Indian Institute Of Technology
Kharagpur

Multiple Majorana modes

Author:
Arnab Barman Ray
13PH20011, 4th year
Integrated
Msc.(Physics), Indian
Institute Of Technology
Kharagpur

Supervisor:
Dr. Ipsita Mandal
Department Of Physics
Indian Institute Of
Technology Kharagpur

Summer Training Report

Abstract

In this report, we first review the Keldysh formalism used to describe non-equilibrium phenomena.Then, we proceed to investigate a Hamiltonian which exhibits multiple Majorana modes in topologically distinct phases and investigate its conductance tomography using a normal and superconducting STM tip. We also provide a derivation of a known analytical result for the differential conductance for a single Majorana mode in a general system.

1 The Keldysh Formalism

1.1 The Gellman-Low theorem

The Keldysh formalism, particularly the idea of contour-ordering allows us to treat non-equilbrium systems with a time-dependent Hamiltonian. The formalism is structurally similar to the Matsubara or the ground-state formalism depending on whether one chooses to retain the initial correlations before the time-dependent part is turned on.

The main difference between equilibrium and non-equilibrium case is the absence of a mathematical certainty called the Gellman-Low theorem which holds true only for the former.

Lets say we have a Hamiltonian, $H = H_0 + H'$ where H_0 is non-interacting and hence simple to solve, while H' is the hard part which is interacting in nature, both being time-independent. The strategy is to use the structure of H' and the known eigenfunctions of H_0 to arrive at the solution for the total Hamiltonian using a series of approximations.

To do this, we take the slightly modified Hamiltonian, $H_0 + e^{-\epsilon|t|}H'$ which clearly reduces to H_0 in the infinite past or future. In the interaction representation, where the second term is treated as a perturbation, the eigenfunction for a particular value of ϵ can be written as: $|\Psi_0(\epsilon)\rangle(t) = \hat{U}_\epsilon(t, -\infty)|\Phi_0\rangle$, where,

$$\hat{U}_\epsilon(t_1, t_2) = \sum_{n=0}^{\infty} (\frac{-i}{\bar{h}})^n \int_{t_1}^{t_2} \dots \int_{t_1}^{t_2} d\tau_1 \dots d\tau_n e^{-\epsilon(|\tau_1| + \dots + |\tau_n|)} T[H'(\tau_1) \dots H'(\tau_n)]$$

and $H_0|\Phi_0\rangle = E_0|\Phi_0\rangle$. $H](\tau)$ is the perturbation in the interaction picture.

The theorem says that[1]:

Gellman-Low Theorem. *If the limit,* $\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = \lim_{\epsilon\to 0} \frac{\hat{U}_\epsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|\hat{U}_\epsilon(0,-\infty)|\Phi_0\rangle}$ *exists, then it is an eigenfunction of the total Hamiltonian,* $H_0 + H'$ *with energy* $E = E_0 + \frac{\langle\Phi_0|H'|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}$.

The same result would hold if $U_\epsilon(0, +\infty)$ was used instead of $U_\epsilon(0, -\infty)$.

This essentially implies that both ways of constructing the eigenfunction for the H would give the same result **in the case of a non-degeneracy**.The concept of contour ordering, as we shall see in the next section, arises because we cannot say the same for time dependent perturbations.

1.2 Necessity of Contour ordering

In our case we deal with a Hamiltonian of the form, $H = H_0 + H'(t)$, where H_0 is non-interacting and $H'(t)$ is a time-dependent single-body potential.

The causal green's function is defined as:

$$iG^c(x, x') = \langle T[\psi_H(x)\psi_H^\dagger(x')]\rangle$$

where $x = (\vec{x}, t)$. The time ordering operator has its usual function and the average is either taken over the ground state(zero temperature formalism), or over the canonical or Grand canonical ensemble, even in the non-equilibrium case where initial correlations are dropped by assuming that the time-dependent perturbation was turned on in the infinite past.

We represent the position arguments with subscript indices for the purpose of writing the function in the form of a perturbation series.

The crucial difference in the non-equilibrium case is that unlike in the case for a time-independent perturbation, we cannot use the interchangeability of the interaction-picture time *evolutors* $\hat{U}(t, \infty)$ and $\hat{U}(t, -\infty)$.[2]

To see this, we the consider the weighted average:

$$iG^c_{i,j}(t, t') = \frac{1}{Z}\sum_n e^{-\beta E_n}\langle\Psi_{H.n}|T[\hat{U}(-\infty, t)\psi_{H_0,i}(t)\hat{U}(t, t')\psi^\dagger_{H_0,j}(t')\hat{U}(t', -\infty)]|\Psi_{H,n}\rangle$$

Since the reference time is taken to be $-\infty$, the Heisenberg states are simply the eigenstates of the original Hamiltonian H_0. This gives,

$$iG^c_{i,j}(t, t') = \frac{1}{Z}\sum_n e^{-\beta E_n}\langle\Phi_{0.n}|T[\hat{U}(-\infty, t)\psi_{H_0,i}(t)\hat{U}(t, t')\psi^\dagger_{H_0,j}(t')\hat{U}(t', -\infty)]|\Phi_{0,n}\rangle \tag{1}$$

However, since temporal symmetry is broken[2], we cannot write the expression above as a simple time ordered integral running from $-\infty$ to $+\infty$.

