Image Analysis Excercise Sheet 3

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1 Convolution in 1-D

Given are an input signal g and a filter h, which will act on the signal:

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$$
$$h(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}.$$

The output signal will be

$$(g * h)(x) = \begin{pmatrix} 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}^{\mathrm{T}},$$

a smoothened, shifted and amplified version of the signal. The circulant matrix of the filter is given by

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The convolution of g and h can be computed as

$$g * h = H \cdot g.$$

Theorem. *Let g be a signal and let h be a filter with*

$$\sum_k (h * g)(k) = \sum_k g(k).$$

In the general case $\sum_k g(k) \neq 0$, we find that the DC term $\sum_k h(k) = 1$.

Proof. Define $s := (1, \dots, 1)^{\mathrm{T}}$.

$$\begin{aligned}\sum_j g(j) &= \sum_k (h * g)(j) = s^{\mathrm{T}} (h * g) \overset{\text{Convolution Theorem}}{=} \\ &= s^{\mathrm{T}} \mathcal{F}^{-1} \left(\mathcal{F} f \odot \mathcal{F} g\right) = s^{\mathrm{T}} F^{\dagger} \left(F f \odot F g\right) \frac{1}{n} = \\ &= (1, 0, 0, \dots, 0) \left(F f \odot F g\right) = \left(s^{\mathrm{T}} f \cdot s^{\mathrm{T}} g\right) = \\ &= \left(\sum_k h(k)\right) \left(\sum_j g(j)\right)\end{aligned}$$

We can divide both sides by the signal's sum and get $\sum_k h(k) = 1$. ∎

2 DFT in 1-D

Given is the transformed function

$$\mathring{g}(k) = \sqrt{n} \frac{i}{2} \left(\delta_{+2}(k) - \delta_{-2}(k)\right),$$

and we want to compute the original function by discrete Fourier transform. Let $F_{l,\cdot}$ denote the l-th row of the Fourier matrix.

$$\begin{aligned} g(l) &= \frac{1}{\sqrt{n}} F_{l,\cdot} \mathring{g} = \frac{i}{2} \sum_{k=0}^{n-1} e^{\frac{2\pi i}{n} lk} \left(\delta_{+2}(k) - \delta_{-2}(k)\right) \\ &= \frac{i}{2} \left(e^{\frac{4\pi i}{n} l} - e^{-\frac{4\pi i}{n} l}\right) = \frac{i}{2} \left(2i \sin\left(\frac{4\pi}{n} l\right)\right) = -\sin\left(\frac{4\pi}{n} l\right)\end{aligned}$$

3 DFT in 2-D

When two matrices have no spectral overlap, the convolution theorem asserts that

$$a * b = \mathcal{F}^{-1} \left(\mathcal{F}(a) \odot \mathcal{F}(b)\right) = \mathcal{F}^{-1}(0) = 0.$$

The python code is found in the appendix.

4 Correlation Theorem

Theorem. *Cross Correlation Theorem*

$$\mathcal{F}\left(a \star g\right) = \mathcal{F}(a)^* \odot \mathcal{F}(g)$$

Proof. We observe that the cross correlation operation can be stated in terms of the circulant matrix A of a as

$$a \star g = A^{\dagger} g.$$

We recall the equation

$$AW = W\mathring{A} \ \Leftrightarrow\ W^\dagger A^\dagger = \mathring{A}^\dagger W^\dagger \ \Leftrightarrow\ W^\dagger A^\dagger W = \mathring{A}^*,$$

where $\mathring{A}$ is the diagonal matrix with entries $\mathring{a} = \mathcal{F}a$ and W is the normalized Fourier matrix ($WW^\dagger = I$). We obtain

$$\begin{aligned}\mathcal{F}\left(a \star g\right) &= \mathcal{F}\left(A^\dagger g\right) = W^\dagger A^\dagger g = W^\dagger A^\dagger W W^\dagger g\\ &= \mathring{A}^* W^\dagger g = \mathcal{F}(a)^* \odot \mathcal{F}(g).\end{aligned}$$

$\square$