$.

Exercise sheets are to be handed in Friday, April 26^{th} in the lecture.