So, we are forced to use the idea of contour ordering as a natural means of writing a smaller expression which when broken up gives the same result.

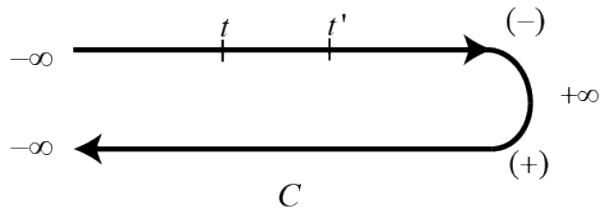


Figure 1: Contour for the causal Green's function.

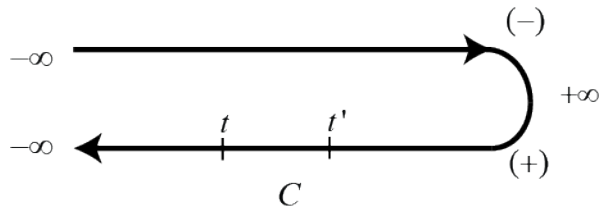


Figure 2: Contour for the anti-causal Green's function.

1.3 Non-equilibrium Green's functions

The causal green's functions defined in the above expression can be written in terms of the contour C given in Fig 1 as:

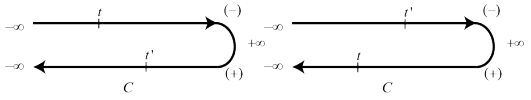
$$iG^{--}_{i,j}(t,t') = \langle T_C[e^{-\int_C H'(\tau)d\tau}\psi_{H_0,i}(t)\psi^{\dagger}_{H_0,j}(t')]\rangle$$

C has two branches, $(+)$ and $(-)$. The contour ordering operator acts in such a way so that operators at times later in the contour are to the left. In this case since both the times are on the upper branch(hence, $--$ is the superscript), the contour ordering acts on them in essentially the same way as the ordinary time-ordering operator.

A rigorous proof of this can be carried out by realizing that the contour can be broken up into 4 distinct pieces, these pieces when separated from one another to give Eqn(1).

Similarly, Fig. 2 shows the contour for the anti-causal Green's function, G^{++}.Here, since both the operators are on the lower branch the contour ordering acts on them in the same way as the anti-time ordering operator.

Fig.3 shows the respective contours for the lesser green's function, $iG^{<} \equiv iG^{-+} = \langle\psi(x)\psi^{\dagger}(x')\rangle$ and the greater green's function, $iG^{>} \equiv iG^{+-} = -\langle\psi^{\dagger}(x')\psi(x)\rangle$.The way contour ordering acts for these cases is quite intuitive.



(a) G^{-+} (b) G^{+-}

Figure 3: Contours for the lesser and greater Green's functions

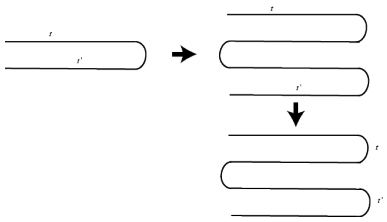


Figure 4: Contour deformation

All the four Green's functions that have been defined are usually put in the form of a matrix:

$$\tilde{G} = \begin{bmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{bmatrix}.$$

In this form, the Dyson's equation for an external classical field can be written as[3]:

$$\tilde{G} = \tilde{G}^0 + \int_{-\infty}^{+\infty} d\tau \tilde{G}^0 \tilde{V(\tau)} \tilde{G}$$

where, $\tilde{V}(\tau) = \begin{bmatrix} V(\tau) & 0 \\ 0 & -V(\tau) \end{bmatrix}$.

The advantage of this is that now, the integral runs over the real time axis.

1.4 Langreth Rules

Another way of writing the Dyson's equation involves certain relations called the Langreth rules which allows us to convert the integrals over the complex contours to integrals over real time. Say, we have a lesser function, $C^<(t,t') = \int_C d\tau A(t^-,\tau)B(\tau,t'^+)$. What we can do in this case is that we can deform the original contour in the form shown in Fig.4.

The integral can be modified accordingly to give:

$$C^<(t,t') = \int_{-\infty}^{+\infty} d\tau A^r B^< + A^< B^a$$

Similar rules exist for greater functions as well as for larger number of terms in the integral[3].

