Excercise Sheet 2

Exercise 1 (Fourier transform of radial functions ($d = 2$))

Consider the Bessel function of the first kind:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in(\tau - x\cos(x))} dx \,.$$

a) Assume a radially symmetric function $f : \mathbb{R}^2 \to \mathbb{R}$ is given, which means $f(x) = f(\|x\|)$. Express the Fourier transform $\mathcal{F}(f)$ as an integral of f and the Bessel function J.

b) Assume a radially symmetric function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ is given, which means $\hat{f}(x) = \hat{f}(\|x\|)$. Express the inverse Fourier transform $\mathcal{F}^{-1}(\hat{f})$ as an integral of f and the Bessel function J.

c) Show that the Bessel function solves the differential equation

$$x^2 \frac{d^2 y(x)}{dx^2} + x\frac{df(x)}{dx} + (x-n)^2 f(x) = 0$$

Exercise 2 (Preserving the mean and the first moment)

Assume we want to design a filter h such that for $h * f$ we preserve the mean and first moment of f, that is

$$\int (h * f)(x) dx = \int f(x) dx$$

and

$$\int x((h * f)(x)) dx = \int x f(x) dx$$

for arbitrary f.

Which equalities does $\mathcal{F}(h)$ need to obey?

Exercise 3 (Fourier transforms of translating patterns)

Asume we are given a function $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x,t) = h(x - ut),$$

where $u \in \mathbb{R}^d$. Show that $\mathcal{F}(f)$ is concentrated on the plane $\langle \omega_x, u \rangle + \omega_t = 0$, which means

$$\mathcal{F}(f)(\omega) = \hat{h}(\omega_x)\delta(\langle \omega_x, u \rangle + \omega_t) \,.$$

Remark: Note that a moving camera will produce a sequence of images which are of the form $f(x,t) = h(x-ut)$. Under some assumptions on h we can analyze its Fourier transform and determine the direction of the camera movement by the formulas above.

Exercise sheets are to be handed in Friday, May 3^{rd} in the lecture.