2 The system and transport properties

2.1 The Hamiltonian and related properties

In this work, we consider a mesoscopic system consisting chains of (usually InSb or InAs) nanonwires placed on top of a s-wave superconductor. The combined effect of Rashba SOI, proximity-induced superconductivity as well as in-plane Zeeman splitting gives rise to localized Majorana modes at the ends of the chain[4]. These modes, being topologically protected are a candidate for fault-tolerant quantum computation. One way to prove their existence is through measurements of the differential conductance using a STM tip.

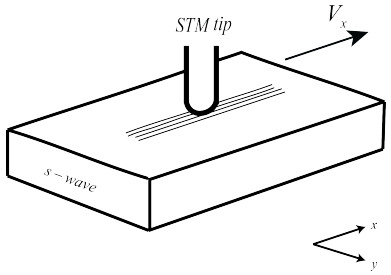


Figure 5: Nanowire system

Both numerical as well as experimental results are known for a single chain.[5, 6]. However, it seems it was too early to celebrate as it has been

shown that the experimental results could also have been caused by disorder among other reasons.

In this case, we try to find the numerical and theoretical results for a setup of 4 such chains placed close and parallel to each other so as to allow coupling between adjacent sites of neighbouring chains through a hopping term. Such systems are found to host multiple Majorana modes in topologically distinct phases. The system is treated to be quasi-one dimensional in the sense that the STM tip is coupled equally to all the neigbouring sites on different chains.

The Hamiltonian of the system is:

$$H_C = \frac{1}{2}\sum_{j=1}^{nN} \tilde{\psi}_j^\dagger \left\{-\mu\tau_z + \Delta\tau_x + V_y\sigma_x\right\}\tilde{\psi}_j +$$

$$\frac{1}{2}\sum_{k=0}^{N-1}\sum_{j=1}^{n}\left\{\tilde{\psi}_{j+kn+1}^\dagger(-t_x - i\alpha\sigma_y)\tau_z\tilde{\psi}_{j+kn} + \tilde{\psi}_{j+kn}^\dagger(-t_y)\tau_z\tilde{\psi}_{j+(k+1)n}\right\}$$

where σ_m and τ_m ($m = (x, y, z)$) are Pauli matrices which act respectively in the spin and particle-hole spaces, μ is the chemical potential, V_x the Zeeman field, α the spin-orbit coupling, Δ is the superconducting gap, and t_y is the hopping strength between sites in neighbouring chains while t_x is the hopping strength between neighbouring sites on the same chain.The index j is a label that identifies the sites and can have values from 1 to nN, where n is the number of sites in a single chain and N is the total number of chains. The operator $\tilde{\psi}_j^\dagger = (\psi_{j,\uparrow}^\dagger, \psi_{j,\downarrow}^\dagger, \psi_{j,\uparrow}, -\psi_{j,\downarrow})$.

The Hamiltonian of the superconducting STM tip is $H_S = \sum_{k,\sigma}\epsilon_{k,\sigma}\tilde{\Psi}_{k,\sigma}^\dagger\tilde{\Psi}_{k,\sigma} + \sum_k(\Delta_{s\tilde{\Psi}_{k,\uparrow}\tilde{\Psi}_{-k,\downarrow}})$ where the symbols have their usual meanings[5]. The normal tip is obtained by setting $\Delta_s = 0$.

A diagonalization of the Hamiltonian reveals the spectrum of the system in Fig. 2. There are 5 distinct phases. As we go along increasing t_y, it causes 7 phase transitions, always accompanied by a closing of the gap, except once near $t_y \approx 3.55$, where a transition happens without a gap closure. As we increase t_y, the number of Majorana modes progresses in the sequence: $4 \to 2 \to 3 \to 1 \to 0 \to 1 \to 0$. Due to closely lying *ring states*, these too contribute to the current even at very low temperatures as we shall see later on.

Calculating the LDOS can reveal that these zero modes are indeed Majorana modes localized at the ends.

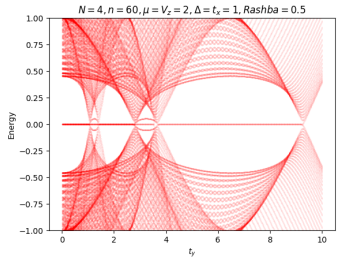


Figure 6: Spectrum for $\mu = V_x = 2, \Delta = t_x = 1, \alpha = 0.5$

The phase diagram of the system in $\mu - t_y$ space is given below:

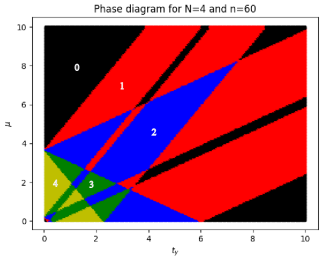


Figure 7: Color-coded Phase diagram for $V_x = 2, \Delta = t_x = 1, \alpha = 0.5$

2.2 Transport properties

We consider the tunneling Hamiltonian when the tip is at site i along the x-axis,

$$H_{T,i} = \sum_{k,j} \tilde{\psi}^{\dagger}_{jn+i} \tilde{\gamma} \tilde{\Psi}_k + h.c \tag{2}$$

where $\tilde{\gamma} = \text{diag}(\gamma, \gamma, -\gamma, -\gamma)$. We assume the tip to be coupled equally along adjacent sites on neighbouring sites.

The total Hamiltonian is now given by $H = H_C + H_S + H_{T,i}$. Under the application of a potential V the Hamiltonian of the tip acquires a time dependence occurring alongside its superconducting parameter, $H_S(t) = \sum_{k,\sigma}(\epsilon_{k,\sigma} + \text{e}V)\tilde{\Psi}^{\dagger}_{k,\sigma}\tilde{\Psi}_{k,\sigma} + \sum_k \Delta_s e^{-i\phi - \frac{2i}{\hbar}Vt}\tilde{\Psi}_{k,\uparrow}\tilde{\Psi}_{-k,\downarrow}$

To transfer the time-dependence to the tunneling Hamiltonian we perform

a Unitary transformation[7] with the matrix, $U = \exp\{-\frac{i}{h}\sum_{k,\sigma}[\phi/2 + \frac{Vt}{h}]\Psi^{\dagger}_{k,\sigma}\Psi_{k,\sigma}\}$ to give the Hamiltonian, $H' = UHU^{\dagger} - i\dot{U}U^{\dagger}$.

The new Hamiltonian is:

$$H' = H_C + H_S + \sum_{k,j,\sigma} \gamma e^{\frac{iVt}{h}} \Psi^{\dagger}_{k,\sigma}\psi_{i+nj,\sigma} + h.c$$

From now on, we move to the natural for simplicity.(so, e $= \bar{h} = 1$.) In the Heisenberg picture, The current flowing from the tip to the wire is given by:

$$I_i(t) = i\langle[N_s, H']\rangle$$

where $N_S = \sum_{k,\sigma} \Psi^{\dagger}_{k,\sigma}(t)\Psi_{k,\sigma}(t)$ is the number operator for the tip.

Performing the commutation reveals the expression:

$$I_i(t) = i\sum_{k,\sigma,j}[\gamma e^{iVt}\langle\Psi^{\dagger}_{k,\sigma}(t)\psi_{i+nj,\sigma}(t)\rangle - \gamma^* e^{-iVt}\langle\psi^{\dagger}_{i+nj,\sigma}(t)\Psi_{k,\sigma}(t)\rangle]$$

.

We express this relation in terms of the lesser green's functions:

$$I_i(t) = \sum_{k,\sigma,j}[\gamma e^{iVt}G^{<}_{i+nj,\sigma;k,\sigma}(t,t) - \gamma^* e^{-iVt}G^{<}_{k,\sigma;i+nj,\sigma}(t,t)] \tag{3}$$

.

The system of the tip and the central region are at equilibrium in the infinite past. Using the perturbation expansion technique outlined in [8], we have the contour integral expression:

$$G^{<}_{i+jn,\sigma;k,\sigma}(t,t') = \sum_l \int_C d\tau\{G_{i+nj,\sigma;i+nl,\sigma}(t,\tau)\gamma^* e^{-iV\tau} g_{k\sigma}(\tau,t')\}$$

Similarly,

$$G^{<}_{k,\sigma;i+jn,\sigma}(t,t') = \sum_l \int_C d\tau\{\gamma e^{iV\tau} g_{k\sigma}(t,\tau)G_{i+nj,\sigma;i+nl,\sigma}(\tau,t')\}$$

Using Langreth rules, the contour integrals are rewritten on the real time axis as:

$$G^{<}_{i+jn,\sigma;k,\sigma}(t,t') = \sum_l \int_{-\infty}^{+\infty} d\tau \{ G^{<}_{i+nj,\sigma;i+nl,\sigma}(t,\tau)\gamma^* e^{-iV\tau} g^a_{k\sigma}(\tau,t') + G^r_{i+nj,\sigma;i+nl,\sigma}(t,\tau)\gamma^* e^{-iV\tau} g^{<}_{k\sigma}(\tau,t') \}$$

and

$$G^{<}_{k,\sigma;i+jn,\sigma}(t,t') = \sum_l \int_{-\infty}^{+\infty} d\tau \{ \gamma e^{iV\tau} g^{<}_{k\sigma}(t,\tau) G^a_{i+nj,\sigma;i+nl,\sigma}(\tau,t') + \gamma e^{iV\tau} g^r_{k\sigma}(t',\tau) G^{<}_{i+nj,\sigma;i+nl,\sigma}(\tau,t) \} \quad (4)$$

g^a, $g^<$ and g^r are the non-interacting equilibrium advanced, lesser and retarded Green's functions for the STM tip respectively and they obey time translation invariance meaning that $g(t_1,t_2) \equiv g(t_1 - t_1)$

The total current $I(t)$ can be expanded in a Fourier series with a period of $\frac{2\pi}{V}$,

$$I_i(t) = \sum_m I_m e^{imVt}$$

.

In finding the DC current, we follow a similar approach as in [11]. Since time-translation symmetry no longer holds for non-equilibrium systems, the interacting Green's functions used in (4) can only be written as a generalized Fourier series:

$$G(t_1,t_2) = \frac{1}{2\pi} \sum_m e^{imVt_2} \int d\omega \{ e^{-i\omega(t_1-t_2)} G(\omega, \omega + mV) \} \quad (5)$$

. A new quantity is defined, $G_{m,n}(\omega) = G(\omega + mV, \omega + nV)$, which obeys the rule, $G_{m,n} = G_{m-n,0}$ when the variable ω in the integration is rolled over all real values in the range $(-\infty, +\infty)$.

For the DC current, we consider only the zeroth order component of the total expression (3). We do this by noting that the couplings of the forms $e^{\pm iVt}$ need to be negated by the terms in Fourier expansion (5) to give a time-dependent component. Doing so and using the relations $G^< - G^> = G^a - G^r$, we find,

$$I^{dc}_i = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{l,j,\sigma} \{ G^{>}_{i+jn,\sigma;i+ln,\sigma} \Gamma^{<}_{i+ln,\sigma;i+jn,\sigma}(\omega - V) - G^{>}_{i+jn,\sigma;i+ln,\sigma} \Gamma^{<}_{ln+i,\sigma;jn+i,\sigma}(\omega + V) - G^{<}_{jn+i,\sigma;ln+i\sigma} \Gamma^{>}_{ln+i,\sigma;jn+i,\sigma}(\omega - V) + G^{<}_{jn+i,\sigma;ln+i,\sigma} \Gamma^{>}_{ln+i,\sigma;jn+i\sigma}(\omega + V) \} \quad (6)$$

where $\Gamma^{a/<}_{i+ln,\sigma;i+jn,\sigma}(\omega \pm V) \equiv \Gamma^{a/<}(\omega \pm V) = |\gamma|^2 \sum_k g^{a/<}_{k,\sigma}(\omega \pm V)$. since the non-interacting Green's functions of the tip are not spin-dependent, we can loose those cumbersome indices.

The lesser function of the superconducting tip gives, $\sum_k g^{<}_k(\omega) = \frac{-2i\pi\nu_0|\omega|}{\sqrt{\omega^2-\Delta_s^2}} n_F \theta(|\omega|-\Delta_s)$, $n_F = \frac{1}{e^{\beta\omega}+1}$ is the Fermi function, while the greater function is $\sum_k g^{>}_k(\omega) = \frac{2i\pi\nu_0|\omega|}{\sqrt{\omega^2-\Delta_s^2}}(1-n_F)\theta(|\omega|-\Delta_s)$. In the extended Nambu basis, $(\psi^\dagger_{i,\uparrow}, \psi^\dagger_{i,\downarrow}, \psi_{i,\downarrow}, -\psi_{i,\uparrow})$, the total current can be written neatly in the matrix form with the physical dimensions inserted as:

$$I^{dc}_i = \frac{e}{2h} \int_{-\infty}^{+\infty} d\omega \{ Tr[\tau_z [G^{>}\Sigma^{<} - G^{<}\Sigma^{>}]] \} \tag{7}$$

.

The self-energy matrices Γ are zero everywhere except at 4×4 blocks between site-indices which are coupled to the tip, in the form:

$$\Sigma^{>/<}_{i+ln,i+jn} \equiv \begin{bmatrix} \Gamma^{>/<}(\omega - V) & 0 & 0 & 0 \\ 0 & \Gamma^{>/<}(\omega - V) & 0 & 0 \\ 0 & 0 & \Gamma^{>/<}(\omega + V) & 0 \\ 0 & 0 & 0 & \Gamma^{>/<}(\omega + V) \end{bmatrix}$$

Clearly, the self-energy is zero in all the other site-indices which are not coupled to the tip. Similarly, τ_z acts in the manner of the 4×4 inter-site block matrix, $diag(1, 1, -1, -1)$ on all the pairs of sites coupled to the tip, and is zero otherwise.

The Dyson equations give, $G^{</>} = G^R \Sigma^{</>} G^A$ when we neglect the single particle current[**?**] and $G^R = (\omega - H_C - \Sigma^R)^{-1}$.[9, 5]. These, together with the relation, $G^R = (G^A)^\dagger$ allow us a recipe for calculating the DC response from the Nambu matrix structure of H_C.

The case for the normal tip can be easily found by setting $\Delta_s = 0$.

We hope to present the numerical results at a future time.

3 An analytical result for a general Hamiltonian

3.1 Single wire, single point of contact current

In this section, we merely present the derivation for an analytical expression of the differential conductance for a general Hamiltonian as shown in [11] for the case of a **single wire** with a **single site** coupled to the tip. In the paper, the total Hamiltonian is divided into three parts (as before), a general Hamiltonian which exhibits the Majorana modes: H_R, the Hamiltonian for the superconducting tip, $H_L = \int \frac{d\mathbf{k}}{(2\pi)^3}[\sum_\sigma \epsilon_{k,\sigma}\tilde{\Psi}^\dagger_{k,\sigma}\tilde{\Psi}_{k,\sigma} + (\Delta_s \tilde{\Psi}_{k,\uparrow}\tilde{\Psi}_{-k,\downarrow} + h.c.)]$, and the tunneling, $H_{T,i} = \sum_k \tilde{\psi}^\dagger_i \tilde{\gamma}\tilde{\Psi}_k + h.c.$

The current is expressed in the extended 4×4 Nambu form:

$$I(\tau) = \frac{\mathrm{e}}{2\bar{h}} Tr\{\tau_z[\hat{t}(\tau)G^<_{RL}(\tau,\tau) - G^<_{LR}(\tau,\tau)\hat{t}^*(\tau)]\}$$

The factor of $\frac{1}{2}$ is used here as before because in this case the Nambu Green's function is used, which grants a 4×4 matrix structure and we have to account for double contributions from the diagonal elements since the operators involved in the average anti-commute at equal times.

$$G^<_{\alpha,\beta}(t,t') = -i\theta(t,t') \begin{pmatrix} \langle\Psi_{\uparrow,\beta}(t),\Psi^\dagger_{\uparrow,\alpha}(t')\rangle & \langle\Psi_{\uparrow,\beta}(t),\Psi^\dagger_{\downarrow,\alpha}(t')\rangle & \langle\Psi_{\uparrow,\beta}(t),\Psi_{\downarrow,\alpha}(t')\rangle & -\langle\Psi_{\uparrow,\beta}(t),\Psi_{\uparrow,\alpha}(t')\rangle \\ \langle\Psi_{\downarrow,\beta}(t),\Psi^\dagger_{\uparrow,\alpha}(t')\rangle & \langle\Psi_{\downarrow,\beta}(t),\Psi^\dagger_{\downarrow,\alpha}(t')\rangle & \langle\Psi_{\downarrow,\beta}(t),\Psi_{\downarrow,\alpha}(t')\rangle & -\langle\Psi_{\downarrow,\beta}(t),\Psi_{\uparrow,\alpha}(t')\rangle \\ \langle\Psi^\dagger_{\downarrow,\beta}(t),\Psi^\dagger_{\uparrow,\alpha}(t')\rangle & \langle\Psi^\dagger_{\downarrow,\beta}(t),\Psi^\dagger_{\downarrow,\alpha}(t')\rangle & \langle\Psi^\dagger_{\downarrow,\beta}(t),\Psi_{\downarrow,\alpha}(t')\rangle & -\langle\Psi^\dagger_{\downarrow,\beta}(t),\Psi_{\uparrow,\alpha}(t')\rangle \\ -\langle\Psi^\dagger_{\uparrow,\beta}(t),\Psi^\dagger_{\uparrow,\alpha}(t')\rangle & -\langle\Psi^\dagger_{\uparrow,\beta}(t),\Psi^\dagger_{\downarrow,\alpha}(t')\rangle & -\langle\Psi^\dagger_{\uparrow,\beta}(t),\Psi_{\downarrow,\alpha}(t')\rangle & \langle\Psi^\dagger_{\uparrow,\beta}(t),\Psi_{\uparrow}(t')\rangle \end{pmatrix}$$
$$\equiv \begin{pmatrix} G^{<,ee}_{\alpha,\beta} & G^{<,eh}_{\alpha,\beta} \\ G^{<,he}_{\alpha,\beta} & G^{<,hh}_{\alpha,\beta} \end{pmatrix} \quad (8)$$

where α/β is equal to R or L depending on which of the regions the annihilation/creation operators belong to in the intended Green's function and $\hat{t}(\tau) = diag(te^{iVt}, te^{iVt}, -t^*e^{-iVt}, -t^*e^{-iVt})$.

The current obtained is similar to (6):

$$I = -\frac{\mathrm{e}t^2}{2h}\int d\omega\{Tr[G^{<,ee}_R g^>_L(\omega - \mathrm{e}V) - G^{<,hh}_R g^>_L(\omega+\mathrm{e}V) - G^{>,ee}_R g^<_L(\omega-\mathrm{e}V) + G^{>,hh}_R g^<_L(\omega+\mathrm{e}V)]\}$$

where the trace is being taken only over the spin.

We use the same Langreth rules as used before to write the lesser and greater functions in terms of the retarded and advanced functions using the self-energy $\Sigma^{</>}(\omega) = diag(g_L^{</>}(\omega - \mathrm{e}V), g_L^{</>}(\omega - \mathrm{e}V), g_L^{</>}(\omega - \mathrm{e}V), g_L^{</>}(\omega - \mathrm{e}V))$. to give, $G^{</>} = G^r\Sigma^{</>}G^a$, again this is true because the single-particle contribution to the current is ignored. This approximation is valid because in the superconducting case the conductance is analyzed in the regime $\mathrm{e}V \approx \Delta_s$ and at this scale of energy the processes required for single-particle currents are greatly subdued. From now on, following suit of the original paper, the short form is adopted: $\omega \pm eV \equiv \omega_\pm$.

Exploiting the matrix structure of the Nambu Green's function, it can be written that:

$$\begin{aligned}
&\begin{pmatrix} G^{<,ee} & G^{<,eh} \\ G^{<,he} & G^{<.hh} \end{pmatrix} = t^2 \begin{pmatrix} G^{r,ee} & G^{r,eh} \\ G^{r,he} & G^{r,hh} \end{pmatrix} \times \begin{pmatrix} g^<(\omega_-)G^{a,ee} & g^<(\omega_-)G^{a,eh} \\ g^<(\omega_+)G^{a,he} & g^<(\omega_+)G^{a,hh} \end{pmatrix} \\
&= t^2 \begin{pmatrix} G^{a,ee}g^<(\omega_-)G^{a,ee} + G^{a,eh}g^<(\omega_+)G^{a,he} & G^{a,ee}g^<(\omega_-)G^{a,eh} + G^{a,eh}g^<(\omega_+)G^{a,hh} \\ G^{a,he}g^<(\omega_-)G^{a,ee} + G^{a,hh}g^<(\omega_+)G^{a,he} & G^{a,he}g^<(\omega_-)G^{a,eh} + G^{a,hh}g^<(\omega_+)G^{a,hh} \end{pmatrix} \quad (9)
\end{aligned}$$

Putting the respective results in the expression for the current and canceling out some terms and dropping the R and L labels, we get:

$$\begin{aligned}
I(V) = -\frac{\mathrm{e}t^4}{2h} \int d\omega Tr\{&G^{r,eh}g^<(\omega_+)G^{a,he}g^>(\omega_-) - G^{r,he}g^<(\omega_-)G^{a,eh}g^>(\omega_+) - \\
&G^{r,eh}g^>(\omega_+)G^{a,he}g^<(\omega_-) + G^{r,he}g^>(\omega_-)G^{a,eh}g^<(\omega_+)\} \quad (10)
\end{aligned}$$

.

Now, noting that $G^{a,he} = (G^{r,eh})^*$, the current can be written as:

$$\begin{aligned}
I(V) &= -\frac{\mathrm{e}t^4}{2h} \int d\omega F[G^{r,eh}] g^<(\omega_+)g^>(\omega_-) - g^<(\omega_-)g^>(\omega_+) - g^>(\omega_+)g^<(\omega_-) + g^<(\omega_-)g^>(\omega_+) \\
&= \frac{et^4}{h} \int d\omega F[G^{r,eh}]\{g^<(\omega_+)g^>(\omega_-) - g^<(\omega_-)g^>(\omega_+)\}
\end{aligned}$$

where $F[G^{r,rh}]$ is the Frobenius norm, i.e the sum of the magnitudes of all the elements in the $G^{r,eh}$ block.

Now, using the previous values for the tip functions, $g^<(\omega) = \frac{-2i\pi\nu_0|\omega|}{\sqrt{\omega^2 - \Delta_s^2}} n_F \theta(|\omega| - \Delta_s) = -2\pi i\rho(\omega)n_F$ and $g^>(\omega) = \frac{2i\pi\nu_0|\omega|}{\sqrt{\omega^2 - \Delta_s^2}}(1 - n_F)\theta(|\omega| - \Delta_s) = 2i\pi\rho(\omega)(1 - n_F)$, we have the final expression:

$$I(V) = \frac{4\pi^2\mathrm{e}t^4}{2h} \int d\omega F[G^{r,eh}]\{\rho(\omega_+)\rho(\omega_-)(n_F(\omega_-) - n_F(\omega_+))\} \quad (11)$$

3.2 Contribution from the Majorana modes

The general form of the Majorana wavefunction in the system is taken as: $\Phi_0(x) = (u_\uparrow(x), u_\downarrow(x), u_\downarrow(x)^*, -u_\uparrow(x)^*)^T$. We can see that under a particle-hole transformation which in this case would mean a simple conjugation would leave the wavefunction unchanged.

To calculate the contribution from the Majorana modes, the retarded Green's function is approximated to contain only the zero energy states while neglecting the contribution from states that lie far above the zero point. So, the retarded function looks like:

$$G^r_{eh}(\omega) = \frac{1}{\omega - \Sigma_m}\Phi_0\Phi_0^\dagger$$

.

Σ_m is the original self-energy *projected* into the Majorana wavefunction which is simply the scalar product, $\Phi_0^\dagger\Sigma(\omega)\Phi_0$. So, the block structure is:

$$G^r_{eh}(\omega) = \frac{1}{\omega + i\pi t^2|\epsilon|^2\{\rho(\omega_-) + \rho(\omega_+)\}} \begin{pmatrix} u_\uparrow u_\downarrow^* & -u_\uparrow^2 \\ u_\downarrow^{2*} & -u_\uparrow u_\downarrow^* \end{pmatrix}$$

where $|\epsilon|^2 = |u_\uparrow|^2 + |u_\downarrow|^2$. We use the Frobenius norm obtained from the matrix above in (11) to give:

$$I_m(V) \;=\; \frac{e}{h}\int d\omega \frac{4\pi^2 t^4|\epsilon|^4\rho(\omega_-)\rho(\omega_+)}{\omega^2 + 2\pi t^2|\epsilon|^2(\rho(\omega_+ + \omega_-))/4} \;\times\; [n_F(\omega_-) \;-\; n_F(\omega_+)]$$

The theta functions in the ρ functions restrict the integration intervals to $\omega \in (\Delta_s - \mathrm{e}V, -\Delta_s + \mathrm{e}V)$.

The variable η is defined as : $\eta = \Delta_s - \mathrm{e}V \ll \Delta_s$. Using this, the symmetry, $I(-V) = -I(V)$, and the condition at low temperatures that $n_F(\omega - \mathrm{e}V) - n_F(\omega + \mathrm{e}V) \approx 1$, the current is expressed as an integral over the positive part of the axis as:

$$I_m(V) = \frac{2e}{h}\int_0^\eta d\omega \frac{4\pi^2 t^4|\epsilon|^4\rho(\omega_-)\rho(\omega_+)}{\omega^2 + \pi^2 t^4|\epsilon|^4(\rho(\omega_+) + \rho(\omega_-))}$$

The density of quasi-particles is approximated to the first order : $\rho(\omega_\pm) = \frac{\nu_0|\omega \pm \mathrm{e}V|}{\sqrt{((\omega \pm \mathrm{e}V)^2 - \Delta_s^2}} \approx \nu_0\sqrt{\frac{\Delta_s}{2}}\frac{1}{\sqrt{\eta \pm \omega}}$. Using this the final result is:

$$I_m(V) = \frac{2\mathrm{e}\eta}{h}(4-\pi) \times \frac{4}{4-\pi}\int_0^1 dz \frac{1}{\sqrt{1-z^2}} \frac{1}{z^2x^3 + (\frac{1}{\sqrt{1+z}} + \frac{1}{\sqrt{1-z}})^2} \qquad (12)$$

where $x = \frac{\eta}{\omega_t}$ and $\omega_t = (\pi t^2 \nu_0 |\epsilon|^2 \sqrt{\Delta_s/2})^{2/3}$.

When $\eta = 0 \implies x = 0$, i.e when the voltage applied approaches the value of the superconducting order parameter, the differential conductance $\frac{dI}{dV}$ approaches the value:

$$\frac{dI}{dV} = G_m = \frac{2\mathrm{e}^2}{h}(4 - \pi)$$

.

References

[1] A.L.Fetter and J.D. Walecka. *Quantum Theory Of Many Particle Systems*. Dover Publications Inc., Mineola, New York, 2003.

[2] A.M.Rodero *Dynamics of Interacting Electrons in Quantum Wires, "La Cristalera", Miraflores de la Sierra, Spain.*

[3] Jørgen Rammer *Quantum Field Theory of Non-equilibrium States.*

[4] Tudor D. Stanescu, Roman M. Lutchyn, and S. Das Sarma *Majorana Fermions in Semiconductor Nanowires, arXiv:1302.5433.*

[5] Denis Chevallier and Jelena Klinovaja *Physical Review B 94, 035417 (2016) Tomography of Majorana fermions with STM tips.*

[6] S. M. Albrecht and A. P. Higginbotham *Exponential protection of zero modes in Majorana islands, Nature 531, 206–209, 10 March 2016 doi:10.1038/nature17162.*

[7] Wang Huaiyu *Green's Functions in Condensed Matter Physics, Science Press and Alpha Science Iternational Ltd., 2012.*

[8] Hartmut Haug, Antti-Pekka Jauho *Quantum Kinetics in Transport and Optics of Semiconductors, Springer Series in solid-state sciences,Second edition, 2008*

[9] Liliana Arrachea1 and Michael Moskalets *Relation between scattering-matrix and Keldysh formalisms for quantum transport driven by time-periodic fields, Physical Review B 74, 245322, 2006*

[10] Pablo San-Jose, Jorge Cayao, Elsa Prada, Ram´on Aguado *Multiple Andreev reflection and critical current in topological superconducting nanowire junctions, arXiv:1301.4408*

[11] Michael Ruby, Falko Pientka, Yang Peng, Felix von Oppen, Benjamin W. Heinrich, and Katharina J. Franke *Tunneling Processes into Localized Subgap States in Superconductors Phys. Rev. Lett. 115, 087001*

[12] Yang Peng, Falko Pientka, Yuval Vinkler-Aviv, Leonid I. Glazman, Felix von Oppen *Robust Majorana conductance peaks for a superconducting lead, arXiv:1506.06763*